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A DYNAMIC FACTOR MODEL FOR THE ANALYSIS OF MULTIVARIATE TIME SERIES

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As a method to ascertain the structure of intra-individual variation, *P*-technique has met difficulties in the handling of a lagged covariance structure. A new statistical technique, coined dynamic factor analysis, is proposed, which accounts for the entire lagged covariance function of an arbitrary second order stationary time series. Moreover, dynamic factor analysis is shown to be applicable to a relatively short stretch of observations and therefore is considered worthwhile for psychological research. At several places the argumentation is clarified through the use of examples.

The statistical analysis of time series has many ramifications, only some of which have so far become fashionable in psychology. In this article, attention will be drawn to a branch of time series analysis pertaining to dynamic factor modelling of the lagged covariance structure of a vector-valued time series. In order to simplify any introduction to the dynamic factor model, it may be useful to first take a look at a rather well-known precursor. Several decades ago, Cattell (1952) suggested the analysis of an observed trajectory of a vector-valued time series, i.e., repeated measurements on a single subject across many occasions, by means of the usual factor model. Cattell proposed the special application of traditional factor analysis, coined P-technique, as a way of determining whether the covariance structure of the observed time series can be conceived of as being caused by an inferred latent factor series of reduced dimension. Although similar proposals had been made earlier (e.g., Stone, 1947), factor analysis of multivariate time series found general acceptance within psychology under the heading of *P*-technique, and has become a respectable method used to ascertain the structure of intra-individual variation. For all that, *P*-technique has been criticized on several grounds (cf. Holtzman, 1962; Anderson, 1963). The main theme underlying this criticism concerned the proper way in which a lagged covariance structure should be handled. In its original conception, *P*-technique takes into account only simultaneous relations between the components of a multivariate time series. Thus, the timing of the components is lost, i.e., the relations between component series at different times are absent from the analysis. It has been suggested (Cattell, 1957), that for each pair of component series one considers the covariances with different lags and select the numerically largest one. The difficulties with respect to the interpretation of results thus obtained have been illustrated by Anderson (1963). Subsequently, Cattell (1963) proposed his so-called method of iteration of factor-variable displacement, which involves an iterative search for the lag at which each component of a multivariate time series is maximally correlated with a given factor. This method still is too restrictive, because for each component series it yields a relationship with a given factor at one interval of time, while the relationship may exist over several units of time. For instance,

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an effect on a component series that involves the physiology of an individual may take several days to wear itself out (cf. Anderson).

What is needed is a generalization of P-technique in order to accommodate the lagged covariance structure of a multivariate time series. The primary purpose of this article is to outline a dynamic factor analysis that yields the required generalization. It is based upon a model in which the latent factors are conceived of as random time-dependent functions, called factor series. At each time t, then, the communal part of an observed series is represented by a weighted sum of the latent factor series at $t, t-1, \ldots$ In reverse, the effect of a realization of the latent factor series at time t will take several consecutive times $t, t+1, \ldots$ to wear itself out. Particular versions of this model have been discussed by, e.g., Priestley, Subba Rao, and Tong (1973) and Brillinger (1975). In this article, we will describe a time domain dynamic factor analysis that is applicable to a single, relatively short trajectory of a multivariate time series. In addition, a point for point description is given of some special properties of the proposed analysis, including the estimation of a latent factor series. In the closing section, some of the related approaches alluded to earlier will be discussed.

The Dynamic Factor Model

Definition of the Lagged Covariance Structure

A time series or random function $z(t)$, $t = 0, \pm 1, \ldots$, can be conceived of as an ensemble of trajectories which are generated by some random scheme. We can denote this ensemble by $\{z(t, \omega), \omega \in \Omega \text{ and } t = 0, \pm 1, \ldots\}$, where ω denotes a random variable taking values in Ω. Accordingly, the function $z(t, \omega)$ with ω fixed is a trajectory or realization of the time series. Henceforth, a finite trajectory will be denoted by $z(t) = z_t$, $t = 1, 2, \ldots, n$. The finite dimensional distributions which characterize a p vector-valued time series $z(t)$ are defined by:

$$F_{a_1 \cdots a_k}(Z_1, \ldots, Z_k; t_1, \ldots, t_k) = \text{Prob}\,[z_{a_1}(t_1) \leq Z_1, \ldots, z_{a_k}(t_k) \leq Z_k],$$
$$a_1, \ldots, a_k \in \{1, 2, \ldots, p\} \quad \text{and} \quad k = 1, 2, \ldots$$

Accordingly, the mean function and covariance function are defined by, respectively:

$$c_a(t) = \int Z \, dF_a(Z; t), \quad \text{and}$$

$$c_{a_1 a_2}(t_1, t_2) = \iint [Z_1 - c_{a_1}(t_1)][Z_2 - c_{a_2}(t_2)] \, dF_{a_1 a_2}(Z_1, Z_2; t_1, t_2).$$

A time series $z(t)$ is called second order stationary if:

$$c_a(t) = c, \qquad c_{a_1 a_2}(t_1, t_2) = c_{a_1 a_2}(0, t_2 - t_1) = c_{a_1 a_2}(u).$$

This assumption implies that $z(t)$ contains no deterministic trend and that its covariance function is invariant under a translation along the time axis. Hence, the covariance function of a second order stationary time series can be estimated from a single realization by:

$$\hat{C}(u) = \frac{1}{n} \sum_{t=u+1}^{n} [z(t) - \hat{c}][z(t-u) - \hat{c}]^T,$$

where

$$C(u) = \{c_{a_1 a_2}(u)\} \qquad \hat{c} = \frac{1}{n} \sum_{t=1}^{n} z(t),$$

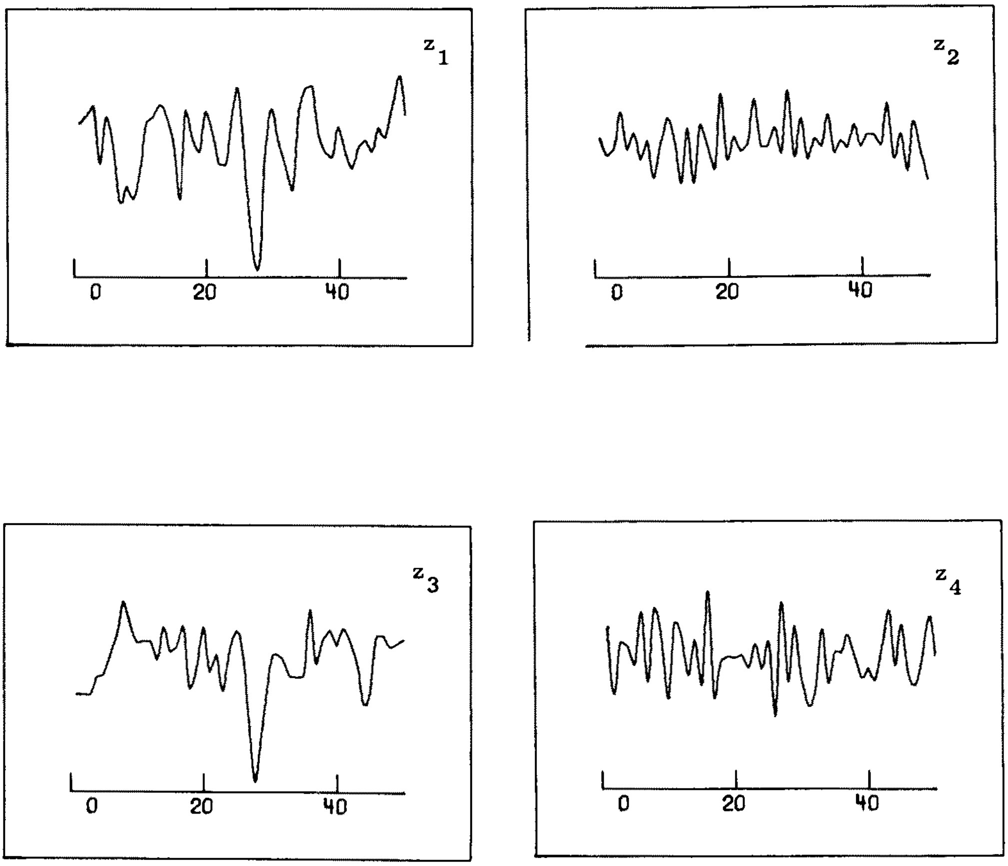


FIGURE 1
Time course of a 4 vector-valued trajectory.

and n is the length of the observed trajectory. Although division by $n - u$ rather than n might seem more rational because it yields an unbiased estimator for $C(u)$, division by n is preferable because it yields an estimate which often has smaller mean square error (Jenkins & Watts, 1968).

Consider, for the purpose of clarification, the 4 vector-valued trajectory $z(t) = z_t$, $t = 1, 2, \ldots, 50$, which is depicted in Figure 1. Some such trajectory might consist of the time course of repeated measurements of various indicatory functions on a deviant subject. A classic paper by Holtzman (1963) discusses a case like this, involving daily measurements with a single schizophrenic patient. The trajectory in Figure 1 is regarded as a realization of a second order stationary random function $z(t)$, $t = 0, \pm 1, \ldots$ The first few estimated coefficient matrices of the corresponding covariance function are given in Table 1. The assumed stationarity, both of the mean function and of the covariance function, can be tested against suitable alternatives involving models with time-varying coefficients (Kashyap & Rao, 1976). Consequently, a situation may be found in which the condition of constant mean function is violated. As to that, Cattell (1963) has drawn attention to the technical difficulties which arise in the handling of trends. Nevertheless, the required stationarity of the mean function can be dismissed without invalidating a dynamic factor analysis in case the time series has a bounded time-varying trend

Table 1

Estimated Covariance Function

of Figure 1

$\hat{\underline{C}}(0)$				$\hat{\underline{C}}(1)$			
2.002				0.889	-0.386	0.878	-0.153
-0.057	1.660			0.440	-0.742	0.045	-0.330
0.965	-0.425	3.756		-0.046	-0.881	1.935	0.002
-0.251	0.504	-0.214	3.031	0.076	-0.582	0.172	-1.271
$\hat{\underline{C}}(2)$				$\hat{\underline{C}}(3)$			
-0.208	-0.258	0.459	-0.024	-0.503	-0.204	-0.235	0.676
-0.174	0.329	-0.266	0.034	-0.273	-0.148	0.031	-0.421
-0.666	-0.064	0.472	0.517	-0.024	-0.013	-0.153	0.827
0.210	0.070	0.001	0.427	-0.168	0.097	0.679	-0.207

(Molenaar, 1984). To approach this extension in further detail would, however, involve a separate study.

Definition of the Dynamic Factor Model

Suppose, that the trajectory plotted in Figure 1 has been obtained from repeated measurements with a deviant subject and the question is asked, whether a common unidimensional aetiological process accounts for the covariance function of the observations. An application of P-technique to this case is based on:

$$z(t) = \Lambda\eta(t) + \varepsilon(t), \qquad t = 0, \pm 1, \ldots \tag{1}$$

where Λ is a 4-dimensional vector of loadings, $\eta(t)$ is a univariate latent factor series which represents the aetiological process, and $\varepsilon(t)$ is a 4 vector-valued residual series. Both $\eta(t)$ and $\varepsilon(t)$ are conceived of as random functions, and for $u = 0, \pm 1, \ldots$ the respective covariance functions are defined by:

$$\text{cov}\ [\eta(t), \eta(t-u)] = \Xi(u),$$
$$\text{cov}\ [\varepsilon(t), \varepsilon(t-u)] = \Theta(u) = \text{diag}\ [\theta_1(u), \ldots, \theta_4(u)].$$

An application of P-technique, then, involves the fit of Equation 1 to the *zero*-lagged coefficient matrix of the covariance function of $z(t)$:

$$C(0) = \Lambda\Xi(0)\Lambda^T + \Theta(0). \tag{2}$$

Hence, for $u \neq 0$ the covariance function $C(u)$ is left unaccounted for. This restriction is of course only justified if all coefficient matrices of the covariance function of $z(t)$ vanish at non-zero lags, i.e., $C(u) = \delta(u)C$, where $\delta(u)$ is the Kronecker delta. As to the present example, however, Table 1 indicates that $C(u) \neq 0$ if $u \neq 0$. As will be shown in the sequel, this can be confirmed by means of a suitable test. Therefore, application of P-technique in this case would not seem warranted.

A reasonable generalization of P-technique in order to arrive at a complete model for the covariance function of a second order stationary time series is accomplished by proceeding from the dynamic factor model:

$$z(t) = \sum_{u=0}^{s} \Lambda(u)\eta(t-u) + \varepsilon(t), \qquad t = 0, \pm 1, \ldots, \tag{3}$$

where the lag $s \geq 0$ is an unknown parameter and $\Lambda(u)$, $u = 0, 1, \ldots, s$, is a causal filter which in our example consists of a sequence of 4-dimensional vectors of lagged loadings. Evidently, the determination of $z(t)$ by a realization of $\eta(t)$ at time t is not instantaneous, but is manifest over consecutive time points t, $t + 1, \ldots$ In reverse, the communal part of $z(t)$ at time t is a weighted sum of $\eta(t)$, $\eta(t-1), \ldots$ Strictly speaking, the lag beyond which $\Lambda(u)$ is zero should be taken to be arbitrarily large. However, the assumed stationarity of $z(t)$ generally will give rise to a decaying filter as $u \to \infty$ (cf. Hannan, 1970, p. 151). Therefore, $\Lambda(u)$ can be truncated at a suitable finite lag $u = s$. The dynamic factor model given by (3) covers the entire covariance function of $z(t)$:

$$C(u) = \sum_{v=0}^{s} \sum_{w=0}^{s} \Lambda(v)\Xi(u + w - v)\Lambda(w)^T + \Theta(u), \qquad u = 0, \pm 1, \ldots \tag{4}$$

In general, $\eta(t)$ is a q vector-valued latent factor series, $q < p$, whence $\Lambda(u)$, $u = 0, 1, \ldots$ is a sequence of $(p \times q)$ dimensional matrices of lagged loadings, and $\Xi(u)$, $u = 0, 1, \ldots$ is a sequence of $(q \times q)$ dimensional matrices of lagged covariances.

Notwithstanding important differences in development, the same basic rationale underlies both the traditional static factor model (including P-technique) as well as the dynamic factor model. In each case the covariance structure of observables is conceived of as being due to a common latent source. However, the observed variables in a dynamic factor model are random time-dependent functions with a lagged covariance structure, and the common latent sources also are random functions that have a lagged functional relationship with the observed variables. For all that, it will be shown that the causal filter which constitutes this lagged functional relationship plays the same part in the interpretation of a common latent source as the matrix of loadings in a static factor analysis.
time-dependent functions with a lagged covariance structure, and the common latent sources also are random functions that have a lagged functional relationship with the observed variables. For all that, it will be shown that the causal filter which constitutes this lagged functional relationship plays the same part in the interpretation of a common latent source as the matrix of loadings in a static factor analysis.

Obviously, it will be a rather tedious matter to fit an intricate scheme such as given by (4). As a matter of fact, a rather elegant frequency domain analysis exists for the case in which an observed trajectory of sufficient length is available (Brillinger, 1975). For this type of analysis to be rewarding, a reasonable amount of data is likely to be required—one hundred to three hundred points is close to the low end (Tukey, 1978). However, trajectories which are encountered in psychological research are generally too short to justify the application of spectral analysis. In order to advance the use of dynamic factor analysis in psychology, it will be necessary to arrive at a statistical method which is robust against small sample size. Such a method is shortly presented.

Analysis of a Short Trajectory

Suppose, for the sake of argument, that the dimension q of the latent factor series $\eta(t)$ in (3) is known. Suppose also, that the truncation lag s of the corresponding causal filter is known. It is then possible to rewrite the dynamic factor model as a simultaneous structural equation system which, for arbitrary t, covers a subset of $a + 1$ consecutive time-points $t - a, t - a + 1, \ldots, t$, where $a \geq s$. This is accomplished by letting:

$$z^T = [z(t)^T, \ldots, z(t-a)^T],$$
$$\eta^T = [\eta(t)^T, \ldots, \eta(t-a-s)^T],$$
$$\varepsilon^T = [\varepsilon(t)^T, \ldots, \varepsilon(t-a)^T],$$

which gives:

$$z = \Lambda\eta + \varepsilon, \tag{5}$$

where:

$$\Lambda = \begin{pmatrix} \Lambda(0) & \Lambda(1) & \cdots & \Lambda(s) & 0 & \cdots & 0 & 0 \\ 0 & \Lambda(0) & \cdots & \Lambda(s-1) & \Lambda(s) & \cdots & 0 & 0 \\ \vdots & & & & & & & \\ 0 & 0 & \cdots & & & \cdots & \Lambda(s-1) & \Lambda(s) \end{pmatrix}.$$

Accordingly:

$$C = \Lambda\Xi\Lambda^T + \Theta \tag{6}$$

where:

$$C = \{C(i-j);\, i, j = 1, \ldots, a+1\} = \begin{pmatrix} C(0) & & & \\ C(1) & C(0) & & \\ \vdots & & & \\ C(a) & C(a-1) & \cdots & C(0) \end{pmatrix}.$$

Likewise:

$$\Xi = \{\Xi(i-j);\, i, j = 1, \ldots, a+s+1\},$$
$$\Theta = \{\Theta(i-j);\, i, j = 1, \ldots, a+1\}.$$

According to the Wold decomposition theorem (cf. Hannan, 1970) each nondeterministic q vector-valued stationary process $\eta(t)$ can be represented by:

$$\eta(t) = \sum_{u=0}^{\infty} \Gamma(u)B^u\xi(t),$$

where B is the backward shift operator defined by $B\xi(t) = \xi(t-1)$ (Box & Jenkins, 1970), and where

$$\text{cov}\,[\xi(t), \xi(t-u)] = \delta(u)I_q.$$

Substitution in (3) gives:

(*a*)
$$z(t) = \sum_{u=0}^{\infty} \Lambda(u)B^u\eta(t),$$
$$= \sum_{u=0}^{\infty} \Lambda(u)B^u \sum_{v=0}^{\infty} \Gamma(v)B^v\xi(t), \quad \text{and}$$

(b) $$= \sum_{u=0}^{\infty} \Lambda^*(u) B^u \xi(t),$$

where the coefficient matrices $\Lambda^*(u)$ are obtained by equating coefficients of like powers of B, yielding:

$$\Lambda^*(0) = \Lambda(0)\Gamma(0),$$

$$\Lambda^*(1) = \Lambda(0)\Gamma(1) + \Lambda(1)\Gamma(0), \quad \text{etc.}$$

As the causal filter is unconstrained, the equivalence of the representations (a) and (b) implies that it will not be possible to find a consistent estimator of the covariance function of $\eta(t)$ (Jöreskog, 1976). In order to obtain an identified system, this covariance function has to be fixed, e.g., at $\Xi(u) = I_q$, where I_q is the $(q \times q)$ dimensional identity matrix.

Thus, conditional to fixed values of q and s the dynamic factor model given by (4) for a p vector-valued second order stationary time series $z(t)$ has been rewritten as a simultaneous structural equation system:

$$C = \Lambda\Lambda^T + \Theta, \tag{7}$$

where C and Θ are $(p(a + 1) \times (a + 1)p)$ dimensional covariance matrices, and Λ is a $(p(a + 1) \times (a + s + 1)q)$ dimensional matrix of loadings. The system defined by (7) can be fitted by using the method of maximum likelihood (Jöreskog & Sörbom, 1978). Unfortunately, the only way to remove the conditions on which (7) has been obtained is to carry out a search on a grid $[q \in \{1, 2, \ldots, Q\}, s \in \{0, 1, \ldots, S\}]$ of a priori specified feasible values of Q and S. Moreover, although the number a of consecutive time points at which the dynamic factor model is considered in the derivation of (7) is bounded at the lower end, i.e., $a \geq s$, this still leaves open the question about the most efficient value of a. These matters will be considered more fully in the next section. At this juncture we will illustrate the fit of a specific instance of (7) to the covariance function in Table 1. For reasons of conciseness, let $q = 1$, $s = 1$, and $a = 1$. Conditional to these fixed values, the corresponding coefficient matrices in (7) are:

$$C = \begin{pmatrix} C(0) & \\ C(1) & C(0) \end{pmatrix} \quad \Lambda = \begin{pmatrix} \Lambda(0) & \Lambda(1) & 0 \\ 0 & \Lambda(0) & \Lambda(1) \end{pmatrix} \quad \Theta = \begin{pmatrix} \Theta(0) & \\ \Theta(1) & \Theta(0) \end{pmatrix},$$

where estimates of $C(0)$ and $C(1)$ are obtained from Table 1. This system is fitted by means of the method of maximum likelihood. Let Ω denote the set of all matrices C that are of order $(p(a + 1) \times (a + 1)p)$, symmetric and positive definite. Let ω be the subset for which (7) holds. Let $L(\Omega)$ and $L(\omega)$ denote the maxima of the likelihood function in Ω and ω, respectively. Then it is known that for large samples $-2 \ln [L(\omega)/L(\Omega)]$ is distributed approximately as χ^2 if (7) is true. Henceforth, we will refer to this χ^2-approximation as the chi-square goodness-of-fit. Proceeding with our illustrative example, the results of the fit are given in Table 2. Apparently, conditional to $q = 1$, $s = 1$, and $a = 1$, the fit of the dynamic factor model is fairly satisfactory. We will shortly discuss the merits of this result.

Further Aspects of the Proposed Analysis

An Optimum Value of a

In the foregoing section, the fit of a dynamic factor model with $s = 1$, $q = 1$, and $a = 1$ was found to be satisfactory. Referring to (4), this dynamic factor model should account for the entire covariance function. Hence, for arbitrary $a > 1$, the corresponding simultaneous equation system, with parameters fixed at the values that have been obtained with $a = 1$, should also give rise to a satisfactory fit. However, if such a model check is carried out by taking, e.g., $a = 3$ and, accordingly, with $\Theta(u)$, $u = 2, 3$, as the sole

Table 2

Parameter Estimates of the

Preliminary Model for Figure 1*

$\underline{z}(t)=\Lambda(0)\eta(t)+\Lambda(1)\eta(t-1)+\varepsilon(t)$		$\Theta(u)=\mathrm{diag}\,[\theta_1(u),\ldots,\theta_4(u)]$	
$\hat{\Lambda}(0)$	$\hat{\Lambda}(1)$	$\hat{\Theta}(0)$	$\hat{\Theta}(1)$
.327	.447	1.453	.695
.354	-.799	.382	-.430
.959	.925	1.933	.925
.192	-.354	2.821	-1.080

*Chi-square goodness-of-fit = 5.98 ($\underline{df}$ = 10)

free coefficient matrices, this results in a substantially reduced chi-square goodness-of-fit = 87.597 ($df = 50$). Similarly, if the same dynamic factor model (i.e., with $a = 3$, while $s = 1$ and $q = 1$) is fitted from scratch (i.e., with all coefficient matrices free), then this results in a chi-square goodness-of-fit = 82.020 ($df = 34$). Clearly, the outcome of a dynamic factor analysis is dependent upon the particular value of a at which it is carried out. This raises the problem of choosing an optimum value of a.

A similar problem is well-known from spectral analysis and concerns the choice of a suitable degree of smoothing. In a nutshell, a small value of a corresponds to a large degree of smoothing, i.e., implies an increased bias and a decreased sampling variability of parameter estimates. There exist several approaches in the search for an optimum degree of smoothing (Jenkins & Watts, 1968; Haykin, 1979), which may serve as guidelines for the choice of a. Presently, optimization of a with respect to bias and sampling variability awaits further elaboration.

Model Evaluation

The simultaneous structural equation system given by (5) is a special instance of the class of covariance structure models. For an extensive discussion of the specific computing formulas in the analysis of covariance structure models, e.g., for the likelihood function, chi-squared statistics, etc., we refer to Jöreskog (1978). Recalling our previous remark on the removal of the conditions on which (5) has been derived, if we carry out a search on a grid of a priori feasible values of q and s, the corresponding covariance structure models have to be compared with each other as to their adequacy. Moreover, (5) has been proposed with intent to analyze short trajectories, taking into account that statistical theory in covariance structure analysis has been developed primarily for large samples. Recently, Bentler and Bonett (1980) have summarized the somewhat unsatisfactory state of affairs in relation to model evaluation in this area. Furthermore, they propose incremental fit indices comparable to Tucker's reliability coefficient. These indices are quite useful due to their relative independence from sample size. On the other hand, the usual chi-square goodness-of-fit test is heavily dependent upon sample size, i.e., in small samples virtually any model tends to be accepted as adequate. Drawing upon similar observations, Eiting and Mellenberg (1980) propose a sophisticated decision theoretic procedure for the comparison of alternative models. From a somewhat different point of view, analogous indices for the comparison of alternative models have been discussed in the time series literature, a particular case in point being Akaike's information criterion (see Kashyap and Rao, 1976, for an extensive overview). At present, we will concentrate upon the incremental fit indices, which serve their purpose within a hierarchical model comparison scheme. In order to obtain the proper application of this scheme to the selection of an adequate simultaneous equation system, see (7), it is a prerequisite that the same value of a be used throughout all comparisons. Specifically, after a grid G of feasible values of q and s has been chosen, it follows that $a \geq S$, where S is the maximum $s \in G$. Proceeding in this way, the comparison of several alternative dynamic factor models can be made in relation to the same baseline.

Estimation of the Latent Trajectory

Whenever an adequate dynamic factor model has been selected, it will be of interest to estimate the time course of the particular realization of the latent factor series $\eta(t)$ underlying the observed trajectory. This endeavour can be conceived of as the dynamic analogue of the estimation of factor scores in a traditional static factor model. Whereas in the latter case the factor scores pertain to subjects, in a dynamic factor model the "factor scores" pertain to consecutive time points. Unfortunately, estimation according to the usual methods of factor analysis, e.g., the regression method, leads to nonsensical results—the estimate of the realization of $\eta(t)$ at each time t is not unique, but consists of a set of different values. One could explain the breakdown by saying that it is not legitimate to conceive of (5) as a single covariance structure model. Specifically, we may say that (5) holds for arbitrary t and hence refers to an ensemble of equivalent models. As is explained more fully in the Appendix, the correct method requires the translation of a dynamic factor model into a Markovian state model. Subsequently, the latent trajectory can be estimated by means of the Kalman filter (Jazwinsky, 1970).

A Substantive Application

At this juncture, it is time to reveal the true nature of the trajectory in Figure 1. In fact, this trajectory has been simulated, using the dynamic factor model shown in Table 3. As the true model underlying the observed trajectory is known, we are in a comfortable position when it comes to an evaluation of the results from the point of view of application of the proposed analysis to this case. Proceeding, a grid of feasible values $[q \in Q, s \in S]$ is determined. Assuming that a priori information about the parameters of the dynamic factor model is lacking, we will, with a view to the rather small dimension $p = 4$ of $z(t)$, take $Q = \{1\}$. In addition, considering the relative shortness of the observed trajectory ($n = 50$), we will take $S = \{0, 1, 2\}$. Accordingly, $C = \{C(i-j);\ i, j = 1, \ldots, 4\}$ is a (16×16) dimensional so-called block Toeplitz matrix with 58 degrees of freedom. Consecutively, our null model based on modified independence among variates (Bentler & Bonett, 1980) is introduced:

$$z(t) = \Lambda(0)\eta(t) + \varepsilon(t),$$

Table 3

Dynamic One-factor Simulation Model

$z(t)=\Lambda(0)\eta(t)+\Lambda(1)\eta(t-1)+\varepsilon(t)$		$\varepsilon(t)=\Gamma(1)\varepsilon(t-1)+\alpha(t)$	
$\Lambda(0)$	$\Lambda(1)$	$\Gamma(1)$	diag T^*
1	.8	.5	1
1	-.8	-.5	1
1	.8	.5	2
1	-.8	-.5	2

*Cov $[\alpha(t),\alpha(t-u)] = \delta(u)\ T$

M_0:

$$\Xi(u) = \delta(u) \qquad \Theta(u) = \text{diag}\ [\theta_1\delta(u), \ldots, \theta_4\delta(u)].$$

Notice, that if M_0 is rewritten as the simultaneous structural equation system defined by (5), then the latter can be conceived of as a concatenation of P-techniques. Notice also, that the expected covariance function of our null model is: $C(u) = \delta(u)C$, where C is an arbitrary positive-definite matrix. Hence, incremental fit indices based upon this null model will have lower values than would have obtained by choosing the most extreme null model with expected covariance function $C(u) = \text{diag}\ [c_1\delta(u), \ldots, c_4\delta(u)]$. However, in the present context we find the current choice of M_0 more appealing because it yields values of incremental fit relative to P-technique. Next, a convenient sequence of increasingly less restricted dynamic one-factor models is obtained by allowing the components of $\varepsilon(t)$ to become arbitrarily autocorrelated processes, and by taking $s = 0, 1, 2$, respectively:

M_1: $\quad z(t) = \Lambda(0)\eta(t) + \varepsilon(t),$

M_2: $\quad z(t) = \Lambda(0)\eta(t) + \Lambda(1)\eta(t-1) + \varepsilon(t),$ and

M_3: $\quad z(t) = \Lambda(0)\eta(t) + \Lambda(1)\eta(t-1) + \Lambda(2)\eta(t-2) + \varepsilon(t),$

where in each case

$$\Xi(u) = \delta(u), \qquad \Theta(u) = \text{diag}\ [\theta_1(u), \ldots, \theta_4(u)].$$

The incremental fit indices which will be used in the comparison of these models are defined by:

$$\rho_{kt} = \frac{(Q_k - Q_t)}{(Q_0 - 1)},$$

Table 4

Incremental Fit Indices

	Model Comparisons			
	M_0	M_1	M_2	M_3
M_0		.552	.725	.779
M_1	.494		.173	.227
M_2	.716	.222		.054
M_3	.760	.266	.044	

where Q_j is the ratio $(v/df)_j$ of the chi-square goodness-of-fit v to df corresponding to M_j, and

$$\Delta_{kt} = \frac{(F_k - F_t)}{F_0},$$

where F_j is the maximum of the log likelihood function corresponding to M_j. The model comparisons associated with this application are shown in Table 4. Below the main diagonal are estimated ρ_{kt} values, above the main diagonal are estimated Δ_{kt} values. Notice that the redundancy in the block Toeplitz pattern of C implies that the chi-square goodness-of-fit statistic, which involves a comparison with the completely saturated model M_s, is inflated. Hence, the corresponding model tests have been omitted. In fact, one of the virtues of the incremental fit indices in conjunction with the use of the same value of a is their insensitivity to this inflating redundancy.

Inspection of Table 4 reveals the substantial improvement of M_2 and M_3 over M_0. Since M_0 corresponds to the model underlying P-technique, the latter approach would not seem preferable in this case. As the distinction between M_2 and M_3 is negligible, M_2 is the preferred model by appeal to Ockham's razor. The corresponding parameter estimates are shown in Table 5. With regard to the estimation of the trajectory of the latent factor series, we already alluded to the inapplicability of the usual methods such as the regression method. As is indicated in the Appendix, the correct method involves the translation of the dynamic factor model into a Markovian state model enabling the use of the Kalman filter. An appeal to the Kalman filter in order to estimate the latent trajectory of the factor series requires that a suitable process model be fitted to the covariance function of $\varepsilon(t)$, the result of which is also shown in Table 5. With these specifications, the Kalman filter can be invoked and the resulting estimated trajectory of the latent factor series is depicted in Figure 2. Having obtained the estimated trajectory $\eta(t) = \hat{\eta}_t$, $t = 1, \ldots, 50$, the possibility arises of estimating the common component

$$\zeta(t) = \Lambda(0)\eta(t) + \Lambda(1)\eta(t-1)$$

of $z(t)$ by substitution.

Table 5

Parameter Estimates of M_2

$\underline{z}(t)=\Lambda(0)\eta(t)+\Lambda(1)\eta(t-1)+\varepsilon(t)$		$\varepsilon(t)=\Gamma(1)\varepsilon(t-1)+\Gamma(2)\varepsilon(t-2)+\alpha(t)$		
$\hat{\Lambda}(0)$	$\hat{\Lambda}(1)$	$\hat{\Gamma}(1)$	$\hat{\Gamma}(2)$	diag $\hat{T}^*$
.56	.56	.63	-.39	.96
.41	-.81	-.44	0	.64
.90	.76	.50	0	1.76
.28	-.30	-.40	0	2.40

$^*\text{cov}\,[\alpha(t),\alpha(t-u)] = \delta(u)\,T$

This completes the description of the application of dynamic factor analysis to the simulated trajectory in Figure 1. An evaluation of the results thus obtained with respect to the true model shows that:

— the correct dimension $q = 1$ of the latent series $\eta(t)$ has been identified,
— the correct truncation lag $s = 1$ of the causal filter has been identified.
— in view of the short length of the observed trajectory ($n = 50$), the estimated parameters of the dynamic factor model are fairly close to their true values.

An Application to Real Data

At this juncture, it will be convenient to consider an application to real data in order to emphasize the major steps in an applied dynamic factor analysis and to illustrate the interpretation of a latent factor series. The data set is due to Hutt, Lenard, and Prechtl (1969). While investigating the time course of behavioral states of a single 8-day-old

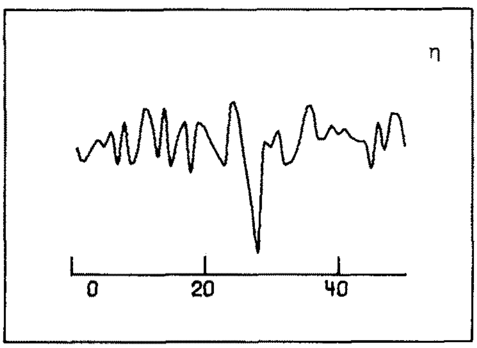


FIGURE 2
Kalman filtered estimate of the latent trajectory.

infant, a $6\frac{1}{2}$-hour polygraphic record yielded the following measurements per 3 minutes:

$z_1(t)$: power of the EEG ($\mu V^2/3$ min.),

$z_2(t)$: heart rate (beats/min.),

$z_3(t)$: respiration rate (cycles/min.),

$z_4(t)$: heart rate variability, and

$z_5(t)$: respiration rate variability.

As the component series of $z(t)$ are expressed in different measurement units, the original 5 vector-valued trajectory $z(t) = z_t$, $t = 1, 2, \ldots, 124$, has been standardized and is depicted in Figure 3. The first few coefficient matrices of the corresponding correlation function are shown in Table 6. The interesting question is, whether time-dependent communalities between the observed physiological indices can be conceived of as being due to a single underlying process, i.e., a univariate latent factor series. If so, then it might be attempted to interpret this latent process in terms of, e.g., arousal (Duffy, 1962).

We will consider the sequence of increasingly less restricted dynamic one-factor models M_k, $k = 0, 1, 2, 3$, which has been invoked in the foregoing section. Specifically, M_k, $k = 1, 2, 3$, are dynamic one-factor models with $s = k - 1$, respectively, and with component series of $\varepsilon(t)$ being arbitrarily auto-correlated. Moreover, we will consider a similar sequence of increasingly less restricted dynamic two-factor models M_{kk}, $k = 1, 2, 3$, with $s = k - 1$, respectively, and with component series of $\varepsilon(t)$ being arbitrarily auto-correlated. In order to apply (6) we will, for arbitrary t, consider each dynamic factor model at $a + 1 = 5$ consecutive time points $t, t - 1, \ldots, t - 4$. Accordingly, the ensuing (25×25) dimensional block Toeplitz matrix $C = \{C(i - j); i, j = 1, 2, \ldots, 5\}$ is determined from Table 6, and each dynamic factor model M_k, $k = 0, 1, 2, 3$, M_{kk}, $k = 1, 2, 3$, consecutively fitted to C. The resulting incremental fit indices are given in Table 7. Below the main diagonal are estimated ρ_{kt} values, above the main diagonal are estimated Δ_{kt} values. Table 7 shows that, compared with M_0 (i.e., a concatenation of P-techniques), M_3 and M_{33} yield substantial improvements. As the difference between M_3 and M_{33} is negligible, the former model is preferred because it is simpler. The parameter estimates corresponding to M_3 are shown in Table 8. With these specifications the Kalman filter can be invoked, and the resulting estimated trajectory of the latent factor series corresponding to M_3 is depicted in Figure 4.

Summarizing, the correlation function of the neonatal physiological indices can be satisfactorily accounted for in terms of the dynamic one-factor model M_3. Hence, the time-dependent communalities between these indices can be regarded as being due to a unitary latent process. A tentative interpretation of this latent process might be derived from the parameter estimates presented in Table 8. First, respiration rate, variability of respiration rate and power of the EEG have substantial filter loadings, whereas the filter loadings of heart rate and heart rate variability are much smaller. Hence, the usually found cardiac-respiratory coupling (Porges, Bohrer, Keren, Cheung, Franks, & Drasgow, 1981) seems to be lacking. This may be due to the relatively low sampling rate used by Hutt et al. (1969) ($\Delta t = 3$ min., whereas $\Delta t \approx 250$ msec. is customary in the study of cardiac-respiratory relationships). In so far as cardiac-respiratory coupling occurs on a fast time scale, it cannot be detected in the Hutt et al. study where the focus is on physiological processes that take place on a rather slow time scale. In this connection Hutt et al. refer to slowly evolving metabolic influences that might contribute to the relative autonomy of cardiac activity. Second, a lead-lag pattern can be discerned as to the way in which the latent series affects the rate indices (viz. heart rate and respiration rate) and the variability indices (power of the EEG, heart rate variability and variability of respiration

Figure 3

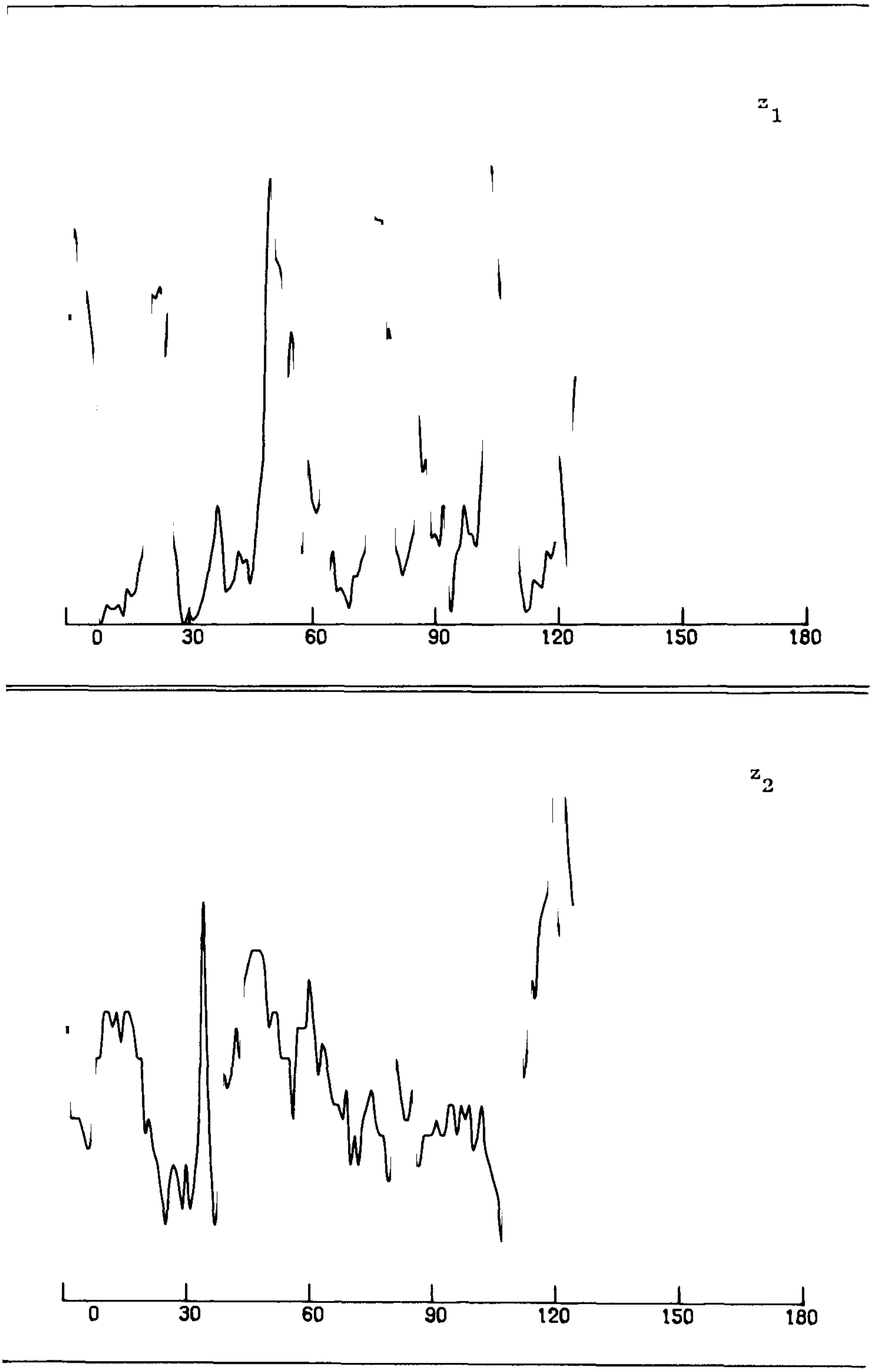
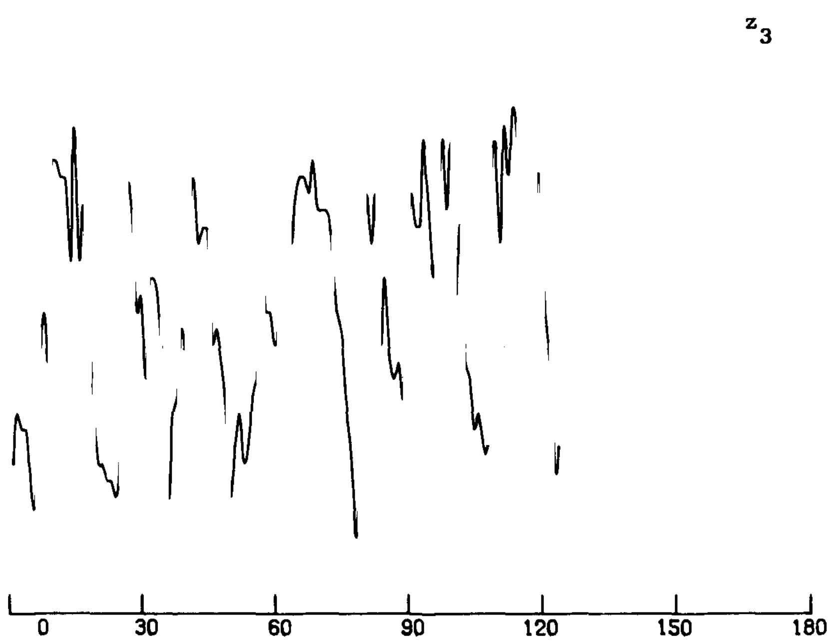


Figure 3



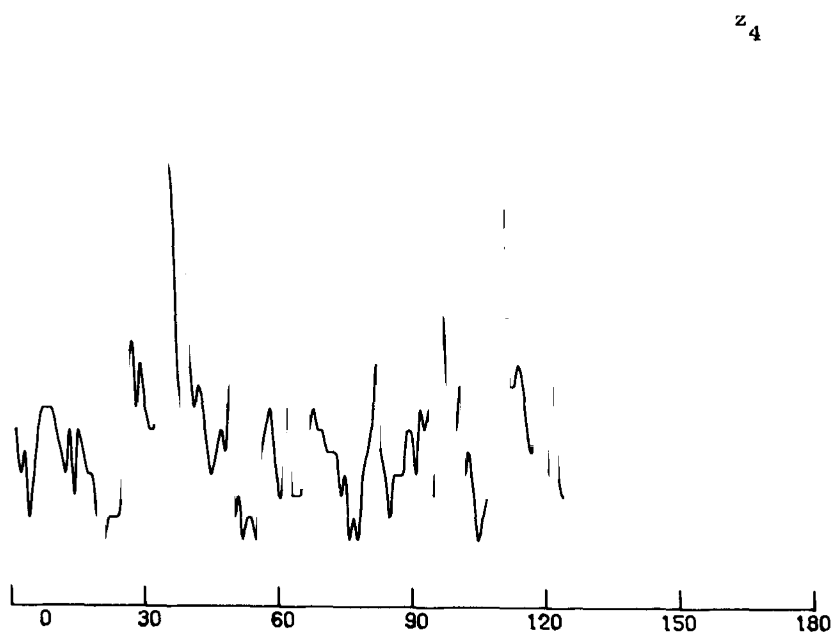


Figure 3

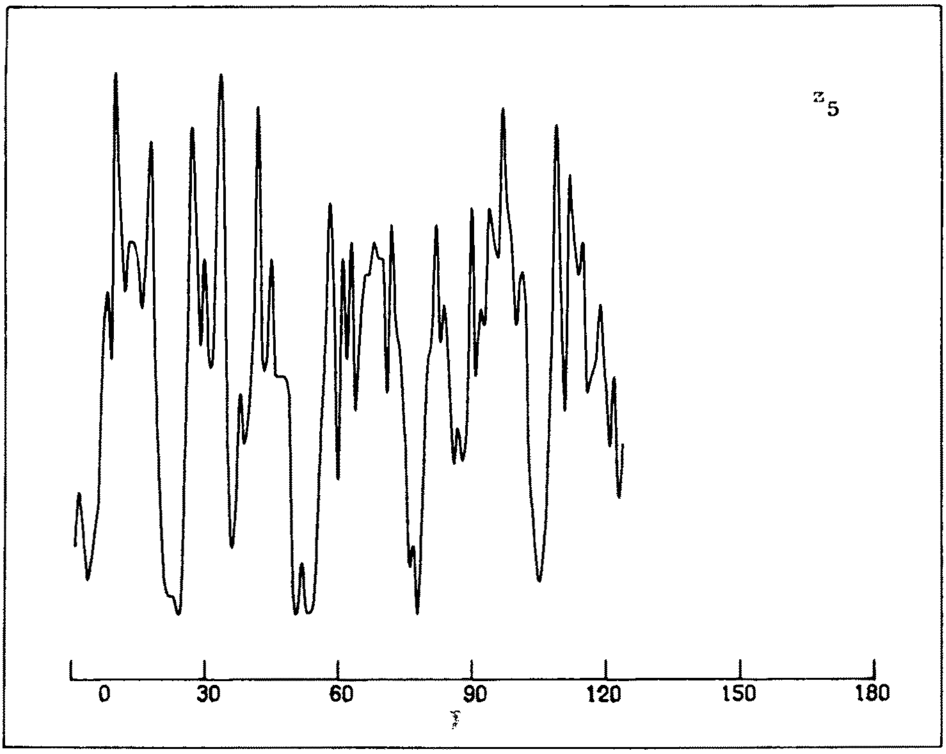


FIGURE 3
Time course of the 5 vector-valued trajectory obtained by Hutt et al.

rate). Specifically, the maximum filter loading of each rate index occurs at lag $u = 2$ ($\hat{\lambda}_2(2) = .342$ and $\hat{\lambda}_3(2) = .522$), whereas the maximum filter loading of each variability index occurs at lag $u = 1$. Grossberg (1982) has pointed out that arousal acts as a tuning mechanism that in the first instance increases the response range (i.e., variability) of physiological processes. Subsequently, the availability of an increased response range allows for the selection of a more adaptive mean response rate. In conclusion, then, a tentative interpretation of the latent factor series in terms of arousal would seem to be justified.

Discussion and Conclusion

We have conceived of dynamic factor analysis as yielding a causal model for an observed vector-valued time series. From a different stance, we might have concentrated upon the approximation of an observed p vector-valued time series by a q vector-valued series of reduced dimension $q < p$. As to this reduction of the dimension of vector-valued time series, there exists an important time domain approach due to Subba Rao (1975, see also Subba Rao & Tong, 1974). This approach pertains to the particular case in which both the input and the output series of a stochastic system are assumed to be given. In contrast, in a dynamic factor analysis only the output series $z(t)$ is assumed to be known, while the input series, viz. the latent factor series, forms part of the hypothesized causal model for $z(t)$. If the approach by Subba Rao would be applied to the case in which only

Table 6

Estimated Correlation Function

of Figure 3

$\hat{C}(0)$					$\hat{C}(1)$				
1.000					.822	-.218	-.634	-.354	-.694
-.195	1.000				-.082	.787	.109	.111	.016
-.690	.239	1.000			-.608	.238	.710	.226	.617
-.409	.159	.247	1.000		-.396	.277	.362	.456	.382
-.718	.216	.800	.384	1.000	-.634	.258	.650	.305	.652

$\hat{C}(2)$					$\hat{C}(3)$				
.608	-.181	-.492	-.271	-.540	.391	-.181	-.338	-.131	-.345
.005	.593	.002	.012	-.125	.060	.484	-.025	-.143	-.129
-.436	.235	.521	.081	.428	-.209	.166	.304	-.033	.266
-.347	.182	.260	.357	.220	-.297	.198	.134	.277	.087
-.423	.170	.430	.215	.393	-.254	.065	.251	.058	.225

$\hat{C}(4)$				
.188	-.145	-.132	-.008	-.126
.130	.438	-.107	-.216	-.197
-.020	.151	.098	-.074	.071
-.149	.174	.056	.169	.029
-.044	.026	.054	-.068	.037

the output series $z(t)$ is given, then one would proceed by fitting an autoregressive model to $z(t)$. Consecutively, a principal component analysis of the variance of the innovations in this autoregression would yield a reduction of the dimension of $z(t)$. Notice, that this principal component analysis would not pertain to the covariance function of $z(t)$, but to the variance of a derived series of innovations. A related time domain approach to the reduction of the dimension of a vector-valued time series is due to Box and Tiao (1977). Their approach yields a transformation of the original time series, where the components

Table 7

Incremental Fit Indices

Model Comparisons

	M_0	M_1	M_{11}	M_2	M_{22}	M_3	M_{33}
M_0		.486	.516	.693	.731	.795	.839
M_1	.323		.030	.207	.245	.309	.353
M_{11}	.389	.066		.177	.215	.279	.323
M_2	.635	.312	.246		.038	.102	.146
M_{22}	.635	.312	.246	.000		.064	.108
M_3	.763	.440	.374	.128	.128		.044
M_{33}	.765	.442	.376	.130	.130	.002	

Table 8

Parameter Estimates of M_3

$z(t)=\Lambda(0)\eta(t)+\Lambda(1)\eta(t-1)+\Lambda(2)\eta(t-2)$			$\varepsilon(t)=\Gamma(1)\varepsilon(t-1)+\alpha(t)$	
$\hat{\Lambda}(0)$	$\hat{\Lambda}(1)$	$\hat{\Lambda}(2)$	$\hat{\Gamma}(1)$	diag $\hat{T}$ *
-.248	-.453	-.316	.655	.222
-.073	.182	.342	.864	.227
.387	.391	.522	.591	.171
.179	.392	-.002	.422	.646
.363	.580	.499	-.528	.058

* $\text{cov}\,[\alpha(t),\alpha(t-u)] = \delta(u)\,T$

Figure 4

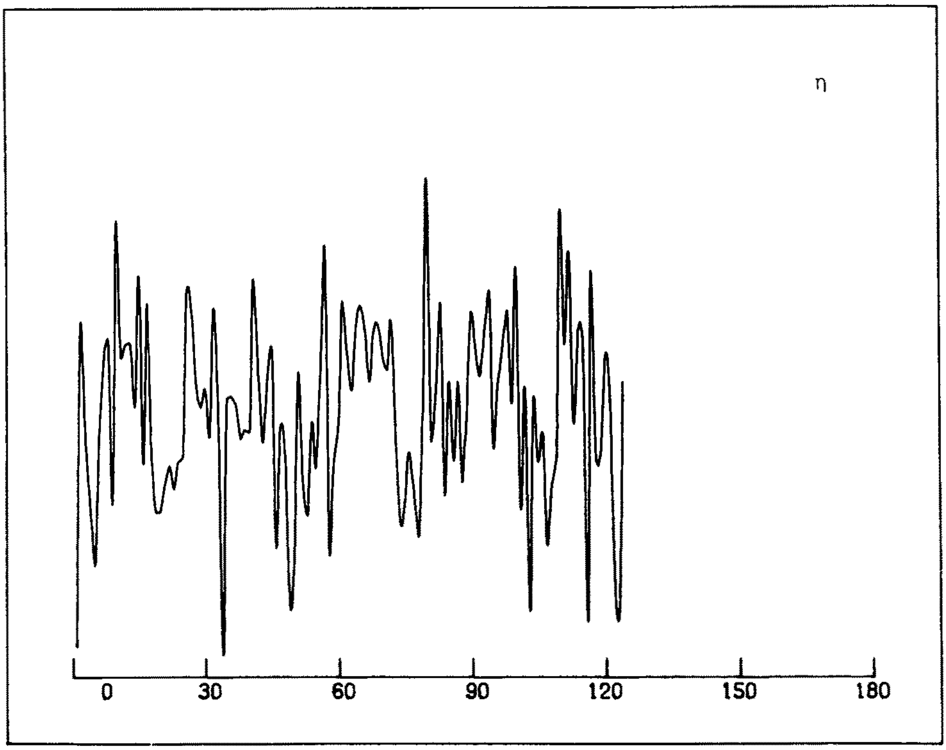


FIGURE 4
Kalman filtered estimate of the realization of the latent factor series corresponding to the trajectory obtained by Hutt et al.

of the transformed series are ordered from least to most predictable. This transformation is based upon a principal component analysis of the variance of the error of the one-step ahead forecasts, whence the covariance function of the original series is only remotely involved.

Contrary to the aforementioned approaches, a dynamic factor analysis is based on a model which explicitly accounts for the entire lagged covariance function of an arbitrary second order stationary time series. Hence, the criticism raised by e.g., Anderson (1963) concerning *P*-technique would not seem to apply to dynamic factor analysis. Moreover, the proposed analysis would seem to be justified in the case where a relatively short trajectory is observed and therefore may be considered worthwhile for application in the field of psychology. It has in fact already been proven to be useful in such diverse areas as single subject research in psychotherapy (Molenaar, 1981), and the investigation of individual differences in EEG topography (Molenaar, 1982a). The proposed analysis involves a reformulation in terms of a simultaneous structural equation system, thereby giving rise to a search on a grid of a priori feasible values both for the dimension of the latent factor series as well as for the truncation lag of the causal filter. It is one of the main qualities of a frequency domain analysis that this search is substantially simplified. Hence, much emphasis is placed on a modified spectral analysis for short trajectories, which becomes possible by appealing to evolutionary spectral analysis (Molenaar, 1982b). Finally, the dynamic factor model can be extended in order to accommodate time-varying trends

(Molenaar, 1984). As will be explained in a separate study, this opens up the possibility to carry out a simultaneous analysis of the lagged covariance function in conjunction with intervention effects on the mean function in a single-subject design.

Appendix

A Recursive Estimator for the Latent Trajectory

As far as the residual series $\varepsilon(t)$ is concerned, the fit of a simultaneous equation system results in an estimate of the covariance function $\Theta(u)$, $u = 0, \pm 1, \ldots, \pm a$. In order to be able to rewrite the dynamic factor model as a Markovian state model, it is necessary that a process model corresponding to this covariance function of $\varepsilon(t)$ is determined. Although there is no inherent necessity to do so, it will be helpful, in order to avoid undue technicalities, to restrict the class of candidate models for $\varepsilon(t)$ to autoregressions:

$$\varepsilon(t) = \sum_{u=1}^{r} \Gamma(u)\varepsilon(t-u) + \alpha(t),$$

where

$$\Gamma(u) = \text{diag}\,[\gamma_1(u), \ldots, \gamma_p(u)],$$
$$\text{cov}\,[\alpha(t), \alpha(t-u)] = \delta(u)\Upsilon, \quad \text{and}$$
$$\Upsilon = \text{diag}\,[\upsilon_1, \ldots, \upsilon_p].$$

If autoregressions of consecutively increasing order $r \in R$, $R = \{0, 1, \ldots, a\}$, are fitted to the estimated covariance function of $\varepsilon(t)$, then the selection of an adequate r^*-th order model can be accomplished by appealing to Akaike's information criterion:

$$r^* = \min_{r \in R} \left[-\frac{n}{2} \ln \det \Upsilon - rp^2 \right].$$

Accordingly, the dynamic factor model thus obtained:

$$z(t) = \sum_{u=0}^{s} \Lambda(u)\eta(t-u) + \varepsilon(t),$$
$$\varepsilon(t) = \sum_{u=1}^{r} \Gamma(u)\varepsilon(t-u) + \alpha(t), \qquad \text{and}$$
$$\Xi(u) = \delta(u)I_q,$$

can be rewritten as a Markovian state model:

$$x(t+1) = Hx(t) + u(t), \qquad \text{and}$$
$$z(t) = Fx(t),$$

where:

$$x(t)^T = [\eta(t)^T, \ldots, \eta(t-s)^T, \varepsilon(t)^T, \ldots, \varepsilon(t-r)^T],$$
$$u(t)^T = [\eta(t)^T, 0, \ldots, 0, \alpha(t)^T, 0, \ldots, 0],$$
$$H = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix}, \qquad \dim H = [(s+1)q + (r+1)p, (s+1)q + (r+1)p],$$

$$H_1 = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ I_q & 0 & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & I_q \end{pmatrix}, \qquad \dim H_1 = [(s+1)q, (s+1)q],$$

$$H_2 = \begin{pmatrix} \Gamma(1) & \Gamma(2) & \cdots & \Gamma(r) \\ I_p & 0 & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & I_p \end{pmatrix}, \qquad \dim H_2 = [(r+1)p, (r+1)p],$$

$$F = [\Lambda(0), \ldots, \Lambda(s), I_p, \ldots, 0], \qquad \dim F = [p, (s+1)q + (r+1)p], \quad \text{and}$$

$$\text{var}\,[u(t)] = W = \text{diag}\,[I_q, 0, \ldots, 0, \Upsilon, 0, \ldots, 0].$$

The Kalman filter corresponding to this model is defined by:

$$\hat{x}(t) = H\hat{x}(t-1) + K(t)[z(t) - FH\hat{x}(t-1)],$$

$$K(t) = [HR(t-1)H^T + W]F^T\{F[HR(t-1)H^T + W]F^T\}^{-1},$$

$$R(t) = [I_m - K(t)F][HR(t-1)H^T + W],$$

$$m = (s+1)q + (r+1)p, \qquad \hat{x}(0) = x_0, \quad \text{and} \quad R(0) = W.$$

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