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# A COMPARATIVE STUDY OF ASSOCIATION MEASURES

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This paper discusses the general problem of measuring the association between an independent nominal-scaled variable $X$ and a dependent variable $Y$ whose scale of measurement may be interval, ordinal or nominal. The theoretical foundations of a wide range of asymmetric association measures are discussed. Some new measures are also suggested. Fifteen of these association measures, some previously suggested, some new, are singled out for a computer-assisted numerical study in which we compute the value actually taken by each measure under a wide variety of conditions. This comparative study provides important insights into the behavior of the measures.

## 1. *Introduction*

We shall study conceptual foundations and properties of a number of association measures. Some of these measures have been previously suggested and are in frequent use, others are new suggestions or modifications of existing ones. By means of a simulation approach, the behavior of the association measures is studied in a variety of situations as the association (as measured by one of them) moves from "independence" to "perfect association".

The delimitations of this paper are: We consider alternative measures of the association between the "independent" or "explanatory" variable $X$ and the "dependent" or "response" variable $Y$. $X$ precedes $Y$ causally, temporally or otherwise. Hence, we are only concerned with association in the asymmetric sense. The scale of measurement of $X$ is nominal; there are $I \geq 2$ levels or categories representing the $X$-dimension. A fixed-size sample of values of $Y$ is available at each level of $X$. Three cases are distinguished according to the scale of measurement of $Y$: Interval (or possibly ratio), ordinal and nominal.

Discussion of the conceptual bases and the behavior of association measures forms the main body of the paper. There is little of sampling theory, and no tests of zero association are proposed.

Important sources of information in the area of association are the text by Hays [1963], especially for analysis of variance oriented measure-

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ment of "strength of relationship", the review papers by Goodman and Kruskal [1954, 1959], especially for association between categorical variables, and the text by Freeman [1965] as a practical source book on association. Upon consulting these references, it is evident that the present paper deals only with a small number of all association measures available in the literature. Excepting our new suggestions, we can possibly claim that the measures considered below either are in common use or have merited some attention in recent literature.

Association measures typically take values between zero and one inclusive, where zero usually indicates "independence" and unity, if ever attained, means "perfect association" between $X$ and $Y$. In a given situation there is often a multitude of alternative association measures of this kind to choose from. It is sometimes difficult to maintain that use of one measure is more justified from a conceptual or operational point of view than another. Little is known about the behavior of the various association measures in the interior of the [0, 1] interval: The association as measured by one measure may take a drastically different value from that produced by another measure for the same set of data. The simulation study in this paper throws some light on such differences among association measures, as well as on the effect of changes in the definition of the response categories. We also discuss a general approach to the problem of correcting for the inflated values often produced by association measures under conditions close to independence.

Section 2 introduces the notation to be used. In Section 3 we discuss the meanings to be given to "independence" and "perfect association" for different types of scale of measurement of $Y$, as well as the objectives of "correction" of an association measure. In Sections 4, 5 and 6 we review certain traditional association measures, discuss theoretically their properties, and suggest a few new association measures. Finally, in Section 7 we report the results of the computer simulation study of the performance of 15 association measures.

## 2. *Presentation of Models to be Considered*

We introduce the notation and the statistical models to be used in the three cases: $Y$ interval-, ordinal-, or nominal-scaled.

First, if $Y$ is interval-scaled, we shall consider the customary random effects ANOVA model

$$(2.1) \qquad Y_{ik} = \mu + x_i + \epsilon_{ik} \ (i = 1, \cdots, I;\ k = 1, \cdots, n_{i.}),$$

where the $n_{i.}$ are fixed, positive integers, where $x_i$ has mean 0, variance $\sigma_x^2$ and $\epsilon_{ik}$ has mean 0, variance $\sigma^2$, and the $x_i$ and $\epsilon_{ik}$ are mutually independent. (Since the distribution theory plays a subordinated role in the paper, we do not generally need to make the usual normality assumption.) Let

$$
(2.2) \quad
\begin{aligned}
n &= \sum_i n_{i.}\,, \qquad \bar{Y} = n^{-1} \sum_i \sum_k Y_{ik}\,, \qquad \bar{Y}_i = n_{i.}^{-1} \sum_k Y_{ik}\,,\\
SS_T &= \sum_i \sum_k (Y_{ik} - \bar{Y})^2, \qquad SS_W = \sum_i \sum_k (Y_{ik} - \bar{Y}_i)^2\\
SS_H &= \sum_i n_{i.} (\bar{Y}_i - \bar{Y})^2,\\
MS_H &= (I - 1)^{-1} SS_H\,, \qquad MS_W = (n - I)^{-1} SS_W
\end{aligned}
$$

where $\sum_i$ and $\sum_k$ denote summations from 1 to $I$ and from 1 to $n_{i.}$, respectively. Under the random effects model (2.1) with $n_{i.} = m$ $(i = 1, \cdots, I)$, we have [see, for example, Scheffé, 1959] $E(MS_H) = \sigma^2 + m\sigma_x^2$, $E(MS_W) = \sigma^2$.

Alternatively, we shall sometimes treat (2.1) as a model with fixed effects $x_i$ obeying the constraint $\sum_i n_{i.} x_i = 0$. In that case the $\epsilon_{ik}$ are assumed to have mean 0, variance $\sigma^2$. (Normality is generally not assumed.) We have $E(MS_H) = \sigma^2 + (I - 1)^{-1} \sum_i n_{i.} x_i^2$, $E(MS_W) = \sigma^2$.

Secondly, if $Y$ is measured on a nominal scale, we assume that there exist two polytomies with $I$ categories of the $X$-dimension (forming the $I$ rows of a contingency table) and $J$ response categories of the $Y$-dimension (forming the $J$ columns of the contingency table). The observed frequency in cell $i$, $j$ of the contingency table is denoted by $n_{ij}$ $(i = 1, \cdots, I;\ j = 1, \cdots, J)$, and $n_{i.} = \sum_j n_{ij}$, $n_{.j} = \sum_i n_{ij}$, $n = \sum_i n_{i.} = \sum_j n_{.j}$, $p_{i.} = n_{i.}/n$, $p_{.j} = n_{.j}/n$, $p_{j|i} = n_{ij}/n_{i.}$, $p_{ij} = n_{ij}/n$. We assume that the $n_{i.}$ are fixed, non-random positive integers, and that (possibly after pooling of categories) $n_{.j} > 0$ $(j = 1, \cdots, J)$. We assume, for a given $X$-category, $i$, the model:

$$
(2.3) \qquad E(p_{j|i}) = \pi_{j|i} = \pi_{.j} + \delta_{ij} \quad (i = 1, \cdots, J),
$$

with $\sum_j \delta_{ij} = 0$ $(i = 1, \cdots, I)$, where $\pi_{.j} = E(p_{.j}) = E(\sum_i p_{j|i}\, p_{i.})$ is an unknown marginal probability. It follows that we also have the constraints $\sum_i p_{i.}\, \delta_{ij} = 0$ $(j = 1, \cdots, J)$.

Two well-known statistics (see, for example, Goodman [1971]) for the contingency table are

$$
(2.4) \qquad S_H = 2n \sum_i p_{i.} \sum_j p_{j|i} \log (p_{j|i}/p_{.j}),
$$

$$
(2.5) \qquad \tilde{S}_H = n\{ \sum_i p_{i.} \sum_j p_{j|i}^2/p_{.j} - 1\};
$$

they approximate each other and each has an approximate $\chi^2$ distribution with $(I - 1)(J - 1)$ degrees of freedom when the two polytomies are independent. We can partition $S_H$ as $S_H = S_{H1} - S_{H0}$, where

$$
\begin{aligned}
S_{H1} &= 2n \sum_i p_{i.} \sum_j p_{j|i} \log (p_{j|i}/\pi_{.j}),\\
S_{H0} &= 2n \sum_j p_{.j} \log (p_{.j}/\pi_{.j}).
\end{aligned}
$$

These terms can be approximated, respectively, by

$$\tilde{S}_{H1} = n\{ \sum_i p_{i.} \sum_j p_{j|i}^2/\pi_{.j} - 1\},$$

$$\tilde{S}_{H0} = n\{ \sum_j p_{.j}^2/\pi_{.j} - 1\}.$$

The difference $\tilde{S}_{H1} - \tilde{S}_{H0}$ will, however, only approximate $\tilde{S}_H$ . Under the model (2.3), we have $E(S_{H1}) \simeq E(\tilde{S}_{H1}) = I(J - 1) + R; E(S_{H0}) \simeq E(\tilde{S}_{H0}) = J - 1$, and

$$E(S_H) \simeq E(\tilde{S}_H) \simeq (I - 1)(J - 1) + R, \tag{2.6}$$

where

$$R = \sum_i \sum_j \delta_{ij}/\pi_{.j} + \sum_i \sum_j (n_{i.} - 1)\, \delta_{ij}^2/\pi_{.j}\,.$$

When $n$ is large, $R$ can be approximated as

$$\underline{R} \simeq n \sum_i \sum_j p_{i.}\ \delta_{ij}^2/\pi_{.j}\,, \tag{2.7}$$

the other terms being small by comparison. Formulas (2.6) and (2.7) will be subject to further analysis below, since they provide the rationale for a new association measure to be suggested in Section 5.

Thirdly, consider the case where the $J$ categories of $Y$ are ordinally arranged, wearing labels such as "low", "medium", "high", etc. Two subcases will be distinguished according as there exists a relevant continuum underlying $Y$ or not. In either case we shall use the same notation as when $Y$ is nominal, *i.e.* $n_{ij}$ denotes the observed frequency in the nominal $X$-category $i$ and the ordinal $Y$-category $j$, etc.

If there does exist a relevant continuum underlying the $Y$-dimension, the two models (2.1) and (2.3) can be related in a formal way. Assume that the ordinal classification of the $Y$-dimension has been achieved by grouping together the $Y_{ik}$ — values falling into one and the same of a set of $J$ mutually exclusive and exhaustive intervals formed by the points $y_j$ ($j = 0, 1, \cdots, J$) such that $-\infty = y_0 < y_1 < \cdots < y_{J-1} < y_J = \infty$. The observed number of $Y_{ik}$ in the interval $(y_{j-1}, y_j)$ equals $n_{ij}$ .

Let $F(z)$ denote the cumulative distribution function of $Z = (Y - \mu_{Y|X})/\sigma$ with density function $f(z) = dF(z)/dz$ tending to zero as $z \to \pm\infty$. Conditional upon $X = x_i$ , the probability of an observation falling in cell $i$, $j$ can, for $i = 1, \cdots, I; j = 1, \cdots, J$, be written in two ways which relate the parameters of models (2.1) and (2.3):

$$\pi_{.j} + \delta_{ij} = F\left(z_j - \frac{x_i}{\sigma}\right) - F\left(z_{j-1} - \frac{x_i}{\sigma}\right) \tag{2.8}$$

where $z_j = (y_j - \mu)/\sigma$ for $j = 1, \cdots, J - 1$, and $z_0 = -\infty$, $z_J = \infty$. The marginal probability of the $j$:th $Y$-category is, for $j = 1, \cdots, J$,

$$\pi_{.j} = \sum_i p_{i.}\left[F\left(z_j - \frac{x_i}{\sigma}\right) - F\left(z_{j-1} - \frac{x_i}{\sigma}\right)\right].$$

We expand $F(z_j - x_i/\sigma)$ in a Taylor series around $z_j$, thereby obtaining as a first approximation $\pi_{.j} \simeq F(z_j) - F(z_{j-1})$. (Terms of order $x_i/\sigma$ cancel, terms of order $(x_i/\sigma)^2$ and higher are dropped.) This approximation can be expected to be close only when the relative effects $x_i/\sigma$ are small, which is likely to occur if $\sigma_x/\sigma$ is small, *i.e.*, when the association is weak. Inserting the approximation into (2.8), another expansion yields $\delta_{ij} \simeq -x_i[f(z_j) - f(z_{j-1})]/\sigma$, where $f(z) = dF(z)/dz$. Inserting the two approximations into $R$ given by (2.7), we obtain

$$R \simeq nK_J \textstyle\sum_i p_{i.}x_i^2/\sigma^2, \tag{2.9}$$

where

$$K_J = \sum_{j=1}^{J} \frac{(f_j - f_{j-1})^2}{F_j - F_{j-1}} \tag{2.10}$$

with $f_j = f(z_j)$, $F_j = F(z_j)$ $(j = 1, \cdots, J-1)$, $F(z_0) = f(z_0) = f(z_J) = 0$, $F(z_J) = 1$. The end result (2.9) will be used in Section 5. The quantity $K_J$ is well-known in statistical inference; it approaches, as $J \to \infty$ and the interval-widths approach zero, the amount of Fisher-information,

$$E\left[\left(\frac{\partial \log f(z)}{\partial z}\right)^2\right].$$

## *3. Independence and Perfect Association*

All association measures considered below are constructed with the idea in mind that an association of zero means that $X$ and $Y$ are "independent". At the other extreme, the majority of the association measures to be considered are normed to take the value unity when $X$ and $Y$ are "perfectly associated" in the sense that knowledge of $X$ will remove all uncertainty as to what value $Y$ will take.

First, it will be necessary to state the exact meanings to be given in this paper to the terms "independence" and "perfect association". Each of the two terms may be employed either in a theoretical (or model, or population) context or in a sample context. Each of the eight entries of the following table denotes in a compact form (for convenient future reference) a condition defined in full detail below the table. For example, when the condition denoted by Ind $(\sigma)$, is satisfied, we shall say that $X$ and $Y$ are theoretically independent, provided $Y$ is being measured on an interval scale. Conditions referring to the theoretical context are expressed in terms of unknown population parameters, and conditions referring to the sample context are expressed in terms of statistics computed from the sample. The eight conditions are defined as follows in terms of notation introduced in Section 2.

| | Scale of measurement of $Y$ | |
|---|---|---|
| | Interval (or ratio) | Ordinal Or Nominal |
| Theoretical independence | Ind ($\sigma$) | Ind ($\pi$) |
| Sample independence | Ind ($s$) | Ind ($p$) |
| Theoretical perfect association | Pas ($\sigma$) | Pas ($\pi$) |
| Sample perfect association | Pas ($s$) | Pas ($p$) |

Ind ($\sigma$) $\Leftrightarrow \sigma_x^2 = 0$ (or, in the fixed effects model: $x_i = 0, i = 1, \cdots, I$)
Pas ($\sigma$) $\Leftrightarrow \sigma^2 = 0$
Ind ($s$) $\Leftrightarrow \bar{Y}_i = \bar{Y} \ (i = 1, \cdots, I)$
Pas ($s$) $\Leftrightarrow Y_{ik} = \bar{Y}_i \ (k = 1, \cdots, n_{i.}\ ; i = 1, \cdots, I)$
Ind ($p$) $\Leftrightarrow p_{j|i} = p_{.j} \ (j = 1, \cdots, J; i = 1, \cdots, I)$
Pas ($p$) $\Leftrightarrow$ For each $i$ ($i = 1, \cdots, I$), there exists a $j = j_0(i)$, say, such that $p_{j|i} = 1$ for $j = j_0(i)$ and zero otherwise.

The conditions Ind ($\pi$) and Pas ($\pi$) are defined in the same way as Ind ($p$) and Pas ($p$), respectively, except that $\pi_{j|i}$ is substituted for $p_{j|i}$ and $\pi_{.j}$ for $p_{.j}$.

One of the conditions may imply one or several of the other conditions. For example, perfect association in the theoretical sense implies perfect association in the sample sense, *i.e.* Pas ($\sigma$) $\Rightarrow$ Pas ($s$) and Pas ($\pi$) $\Rightarrow$ Pas ($p$). Conversely, Pas ($s$) $\Rightarrow$ Pas ($\sigma$) with probability one, and Pas ($p$) $\Rightarrow$ Pas ($\pi$) with probability tending to one when $n$ becomes large.

Turning to the case of independence, we see that Ind ($\sigma$) does not imply, nor is it implied by, Ind ($s$). The same conclusion holds for the pair Ind ($\pi$) and Ind ($p$). Theoretical independence does, however, have some important implications for the expected values of certain statistics. When $Y$ is interval-scaled, Ind ($\sigma$) implies that $SS_H$ and $SS_W$ have central $\chi^2$-distributions (if the normality assumption is added) and that

$$E(MS_H - MS_W) = 0. \tag{3.1}$$

When $Y$ is ordinal or nominal, Ind ($\pi$) implies that $\tilde{S}_H$ and $S_H$ have approximate $\chi^2$-distributions with $f_H = (I - 1)(J - 1)$ degrees of freedom each, hence

$$E(S_H - f_H) \simeq E(\tilde{S}_H - f_H) \simeq 0. \tag{3.2}$$

The behavior of association measures at the lower end of the range (under conditions of independence or near independence) suggests that one should dichotomise association measures into *uncorrected measures* and *corrected measures*. In this paper, the first kind is defined as follows:

*Uncorrected association measure*: an association measure that takes the value zero under conditions of sample independence, Ind ($s$) or Ind ($p$).

The term corrected measure will be defined below following a discussion of the objectives of "correction".

An often used method of constructing a normed uncorrected association measure consists in forming the ratio

$$\frac{U(Y) - U(Y \mid X)}{U(Y)} \tag{3.3}$$

where the uncertainties in $Y$ with and without knowledge of $X$, $U(Y \mid X)$ and $U(Y)$, respectively, are such that $U(Y) = U(Y \mid X)$ under Ind ($s$) or Ind ($p$), and $U(Y \mid X) = 0$ under Pas ($s$) or Pas ($p$). (We are using the term "uncertainty" in a general sense, *e.g.*, more general than in the context of uncertainty analysis, Garner and McGill [1956].) The interpretation of any association measure of this type is that of "relative reduction in uncertainty about $Y$ from getting to know $X$". The value taken by (3.3) is zero under Ind ($s$) or Ind ($p$) and unity under Pas ($s$) or Pas ($p$). The difficulty is that there are infinitely many association measures of the type (3.3) corresponding to the infinite variety of definitions of the concept "uncertainty". Or, in more concrete terms, how should one interpret a computed association of 0.60 in a certain situation, is it "strong", "weak", etc.? The answer depends a great deal on the definition of uncertainty used in defining the association measure that gave the 0.60 value.

Frequently, formula (3.3) gives an inflated value of the degree of association under conditions approaching independence. Denote by $E$ the expected value of $U(Y) - U(Y \mid X)$ under Ind ($\sigma$) or Ind ($\pi$). Let $\hat{E}$ be an unbiased estimate of $E$. Consider the association measure

$$\frac{U(Y) - \hat{E} - U(Y \mid X)}{U(Y) - \hat{E}} \tag{3.4}$$

Its numerator has the expected value zero under theoretical independence, and it preserves the property of attaining the value unity under Pas ($s$) or Pas ($p$).

One may wish to further adjust the denominator of (3.4). For example, the measure (3.4) still attains the value unity under Pas ($s$) and Pas ($p$) if any multiple of $U(Y \mid X)$ were to be added to the denominator. Or, one may wish to make the further correction of (3.4) in such a way that the denominator has a specified expected value.

As an example consider the association measure (4.3) below, a variation of Hays' (1963) $\omega^2$, which is constructed to give a ratio of expected values equal to the intraclass correlation $\sigma_x^2/(\sigma_x^2 + \sigma^2)$. The true objective is, of course, to construct an unbiased estimate of $\sigma_x^2/(\sigma_x^2 + \sigma^2)$. However, since the expected value of the ratio is difficult to handle, one chooses to correct in such a way that the ratio of expected values equals the desired parametric function. The bias is likely to be small for most practical purposes.

We shall use the term "corrected association measure" in a fairly wide sense in this paper, a sense which implies that the main purpose of correction is to prevent obtaining an inflated value of the association under conditions of independence or near-independence:

*Corrected association measure*: an uncorrected association measure adjusted in such a way that the expected value of its numerator is zero under conditions of theoretical independence, Ind $(\sigma)$ or Ind $(\pi)$.

In the case where $Y$ is an interval-scaled variable, one can find several examples in the literature of corrected association measures, notably the several "$\omega^2$-like" measures discussed in Section 4; all of these achieve correction of the correlation ratio (4.1) by utilizing (3.1). The practice of "correcting" seems, however, to be virtually non-existent when $Y$ is ordinal or nominal, even though in these cases the methodological reasons for correction seem as well-founded as in the case where $Y$ is interval-scaled. When the approximate $\chi^2$-variables $S_H$ and $\tilde{S}_H$ are involved, correction can be achieved by utilizing (3.2); see several suggestions in this direction in Sections 5 and 6.

Consider now the behavior of association measures at the upper end of their respective ranges (*i.e.*, under conditions of perfect association or near-perfect association). All association measures considered in this paper take values in the interval [0, 1]. Most of those measures attain their maximum value of unity (under Pas $(s)$ or Pas $(p)$), but there are a few that do not. It is difficult to find in the literature any extensive discussion of the pros and cons of having an association measure actually attain the upper limit of unity. Cohen [1965, p. 105] implies that it is a deficiency of a measure if it fails to do so, which is the case, for example, with the Pearson mean square contingency coefficient $\tilde{S}_H/(\tilde{S}_H + n)$.

We offer one argument in justification of using an association measure that falls short of the value unity under Pas $(p)$: The condition Pas $(s)$ implies Pas $(p)$, but the reverse is not true. Information is lost when measurement is in terms of the weaker scale. Assume that nominal or ordinal data on $Y$ have been gathered by counting frequencies in non-overlapping intervals of an underlying interval-scaled variable, and assume that the data, in the form of a contingency table, reveal that Pas $(p)$ holds. Had we also been able to go back and observe the underlying interval-scaled $Y$-data, chances are we would have found that Pas $(s)$ does not hold; let us assume that this is the case. Hence, even though Pas $(p)$ holds, the association would be less than unity as measured by any association measure requiring interval-scaled $Y$ and such that its upper limit of unity is attained only if Pas $(s)$ holds.

Thus if we appeal to a desire for comparability of association measures across different scales and allow for the possibility of loss of information due to the weaker scale, then in the case where $Y$ is ordinal or nominal, we should measure association by a measure that attains an upper limit of less than unity under Pas ($p$). Two such measures are suggested in Section 5, formulas (5.5) and (5.6); Freeman's [1965] $\theta$ is also of this kind, formula (5.1), as well as the contingency coefficients $C_1^2$ and $C_2^2$, (6.1) and (6.2).

It is a weakness of this argument that it is impossible to know to what extent one should allow for loss of information. It is also difficult to give wholehearted support to the idea that association measures ought to have comparability across types of scale; most of the time the nature of the data precludes any kind of comparability. Nevertheless, some commonly used association measures do fail to attain the value unity.

### *4. Association Measures when Y is Interval-scaled*

In this and the following two sections we discuss the foundations of the association measures to be studied numerically in Section 7 by simulation techniques. Several new measures are introduced in Sections 5 and 6.

The well-known correlation ratio, $\hat{\eta}^2$, is an uncorrected association measure of the form (3.3) with the uncertainties $U(Y) = SS_T$, $U(Y \mid X) = SS_W$ as defined by (2.2). Hence, letting $F = MS_H/MS_W$,

$$\hat{\eta}^2 = \frac{SS_H}{SS_T} = \frac{F(I-1)}{F(I-1)+n-I} \tag{4.1}$$

It takes the value zero under Ind ($s$) and attains the value unity under Pas ($s$). There is, of course, a multitude of other measures with the same properties.

Instead of the sums, $SS_T$ and $SS_W$, of *squared* deviations around $\bar{Y}$ and $\bar{Y}_i$, one could define the uncertainties $U(Y)$ and $U(Y \mid X)$ in (3.3) as the corresponding sums of some other form of non-negative deviations around $\bar{Y}$ and $\bar{Y}_i$ : The *absolute* deviations is but one example. The rather lively debate around association measures in recent applied psychology literature (see references at the end of this section) has never questioned the legitimacy of using squared deviations in defining uncertainty, but has centered around the question of finding the most suitable correction. Hence, $\hat{\eta}^2$ and its corrected versions to be discussed below are the only measures in common use when $Y$ is interval-scaled. This may be due to the strong position of traditional normal-theory based analysis of variance as a technique of statistical inference in this situation. On the other hand, when $Y$ is nominal (Section 6), there is much less agreement on the preferred quantification of uncertainty.

Various corrected versions of the correlation ratio have been suggested, *e.g.*, (4.2) $$\omega^2 = \frac{SS_H - (I-1)MS_W}{SS_T + MS_W} = \frac{(F-1)(I-1)}{(F-1)(I-1)+n}$$

see Hays [1963], Vaughan and Corballis [1969], and

$$\epsilon^2 = \frac{SS_H - (I-1)MS_W}{SS_T} = \frac{(F-1)(I-I)}{(F-1)(I-1)+n-1}$$

suggested by Kelly [1935], and discussed by Cohen [1965]. Both $\omega^2$ and $\epsilon^2$ accomplish a correction suitable under the fixed effects model: In the case of $\omega^2$, the ratio of expected values of numerator and denominator (when expressed in terms of $SS$'s and $MS$'s) is $\sum_i p_{i.} x_i^2/(\sum_i p_{i.} x_i^2 + \sigma^2)$. A slightly different expected denominator obtains for $\epsilon^2$.

The corrected measure

(4.3) $$\omega_1^2 = \frac{I(MS_H - MS_W)}{SS_T + MS_H} = \frac{(F-1)I}{(F-1)I+n}$$

discussed by, for example, Hays [1963, p. 423], Vaughan and Corballis [1969] is suited for the random effects model; it produces a ratio of expected values equal to $\sigma_x^2/(\sigma_x^2 + \sigma^2)$.

Hence, $\omega^2$, $\epsilon^2$ and $\omega_1^2$, which satisfy $\omega^2 < \epsilon^2 < \omega_1^2$, capitalize on the same correction idea: In each case, the expected value of the numerator is zero under Ind ($\sigma$) due to (3.1). Each attains the value unity under Pas ($s$).

One should have equal group frequencies $n_{i.}$ $(i = 1, \cdots, I)$ in order to be able to meaningfully apply any of the measures $\omega^2$, $\epsilon^2$ and $\omega_1^2$, see, for example, Vaughan and Corballis [1969]. Further recent discussions of these "$\omega^2$-like" measures are found in Fleiss [1969], Friedman [1968], Halderson and Glasnapp [1972], Kennedy [1970]. Finally, Glass and Hakstian [1969] discuss problems with the interpretation of "$\omega^2$-like" measures: When the levels of $X$ are not randomly representative of the $X$-dimension, as could be the case in the fixed-effect ANOVA design, it may not make sense at all to try to attach a coefficient of association as a measure of the strength of relationship between $X$ and $Y$. The measures $\hat{\eta}^2$, $\omega^2$ and $\omega_1^2$ will be studied numerically in Section 7.

## *5. Association Measures When Y is Ordinal*

The case of $X$ nominal, $Y$ ordinal contains two subcases depending on whether a relevant continuum underlies the $Y$-dimension or not. We start with the latter case. All association measures considered in this section are left unchanged if the rows of the contingency table are permuted (because $X$ is nominal).

One possibility (not to be considered in this paper) is to assign a set of scores, *e.g.*, 1, 2, $\cdots$, to the rank ordered $Y$-categories, and construct association measures that can be computed from the scores. A difficulty with this approach is that the choice of scores would have to be arbitrary in most situations.

The basic idea of Freeman's [1965, pp. 108–199] $\theta$ relates to guessing the ordinal number of $Y$ (with no underlying continuum), given knowledge of the nominal $X$ category. Kendall's tau and Goodman–Kruskal's $\gamma$ are based on a similar idea but require that both $X$ and $Y$ be ordinal and hence do not qualify under the auspices of this study. Freeman's $\theta$ can be written as

$$\theta = \frac{\sum_i \sum_{i'>i} p_{i.}p_{i'.} \, |1-B_{ii'} - 2D_{ii'}|}{\sum_i \sum_{i'>i} p_{i.}p_{i'.}} \tag{5.1}$$

where

$$B_{ii'} = \textstyle\sum_j p_{j|i} \, p_{j|i'} \; ; \; D_{ii'} = \sum_j \sum_{j'>j} p_{j|i} \, p_{j'|i'}. \tag{5.2}$$

Under Ind $(p)$, $B_{ii'} + 2D_{ii'} = 1$, hence $\theta$ takes the value zero. However, Pas $(p)$ will not make $\theta$ equal to unity (the extent to which $\theta$ can fall short of unity is indicated by the numerical illustrations in Section 7.) We suggest therefore a modified version of Freeman's $\theta$ that does attain the value unity under Pas $(p)$.

As a starting point, consider Goodman and Kruskal's [1954, p. 749] $\gamma$, a symmetrical association measure for two ordinal polytomies $X$ and $Y$ with no relevant underlying continua; it takes values between $-1$ and 1 inclusive. For $I = 2$ categories of the $X$-dimension it can be written as $\gamma = -(1 - B_{12} - 2D_{12})/(1 - B_{12})$, where $B_{12}$ and $D_{12}$ are given by (5.2) with $i = 1$, $i' = 2$.

We construct an association measure for an arbitrary number, $I$, of nominal $X$-categories as follows: Denote by $g_{ii'}$ the absolute value of the Goodman–Kruskal $\gamma$ for the pair, $i$, $i'$ $(i \neq i' = 1, \cdots, I)$ of $X$-categories; hence $g_{ii'} = |1 - B_{ii'} - 2\,D_{ii'}|/(1 - B_{ii'})$. Take a weighted average, called $\kappa$, of the $g_{ii'}$ with the weights $w_{ii'} = p_{i.}\, p_{i'.}\,(1 - B_{ii'})$. The resulting association measure is

$$\kappa = \frac{\sum_i \sum_{i'>i} p_{i.}p_{i'.} \, |1 - B_{ii'} - 2D_{ii'}|}{\sum_i \sum_{i'>i} p_{i.}p_{i'.}(1 - B_{ii'})} . \tag{5.3}$$

The measure $\kappa$ has the following properties:

(1) for $I = 2$, $\kappa$ equals the absolute value of Goodman–Kruskal's $\gamma$,
(2) $\kappa$ differs from Freeman's $\theta$, (5.1), in that the denominators are somewhat different.
(3) $\kappa$ is zero under Ind $(p)$ and unity under Pas $(p)$,
(4) $\kappa$ is undefined (of the form 0/0) if Pas $(p)$ holds with $j_0(i) = j_0$ $(i = 1, \cdots, I)$.

Pas $(p)$ is a sufficient but not necessary condition to render $\kappa$ the value unity. Consider, for example, the following table of relative sample frequencies $p_{ij}$ with $I = 3$ $X$-categories and $J = 5$ $Y$-categories:

$$(5.4) \qquad \begin{matrix} (1-a)p_{1.} & ap_{1.} & 0 & 0 & 0 \\ 0 & 0 & 0 & (1-b)p_{2.} & bp_{2.} \\ 0 & 0 & p_{3.} & 0 & 0 \end{matrix}$$

where $0 < a < 1$, $0 < b < 1$. Pas ($p$) is clearly not satisfied, yet $\kappa$ takes the value unity. This drawback can be overcome if we use instead, say, $\kappa' = \kappa(L - 1)/(J - 1)$ where $L = \min(I, J)$.

Freeman's $\theta$ and the new suggestion $\kappa$ will be included in the numerical study of association measures in Section 7.

Consider next the case where a $Y$ is ordinal with a continuum (an interval or ratio scale) underlying the $Y$-dimension with a distribution function of $Y$ represented by $F[(y - \mu_{Y|X})/\sigma]$. According to (2.6) and (2.9),

$$E(S_H) - f_H \simeq E(\tilde{S}_H) - f_H \simeq nK_J \{\textstyle\sum_i p_{i.} x_i^2\}/\sigma^2,$$

where $f_H = (I - 1)(J - 1)$. This suggests two measures of association which can be expected to produce values very close to Hays' [1963] $\omega^2$ for small values of $\omega^2$, namely,

$$(5.5) \qquad G_1 = \frac{\tilde{S}_H - f_H}{\tilde{S}_H - f_H + K_J n},$$

and the closely related measure

$$(5.6) \qquad G_2 = \frac{S_H - f_H}{S_H - f_H + K_J n},$$

where $f_H = (I - 1)(J - 1)$, and $\tilde{S}_H$, $S_H$ and $K_J$ are given by (2.5), (2.4) and (2.10), respectively. By imposing an assumed continuous, zero mean, unit variance shape on the marginal $Y$-distribution, $K_J$ can be computed from the observed marginal relative frequencies $p_{.j}$ as follows: In formula (2.10), let $F_j = \sum_{v=1}^{j} p_{.v}$ $(j = 1, \cdots, J)$, $F_0 = 0$, and let $f_j$ be the ordinate of the probability density function of $F$ at a point $z_j$ point such that the area under the curve from $-\infty$ to $z_j$ is $F_J$, and $f_0 = f_J = 0$. If the assumed shape is a $N(0, 1)$ curve, as in the numerical studies of $G_1$ and $G_2$ in Section 7, even a fairly small $J$ will usually give a $K_J$ above 0.8. Unity is the upper limit of $K_J$ for the normal distribution.

$G_1$ and $G_2$, which are studied numerically in Section 7, have the following properties:

(1) both are corrected measures: the expected value of the numerator is (approximately) zero under Ind ($\pi$),

(2) both measures are bounded from above by $(L - 1)/(L - 1 + K_J)$, where $L = \min(I, J)$. This value is attained (approximately) by $G_1$ and $G_2$ if Pas ($p$) holds.

(3) The ratio of expected values of the numerator and of the denominator of both $G_1$ and $G_2$ can be approximated as $\sum_i p_{i.} x_i^2/(\sum_i p_{i.}x_i^2 + \sigma^2)$ when $\sum_i p_{i.}x_i^2$ is small in relation to $\sigma^2$. The measure can therefore be expected to behave like Hays' $\omega^2$ when the association is weak and the distribution of the continuum assumed to underlie $Y$ agrees reasonably well with the true one.

## *6. Association Measures when Y is Nominal*

With both $X$ and $Y$ nominal, each association measure considered in this section will be left unchanged by any permutation of rows or columns in the contingency table. It is important to realize that the values produced by any association measure must in this case be interpreted in relation to the particular definition of $X$- and $Y$-categories used, see Section 7. Many association measures are geared particularly towards the $2 \times 2$ table. These special cases are not considered here.

Karl Pearson's coefficient of contingency is a "traditional" measure. We consider its square,

$$C_1^2 = \frac{\tilde{S}_H}{\tilde{S}_H + n}, \tag{6.1}$$

where $\tilde{S}_H$ is the $\chi^2$ statistic (2.5). A minor variation on this idea is

$$C_2^2 = \frac{S_H}{S_H + n}, \tag{6.2}$$

with $S_H$ given by (2.4); very similar values will be produced by $C_1^2$ and $C_2^2$ in a given situation, see Section 7. $C_1^2$ and $C_2^2$ become zero under Ind $(p)$ but take a value of less than unity under Pas $(p)$. Correction of $C_1^2$ and $C_2^2$ is desirable, since highly inflated values are obtained under conditions at or near Ind $(\pi)$, see Section 7. Letting $f_H = (I - 1)(J - 1)$, a corrected version of $C_1^2$ is

$$C_{1c}^2 = \frac{\tilde{S}_H - f_H}{\tilde{S}_H - f_H + n}.$$

Goodman and Kruskal [1954, p. 740], who complained about the "non-interpretability" of traditional measures like $C_1^2$, favor measures that can be given an "operational definition" (like $\lambda$ and $\tau$ below). We shall discuss two families of association measures each of which has, if not an operational basis at least a "conceptual definition". The two concepts used are "information" and "discrimination information", both taken in a generalized sense as developed in Särndal [1970].

The information-based measures can be explained within the framework of a multinominal experiment [Särndal, 1970], where the possible outcomes in each trial of the experiment are $E_1, \cdots, E_J$ with probabilities $\pi_1, \cdots, \pi_J$, respectively. Let $\psi(\pi) > 0$ be a non-increasing function of $\pi$ in the interval $0 \leq \pi \leq 1$ such that $\psi(1) = 0$, where $\psi(\pi)$ denotes the information in the experiment if the realized outcome of the experiment has probability $\pi$ of occurring. The expected information in one trial of this experiment is

$$\sum_i \pi_i \psi(\pi_i), \tag{6.3}$$

which is zero if one of the outcomes is realized with certainty. The unknown theoretical probabilities $\pi_i$ will be replaced by their unbiased sample estimates for purposes of computing (6.3) from a given sample.

The construction of an association measure by means of the general formula (3.3) requires a specification of $U(Y)$, $U(Y \mid X)$. Using (6.3), we define them as, respectively, the expected marginal information and the weighted expected conditional (with the $p_{i.}$ as weights) information, *i.e.*,

$$U(Y) = \sum_j p_{.j} \psi(p_{.j}); \quad U(Y \mid X) = \sum_i p_{i.} \sum_j p_{j \mid i} \psi(p_{j \mid i}). \tag{6.4}$$

Inserting these expressions into (3.3) we obtain an association measure belonging to a family of such with infinitely many members corresponding to the infinitely many possible choices of the function $\psi$. Cases (1)–(3) below are commonly used association measures belonging to this family.

(1) Let $\psi(\pi) = 1 - \pi$ in (6.4). The association measure (3.3) takes the form

$$\tau = \frac{\sum_i p_{i.} \sum_j p_{j \mid i}^2 - \sum_j p_{.j}^2}{1 - \sum_j p_{.j}^2} \tag{6.6}$$

Goodman and Kruskal [1954], who attribute this measure to a suggestion by W. Allen Wallis, justify the measure in terms of a strategy for "proportional prediction" of the $Y$-category (see below). Light and Margolin [1971], who derive (6.6) as an expression of the "proportion of variation explained" by use of a variation measure for categorical data due to Gini, show that $(n - 1)(J - 1)\tau$ has an approximate $\chi^2$-distribution with $(I - 1)(J - 1)$ degrees of freedom under Ind $(\pi)$.

(2) In (6.4), let $\psi(\pi) = 1$ if $0 \leq \pi < \pi_0$ and $\psi(\pi) = 0$ if $\pi_0 \leq \pi < 1$, where $\pi_0 = \max_j \pi_j$. Formula (3.3) gives the association measure

$$\lambda = \frac{\sum_i p_{i.} \max_j p_{j \mid i} - \max_j p_{.j}}{1 - \max_j p_{.j}} \tag{6.7}$$

This is Guttman's [1941] measure of predictive association. The asymptotic distribution theory of $\lambda$ and $\tau$ is discussed by Goodman and Kruskal [1963, 1972].

(3) Let finally $\psi(\pi) = -\log \pi$ in (6.4). The information measure (3.3) becomes

$$H = \frac{I(Y) - I(Y \mid X)}{I(Y)}, \tag{6.8}$$

where $I(Y) = -\sum_j p_{.j} \log p_{.j}$ ; $I(Y \mid X) = -\sum_i p_{i.} \sum_j p_{j|i} \log p_{j|i}$ are information statistics of the Shannon–Weaver type. Alternatively, (6.8) can be expressed in terms of the approximate $\chi^2$-statistic (2.4) as $H = S_H/[S_H + 2nI(Y \mid X)]$.

The operational (= prediction strategical) framework favored by Goodman and Kruskal [1954, 1959, 1963, 1972] and discussed by Greeno [1973] can also be used to define and interpret a family of association measures to which $\kappa$ and $\tau$ (but not $H$) belong. Let the strategy consist in predicting that an observation belongs in the $j$:th $Y$-category in a proportion of cases equal to $p_{j|i}^q/\sum_j p_{j|i}^q$ (when the $X$-category is known) and $p_{.j}^q/\sum_j p_{.j}^q$ (when knowledge of $X$ is lacking), where $q \geq 0$ is an arbitrary constant. The expected proportion of incorrect guesses is $1 - \sum_j p_{j|i}^{q+1}/\sum_j p_{j|i}^q$ when $X$ is known, and $1 - \sum_j p_{.j}^{q+1}/\sum_j p_{.j}^q$ without knowledge of $X$. Expressing the difference between the latter and the weighted average of the former as a fraction of the latter, we obtain an association measure of the type (3.3) of which (6.6) and (6.7) are the special cases corresponding to $q = 1$ and $q = \infty$, respectively. In the case of $q = \infty$, the strategy predicts with probability one that an observation belongs in the $Y$-category with the highest probability; this strategy minimizes the number of incorrect decisions in the long run.

The idea of basing an association measure on Shannon–Weaver information (and, presumably, other types of information measures) does not appeal to Goodman and Kruskal [1959, p. 147]; they conclude ". . . we were unable to satisfy ourselves that such measures would have reasonable interpretations for many contexts in which cross classifications appear." Nevertheless, the measure (6.8), sometimes called the asymmetric uncertainty coefficient, seems to be frequently used and was recommended by Hays [1963, p. 612], on the basis of the exploration of information-theoretical ideas in psychology presented in texts by Attneave [1959] and Garner [1962]. Earlier, Linfoot [1957] discussed a "logarithmic index of correlation", $r_0$ (the term derives from a Spanish publication by Castañs, [1955]), which after simplification can be written as $r_0 = I(Y) - I(Y \mid X)$. Thus $r_0 = 0$ under Ind ($p$), and $r_0$ differs from (6.8) only in that $r_0$ has not been normed to take the value unity under Pas ($p$). In the area "uncertainty analysis", see McGill [1954], Garner and McGill [1956], Attneave [1959], Garner [1962], the quantities $I(Y)$, $I(Y \mid X)$ and $I(Y) - I(Y \mid X)$ are called total uncertainty, conditional uncertainty and contingent uncertainty, respectively.

Each of the uncorrected measures constructed from (3.3) by means of the uncertainties (6.4) can be transformed into a corrected measure, *e.g.* by formula (3.4), following an analysis of the expected value of the numerator under Ind ($\pi$). For categorical data, corrected association measures do not seem to be in common use, a situation that should be rectified. As examples, corrected versions of (6.6) and (6.8) are given below.

Under the model (2.3), we find

$$nE[1 - \sum_i p_{i.} \sum_j p_{j|i}^2] = (n - I)(1 - \sum_j \pi_{.j}^2) + R_1 ,$$

$$nE[1 - \sum_j p_{.j}^2] = (n - 1)(1 - \sum_j \pi_{.j}^2),$$

where $R_1$ is a function of the $\delta_{ij}$ such that $R_1 = 0$ when all $\delta_{ij} = 0$, as is the case under condition Ind ($\pi$). Therefore, letting $g = (n - I)/(n - 1)$, a corrected version of (6.6) is obtained from (3.4):

$$\tau_c = \frac{g(1 - \sum_j p_{.j}^2) - (1 - \sum_i p_{i.} \sum_j p_{j|i}^2)}{g(1 - \sum_j p_{.j}^2)} \tag{6.9}$$

Use of (3.2) with $f_H = (I - 1)(J - 1)$ gives a corrected version of (6.8),

$$H_c = \frac{S_H - f_H}{S_H - f_H + 2nI(Y \mid X)}. \tag{6.10}$$

In summary, (6.9) and (6.10) have been corrected so that their numerators have zero expected value under Ind ($\pi$) (in the case of (6.10) only approximately zero). Both measures attain the value unity under Pas ($p$).

We explore briefly another idea by which a family of association measures can be constructed: By means of the concept of "discrimination information" [Kullback, 1957] and generalizations thereof [Särndal, 1970], sometimes called "distance", "separation" or "discrepancy".

Let $\beta$ be a constant such that $0 \leq \beta \leq 1$ and define the distance between the conditional distribution of $Y$, given $X = x_i$, and the marginal distribution of $Y$ as

$$D_{i\beta} = \begin{cases} \sum_j (p_{j|i}/p_{.j})^\beta p_{j|i} - 1 & \text{if} \quad 0 < \beta \leq 1 \\ \sum_j p_{j|i} \log (p_{j|i}/p_{.j}) & \text{if} \quad \beta = 0 \end{cases}$$

A weighing of the distances $D_{i\beta}$, using the $p_{i.}$ as weights, gives a measure with the interpretation of average distance for the whole contingency table, $D_\beta = \sum_i p_{i.} D_{i\beta}$, which is zero under Ind ($p$). In order to scale $D_\beta$ to taking values in the [0, 1] interval, we divide $D_\beta$ by a suitable constant $D_\beta^0$, hence obtaining the association measure

$$V_\beta = D_\beta/D_\beta^0,$$

where $\beta$ is any constant such that $0 \leq \beta \leq 1$. The case $\beta = 0$ leads to the uncertainty coefficient $H$, (6.8). If $0 < \beta \leq 1$ and $I = J$, we may take $D_\beta^0$

to be the value of $D_\beta$ under Pas $(p)$, $D_\beta^0 = \sum_i p_{i.}^{1-\beta} - 1$; hence $V_\beta$ becomes zero under Ind $(p)$ and unity under Pas $(p)$. If $I \neq J$, the choice of a suitable $D_\beta^0$ is not straightforward except when $\beta = 1$: If we take $D_1^0 = L - 1$, where $L = \min(I, J)$ we obtain Cramer's [1946, p. 282] association measure

$$(6.11) \qquad V_1 = \frac{\sum_i p_{i.} \sum_j p_{j|i}^2/p_{.j} - 1}{L - 1} = \frac{\tilde{S}_H}{n(L-1)},$$

which is zero under Ind $(p)$ and unity under Pas $(p)$. A weakness of this measure is that it takes the value unity also in some cases where Pas $(p)$ does not hold, such as Table (5.4). To avoid this, one might do better to divide the numerator of $V_1$ by $D_1^0 = J - 1$, which measure still attains unity under Pas $(p)$. If $0 < \beta < 1$ and $I \neq J$, one may divide $D_\beta$ by $D_\beta^0 = J^\beta - 1$; this gives an association measure $V_\beta$ whose maximum value, under Pas $(p)$, is less than unity unless all marginal $Y$-frequencies are equal.

We have shown that a conceptual definition such as "information" or "discrimination information" can be used to define broad families of association measures. Commonly used measures such as $\tau$, $\lambda$, $H$ and $V_1$ belong to these families, but are not necessarily "better" than other members of the families. (An example of "standard usage" of association measures: The computer package SPSS [Statistical Package for the Social Sciences; Nie, Bent and Hull, 1970] computes $\lambda$, $H$, $C_1^2$, $V_1$ as well as others, *e.g.*, Kendall's tau and Goodman–Kruskal's $\gamma$.) Members of the same family of association measures may, of course, still display considerable dissimilarity in the values they take for a given contingency table. This can be seen from the empirical study in Section 7, where $\tau$, $\tau_c$, $H$, $H_c$, $\lambda$, $V_1$, $C_1^2$ and $C_2^2$ are compared.

## 7. *Empirical Study*

In order to compare the various association measures, we conducted a small-scale computer simulation study. Samples of artificial data were generated according to the random effects ANOVA model

$$(7.1) \qquad Y_{ik} = X_i + \epsilon_{ik}$$

where $i = 1, \cdots, I; k = 1, \cdots, n_{i.}; X_i = 6 + \sqrt{\alpha}\, Z_i$; $\epsilon_{ik} = \sqrt{1-\alpha}\, \eta_{ik}$; and $Z_i$ $(i = 1, \cdots, I)$ is a vector of independent $N(0, 1)$ random numbers, $\eta_{ik}$ $(i = 1, \cdots, I; k = 1, \cdots, n_{i.})$ are $I$ vectors of independent $N(0, 1)$ random numbers, also independent of the $Z_i$, and $\alpha$ is a constant such that $0 \leq \alpha \leq 1$.

In our study, we used $I = 8$ $X$-categories with $n_{i.} = 50$ observations each, for a total of $n = 400$ observations. The constant 6 was added merely to ensure that all $Y_{ik}$ be positive. Eleven data sets of 400 observations each were created by setting $\alpha = 0\ (0.1)1$. The extreme cases of $\alpha = 0$ and $\alpha = 1$

are equivalent to, respectively, conditions Ind ($\sigma$) and Pas ($\sigma$), which in turn ensure that Ind ($\pi$) and Pas ($\pi$), respectively, hold.

For each value of $\alpha$, the continuous data on $Y$ were converted into a contingency table with ordinally arranged $Y$-categories by counting, for each $X$-category, the number of $Y_{ik}$ in each of the intervals $6 + d(r - 0.5)$ to $6 + d(r + 0.5)$, for $r = 0, \pm 1, \pm 2, \cdots$. For each value of $\alpha$, we tried two different class-widths, $d = 0.5$ and $d = 1$ in order to get some indication of the effect that a change in definition of the response categories might produce in the values taken by the various association measures. The number, $J$, of $Y$-categories required will be around 13 when $d = 0.5$ and around 7 when $d = 1.0$. The same contingency tables were then used as representing the case of nominal $Y$-categories, pretending that the ordinal information had been lost.

For each of the two values of $d$ and each of the eleven $\alpha$-values, the value taken by each of the 15 association measures $\hat{\eta}^2(4.1)$, $\omega^2(4.2)$, $\omega_1^2(4.3)$, $\theta(5.1)$, $\kappa(5.3)$, $G_1(5.5)$, $G_2(5.6)$, $C_1^2(6.1)$, $C_2^2(6.2)$, $\tau(6.6)$, $\lambda(6.7)$, $H(6.8)$, $\tau_c(6.9)$, $H_c(6.10)$, $V_1(6.11)$ was computed.

These computations were then repeated five times, using each time a new random vector $Z_i$ ($i = 1, \cdots, 8$) and eight new random vectors $\eta_{ik}$ ($k = 1, \cdots, 50$). Hereby, one can obtain at least a rudimentary notion of the variability of each association measure. The mean of the five values of each association measure was also computed.

The results of the study are reported in Tables 1 and 2 for $d = 0.5$, 1.0 and for $\alpha = 0, 0.2, 0.5, 0.8, 1.0$. We have also included the values of

$$s_z = \sqrt{\frac{\sum_i (Z_i - \bar{Z})^2}{I - 1}}, \qquad s = \sqrt{\frac{\sum_i \sum_k (\eta_{ik} - \bar{\eta}_i)^2}{n - I}},$$

since the association measures are, as one would expect, sensitive to $s_z$ and to $s$.

Under the random effects model (7.1), $\sigma_z^2 = \alpha$ and $\sigma^2 = 1 - \alpha$. Hence, in the case of $\omega_1^2$, the ratio of expected values is $\sigma_z^2/(\sigma_z^2 + \sigma^2) = \alpha$, but $\omega_1^2$ has a certain bias as an estimate of $\alpha$. From sample to sample, the values of $\omega_1^2$ fluctuate on both sides of the value $\alpha$, as seen in the tables, but $\omega_1^2$ increases from 0 to 1 with $\alpha$ in an almost linear fashion. Hence, $\omega_1^2$ forms a natural point of reference against which the behavior of the other measures can be gauged.

Limited though it is, the study reported here provides valuable insights into the behavior of the association measures. A significant amount of additional information would probably require a comprehensive, hence costly and time-consuming computer simulation study with a large number of repetitions, a wide range of different number of groups $I$, sample sizes $n$, interval widths $d$, etc. While necessary for drawing more specific conclusions, such a study would have to be the topic of a future report.

Table 1. Values of 15 association measures for some artificial data sets. Class width d = 0.5

| | | | | Y interval or ratio | | | Y ordinal | | | | Y nominal | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| α | Sample no. | $s_z$ | s | $\hat{\eta}^2$ | $\omega^2$ | $\omega_1^2$ | θ | κ | $G_1$ | $G_2$ | τ | $\tau_c$ | H | $H_c$ | λ | $V_1$ | $C_1^2$ | $C_2^2$ |
| | 1 | 0.77 | 0.97 | 0.024 | 0.006 | 0.007 | 0.110 | 0.129 | 0.074 | 0.074 | 0.037 | 0.019 | 0.070 | 0.020 | 0.116 | 0.041 | 0.224 | 0.224 |
| | 2 | 1.07 | 1.04 | 0.010 | -0.008 | -0.009 | 0.079 | 0.091 | -0.017 | -0.009 | 0.015 | -0.003 | 0.043 | -0.002 | 0.037 | 0.025 | 0.150 | 0.156 |
| 0 | 3 | 0.62 | 1.01 | 0.019 | 0.001 | 0.001 | 0.095 | 0.109 | -0.001 | 0.013 | 0.016 | -0.002 | 0.052 | 0.003 | 0.019 | 0.030 | 0.173 | 0.182 |
| | 4 | 0.71 | 1.03 | 0.025 | 0.007 | 0.008 | 0.107 | 0.124 | -0.054 | -0.048 | 0.014 | -0.004 | 0.039 | -0.011 | 0.028 | 0.023 | 0.139 | 0.143 |
| | 5 | 1.42 | 1.02 | 0.016 | -0.001 | -0.001 | 0.096 | 0.111 | -0.008 | -0.003 | 0.022 | 0.005 | 0.041 | -0.001 | 0.044 | 0.024 | 0.143 | 0.147 |
| | mean | | | 0.019 | 0.001 | 0.001 | 0.097 | 0.113 | -0.001 | 0.006 | 0.021 | 0.003 | 0.049 | 0.002 | 0.049 | 0.029 | 0.166 | 0.170 |
| | 1 | 0.77 | 0.97 | 0.130 | 0.114 | 0.128 | 0.251 | 0.297 | 0.087 | 0.099 | 0.031 | 0.013 | 0.078 | 0.028 | 0.060 | 0.043 | 0.232 | 0.240 |
| | 2 | 1.07 | 1.04 | 0.196 | 0.181 | 0.202 | 0.327 | 0.376 | 0.144 | 0.160 | 0.034 | 0.017 | 0.086 | 0.046 | 0.082 | 0.049 | 0.255 | 0.267 |
| 0.2 | 3 | 0.62 | 1.01 | 0.081 | 0.064 | 0.073 | 0.197 | 0.231 | 0.103 | 0.105 | 0.027 | 0.010 | 0.075 | 0.030 | 0.078 | 0.044 | 0.235 | 0.237 |
| | 4 | 0.71 | 1.03 | 0.060 | 0.044 | 0.050 | 0.174 | 0.205 | -0.007 | -0.009 | 0.020 | 0.002 | 0.045 | -0.002 | 0.049 | 0.027 | 0.157 | 0.155 |
| | 5 | 1.42 | 1.02 | 0.329 | 0.316 | 0.346 | 0.420 | 0.477 | 0.239 | 0.255 | 0.047 | 0.030 | 0.120 | 0.076 | 0.083 | 0.072 | 0.335 | 0.346 |
| | mean | | | 0.159 | 0.144 | 0.160 | 0.274 | 0.317 | 0.113 | 0.122 | 0.032 | 0.014 | 0.081 | 0.036 | 0.070 | 0.047 | 0.243 | 0.249 |
| | 1 | 0.77 | 0.97 | 0.359 | 0.347 | 0.377 | 0.439 | 0.523 | 0.251 | 0.280 | 0.064 | 0.047 | 0.148 | 0.104 | 0.084 | 0.075 | 0.343 | 0.366 |
| | 2 | 1.07 | 1.04 | 0.488 | 0.478 | 0.512 | 0.514 | 0.590 | 0.371 | 0.371 | 0.073 | 0.057 | 0.178 | 0.142 | 0.103 | 0.107 | 0.429 | 0.429 |
| 0.5 | 3 | 0.62 | 1.01 | 0.246 | 0.232 | 0.256 | 0.361 | 0.434 | 0.165 | 0.190 | 0.040 | 0.023 | 0.103 | 0.064 | 0.087 | 0.051 | 0.262 | 0.281 |
| | 4 | 0.71 | 1.03 | 0.239 | 0.225 | 0.250 | 0.349 | 0.419 | 0.181 | 0.183 | 0.039 | 0.022 | 0.097 | 0.058 | 0.060 | 0.053 | 0.271 | 0.272 |
| | 5 | 1.42 | 1.02 | 0.647 | 0.640 | 0.670 | 0.630 | 0.702 | 0.442 | 0.473 | 0.101 | 0.085 | 0.239 | 0.202 | 0.139 | 0.140 | 0.494 | 0.520 |
| | mean | | | 0.396 | 0.384 | 0.413 | 0.458 | 0.533 | 0.282 | 0.300 | 0.064 | 0.047 | 0.153 | 0.114 | 0.094 | 0.085 | 0.360 | 0.374 |
| | 1 | 0.77 | 0.97 | 0.692 | 0.685 | 0.714 | 0.656 | 0.779 | 0.482 | 0.505 | 0.176 | 0.161 | 0.302 | 0.275 | 0.269 | 0.147 | 0.507 | 0.527 |
| | 2 | 1.07 | 1.04 | 0.792 | 0.788 | 0.809 | 0.697 | 0.798 | 0.562 | 0.562 | 0.179 | 0.164 | 0.349 | 0.323 | 0.210 | 0.203 | 0.587 | 0.587 |
| 0.8 | 3 | 0.62 | 1.01 | 0.567 | 0.559 | 0.591 | 0.576 | 0.706 | 0.402 | 0.425 | 0.128 | 0.112 | 0.232 | 0.204 | 0.176 | 0.106 | 0.426 | 0.445 |
| | 4 | 0.71 | 1.03 | 0.593 | 0.585 | 0.617 | 0.589 | 0.713 | 0.447 | 0.439 | 0.143 | 0.128 | 0.252 | 0.221 | 0.221 | 0.133 | 0.481 | 0.474 |
| | 5 | 1.42 | 1.02 | 0.877 | 0.875 | 0.889 | 0.782 | 0.865 | 0.646 | 0.630 | 0.196 | 0.182 | 0.398 | 0.371 | 0.219 | 0.272 | 0.655 | 0.641 |
| | mean | | | 0.704 | 0.698 | 0.724 | 0.660 | 0.772 | 0.508 | 0.512 | 0.164 | 0.149 | 0.307 | 0.279 | 0.219 | 0.172 | 0.532 | 0.535 |
| | 1 | 0.77 | 0.97 | 1.000 | 1.000 | 1.000 | 0.893 | 1.000 | 0.813 | 0.771 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 0.800 | 0.757 |
| | 2 | 1.07 | 1.04 | 1.000 | 1.000 | 1.000 | 0.857 | 1.000 | 0.813 | 0.764 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 0.800 | 0.749 |
| 1.0 | 3 | 0.62 | 1.01 | 1.000 | 1.000 | 1.000 | 0.821 | 1.000 | 0.780 | 0.757 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 0.750 | 0.725 |
| | 4 | 0.71 | 1.03 | 1.000 | 1.000 | 1.000 | 0.857 | 1.000 | 0.813 | 0.764 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 0.800 | 0.749 |
| | 5 | 1.42 | 1.02 | 1.000 | 1.000 | 1.000 | 0.893 | 1.000 | 0.813 | 0.771 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 0.800 | 0.757 |
| | mean | | | 1.000 | 1.000 | 1.000 | 0.864 | 1.000 | 0.806 | 0.765 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 0.790 | 0.748 |

Table 2. Values of 15 association measures for some artificial data sets. Class-width d = 1.0

| | | | | Y interval or ratio | | | Y ordinal | | | | Y nominal | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\alpha$ | Sample no. | $s_z$ | $s$ | $\hat{\eta}^2$ | $\omega^2$ | $\omega_1^2$ | $\theta$ | $\kappa$ | $G_1$ | $G_2$ | $\tau$ | $\tau_c$ | $H$ | $H_c$ | $\lambda$ | $V_1$ | $C_1^2$ | $C_2^2$ |
| 0 | 1 | 0.77 | 0.97 | 0.024 | 0.006 | 0.007 | 0.077 | 0.108 | 0.036 | 0.027 | 0.034 | 0.016 | 0.047 | 0.010 | 0.042 | 0.023 | 0.123 | 0.116 |
| | 2 | 1.07 | 1.04 | 0.010 | -0.008 | -0.009 | 0.066 | 0.090 | -0.016 | -0.020 | 0.015 | -0.002 | 0.030 | -0.006 | 0.024 | 0.015 | 0.083 | 0.080 |
| | 3 | 0.62 | 1.01 | 0.019 | 0.001 | 0.001 | 0.096 | 0.131 | -0.009 | -0.001 | 0.016 | -0.002 | 0.035 | 0.000 | 0.000 | 0.016 | 0.089 | 0.094 |
| | 4 | 0.71 | 1.03 | 0.025 | 0.007 | 0.008 | 0.116 | 0.158 | -0.011 | -0.009 | 0.017 | -0.001 | 0.033 | -0.003 | 0.016 | 0.016 | 0.087 | 0.088 |
| | 5 | 1.42 | 1.02 | 0.016 | -0.001 | -0.001 | 0.077 | 0.105 | -0.005 | -0.012 | 0.018 | 0.000 | 0.026 | -0.004 | 0.012 | 0.017 | 0.077 | 0.071 |
| | mean | | | 0.019 | 0.001 | 0.001 | 0.087 | 0.118 | -0.001 | -0.003 | 0.020 | 0.002 | 0.034 | -0.001 | 0.019 | 0.017 | 0.092 | 0.090 |
| 0.2 | 1 | 0.77 | 0.97 | 0.130 | 0.114 | 0.128 | 0.237 | 0.340 | 0.106 | 0.117 | 0.039 | 0.022 | 0.082 | 0.046 | 0.014 | 0.036 | 0.176 | 0.184 |
| | 2 | 1.07 | 1.04 | 0.196 | 0.181 | 0.202 | 0.300 | 0.404 | 0.156 | 0.161 | 0.057 | 0.040 | 0.093 | 0.064 | 0.085 | 0.052 | 0.208 | 0.212 |
| | 3 | 0.62 | 1.01 | 0.081 | 0.064 | 0.073 | 0.178 | 0.247 | 0.069 | 0.072 | 0.030 | 0.013 | 0.063 | 0.027 | 0.029 | 0.029 | 0.149 | 0.151 |
| | 4 | 0.71 | 1.03 | 0.060 | 0.044 | 0.050 | 0.177 | 0.251 | 0.025 | 0.019 | 0.027 | 0.010 | 0.044 | 0.006 | 0.031 | 0.021 | 0.114 | 0.109 |
| | 5 | 1.42 | 1.02 | 0.329 | 0.316 | 0.346 | 0.407 | 0.531 | 0.265 | 0.277 | 0.086 | 0.069 | 0.146 | 0.117 | 0.095 | 0.072 | 0.303 | 0.312 |
| | mean | | | 0.159 | 0.144 | 0.160 | 0.260 | 0.354 | 0.124 | 0.129 | 0.048 | 0.031 | 0.086 | 0.052 | 0.051 | 0.042 | 0.190 | 0.194 |
| 0.5 | 1 | 0.77 | 0.97 | 0.359 | 0.347 | 0.377 | 0.404 | 0.586 | 0.244 | 0.276 | 0.114 | 0.098 | 0.172 | 0.137 | 0.155 | 0.066 | 0.282 | 0.308 |
| | 2 | 1.07 | 1.04 | 0.488 | 0.478 | 0.512 | 0.480 | 0.645 | 0.371 | 0.381 | 0.142 | 0.127 | 0.228 | 0.204 | 0.167 | 0.127 | 0.388 | 0.397 |
| | 3 | 0.62 | 1.01 | 0.247 | 0.232 | 0.256 | 0.340 | 0.490 | 0.187 | 0.202 | 0.066 | 0.049 | 0.115 | 0.090 | 0.031 | 0.069 | 0.217 | 0.229 |
| | 4 | 0.71 | 1.03 | 0.239 | 0.225 | 0.250 | 0.311 | 0.461 | 0.191 | 0.194 | 0.069 | 0.053 | 0.110 | 0.085 | 0.053 | 0.069 | 0.216 | 0.218 |
| | 5 | 1.42 | 1.02 | 0.647 | 0.640 | 0.670 | 0.604 | 0.754 | 0.454 | 0.483 | 0.188 | 0.173 | 0.302 | 0.279 | 0.266 | 0.147 | 0.468 | 0.494 |
| | mean | | | 0.396 | 0.384 | 0.413 | 0.428 | 0.587 | 0.289 | 0.307 | 0.116 | 0.100 | 0.185 | 0.159 | 0.135 | 0.095 | 0.314 | 0.329 |
| 0.8 | 1 | 0.77 | 0.97 | 0.692 | 0.685 | 0.714 | 0.577 | 0.846 | 0.452 | 0.496 | 0.307 | 0.295 | 0.386 | 0.368 | 0.404 | 0.196 | 0.439 | 0.479 |
| | 2 | 1.07 | 1.04 | 0.792 | 0.788 | 0.809 | 0.648 | 0.860 | 0.559 | 0.570 | 0.358 | 0.347 | 0.457 | 0.440 | 0.426 | 0.244 | 0.550 | 0.560 |
| | 3 | 0.62 | 1.01 | 0.567 | 0.559 | 0.591 | 0.491 | 0.751 | 0.377 | 0.408 | 0.209 | 0.195 | 0.288 | 0.265 | 0.204 | 0.143 | 0.364 | 0.391 |
| | 4 | 0.71 | 1.03 | 0.593 | 0.585 | 0.617 | 0.499 | 0.761 | 0.431 | 0.425 | 0.253 | 0.240 | 0.307 | 0.286 | 0.277 | 0.181 | 0.420 | 0.415 |
| | 5 | 1.42 | 1.02 | 0.877 | 0.875 | 0.889 | 0.743 | 0.912 | 0.647 | 0.636 | 0.405 | 0.394 | 0.521 | 0.508 | 0.473 | 0.347 | 0.635 | 0.624 |
| | mean | | | 0.704 | 0.698 | 0.724 | 0.592 | 0.826 | 0.493 | 0.507 | 0.306 | 0.294 | 0.392 | 0.373 | 0.357 | 0.222 | 0.482 | 0.494 |
| 1.0 | 1 | 0.77 | 0.97 | 1.000 | 1.000 | 1.000 | 0.679 | 1.000 | 0.722 | 0.717 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 0.667 | 0.661 |
| | 2 | 1.07 | 1.04 | 1.000 | 1.000 | 1.000 | 0.714 | 1.000 | 0.700 | 0.708 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 0.667 | 0.675 |
| | 3 | 0.62 | 1.01 | 1.000 | 1.000 | 1.000 | 0.750 | 1.000 | 0.708 | 0.724 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 0.667 | 0.684 |
| | 4 | 0.71 | 1.03 | 1.000 | 1.000 | 1.000 | 0.714 | 1.000 | 0.700 | 0.708 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 0.667 | 0.675 |
| | 5 | 1.42 | 1.02 | 1.000 | 1.000 | 1.000 | 0.821 | 1.000 | 0.778 | 0.755 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 0.750 | 0.725 |
| | mean | | | 1.000 | 1.000 | 1.000 | 0.736 | 1.000 | 0.722 | 0.722 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 0.683 | 0.684 |

We proceed to making a series of comments on our data. The conclusions are drawn only from our limited study and cannot be construed as being of general scope unless supported by further research.

(1) *The need for correction of association measures.* The theoretical independence when $\alpha = 0$ should, ideally, make each association measure produce a value close to zero. This aim is achieved very well, on the average, by the corrected measures, $\omega^2$, $\omega_1^2$, $G_1$, $G_2$, $\tau_c$ and $H_c$. As a result of the correction, negative values are, of course, produced occasionally. The remaining, uncorrected measures are inflated to varying extents when $\alpha = 0$. The most severe bias is found in $C_1^2$ and $C_2^2$, whose average values are about 0.17 when $d = 0.5$. Considerable bias is also found in $\theta$ and $\kappa$, while it is less serious for $\hat{\eta}^2$, $\tau$, $H$, $\lambda$ and $V_1$ whose values are usually less than 0.05. As a general rule, it seems to make good sense always to use an appropriately corrected association measure.

(2) *The effect due to changing the definition of the Y-categories.* The number of response categories is reduced roughly by half as $d$ is increased from 0.5 to 1.0. By definition, $\hat{\eta}^2$, $\omega^2$ and $\omega_1^2$ are left unchanged hereby. Furthermore, $G_1$ and $G_2$ are remarkably insensitive (due to the presence of the factor $K_J$) to the change. Some rather small effects are observed in $\theta$ and $\kappa$. When $Y$ is nominal, no a priori reason exists for believing that the measures would stay roughly the same. In fact, several of them appear to be highly sensitive to the change in $d$. A blatant example: When $\alpha = 0.8$, the values of $\tau$ and $\tau_c$ are approximately doubled when the number of $Y$-categories is cut in half. Not all measures increase by the decrease in the number of categories: For $\theta$, $C_1^2$ and $C_2^2$, the tendency is to decrease. This emphasizes the importance for the empirical researcher to report not only the association measure used and its value, but also the definition of his categories; if this is not done, there is little grounds for comparability from one research study to the next.

(3) *When Y is interval-scaled: Comparisons.* The difference between the two corrected measures $\omega^2$ and $\omega_1^2$ is small, as expected; the always positive difference $\omega_1^2 - \omega^2$ is usually less than 0.03. In line with theory, $\omega_1^2$ (and thus $\omega^2$) take values fluctuating in close neighborhood of $\alpha$.

(4) *When Y is ordinal-scaled: Comparisons.* The differences in value between $G_1$ and $G_2$ are generally small, *i.e.*, substituting $S_H$ for $\tilde{S}_H$ produces no great changes for any $\alpha$ (this can be seen also by comparing $C_1^2$ and $C_2^2$). On theoretical grounds developed in Section 2, one can expect $G_1$ and $G_2$ to closely approximate $\omega^2$ when $\alpha$ is small. For greater values of $\alpha$, $G_1$ and $G_2$ underestimate $\omega^2$ considerably, when $\alpha = 0.5$ and 0.8 by roughly 25%. On the other hand, there is no theoretical reason to expect $\theta$ and $\kappa$ to be close to $\omega^2$ or $\omega_1^2$. Both $\theta$ and $\kappa$ ought to have been corrected, since they show considerably inflated values for small $\alpha$. One observes that $\kappa$ consistently exceeds $\omega_1^2$, while $\theta$ exceeds $\omega_1^2$ only for small $\alpha$.

(5) *When Y is nominal-scaled: Comparisons.* Again, there is no theo-

retical basis for expecting the measures in this category to be close to $\omega^2$ or $\omega_I^2$. One observes the pattern of values of $\tau$, $\tau_c$, $H$, $H_c$, $\lambda$ and $V_1$ as $\alpha$ goes from 0 to 1: All six are slow in "taking off" towards the value of unity which they all attain when $\alpha = 1$. For example, when $\alpha = 0.8$, which means that $\omega_I^2$ takes values in the 0.8 area, we find when $d = 0.5$, the following average values: $\tau = 0.164$, $H = 0.307$, $\lambda = 0.219$, $V_1 = 0.172$. There is no consistency in the order relationship among the values of $\tau$, $H$, $\lambda$ and $V_1$ from sample to sample. But when $d = 0.5$, $\tau < V_1 < \lambda < H$ holds fairly well on the average for all values of $\alpha$. Finally, $C_1^2$ and $C_2^2$ seem to be the least appealing measures in this group: Severely inflated when $\alpha$ is small and well below unity when the association is perfect.

## REFERENCES

Attneave, F. *Applications of Information Theory to Psychology.* New York: Holt, Rinehart and Winston, 1959.

Cohen, J. Some statistical issues in psychological research. In B. B. Wolman (Ed.): *Handbook of Clinical Psychology.* New York: McGraw-Hill, 1965, 95–121.

Cramér, H. *Mathematical Methods of Statistics.* Princeton: Princeton University Press, 1946.

Fleiss, J. L. Estimating the magnitude of experimental effects. *Psychological Bulletin,* 1969, **72**, 273–276.

Freeman, L. C. *Elementary Applied Statistics.* New York: Wiley, 1965.

Friedman, H. Magnitude of experimental effect and a table for its rapid estimation. *Psychological Bulletin,* 1968, **70**, 245–251.

Garner, W. R. *Uncertainty and Structure as Psychological Concepts.* New York: Wiley, 1962.

Garner, W. R. and McGill, W. J. The relation between information and variance analyses. *Psychometrika,* 1956, **21**, 219–228.

Glass, G. V., & Hakstian, A. R. Measures of association in comparative experiments: Their development and interpretation. *American Educational Research Journal,* 1969, **6**, 403–414.

Goodman, L. A. Partitioning of chi-square, analysis of marginal contingency tables, and estimation of expected frequencies in multidimensional contingency tables. *Journal of the American Statistical Association,* 1971, **66**, 339–344.

Goodman, L. A. and Kruskal, W. H. Measures of association for cross classifications. *Journal of the American Statistical Association,* 1954, **49**, 1732–1769.

Goodman, L. A. and Kruskal, W. H. Measures of association for cross classifications II: Further discussion and references. *Journal of the American Statistical Association,* 1959, **54** (1969), 123–163.

Goodman, L. A. and Kruskal, W. H. Measures of association for cross classifications III: Approximate sampling theory. *Journal of the American Statistical Association,* **58**, 1963, 310–364.

Goodman, L. A. and Kruskal, W. H. Measures of association for cross classifications IV: Simplification of asymptotic variances. *Journal of American Statistical Association,* **67**, 1972, 415–421.

Greeno, J. C. Explanation and information. In W. C. Salmon: *Statistical Explanation and Statistical Relevance.* Pittsburgh: University of Pittsburgh Press, 1971.

Guttman, L. An outline of the statistical theory of prediction. In P. Horst et. al. (editors): *The Prediction of Personal Adjustment,* Bulletin 48, Social Science Research Council, New York. 1941, 253–318.

Halderson, J. S. and Glasnapp, D. R. Generalized rules for calculating the magnitude of an effect in factorial and repeated measures ANOVA designs. *American Educatorial Research Journal*, 1972, **9**, 301–310.

Hays, W. L. *Statistics for Psychologists*. New York: Holt, Rinehart and Winston, 1963.

Kelley, T. L. An unbiased correlation ratio measure. *Proceedings of the National Academy of Sciences*, 1935, **21**, 554–559.

Kennedy, J. J. The eta coefficient in ANOVA designs. *Educational Psychological Measurement*, 1970, **30**, 885–889.

Kullback, S. *Information Theory and Statistics*. New York: Wiley, 1959.

Light, R. J. and Margolin, B. H. An analysis of variance for categorical data. *Journal of the American Statistical Association*, 1971, **66**, 534–544.

Linfoot, E. H. An informational measure of correlation. *Information and Control*, 1957, **1**, 85–89.

McGill, W. J. Multivariate information transmission. *Psychometrika*, 1954, **19**, 97–115.

Nie, N. H., Bent, D. H. and Hull, C. H. *Statistical Package for the Social Sciences* (SPSS). New York: McGraw–Hill, 1970.

Särndal, C. E. A class of explicata for "information" and "weight of evidence". *Review of the International Statistical Institute*, 1970, **38**, 223–235.

Scheffé, H. *The Analysis of Variance*. New York: Wiley, 1959.

Vaughan, G. M. and Corballis, M. C. Beyond tests of significance: Estimating strength of effects in selected ANOVA designs. *Psychological Bulletin*, 1969, **72**, 204–213.

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