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IMAGE AND ANTI-IMAGE COVARIANCE MATRICES FROM A CORRELATION MATRIX THAT MAY BE SINGULAR

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Following the work of Tucker, Cooper, and Meredith, image and anti-image covariance matrices from a correlation matrix that may be singular are derived.

Key words: covariance, singular matrix, images, anti-images.

This paper extends some results of a recent paper of Tucker, Cooper, and Meredith (*TCM*) [1972]. In that paper *TCM* develop a method for detecting which variates represented in a correlation matrix R of order p have squared multiple correlations (*SMC*s) of one on the remaining $p - 1$ variates. (If such an R has one or more unit *SMC*s, it is singular, of course, and the usual formulas involving the inverse of R for finding *SMC*s, regression equations, image and anti-image covariance matrices, etc., cannot be used.) *TCM* then implicitly give a formula for determining an image and its *SMC* (less than one) for a variate that is represented in a singular R and that is not linearly dependent on the remaining $p - 1$ variates. Using this formula, and observing that the image v_i of a variate z_i which is linearly dependent upon the remaining $p - 1$ variates is simply that variate (*i.e.*, for this case $v_i = z_i$), we develop formulas for the complete set of image and anti-image covariance matrices when R may be singular. It will be useful first to review briefly the algebra of image and anti-image covariance matrices when R is of full rank (*i.e.*, non-singular).

Image And Anti-Image Covariance Matrices When R Is Of Full Rank

Let z be a vector of p variates, centered, *i.e.*, $\mathcal{E}(z) = 0$, and scaled such that $\mathcal{E}(zz') = R$, their population correlation matrix, where $\mathcal{E}$ is the expected value operator. Then the vector variate v of the p images is given by Guttman [1953] as

$$v = (I - S^2R^{-1})z, \tag{1}$$

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where $S^2 = (\text{diag } R^{-1})^{-1}$. The vector variate a of anti-images is

$$
\begin{aligned}
a &= z - v, \\
&= z - (I - S^2R^{-1})z, \qquad (2) \\
&= S^2R^{-1}z.
\end{aligned}
$$

We then write the complete set of covariance matrices—for original variates, images, and anti-images—for the case when R is of full rank:

$$
\begin{aligned}
\mathcal{E}(zz') &= R, \\
\mathcal{E}(zv') &= \mathcal{E}zz'(I - R^{-1}S^2), \\
&= R - S^2, \\
\mathcal{E}(za') &= \mathcal{E}zz'R^{-1}S^2, \\
&= S^2, \\
\mathcal{E}(vv') &= \mathcal{E}(I - S^2R^{-1})zz'(I - R^{-1}S^2), \qquad (3) \\
&= R - 2S^2 + S^2R^{-1}S^2, \\
\mathcal{E}(va') &= \mathcal{E}(I - S^2R^{-1})zz'R^{-1}S^2, \\
&= S^2 - S^2R^{-1}S^2, \\
\mathcal{E}(aa') &= \mathcal{E}S^2R^{-1}zz'R^{-1}S^2, \\
&= S^2R^{-1}S^2.
\end{aligned}
$$

The TCM Results

Let R be of rank $r \leq p$, and let $R = EM^2E'$ be the basic structure of R, where E is the $p \times r$ orthonormal-by-columns matrix of unit-length column eigenvectors of R, and M^2 is the $r \times r$ diagonal matrix of non-zero eigenvalues of R. Consider the symmetric idempotent matrix T:

$$T = EE'. \qquad (4)$$

TCM prove that $0 \leq t_{jj} \leq 1$ for all j and that if $t_{jj} = 1$, then $SMC(J) < 1$, and if $t_{jj} < 1$, then $SMC(J) = 1$. In this determination, they derive implicitly the following formula for the images v_0 associated with *SMC*s less than one.

$$v_0 = D(I - S_*^{\,2}R^*)z, \qquad (5)$$

where the generalized inverse of R is

$$R^* = EM^{-2}E', \qquad (6)$$

$$S_*^{\,2} = (\text{diag } R^*)^{-1}, \qquad (7)$$

and D is a $p \times p$ diagonal matrix constructed as follows:

$$(8) \qquad \begin{aligned} d_{ii} &= 1 \quad \text{if} \quad t_{ii} = 1, \\ d_{ii} &= 0 \quad \text{if} \quad t_{ii} < 1, \end{aligned}$$

i.e., $d_{ii} = 1$ when $SMC(J) < 1$ and $d_{ii} = 0$ when $SMC(J) = 1$. Note that (5) gives *only* the images v_0 associated with SMCs less than one.

Image And Anti-Image Covariance Matrices When R May Be Singular

We observe again that the image v_i of the variate z_i which has unit SMC is simply that variate itself, *i.e.*, $v_i = z_i$, a fact that is obvious geometrically. Then the images v_1 which are associated with SMCs of one are given by

$$(9) \qquad v_1 = (I - D)z.$$

We now combine (5) and (9) to give all p images v:

$$(10) \qquad \begin{aligned} v &= D(I - S_*^2R^*)z + (I - D)z, \\ &= (I - DS_*^2R^*)z. \end{aligned}$$

Let the diagonal matrix

$$(11) \qquad S_\#^2 = DS_*^2,$$

so that $S_\#^2$ has zero diagonals where D has zero diagonals, and rewrite (10) as

$$(12) \qquad v = (I - S_\#^2R^*)z.$$

As we shall see below, this $S_\#^2$ is also the diagonal matrix of anti-image variances, as in the case when R was of full rank, and is equal to S^2 in that case.

The anit-images a are given by

$$(13) \qquad \begin{aligned} a &= z - v, \\ &= z - (I - S_\#^2R^*)z, \\ &= S_\#^2R^*z. \end{aligned}$$

Before developing the complete set of image and anti-image covariance matrices, as in (3) above, we need the following results:

$$(14) \qquad \begin{aligned} RR^* &= EM^2E'EM^{-2}E', \\ &= EE', \\ &= T, \end{aligned}$$

since E is orthonormal by columns; similarly,

$$(15) \qquad R^*R = T;$$

and also,

$$R^*RR^* = EM^{-2}E'EM^2E'EM^{-2}E',$$
$$(16) \qquad = EM^{-2}E',$$
$$= R^*.$$

And, as *TCM* show, if

$$t_{jj} = 1, \quad \text{then} \quad t_{jk} = 0 \quad \text{for all} \quad k, k \neq j,$$

because T is idempotent. Then from (11) and (8), a little effort shows that

$$(17) \qquad TS_*^2 = S_*^2T = S_*^2.$$

Using (12), (13), (14), (15), (16), and (17), we now write out the complete set of covariance matrices—for original variates, images, and anti-images—for the general case where R may be singular:

$$(18) \qquad \mathcal{E}(zz') = R,$$

$$(19) \qquad \mathcal{E}(zv') = \mathcal{E}zz'(I - R^*S_*^2),$$
$$= R - RR^*S_*^2,$$
$$= R - TS_*^2,$$
$$= R - S_*^2,$$

$$(20) \qquad \mathcal{E}(za') = \mathcal{E}zz'R^*S_*^2,$$
$$= RR^*S_*^2,$$
$$= TS_*^2,$$
$$= S_*^2,$$

$$(21) \qquad \mathcal{E}(vv') = \mathcal{E}(I - S_*^2R^*)zz'(I - R^*S_*^2),$$
$$= R - S_*^2R^*R - RR^*S_*^2 + S_*^2R^*RR^*S_*^2,$$
$$= R - S_*^2T - TS_*^2 - S_*^2R^*S_*^2,$$
$$= R - 2S_*^2 + S_*^2R^*S_*^2,$$

$$(22) \qquad \mathcal{E}(va') = \mathcal{E}(I - S_*^2R^*)zz'R^*S_*^2,$$
$$= RR^*S_*^2 - S_*^2R^*RR^*S_*^2,$$
$$= TS_*^2 - S_*^2R^*S_*^2,$$
$$= S_*^2 - S_*^2R^*S_*^2,$$

$$(23) \qquad \mathcal{E}(aa') = \mathcal{E}S_*^2R^*zz'R^*S_*^2,$$
$$= S_*^2R^*RR^*S_*^2,$$
$$= S_*^2R^*S_*^2.$$

These covariance matrices are shown in Table 1. Remember that $R^* = EM^{-2}E'$, the generalized inverse of R, and $S_\#^2$, the diagonal matrix of anti-image variances, are constructed from (7) and (11), remembering (4) and (8).

It is interesting to permute the rows and columns of R, maintaining symmetry, so that all variates represented with *SMC*s of one follow, say, those with *SMC*s less than one. We then partition the above covariance matrices between the row and columns representing variates with *SMC*s less than one and those with *SMC*s equal to one. We see then, for example, that $\mathcal{E}(v_j v_k) = r_{jk}$, when either $SMC(J)$ or $SMC(K)$ is equal to one. Other results of this kind are left to the reader.

The formulas (18)–(23) and Table 1 apply for any R, singular or of full rank. Although computing the generalized inverse of R via the basic structure of R is more time consuming than directly computing the inverse of R when R is of full rank, it may be desirable to always compute the basic structure and the generalized inverse of R to find image covariance matrices, as it may not always be known ahead of time that R is of full rank.

An anonymous referee has pointed out that some of the results of this paper occur in a somewhat different form in an article by Professor Bentler [1969]. Thus, (18)–(23) are anticipated by Bentler, although he does not develop (17), which simplifies the results given in Table 1.

Table 1

Image And Anti-Image Covariance Matrices

From A Correlation Matrix That May Be Singular

	z'	v'	a'
z	R	$R - S_\#^2$	$S_\#^2$
v	$R - S_\#^2$	$R - 2S_\#^2 + S_\#^2 R^* S_\#^2$	$S_\#^2 - S_\#^2 R^* S_\#^2$
a	$S_\#^2$	$S_\#^2 - S_\#^2 R^* S_\#^2$	$S_\#^2 R^* S_\#^2$

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