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FINITE MARKOV PROCESSES IN PSYCHOLOGY*

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Finite Markov processes are reviewed and considered for their usefulness in the description of behavioral data. The various alternative responses in an experimental situation define a vector space, and changes in the probabilities of these alternatives are represented by movements in this space. Methods of fitting the theory to experimental data are considered.

The simplest process, with a constant matrix of transitional probabilities that is applied repeatedly to represent the effect of successive trials, seems inadequate for most learning data. A matrix function that may be useful for learning theory is presented.

In the two general areas where psychology has been relatively successful as a quantitative science, i.e., sensory psychology and test construction, probabilistic considerations long ago proved their worth. It is characteristic of these two areas, however, that the observations are relatively invariant in time. The basic parameters can be explored at length because sequential effects of measurement are secondary and can be ignored or randomized. This fortunate situation makes it possible to use familiar probability models based upon independent random variables.

With the more dynamic problems of psychology, however, this familiar model has not often led to profitable results. For example, it is intrinsic in the very notion of learning that successive measurements are not independent; attempts to use a theory of independent variables must either fail or misrepresent the basic process. Such failures may lead to a rejection of statistical concepts as inadequate; a more proper attitude is to abandon the assumption of independence and ask what help can be had from dependent probabilities. The simplest mathematical models incorporating dependent probabilities are the finite Markov processes. In this paper such processes are examined for their usefulness and their limitations for describing psychological data.

1. *Simple Markov Chains with Two Alternatives.* The data from psychological experiments usually come in the form of sequences of choices embedded in the time continuum. Often it is possible to ignore the temporal order in which alternative choices occur. The purpose of this discussion,

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however, is to examine situations in which the temporal sequence should not be ignored. We shall adopt the Markovian model of dependent probabilities to discuss such sequences. We begin, therefore, with the simplest possible example of a Markov chain.

Consider an experiment in which only two alternative responses are possible. A trial consists of a choice of one of these two alternatives. If the letters A and B designate these choices, then a sequence of trials might produce the sequence of responses $ABBAAABA \ldots$, where the durations and latencies are ignored. We shall assume that this sequence is produced by a Markov process; i.e., that the distribution of probabilities at trial $n + 1$ depends upon the outcome of trial n. However, the knowledge of outcomes prior to n does not change our description of the system if we know the outcome of trial n. In other words, the present state of the system governs its future development.

We adopt the following notation:

n	number of the trial: 0, 1, 2, $\ldots$.
A and B	the two alternative responses.
$p^{(n)}(A)$	probability of alternative A at trial n.
$p(A)$	asymptotic value of $p^{(n)}(A)$ as $n \to \infty$.
d_n	the set of absolute probabilities at trial n, considered as a vector; $[p^{(n)}(A), p^{(n)}(B)]$.
$p_A(B)$	given A at n, the conditional probability of B at $n + 1$.
$p_A^{(m)}(B)$	given A at n, the conditional probability of B at $n + m$, $m = 2, 3, \ldots$.
T	matrix of transitional probabilities.
λ_i	characteristic roots of the matrix T.

Alternative A can occur at trial $n + 1$ in either of two ways. Either it follows an A on trial n, or it follows a B on trial n. Similarly, B can occur at $n + 1$ in either of two ways. This obvious fact leads to the following equations:

$$\begin{aligned} p^{(n)}(A)p_A(A) + p^{(n)}(B)p_B(A) &= p^{(n+1)}(A) \\ p^{(n)}(A)p_A(B) + p^{(n)}(B)p_B(B) &= p^{(n+1)}(B). \end{aligned} \tag{1}$$

In matrix notation these equations can be written

$$\begin{Vmatrix} p_A(A) & p_B(A) \\ p_A(B) & p_B(B) \end{Vmatrix} \begin{Vmatrix} p^{(n)}(A) \\ p^{(n)}(B) \end{Vmatrix} = \begin{Vmatrix} p^{(n+1)}(A) \\ p^{(n+1)}(B) \end{Vmatrix}. \tag{2}$$

The reader is assumed to be familiar with the elements of matrix theory. If the distribution of probabilities on trials n and $n + 1$ is regarded as the vectors d_n and d_{n+1} in a two-dimensional space, then the square matrix of

transitional probabilities is a linear transformation or operator mapping d_n into d_{n+1} . Thus we can write Eq. (2) as

$$Td_n = d_{n+1} . \tag{3}$$

Any sequence of distributions can be produced by operating upon the successive d_i by appropriate transformations. For the moment, however, we shall consider a special case. We shall assume that repeated trials can be represented as repeated transformations by the same operator. Thus we can write for the initial trial:

$$Td_0 = d_1 .$$

A second trial carries d_1 into d_2 :

$$Td_1 = d_2 .$$

In terms of d_0 , therefore, we can write:

$$Td_1 = T(Td_0) = T^2d_0 = d_2 .$$

Or more generally,

$$T^nd_0 = d_n . \tag{4}$$

Since the probabilities of A and B on successive trials are given by T^nd_0 , we proceed to examine the powers of T. The elements of T^n are $p_i^{(n)}(j)$, where $i = A,B$; $j = A,B$. We wish to find a general expression for T^n in terms of $p_i(j)$ and n. From matrix theory we know that every square matrix with distinct roots is similar* to a diagonal matrix whose diagonal elements are the characteristic roots λ_i of T. We designate this similar diagonal matrix by Λ, and write

$$\Lambda = S^{-1}TS,$$

where S is a matrix whose columns are the characteristic vectors of T. From this we obtain

$$T = S\Lambda S^{-1}.$$

To obtain the powers of T we note that

$$T^2 = S\Lambda S^{-1}S\Lambda S^{-1} = S\Lambda^2S^{-1},$$

or more generally,

$$T^n = S\Lambda^nS^{-1}. \tag{5}$$

Powers of Λ are simply calculated, for since Λ is a diagonal matrix, its powers are given by the powers of the diagonal elements λ_i .

To find Λ for the matrix of Eq. (2) we first write the characteristic equation for the matrix T. If we use the fact that $p_A(A) + p_A(B) = 1$ (and

*Two matrices are said to be similar when they have the same characteristic roots.

similarly for B subscripts), the determinantal equation can be written in the convenient form

$$\det (T - \lambda I) = \lambda^2 - [p_A(A) + p_B(B)]\lambda + [p_A(A) - p_B(A)] = 0.$$

The roots of this equation are the characteristic roots of the matrix:

$$\lambda_1 = 1 \qquad \text{and} \qquad \lambda_2 = p_A(A) - p_B(A).$$

Since the sums of all the columns of T are unity, we note that unity is always a root of these matrices. Substituting these roots into $Tv_i = \lambda_i v_i$ and solving for the characteristic vectors, v_i , we obtain the vectors $[1,\ p_A(B)/p_B(A)]$ and $(1, -1)$. These vectors comprise the columns of S, and so from Eq. (5) we obtain, after inverting S,

$$T^n = \begin{Bmatrix} 1 & 1 \\ \dfrac{p_A(B)}{p_B(A)} & -1 \end{Bmatrix} \begin{Bmatrix} 1^n & 0 \\ 0 & [p_A(A) - p_B(A)]^n \end{Bmatrix} \frac{1}{p_A(B) + p_B(A)} \cdot \begin{Bmatrix} p_B(A) & p_B(A) \\ p_A(B) & -p_B(A) \end{Bmatrix}. \quad (6)$$

Eq. (6) can be written more conveniently

$$T^n = \frac{1}{p_A(B) + p_B(A)} \begin{Bmatrix} p_B(A) & p_B(A) \\ p_A(B) & p_A(B) \end{Bmatrix} + \frac{[p_A(A) - p_B(A)]^n}{p_A(B) + p_B(A)} \begin{Bmatrix} p_A(B) & -p_B(A) \\ -p_A(B) & p_B(A) \end{Bmatrix}. \quad (7)$$

Since $| p_A(A) - p_B(A) | < 1$, the second term on the right of Eq. (7) goes to zero as $n \to \infty$, so the first term represents the asymptotic form of T^n.

With Eq. (7) we can calculate $T^n d_0$, and so obtain the probability of A on successive trials:

$$p^{(n)}(A) = \frac{p_B(A)}{p_A(B) + p_B(A)} + [p(A) - p(A)]^n \frac{p^{(0)}(A)p_A(B) - p^{(0)}(B)p_B(A)}{p_A(B) + p_B(A)}. \quad (8)$$

The value of

$$p^{(n)}(A) \to \frac{p_B(A)}{p_A(B) + p_B(A)} \text{ as } n \to \infty.$$

It is apparent that Eq. (8) can be written

$$p^{(n)}(A) = a(1 - be^{-cn}), \quad (9)$$

where

$$a = \frac{p_B(A)}{p_A(B) + p_B(A)},$$

$$b = -p^{(0)}(A)\,\frac{p_A(B)}{p_B(A)} + p^{(0)}(B),$$

$$c = -\ln [p_A(A) - p_B(A)].$$

Eq. (9) is an exponential growth function—a form frequently used to describe data from learning experiments. It should be noted, however, that while the average subject may follow such a learning function, the individual subjects are generating stationary time series that do not represent learning. The term "learning" probably should be reserved for those cases in which the matrix operator changes on successive trials.

We shall illustrate the use of the Markov chain with a numerical example. Suppose that two alternative responses are called right (R) and wrong (W), that $p^{(n)}(R)$ and $p^{(n)}(W)$ are measured by the percentage of subjects in a large sample that choose R and W on trial n, and that the transitional probabilities observed on successive pairs of trials are constant. Assume the following numerical values for T $d_0 = d_1$:

$$\begin{Bmatrix} .97 & .27 \\ .03 & .73 \end{Bmatrix} \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} = \begin{Bmatrix} .27 \\ .73 \end{Bmatrix}.$$

A right response is followed by another right response 97 per cent of the time; wrong follows wrong 73 per cent of the time. From Eq. (8) we calculate that the successive values of $p^{(n)}(R)$ are 0, .27, .46, .59, .68, etc., approaching the asymptote of .90. The equation is

$$p^{(n)}(R) = .9(1 - .7^n) \qquad (n = 0, 1, 2, \ldots)$$

If we know that on a particular trial a W occurred, this equation gives the probability of R on the nth succeeding trial.

2. *Autocorrelation Function.* A simple parameter of such Markov chains is the autocorrelation function. We will mention it now because for the more complex cases we wish to consider next the autocorrelation function is either not defined or is most tedious to compute from the matrix of transitional probabilities.

The autocorrelation function is the correlation of a time series with itself displaced 0, 1, 2, ... steps. With zero displacement the correlation of the series with itself is, of course, +1. With a displacement of one step, the responses on trials 1, 2, 3, ... are correlated with the responses on trials

2, 3, 4, ... If the series of binary choices is fairly long, the autocorrelation after a displacement of one step is given by

$$r_1 = p_A(A) - p_B(A). \tag{10}$$

We note that r_1 is a characteristic root of the matrix of transitional probabilities. More generally,

$$r_m = p_A^{(m)}(A) - p_B^{(m)}(A), \tag{11}$$

where $p_A^{(m)}(A)$ and $p_B^{(m)}(A)$ are elements of T^m. From Eq. (7) we observe that these elements of T^m are

$$p_A^{(m)}(A) = \frac{p_B(A) + p_A(B)[p_A(A) - p_B(A)]^m}{p_A(B) + p_B(A)}$$

and

$$p_B^{(m)}(A) = \frac{p_B(A) - p_B(A)[p_A(A) - p_B(A)]^m}{p_A(B) + p_B(A)}.$$

When these values are substituted in Eq. (11), we obtain

$$r_m = [p_A(A) - p_B(A)]^m = r_1^m. \tag{12}$$

In short, for a simple Markov chain, the autocorrelation between positions n and $n + m$ is the mth power of the autocorrelation between n and $n + 1$. If $| r_1 | < 1$, then $| r_m |$ declines monotonically toward zero.

A simple example is provided by the Samoan language. E. B. Newman has noted that the sequence of consonants (C) and vowels (V) in Samoan writing is adequately described as a Markov chain with the following matrix of transitional probabilities:

$$\begin{Bmatrix} p_C(C) & p_V(C) \\ p_C(V) & p_V(V) \end{Bmatrix} = \begin{Bmatrix} 0 & .49 \\ 1 & .51 \end{Bmatrix}.$$

Consonants never follow consonants in written Samoan. The autocorrelation function is easily computed from this matrix. For successive displacements of one letter the value of the correlation coefficient is 1, −.49, .24, −.12, .06, −.03, etc.

The autocorrelation function for this simple process can also be described as the determinant of T^m. Thus r_0 is the determinant of $T^0 = I$, r_1 is the determinant of T, r_2 is the determinant of T^2, etc.

When the distribution of probabilities at $n + 1$ depends upon events prior to n as well as upon n itself, Eq. (10) still holds as a definition of the autocorrelation function, but Eq. (11) does not hold. When more than two unscaled alternatives are used, the autocorrelation function is not defined.

3. *Extension to More than Two Alternatives.* The extension of the matrix equations to experiments involving more than two alternative responses is straightforward. Designate the alternatives A, B, C, . . . , N. Then we have

$$\begin{Bmatrix} p_A(A) & p_B(A) & \cdots & p_N(A) \\ p_A(B) & p_B(B) & \cdots & p_N(B) \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ p_A(N) & p_B(N) & \cdots & p_N(N) \end{Bmatrix} \begin{Bmatrix} p^{(n)}(A) \\ p^{(n)}(B) \\ \cdot \\ \cdot \\ \cdot \\ p^{(n)}(N) \end{Bmatrix} = \begin{Bmatrix} p^{(n+1)}(A) \\ p^{(n+1)}(B) \\ \cdot \\ \cdot \\ \cdot \\ p^{(n+1)}(N) \end{Bmatrix}. \quad (13)$$

General solutions are known for certain types of operators. These are of considerable interest in physics and genetics, where the elements of T are given by theory. The present use of such operators is almost purely descriptive, however, for we do not know what special types of matrices will be of the greatest psychological interest.

It is not always necessary to find a general solution. A qualitative understanding of an experimental situation is often provided by simply transforming the initial distribution five or ten steps by direct matrix multiplication. For example, a learning situation might be analyzed into three kinds of responses: correct (C), slightly wrong (S), and grossly wrong (G). During the course of learning a subject begins by making gross mistakes, then slight mistakes, and finally manages to make correct responses. Such a situation could produce a matrix equation like the following:

$$Td_0 = \begin{Bmatrix} p_C(C) & p_S(C) & p_G(C) \\ p_C(S) & p_S(S) & p_G(S) \\ p_C(G) & p_S(G) & p_G(G) \end{Bmatrix} \begin{Bmatrix} p^{(0)}(C) \\ p^{(0)}(S) \\ p^{(0)}(G) \end{Bmatrix} = \begin{Bmatrix} .9 & .3 & 0 \\ .1 & .6 & .3 \\ 0 & .1 & .7 \end{Bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}.$$

It is tedious to find the general solution of T^n, and it is easy to see by direct multiplication what happens. The proportion of grossly wrong responses declines steadily: 1, .7, .52, .40, .32, .26, ... , .08. The proportion of small errors on successive trials at first increases, then decreases: 0, .3, .39, .40, .38, .35, ... , .23. The proportion of correct responses gives a roughly S-shaped function: 0, 0, .09, .20, .30, .38, .45, ... , .69. This situation is analogous to pouring water from one vessel into a second, which in turn pours the water into a third. The asymptotic distribution can always be found by solving the equation $Td_n = d_n$.

The form of a general solution can be indicated, for finite matrices with distinct roots, as follows. Let λ_i represent the N characteristic roots of the polynomial $\det(T - \lambda I)$. We define a set of matrices $f_i(T)$ by

$$f_i(T) = \frac{(T - \lambda_1 I)(T - \lambda_2 I) \cdots (T - \lambda_{i-1} I)(T - \lambda_{i+1} I) \cdots (T - \lambda_N I)}{(\lambda_i - \lambda_1)(\lambda_i - \lambda_2) \cdots (\lambda_i - \lambda_{i-1})(\lambda_i - \lambda_{i+1}) \cdots (\lambda_i - \lambda_N)}. \quad (14)$$

In terms of these matrices, T can be expressed

$$T = \lambda_1 f_1(T) + \lambda_2 f_2(T) + \cdots + \lambda_N f_N(T). \tag{15}$$

If $g(\lambda)$ is a rational scalar polynomial, then

$$g(T) = g(\lambda_1) f_1(T) + g(\lambda_2) f_2(T) + \cdots + g(\lambda_N) f_N(T). \tag{16}$$

In particular, if $g(\lambda) = \lambda^n$, we have

$$T^n = \lambda_1^n f_1(T) + \lambda_2^n f_2(T) + \cdots + \lambda_N^n f_N(T). \tag{17}$$

The 2×2 transformation is expressed in this form in Eq. (7). Concerning the roots λ_i , we know that λ_1 can be assigned the value 1, and that all the other roots fall between -1 and $+1$. Thus the asymptotic value of T^n is given by $f_1(T)$.

The solution for a particular matrix can always be obtained by (a) finding the roots of the characteristic polynomial, $\det(T - \lambda I)$; (b) determining the $f_i(T)$ according to Eq. (14); (c) substituting into Eq. (17); and (d) solving $T^n d_0$ for the given boundary conditions of d_0 . This procedure has the advantage of avoiding the problem of inverting a large matrix, but if two or more roots are nearly the same, the computations may be quite difficult.

The autocorrelation function is not defined for more than two unordered alternatives, because the value of the correlation coefficient varies according to the various possible assignments of numerical values to the different alternatives. However, the determinant of the matrix of transitional probabilities has many of the characteristics of a correlation coefficient, and in the 2×2 case the determinant and the autocorrelation coefficient are identical. The determinant of T^n, as a function of n, lies between $+1$ and -1, declines toward 0 for the Markov processes, and can reveal periodicities in much the same way as an autocorrelation function. The possible usefulness of this extension to $N \times N$ transformations needs to be explored.

4. *Extension to Compound Responses.* For psychological purposes it is an inconvenience that Markov processes have no memory. We must now remove the restriction that, if the outcome of the trial n is known, events prior to n are irrelevant for predicting the outcome at $n + 1$. We must consider the non-Markovian case. What we must do is to expand the definition of a state of the system in order to make such systems Markovian in a larger space.

If the probabilities at trial $n + 1$ depend upon the outcomes of trials n and $n - 1$, but knowledge of events prior to $n - 1$ does not change our prediction for $n + 1$, we have a non-Markovian system. This system is made to be Markovian by changing the definition of an event. Instead of characterizing the state of the system by the occurrence of a single response, we characterize it by pairs of responses. If there are two atomic alternatives,

A and B, in the original system, then there are four compound alternatives, AA, AB, BA, and BB, in the new system. Thus we must define a distribution d_n over four alternatives, and T is a square matrix of fourth order:

$$Td_n = \begin{Bmatrix} p_{AA}(AA) & 0 & p_{BA}(AA) & 0 \\ p_{AA}(AB) & 0 & p_{BA}(AB) & 0 \\ 0 & p_{AB}(BA) & 0 & p_{BB}(BA) \\ 0 & p_{AB}(BB) & 0 & p_{BB}(BB) \end{Bmatrix} \begin{Bmatrix} p^{(n)}(AA) \\ p^{(n)}(AB) \\ p^{(n)}(BA) \\ p^{(n)}(BB) \end{Bmatrix}$$

$$= \begin{Bmatrix} p^{(n+1)}(AA) \\ p^{(n+1)}(AB) \\ p^{(n+1)}(BA) \\ p^{(n+1)}(BB) \end{Bmatrix} = d_{n+1} . \quad (18)$$

Note that many of the transitional probabilities are zero; it is not possible for the system to move from some state to others in a single step. For example, the system cannot move from AA to BB in less than two steps: $AA \rightarrow AB \rightarrow BB$ as in the sequence $AABB$.

Tabulations of sequences of vowels and consonants in written Hebrew have been made by E. B. Newman. The sequence of consonants (A) and vowels (B) can be adequately represented by a matrix of the form of Eq. (18):

$$\begin{Bmatrix} 0 & 0 & .23 & 0 \\ 1 & 0 & .77 & 0 \\ 0 & .81 & 0 & .90 \\ 0 & .19 & 0 & .10 \end{Bmatrix} \begin{Bmatrix} .095 \\ .410 \\ .410 \\ .085 \end{Bmatrix}.$$

As before, the transformation T can be applied iteratively to carry any initial distribution into a final, unique, stable distribution.

This extension of the Markov process can be carried as far as the data seem to merit. For example, fixed-ratio reinforcement in operant conditioning requires an animal to respond m times in one way, then approach the food tray. In order to keep track of the sequential aspects of this behavior we could define a state of the system to include all the possible sequences of responses and approaches of length $m + 1$. Thus there would be 2^{m+1} alternative states, and the transformation would be of order 2^{m+1}. More complex sequential dependencies arise in human verbal behavior and can be treated in a similar manner. The verbal case is so complex, however, that it cannot be adequately discussed in this paper.

In principle it is possible to extend the Markov definition indefinitely to take into account as much of the past history of the system as one desires.

Cases are known, however, in which the extension would need to be carried infinitely far into the past in order for the Markov model to summarize all the information. Such cases are better handled in other ways. At present, it seems likely that most learning situations will need to be described by these other methods, and that Markov processes using a single matrix of transitional probabilities are most valuable when the behavior has settled into a relatively stable pattern.

5. *Least-Squares Fit to Data.* Under the assumption that a single transformation describes the behavior, every trial can be considered a measurement of the single transformation T. We wish to find a least-squares solution that will give the best estimate for T from the available data. The following procedures may not be the most efficient for Markov processes, but they represent one fairly natural extension of the procedures used with more familiar statistical problems.

We introduce a matrix M to represent the observed data. This matrix is formed by placing in successive columns the distributions observed on successive trials, from trial 1 through trial $n - 1$. If each distribution contains a alternative quantities, and n such distributions are known for successive trials, then M is an $a \times (n - 1)$ matrix. A matrix N is formed analogously by placing in successive columns the distributions observed on the successive trials from 2 through n. Thus N is also an $a \times (n - 1)$ matrix. The matrix $\overline{N}$ represents the best estimate of the successive distributions:

$$\overline{N} = N + C, \tag{19}$$

where the elements of the matrix C are the corrections that must be added to the observed values in N to give the best estimate $\overline{N}$.

We wish to determine $\overline{T}$, the best estimate of the transformation. From the definition of M and $\overline{N}$ and the assumption of a single operator throughout learning, we have the equation:

$$\overline{T}M = \overline{N} = N + C. \tag{20}$$

From Eq. (20) we obtain an expression for C:

$$C = -N + \overline{T}M. \tag{21}$$

For a least-squares solution, CC' must be a minimum. This is obtained by putting the partial derivative with respect to $\overline{T}$ to zero:

$$\frac{\partial}{\partial \overline{T}} CC' = MC' = 0. \tag{22}$$

We now substitute for C' from Eq. (21) into Eq. (22) and obtain

$$M(-N + \overline{T}M)' = -MN' + MM'\overline{T}' = 0.$$

Rearranging terms gives

$$\overline{T}' = (MM')^{-1}MN',$$

or

$$\overline{T} = NM'(MM')^{-1}. \tag{23}$$

Eq. 23 provides a best estimate of T on the basis of the data matrices M and N.

As an example, consider an experiment in a T-maze. We decide from an examination of the data that the learning process can be described by a Markov process with a single transformation. Suppose that 10 rats were run for 20 trials, and that on successive trials the following numbers of rats made the correct choice: 5, 7, 6, 6, 8, 8, 8, 7, 8, 9, 8, 7, 8, 9, 10, 10, 8, 8, 9, 9. From these data we construct the matrices:

$$M = \begin{Bmatrix} .5 & .7 & .6 & .6 & .8 & .8 & .8 & .7 & .8 & .9 & .8 & .7 & .8 & .9 & & & & & \\ .5 & .3 & .4 & .4 & .2 & .2 & .2 & .3 & .2 & .1 & .2 & .3 & .2 & .1 & & & & & \\ & & & & & & & & & & & & 1.0 & 1.0 & .8 & .8 & .9 \\ & & & & & & & & & & & & 0 & .0 & .2 & .2 & .1 \end{Bmatrix}$$

$$N = \begin{Bmatrix} .7 & .6 & .6 & .8 & .8 & .8 & .7 & .8 & .9 & .8 & .7 & .8 & .9 & 1.0 & & & & \\ .3 & .4 & .4 & .2 & .2 & .2 & .3 & .2 & .1 & .2 & .3 & .2 & .1 & 0 & & & & \\ & & & & & & & & & & & & 1.0 & .8 & .8 & .9 & .9 \\ & & & & & & & & & & & & 0 & .2 & .2 & .1 & .1 \end{Bmatrix}.$$

Next we multiply these matrices to obtain

$$NM' = \begin{Bmatrix} 12.16 & 3.14 \\ 2.74 & .96 \end{Bmatrix}, \qquad MM' = \begin{Bmatrix} 11.99 & 2.91 \\ 2.91 & 1.19 \end{Bmatrix}.$$

The matrix MM' is easily inverted, and we have

$$\overline{T} = NM'(MM')^{-1} = \begin{Bmatrix} 12.16 & 3.14 \\ 2.74 & .96 \end{Bmatrix} \begin{Bmatrix} 1.19 & -2.91 \\ -2.91 & 11.99 \end{Bmatrix} \frac{1}{5.8},$$

$$\overline{T} = \begin{Bmatrix} .92 & .39 \\ .08 & .61 \end{Bmatrix}.$$

The initial distribution d_0 is (.5, .5), and from Eq. (8) we obtain

$$p^{(n)}(R) = .83 - .33(.63)^n.$$

The values calculated from this equation are .500, .665, .738, .785, .804, . . . , approaching .83 as the asymptote. Note that we do not have a least-squares

fit of this function, $p^{(n)}(R)$, to the observed data; we have a least-squares fit for the transformation T.

From Eq. (21) we can calculate the corrections that are added to N:

$$\overline{TM} = \begin{Bmatrix} .655 & .761 & .708 & .708 & .814 & .814 & .814 & .761 & .814 & .867 \\ .345 & .239 & .292 & .292 & .186 & .186 & .186 & .239 & .186 & .133 \\ & .814 & .761 & .814 & .867 & .920 & .920 & .814 & .814 & .867 \\ & .186 & .239 & .186 & .133 & .080 & .080 & .186 & .186 & .133 \end{Bmatrix},$$

$$C = \begin{Bmatrix} -.045 & .161 & .108 & -.092 & .014 & .014 & .114 & -.039 \\ .045 & -.161 & -.108 & .092 & -.014 & -.014 & -.114 & .039 \\ -.086 & .067 & .114 & -.039 & -.086 & -.133 & -.080 & \\ .086 & -.067 & -.114 & .039 & .086 & .133 & .080 & \\ & & & & .120 & .014 & -.086 & -.033 \\ & & & & -.120 & -.014 & .086 & .033 \end{Bmatrix}.$$

The squared deviations are given by

$$CC' = \begin{Bmatrix} .144 & -.144 \\ -.144 & .144 \end{Bmatrix}.$$

The best estimate of the dispersion of the calculated from the observed values is

$$\sigma = \sqrt{\frac{cc'}{n - a - 1}} = \sqrt{\frac{.144}{17}} = .092. \tag{24}$$

The variance-covariance matrix V is given by

$$V = \sigma^2(MM')^{-1} = \frac{.00847}{5.8} \begin{Bmatrix} 1.19 & -2.91 \\ -2.91 & 11.99 \end{Bmatrix}. \tag{25}$$

From Eq. (25) we compute the standard deviations of the estimates of $p_A(A)$ and $p_B(B)$:

$$\sigma[p_A(A)] = .092 \sqrt{\frac{1.19}{5.8}} = .04 ,$$

$$\sigma[p_B(B)] = .092 \sqrt{\frac{11.99}{5.8}} = .132.$$

The same procedure can be applied to the data from a single animal. The data matrices M and N then have either 0 or 1 on successive trials; e.g.,

$$M = \begin{Bmatrix} 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & \cdots & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & \cdots & 1 & 0 & 0 \end{Bmatrix}$$

$$N = \begin{Bmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & \cdots & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 \end{Bmatrix}.$$

In order to solve for $\overline{T}$ we determine

$$NM' = \begin{Bmatrix} m(1,1) & m(0,1) \\ m(1,0) & m(0,0) \end{Bmatrix}, \qquad MM' = \begin{Bmatrix} m(1) & 0 \\ 0 & m(0) \end{Bmatrix}.$$

The symbol $m(i,j)$ represents the number of occurrences of the ordered pair i,j; $m(i)$ represents the number of occurrences of i; and $m(0) + m(1) = n - 1$, where n is the number of trials. Next we invert MM' and solve for $\overline{T}$:

$$\overline{T} = NM'(MM')^{-1} = \begin{Bmatrix} m(1,1) & m(0,1) \\ m(1,0) & m(0,0) \end{Bmatrix} \begin{Bmatrix} \frac{1}{m(1)} & 0 \\ 0 & \frac{1}{m(0)} \end{Bmatrix} \tag{26}$$

$$\overline{T} = \begin{Bmatrix} \frac{m(1,1)}{m(1)} & \frac{m(0,1)}{m(0)} \\ \frac{m(1,0)}{m(1)} & \frac{m(0,0)}{m(0)} \end{Bmatrix}.$$

Eq. (26) is the result that would be expected from the definition of the transitional probabilities.

In order to estimate the dispersion we calculate

$$\overline{T}M = \begin{Bmatrix} \frac{m(1,1)}{m(1)} & \frac{m(0,1)}{m(0)} & \frac{m(1,1)}{m(1)} & \cdots & \frac{m(1,1)}{m(1)} \\ \frac{m(1,0)}{m(1)} & \frac{m(0,0)}{m(0)} & \frac{m(1,0)}{m(1)} & \cdots & \frac{m(1,0)}{m(1)} \end{Bmatrix}.$$

Then from Eq. (21) we find

$$C = \overline{T}M - N = \begin{Bmatrix} \frac{m(1,1)}{m(1)} & \frac{-m(0,0)}{m(0)} & \frac{-m(1,0)}{m(1)} & \cdots & \frac{-m(1,0)}{m(1)} \\ \frac{-m(1,1)}{m(1)} & \frac{m(0,0)}{m(0)} & \frac{m(1,0)}{m(1)} & \cdots & \frac{m(1,0)}{m(1)} \end{Bmatrix}$$

The squared deviations are given by

$$CC' = \begin{Bmatrix} c & -c \\ -c & c \end{Bmatrix}$$

where

$$c = m(1,1)\left[\frac{m(1,0)}{m(1)}\right]^2 + m(1,0)\left[\frac{m(1,1)}{m(1)}\right]^2 + m(0,1)\left[\frac{m(0,0)}{m(0)}\right]^2 + m(0,0)\left[\frac{m(0,1)}{m(0)}\right]^2$$

$$= [m(1,0) + m(1,1)]\left[\frac{m(1,1)m(1,0)}{m(1)\ m(1)}\right] + [m(0,0) + m(0,1)]\left[\frac{m(0,1)m(0,0)}{m(0)\ m(0)}\right]$$

$$= m(1)\left[\frac{m(1,1)}{m(1)} \cdot \frac{m(1,0)}{m(1)}\right] + m(0)\left[\frac{m(0,1)}{m(0)} \cdot \frac{m(0,0)}{m(0)}\right].$$

The dispersion is, therefore,

$$\sigma = \sqrt{\frac{c}{n-1-a}} = \left\{\left[\frac{m(1)}{n-3} \cdot \frac{m(1,1)}{m(1)} \cdot \frac{m(1,0)}{m(1)}\right] + \left[\frac{m(0)}{n-3} \cdot \frac{m(0,1)}{m(0)} \cdot \frac{m(0,0)}{m(0)}\right]\right\}^{1/2}. \quad (27)$$

The variance-covariance matrix is

$$V = \sigma^2(MM')^{-1} = \frac{c}{n-3}\begin{pmatrix} \frac{1}{m(1)} & 0 \\ 0 & \frac{1}{m(0)} \end{pmatrix}$$

and from this matrix we compute

$$\sigma[p_A(A)] = \sigma\sqrt{\frac{1}{m(1)}} \quad \text{and} \quad \sigma[p_B(B)] = \sigma\sqrt{\frac{1}{m(0)}}. \quad (28)$$

Although these examples are worked out for the Markov case with two alternatives, the same procedures can be used with more than two alternatives or with Markov processes defined for compound responses. It should be stressed, however, that the statistical properties of Markov chains are neither simple nor well understood. Better techniques will undoubtedly develop as the Markov process becomes more widely applied.

6. *Variable Transformations.* Up to this point we have made the explicit assumption that a single transformation could describe the successive changes in the probabilities of the alternative responses or alternative sequences of responses. This assumption greatly simplifies the theoretical landscape and should be made whenever the data hint that it might be true. Simplicity is not, however, an intrinsic property of the behavior of living organisms, and so we must be prepared to deal with situations that obviously violate the assumption.

The assumption that a single transformation is adequate means that the transitional probabilities are fixed from the first through the last trial. Since the transitional probabilities determine the sequences of responses that are probable or improbable, we are assuming that the animal's course of action or strategy is fixed throughout the experiment. In a certain sense, therefore, such an assumption means that there is no learning at all; as soon as the experimental situation is encountered for the first time, the subject adopts the set of transitional probabilities that will later describe the statistical properties of his behavior after he has had long experience in the situation.

The assumption of a single transformation would be justified, for example, after a long series of alternate conditioning and extinction. In this experiment the subject is able to evolve a single transformation for the reinforcement conditions and another for the extinction conditions. Or if an animal has adopted a stable mode of behavior in a situation and then is temporarily distracted in some way, his return to normal when the impediment is removed might be expected to follow a single transformation. But in most of the situations that are studied experimentally there is no *a priori* reason to expect that a single transformation will be adequate, and there are several reasons to expect that it will not be.

In order to illustrate what is involved in the assumption of a single transformation, Table I has been prepared to show one case where the assumption is correct and another where the assumption is wrong. Once more we consider the data from 10 rats on 20 consecutive choices in a *T*-maze. The symbol 1 represents a correct choice, and 0 represents an incorrect choice. In Tables IA and IB the numbers of rats making the correct choice are the same, and both are the same as the example fitted in the preceding section.

TABLE 1

Hypothetical Data for Ten Rats on Twenty Trials in a T-Maze

IA. Constant Transformation

	Trial																			
Rat	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	1	1	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1
2	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	0	1	1	1
3	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	0	0
4	0	1	1	1	1	1	1	1	1	1	0	0	0	0	1	1	1	1	1	1
5	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	0	1	1
6	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
7	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
8	1	1	1	1	1	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1
9	0	0	0	0	1	1	1	1	1	1	1	0	0	1	1	1	1	1	1	1
10	1	1	1	1	1	1	1	0	0	1	1	1	1	1	1	1	1	1	1	1
Σ	5	7	6	6	8	8	8	7	8	9	8	7	8	9	10	10	8	8	9	9

TABLE 1 (*Continued*)

IB. Variable Transformation

Rat	Trial 1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
11	1	0	1	1	1	1	1	1	0	1	1	1	1	1	1	1	1	1	1	1
12	0	0	0	0	1	1	1	1	1	1	0	1	1	1	1	1	0	1	1	1
13	1	1	1	1	1	0	0	1	1	1	1	1	0	0	1	1	1	0	0	0
14	1	1	0	1	1	1	1	1	1	1	1	0	0	1	1	1	1	1	1	1
15	1	1	1	0	1	1	0	0	1	1	1	1	1	1	1	1	0	0	1	1
16	0	1	0	1	1	1	1	1	0	0	1	1	1	1	1	1	1	1	1	1
17	0	1	1	1	0	1	1	1	1	1	1	0	1	1	1	1	1	1	1	1
18	0	1	1	0	0	1	1	0	1	1	1	1	1	1	1	1	1	1	1	1
19	1	0	0	0	1	0	1	0	1	1	1	1	1	1	1	1	1	1	1	1
20	0	1	1	1	1	1	1	1	1	1	0	0	1	1	1	1	1	1	1	1
Σ	5	7	6	6	8	8	8	7	8	9	8	7	8	9	10	10	8	8	9	9

From the data in Table I we can estimate the values of $p_1(1)$ and $p_0(0)$ on successive pairs of trials by $[m(i,j)]/m(i)$:

IA Trial	$p_1(1)$	$p_0(0)$	IB Trial	$p_1(1)$	$p_0(0)$
1-2	1.00	0.60	1-2	0.60	0.20
2-3	0.86	1.00	2-3	0.72	0.67
3-4	1.00	1.00	3-4	0.60	0.50
4-5	1.00	0.25	4-5	0.83	0.25
5-6	0.88	0.50	5-6	0.75	0.00
6-7	1.00	1.00	6-7	0.88	0.50
7-8	0.88	1.00	7-8	0.75	0.50
8-9	1.00	0.67	8-9	0.72	0.00
9-10	1.00	0.50	9-10	0.88	0.50
10-11	0.89	1.00	10-11	0.78	0.00
11-12	0.88	1.00	11-12	0.75	0.50
12-13	1.00	0.67	12-13	0.86	0.33
13-14	1.00	0.50	13-14	1.00	0.50
14-15	1.00	0.00	14-15	1.00	0.00
15-16	1.00	....	15-16	1.00	....
16-17	0.80	....	16-17	0.80	....
17-18	0.88	0.50	17-18	0.88	0.50
18-19	1.00	0.50	18-19	1.00	0.50
19-20	1.00	1.00	19-20	1.00	1.00

There seems to be a clear trend in IB for $p_1(1)$ to increase on successive trials, whereas no trend for $p_1(1)$ is observable in IA. If we group the trials by fives to secure more reliable estimates, we get

IA Trials	$p_1(1)$	$p_0(0)$	IB Trials	$p_1(1)$	$p_0(0)$
1-6	0.94	0.67	1-6	0.72	0.33
6-11	0.95	0.89	6-11	0.85	0.30
11-16	0.98	0.63	11-16	0.93	0.38
16-20	0.92	0.60	16-20	0.92	0.60

Comparisons such as these show that the assumption of a constant transformation cannot be checked by the successive distributions alone, for IA and IB are identical in this respect. The assumption is justified if the analysis of short sequences of trials shows relatively constant transitional frequencies, as in IA. If the transitional frequencies show a definite trend, as in IB, the assumption is not justified.

The question is what to do when we face variable transformations. Whatever we do, the situation will not be simple. If $\ldots PQRST\ d_0$ cannot be translated into $\ldots TTTTT\ d_0$, the matrix products may get quite complex. If we could choose P, Q, R, S, T as commutative matrices, it would be possible to find a simultaneous solution for all of them; all matrices would have the same characteristic vectors but different characteristic roots. Unfortunately, however, it does not seem possible in general to choose commutative matrices with the properties demanded by the data.

If the complexity of the problem is admitted as inevitable, we can still look for a matrix function of n, $T(n)$, that changes in some reasonable way on successive trials. The following argument illustrates one possible approach. We assume that at the beginning of the experiment the subjects are equipped with transitional preferences given by the matrix U. After long experience in the situation the subjects develop transitional preferences given by the matrix V. As the experiment progresses the tendencies represented by U are slowly extinguished and those represented by V are slowly strengthened. Consider the following sequence of equations:

$$\begin{aligned} T(0) &= U \\ T(1) &= wT(0) + (1 - w)V \\ T(2) &= wT(1) + (1 - w)V \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ T(n) &= wT(n - 1) + (1 - w)V, \end{aligned} \tag{29}$$

where $0 \leq w < 1$. The rationale for this set of equations is that w represents the perseveration of the tendencies on the preceding trial, and $(1 - w)$ represents the ability to adopt the new mode of response symbolized by V. If the extinction of the old pattern of responses is slow, w is near unity; if the old pattern extinguishes rapidly, w is near zero.

Eq. (29) can be written in terms of U and V:

$$\begin{aligned} T(0) &= U && = w^0(U - V) + V \\ T(1) &= wU + (1 - w)V && = w^1(U - V) + V. \\ T(2) &= w^2U + (1 - w^2)V && = w^2(U - V) + V. \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ T(n) &= w^nU + (1 - w^n)V && = w^n(U - V) + V. \end{aligned} \tag{30}$$

In this form it is clear that, since $0 \leq w < 1$, $T(n)$ approaches V as n increases. The importance of U becomes progressively smaller as the subject has more and more experience in the experimental situation. This formulation has the advantage that it is relatively easy to compute the successive values of $T(n)$, given U and V. The initial and final matrices, U and V, can be given theoretically or can be determined from data obtained prior to the first trial and after the learned behavior has stabilized again in the new course of action.

For illustrative purposes, assume that U and V are known to be

$$U = \begin{Bmatrix} .5 & .5 \\ .5 & .5 \end{Bmatrix} \quad \text{and} \quad V = \begin{Bmatrix} .9 & .4 \\ .1 & .6 \end{Bmatrix},$$

and that the weight w is calculated to be 0.8. Then Eq. (30) gives

$$T(n) = .8^n \begin{Bmatrix} -.4 & .1 \\ .4 & -.1 \end{Bmatrix} + \begin{Bmatrix} .9 & .4 \\ .1 & .6 \end{Bmatrix}.$$

Then on successive learning trials we have:

n:	0	1	2	3	4	5	6	7	8	9	10	$\cdots$
$p_A(A)$:	.5	.58	.644	.695	.736	.768	.796	.816	.832	.846	.857	$\cdots$
$p_B(B)$:	.5	.52	.536	.549	.559	.567	.574	.579	.583	.587	.589	$\cdots$

Next we calculate the proportions of right and wrong responses on successive trials. This is given by the equation:

$$\begin{aligned} T(0)d_0 &= d_1 \\ T(1)d_1 &= d_2 \quad = T(1)T(0)d_0 \\ T(2)d_2 &= d_3 \quad = T(2)T(1)T(0)d_0 \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ T(n)d_n &= d_{n+1} = \prod_n^0 T(i)d_0. \end{aligned} \tag{31}$$

It is assumed that $T(0) = U$ and d_0 are known from preliminary experimentation. Assume the boundary condition $d_0' = (.5, .5)$. Then direct computation gives the values:

n:	1	2	3	4	5	6	7	8	9	10	$\cdots$	∞
$p(R)$:	.5	.53	.559	.587	.614	.639	.662	.683	.700	.716	$\cdots$	.800

Considerable care must be taken with such iterated computation, for the errors are cumulative.

It should be noted that if $w = 0$, the variable case reduces to the constant

case, for then $T(n) = V$ and $\Pi T(i) = T^n$. Similarly, if $w = 1$, then $T(n) = U$ and we again have a single transformation.

A special case arises if U and V commute, $UV = VU$, for then $T(n)$ and $T(n + k)$ also commute. If two matrices with distinct roots commute, then one can be written as a polynomial in terms of the other, with scalar coefficients. Thus if the matrices A and B commute, we can write, according to Eq. (15) and (16),

$$\begin{aligned} B &= \lambda_1 f_1(B) + \lambda_2 f_2(B) + \cdots + \lambda_N f_N(B) \\ A &= g(B) = g(\lambda_1) f_1(B) + g(\lambda_2) f_2(B) + \cdots + g(\lambda_N) f_N(B), \end{aligned} \tag{32}$$

where λ_1 is the characteristic root of B; $g(\lambda_1)$ is the characteristic root of A; and for matrices of transitional probabilities $\lambda_1 = g(\lambda_1) = 1$. Thus A and B have different roots, but $f_i(A) = f_i(B)$. Another way of saying the same thing is to note that commutative matrices are transformed into their diagonal form by the same operator. Thus if S transforms A into the diagonal form Λ_A, S also transforms B into its diagonal form Λ_B. The product of A and B is (since the diagonal matrices Λ_A and Λ_B obviously commute)

$$\begin{aligned} AB &= (S\Lambda_A S^{-1})(S\Lambda_B S^{-1}) = S\Lambda_A\Lambda_B S^{-1} = S\Lambda_B\Lambda_A S^{-1} \\ &= (S\Lambda_B S^{-1})(S\Lambda_A S^{-1}) = BA. \end{aligned}$$

If the matrices $T(i)$ commute, then

$$\prod_n^0 T(i) = S\left[\prod_n^0 \Lambda(i)\right] S^{-1}, \tag{33}$$

where the $\Lambda(i)$ are the diagonal matrices similar to $T(i)$. The product of the $T(i)$ reduces to the product of diagonal matrices. If all of the $\Lambda(i)$'s are equal, then Eq. (33) reduces to the constant case given by Eq. (5).

Commutative matrices occur when the distribution over the several alternative responses does not change, although the transitional probabilities do change. If U has been applied repeatedly, U^n approaches $f_1(U)$ as a limit; after V has been applied repeatedly, V^n approaches $f_1(V)$. When U and V commute, $f_1(U) = f_1(V)$, and so both transformations lead to the same stable distribution. Such a situation might arise in learning a simple alternation between left and right. The learning might leave $p(L) = p(R) = .5$, although the transitional probabilities were altered.

This discussion of learning should suggest some of the descriptive possibilities of systems of dependent probabilities. By this general development we arrived at a mathematical description of complex behavioral changes—a description that enables us to talk about the gradual replacement of one pattern of responses by another.

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