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THE DEVELOPMENT OF HIERARCHICAL FACTOR SOLUTIONS*

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Although simple structure has proved to be a valuable principle for rotation of axes in factor analysis, an oblique factor solution often tends to confound the resulting interpretation. A model is presented here which transforms the oblique factor solution so as to preserve simple structure and, in addition, to provide orthogonal reference axes. Furthermore, this model makes explicit the hierarchical ordering of factors above the first-order domain.

The purpose of this paper is to present a procedure for transforming an oblique factor analysis solution containing a hierarchy of higher-order factors into an orthogonal solution which not only preserves the desired interpretation characteristics of the oblique solution, but also discloses the hierarchical structuring of the variables.

Oblique simple structure was proposed by Thurstone as a factor model useful for psychological research because of the simplicity with which interpretation could be made from a set of linear components underlying a set of scores. His argument is convincing when consideration is given to his "box problem" [9, pp. 140–146] for the factor loadings readily identify the dimensions of the boxes. In many studies, correlations among the reference axes make interpretation of simple structure difficult or questionable. In such cases usual methods of transformation from oblique to orthogonal axes fail to clarify the nature of the underlying parameters because many of the vanishing factor loadings become non-vanishing, thereby destroying simple structure. If one is willing to disavow the principle of parsimony of common factors, one may employ the type of factor solution outlined in this paper. This solution not only furnishes simple structure on orthogonal reference axes, but also provides a more complete rationale of the structuring of psychological traits than that given by (i) a conventional oblique solution or, for

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that matter, (ii) a solution in which the number of common factors is equal to the rank of the reduced correlation matrix.

It seems reasonable to assume that psychological behavior may be conceived as functioning at different levels of complexity. That is, a complex behavior activity might be thought of as an assembly of progressively less complex levels of activity—each level may have semantic, psychological, or practical meaning. For example, Vernon [11, pp. 22–24] reports that the mental structure of a group of British Army and Navy recruits was examined with a battery of cognitive tests. As determined from the sign pattern of centroid factor loadings, one general factor was found to be present in all tests. This factor was designated as g. With the elimination of g, the battery could be fractionated into two main groups of tests: academic and practical. In turn, the academic factor could be broken into verbal, numerical, and educational factors; the practical factor could be broken into mechanical, spatial, and physical factors. This structuring of the tests into a hierarchy of factors has many recommendable features—it provides information about the classification of tests and the behaviors measured by them in varying orders of concurrence and dependence. Had this particular centroid solution been rotated to an oblique solution, the hierarchical ordering would have been lost or rendered uncertain.

Structuring of tests into a hierarchical pattern is not a new consideration. Holzinger's bi-factor solution is a special case in which one second-order factor overlays the first-order group factors. Burt [1, 2, 3] has strongly advocated the hierarchical model for many years. His group factor method, which yields this hierarchy, proceeds by successive grouping of variables according to their sign pattern in a centroid solution. The procedure set forth in this paper, however, is an elaboration of the procedure demonstrated by Thompson [8, pp. 297–302] and Thurstone [9, pp. 411–439]. It differs from Burt's not in the product but in the process. The hierarchical solution is shown to be a consequence of successively obtained higher-order factor solutions. A necessary condition is the existence of simple structure at each level. If oblique simple structure exists, it can be recast into a hierarchical pattern similar in kind to that which Vernon inferred from the centroid solution. It will be seen that the characteristics of simple structure are retained not only at the level of the first-order factors but also at all levels.

Mathematical Rationale

The mathematical rationale for the model outlined in the paper is derived from Tucker's [10] generalization of the fundamental factor theorem stated by Thurstone [9, p. 78]. This theorem states that a correlation matrix, R, may be decomposed into correlated common factors and unique factors.

$$R = P\phi P' + U^2, \tag{1}$$

where P represents the coordinates of the vector representation of the variables on oblique Cartesian reference axes or factors, ϕ represents the intercorrelations among the oblique reference axes, and U represents the unique factor coefficients. It is readily seen that if ϕ is the identity matrix, the fundamental factor theorem of Thurstone results.

A second theorem used in this development also stems from Tucker's article. He shows that if the intercorrelations among the factors, ϕ, can be decomposed as

$$\phi = HH', \tag{2}$$

then the oblique factors, P, may be transformed into orthogonal factors, F, according to the operation

$$PH = F. \tag{3}$$

That is, the coordinates of the variables represented as vectors may be transformed from oblique to orthogonal reference axes. Each row of H represents the direction cosines of the oblique axes with respect to the orthogonal axes developed by the decomposition.

Guttman [4] demonstrates that if a matrix of intercorrelations, ϕ, is factored as in (2) no matter how H is built up, the reference axes are orthogonal. The factoring or decomposition may involve any of a variety of procedures, such as the diagonal or square root method of factoring, the centroid procedure, or the method of principal axes.

The development of the hierarchical model utilizes these propositions. In the following discussion, P_i will refer to the primary factor pattern of the ith order variables or factors; that is, the coordinates of the vector representation of the variables on the ith order oblique reference axes. R_i will be used to designate the intercorrelations among the ith primary factor reference axes. U_i will represent the unique ith order variables or factors.

At the outset, the initial correlation matrix, R, is decomposed according to (1) as follows:

$$R = P_1R_1P_1' + U_1^2 . \tag{4}$$

In like manner, R_1 is decomposed

$$R_1 = P_2R_2P_2' + U_2^2 . \tag{5}$$

In turn, R_2 is decomposed

$$R_2 = P_3R_3P_3' + U_3^2 . \tag{6}$$

Each higher-level matrix of intercorrelations among primary factors is decomposed in this fashion until R_i becomes the identity matrix, which implies that the ith order primary factors are orthogonal. That is,

$$R_{i-1} = P_iP_i' + U_i^2 . \tag{7}$$

In many cases, R_i becomes a unit scalar and P_i, therefore, is merely a column matrix. Elementary matrix manipulation permits (7) to be rewritten as a product of a supermatrix and its transpose:

$$R_{i-1} = [P_i : U_i] \cdot [P_i : U_i]'. \tag{8}$$

Designating the supermatrix, $[P_i : U_i]$, by B_i, according to (3), the $(i-1)$th order primary factors, P_{i-1}, can be made orthogonal by the operation $P_{i-1}B_i$.

However

$$R_{i-2} = P_{i-1}R_{i-1}P'_{i-1} + U^2_{i-1}. \tag{9}$$

Therefore, it follows that R_{i-2} may be rewritten as a product of a new supermatrix and its transpose:

$$R_{i-2} = [P_{i-1}B_i : U_{i-1}] \cdot [P_{i-1}B_i : U_{i-1}]'. \tag{10}$$

This new supermatrix may be designated as B_{i-1}. By virtue of (2) and Guttman's demonstration [4], orthogonal reference axes are obtained. Furthermore, B_{i-1} serves to rotate the primary pattern, P_{i-2}, to this orthogonal reference framework. Continuing this process to the lowest-order level, the initial primary or first-order factors, P_1, are orthogonalized by the operation P_1B_2. Designate P_1B_2 as B instead of B_1 since one is usually not concerned with explicitly appending the diagonal matrix of unique factors to the common factor solution. B, then, is the hierarchical solution. Since

$$R \text{ (with communalities)} = BB', \tag{11}$$

B represents coordinates of the test variables on orthogonal axes.

In the development of a hierarchical solution, careful attention should be paid to the distinction between simple structure and primary pattern. This distinction has been clearly drawn and illustrated by Harris and Knoell [5]. The hierarchical solution is contingent upon the development of a primary pattern at each level. This primary pattern, however, may be obtained from the simple structure, which is computed either graphically or analytically. Once simple structure is identified, it may easily be converted to primary pattern [5] by the operation

$$P_i = V_i(R_i^{-1})^{\frac{1}{2}}_d, \tag{12}$$

where P_i is primary pattern, V_i is simple structure, and $(R_i^{-1})^{\frac{1}{2}}_d$ is the matrix of the reciprocals of the direction cosines between each primary axis and its own simple structure reference axis. $(R_i^{-1})^{\frac{1}{2}}_d$ is obtained by taking the square roots of the diagonal elements only of R_i^{-1}.

Procedure

To demonstrate the procedure for rotating an oblique simple structure into a hierarchical factor solution, a correlation model was constructed from

TABLE 1

Correlation Matrix, R*

	1	2	3	4	5	6	7	8	9	10	11	12
1	6400	7200	3136	2688	0983	0491	1290	0369	2903	1613	0645	0753
2		8100	3528	3024	1106	0553	1452	0415	3266	1814	0726	0847
3			4900	4200	0753	0377	0988	0282	2222	1235	0494	0576
4				3600	0645	0323	0847	0242	1905	1058	0424	0494
5					6400	3200	1344	0384	1089	0605	0242	0282
6						1600	0672	0192	0544	0302	0121	014[illegible]
7							4900	1400	1429	0794	0318	0370
8								0400	0408	0227	0091	0106
9									8100	4500	1458	1701
10										2500	0810	0945
11											3600	4200
12												4900

*Communalities appear in the principal diagonal. Decimal points have been omitted.

TABLE 2

Primary Pattern, P_1

	I	II	III	IV	V	VI
1	.8					
2	.9					
3		.7				
4		.6				
5			.8			
6			.4			
7				.7		
8				.2		
9					.9	
10					.5	
11						.6
12						.7

TABLE 3

Intercorrelations of Primary Factors, R_1

	I	II	III	IV	V	VI
I	1.0000	.5600	.1536	.2304	.4032	.1344
II	.5600	1.0000	.1344	.2016	.3528	.1176
III	.1536	.1344	1.0000	.2400	.1512	.0504
IV	.2304	.2016	.2400	1.0000	.2268	.0756
V	.4032	.3528	.1512	.2268	1.0000	.2700
VI	.1344	.1176	.0504	.0756	.2700	1.0000

TABLE 4

Second-Order Primary Factors, P_2

	I	II	III
1	.8		
2	.7		
3		.4	
4		.6	
5			.9
6			.3

TABLE 5

Correlations Among Second-Order Primary Factors, R_2

	I	II	III
I	1.0000	.4800	.5600
II	.4800	1.0000	.4200
III	.5600	.4200	1.0000

a postulated simple structure factor matrix. It should be emphasized, however, that any set of empirical variables which can be rotated to simple structure can also be put in this more interpretable and meaningful hierarchical form. That is, if simple structure exists by any definition for a set of variables, the procedure is applicable. The given correlation matrix is presented in Table 1. An oblique solution was developed by the multiple-group method [6]. This oblique solution consists of a primary pattern, P_1, and intercorrelations among the primary factors, R_1. These two matrices are presented in Tables 2 and 3. An oblique solution could have been produced by rotation from a centroid solution or by some analytic method instead of the multiple-group procedure. However, the method of arriving at the oblique solution is of little consequence for our purposes, and the grouping procedure was thought to be the most expeditious here. If oblique simple structure, V_1, had been produced, P_1 could be obtained quite readily by the operation indicated in (12). Regardless of methodology, the final rotated oblique solution should be transformed into a primary pattern, P_1, as defined by Holzinger and Harman [7, chap. XI].

The intercorrelations of the primary factors, R_1, (with communalities determined and placed in the diagonal elements) are then factored by any method. Usually it is most expeditious to carry out a *common*-factor analysis at each stage to separate the common-factor space from the unique-factor space. Rotation of these second-order factors is then performed to obtain the primary pattern of the second-order factors, P_2, (Table 4) and the intercorrelations of the second-order primary factors, R_2, (Table 5). A check may be made at this point since R_1 (with communalities) $= P_2R_2P_2'$. Again this P_2 may be developed by the construction of an oblique simple structure, V_2, which is then transformed into P_2 by the operation indicated in (12).

Since the second-order factors, P_2, are correlated, it is obvious that a third-order factor exists. Consequently, R_2 is factored. Factoring shows that there is one third-order factor and three unique factors, B_3 (see Table 6). The progressive factoring of higher orders is now complete. This information is used for developing the preferred hierarchical factor solution. To do this, the operation P_2B_3 is performed (Table 7) and the matrix of unique factors of R_1, U_2, is appended as shown in Table 8. That is, $B_2 = [P_2B_3 : U_2]$. It should be noted that $B_2B_2' = R_1$ (with unities in the diagonal of R_1). This matrix, B_2, is used as the transformation matrix for rotating the first-order oblique solution, P_1, into the final hierarchical solution, B (Table 9), according to the operation

$$B = P_1B_2 . \tag{13}$$

This procedure may be extended to higher orders if correlations are found among fourth-order or higher-order factors.

It will be observed that this hierarchical solution contains 10 common

TABLE 6

Third-Order Common and Unique Factors, B_3

	P_3		U_3		
	I		II	III	IV
1	.8000	:	.6000		
2	.6000	:		.8000	
3	.7000	:			.7141

TABLE 7

Orthogonalized Second-Order Common Factors, P_2B_3

	I	II	III	IV
1	.6400	.4800		
2	.5600	.4200		
3	.2400		.3200	
4	.3600		.4800	
5	.6300			.6427
6	.2100			.2142

TABLE 8

Orthogonalized Second-Order Common and Unique Factors, B_2

	P_2B_3					U_2					
	I	II	III	IV		V	VI	VII	VIII	IX	X
1	.6400	.4800			:	.6000					
2	.5600	.4200			:		.7141				
3	.2400		.3200		:			.9165			
4	.3600		.4800		:				.8000		
5	.6300			.6427	:					.4359	
6	.2100			.2142	:						.9539

TABLE 9

Hierarchical Factor Solution, B

	I	II	III	IV	V	VI	VII	VIII	IX	X
1	.5120	.3840			.4800					
2	.5760	.4320			.5400					
3	.3920	.2940				.4999				
4	.3360	.2520				.4285				
5	.1920		.2560				.7332			
6	.0960		.1280				.3666			
7	.2520		.3360					.5600		
8	.0720		.0960					.1600		
9	.5670			.5784					.3923	
10	.3150			.3214					.2180	
11	.1260			.1285						.5723
12	.1470			.1499						.6677

factors, where all tests define factor I. Factors II, III, and IV are the next most complex factors. Each of these in turn can be broken down into the finer composites illustrated by factors V through X. These last six factors identify the six factors of the original oblique solution, P_1. It will be observed that this solution reproduces the communalities and the off-diagonal correlations of the original correlation matrix exactly. Furthermore, it furnishes the same factorial interpretation as is found in the oblique solution, P_1, which is the usual type of solution obtained by researchers. Ease of psychological interpretation has not been sacrificed by the use of the hierarchical solution, and what was concealed in the intercorrelations of the oblique axes now takes on added meaning in terms of the progressive groupings of the variables at higher levels.

It should be emphasized that even though the oblique solution, P_1, contains variables of complexity one only, this is not a restriction. Variables of any complexity may be used.

Discussion

A question arises about the stability of the hierarchical solution upon modification of the battery of tests. Burt concludes [3, p. 70] that the hierarchical solution—designated by him as the group-factor solution—remains "stable, if not absolutely invariant, even when the battery of tests or traits is modified, e.g., when a comparatively small battery is enlarged by the addition of more tests or more groups of tests, or when a large battery is curtailed by the omission of tests." The introduction of a new group of tests which are unrelated to any group already in the battery would, of course, add a new group factor.

In all probability, selection, univariate and multivariate, and sampling variation would affect this model in the same manner as the simple structure model. These points concerning battery modification, selection, and sampling stability need further research for clarification.

Practical applications of this model will be greatly aided as more objective and analytical criteria and techniques for transformation to simple structure are achieved. Nevertheless, even with present methods of attaining simple structure, the hierarchical solution is useful.

Summary of Steps as Applied to Illustration

1. R, with communalities, was factored into P_1 and R_1 (Tables 1, 2, and 3), that is

$$R \text{ (with communalities)} = P_1 R_1 P_1' .$$

2. R_1, with communalities, was factored into P_2 and R_2 (Tables 4 and 5), that is

$$R_1 \text{ (with communalities)} = P_2 R_2 P_2' ,$$

$$R_1 \text{ (with unities)} = P_2 R_2 P_2' + U_2^2 ,$$

where U_2 represents the diagonal matrix of unique factors of R_1.

3. R_2, with communalities, was factored into P_3. (Table 6). One common factor was found, i.e. R_3 was a unit scalar.

$$R_2 \text{ (with communalities)} = P_3 P_3' ,$$

$$R_2 \text{ (with unities)} = P_3 P_3' + U_3^2 .$$

4. When only one common factor remains, as in this illustration, factoring

of the higher-order matrices is completed. Otherwise, the procedure would be continued until R_i becomes an identity matrix or a single highest-order factor is found. At this stage, these intermediate matrices are used for constructing a rotation matrix for transforming the primary pattern, P_1, into a hierarchical solution, B.

5. Form matrix B_3 by appending the unique-factor loadings of R_2 to P_3, that is

$$B_3 = [P_3 : U_3]. \text{ (Table 6).}$$

It follows that

$$R_2 \text{ (with communalities)} = P_3P_3',$$

$$R_2 \text{ (with unities)} = B_3B_3'.$$

6. Carry out the matrix operation P_2B_3. (Table 7).

7. Form matrix B_2 by appending the unique-factor loadings of R_1 to P_2B_3, that is

$$B_2 = [P_2B_3 : U_2]. \text{ (Table 8).}$$

It follows that

$$R_1 \text{ (with communalities)} = P_2B_3B_3'P_2',$$

$$R_1 \text{ (with unities)} = B_2B_2'.$$

8. The hierarchical solution, B, then is constructed by the operation

$$B = P_1B_2. \text{ (Table 9).}$$

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