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SECOND-ORDER FACTORS

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Second-order factors are defined and illustrated in terms of a literal notation, a physical example, a diagrammatic representation, a geometrical example, and the matrix equations relating the first-order and second-order domains. Both kinds of factors are discussed as parameters which may be not only descriptive of the individual objects in a statistical population but also descriptive of the restrictive conditions under which the objects were generated or selected. Second-order factors may be of significance in reconciling the several theories of intelligence. This paper is concerned with test configurations that show simple structure. If such a structure is not revealed, then the second-order domain is indeterminate.

1. *First-Order and Second-Order Factors*

Most of the work that has been done so far in the development of factorial theory has been concerned with the factors obtained from test correlations with or without rotation of axes for the selection of a suitable reference frame. *Factors that are obtained from the test correlation will be called first-order factors* whether they are selected so as to be orthogonal or oblique. We shall now consider the factors that may be determined from the correlations of the first-order factors. *Factors that are obtained from the correlations of the first-order factors will be called second-order factors.* Factors of this type seem to be of fundamental significance in the interpretation of correlated variables.*

Analysis of second-order factors and their relations to those of first-order can be presented in several different ways. We shall describe these two factorial domains in terms of a literal notation, a physical example, a diagrammatic representation, a geometrical example, and the matrix equations relating the two domains.

Consider first a reduced correlation matrix for the tests whose rank is, say, five. The factoring of this correlation matrix determines five arbitrary orthogonal unit reference vectors which may be denoted *I, II, III, IV,* and *V.* This orthogonal reference frame is arbitrary in

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the sense that it is defined by the method of factoring which happens to be used. This reference frame will be regarded as fixed and all other vectors will be defined in terms of this fixed orthogonal frame, which is designated by the subscript m. Let it be assumed that a complete simple structure can be found in the test configuration and let the corresponding primary vectors be denoted A, B, C, D, and E. (These are ordinarily denoted T_A, T_B, etc.) We shall assume that these primary traits are correlated in the experimental population. Then the primary vectors in the test configuration will be separated by acute angles whose cosines are the correlations between the primary traits in the particular group of subjects studied. Let these correlations be listed in a new correlation matrix of order 5×5 showing the correlations between primary factors. This correlation matrix defines the second-order domain just as the correlation matrix for the tests defines the first-order domain.

The simplest case is that in which the five primary factors are uncorrelated, in which case their correlation matrix is a unit matrix so that an analysis of a second-order domain is not immediately indicated. Next would be the case in which the reduced correlation matrix for the primary factors A, B, C, D, and E is of unit rank. There are two types of interpretation for such a situation. The correlations between the primary factors in a particular experimental population may be due to conditions of selection of the subjects, and in this case the correlations would be of no more theoretical importance than the conditions of selection of the subjects. If, on the other hand, the five primary functions A, B, C, D, and E actually do have some parameter in common, then one would expect their intercorrelations to be of unit rank for different experimental groups of subjects that are selected in different ways. In other words, the mere fact that a set of variables, or a set of factors, are correlated does not imply any scientific obligation to find "the" factors that account for the correlations because the factors, if found, might turn out to be as incidental in significance as the conditions by which the subjects happened to be selected. On the other hand, the fact that correlations between variables, or between factors, can be caused by scientifically trivial circumstances does not guarantee that all correlations between variables are of trivial significance. If the correlations between the five primary factors in the present example should turn out to be of unit rank, then this circumstance merits a closer look because such a simplification would not often happen by chance. If the correlations between the primary factors should turn out to be of unit rank for several different experimental groups, then we should have an obligation to ascertain the cause which must transcend the selective conditions.

In order to avoid misunderstanding, perhaps it should be remarked that in factor analysis we are using the term *parameter* in its universal meaning in science. *A parameter is one of the measurements that are used for describing or defining an object or event.* In statistical theory the term *parameter* is frequently used in a more restricted sense as descriptive of the universe as contrasted with a *statistic* which is the corresponding measurement on a sample. We are not using the term in this restricted sense.

Let it be assumed that the five primary factors do have a parameter p in common. Then the five primaries could be expressed in the form

$$\begin{aligned} A &= f(p, a), \\ B &= f(p, b), \\ C &= f(p, c), \\ D &= f(p, d), \\ E &= f(p, e), \end{aligned}$$

where each primary function is defined in terms of a parameter such as a, b, c, d, or e, which is unique to itself and also in terms of another parameter p, which it shares with the functions that define the other primaries. If there should happen to be conspicuous correlation between the parameters a, b, c, d, and e in the particular group of subjects, then the unit rank of the second-order domain would be disturbed. If the correlations of the primaries show unit rank, then, in addition to the parameters a, b, c, d, and e, a second-order parameter or factor p can be postulated.

It should be noted that we now have six parameters, namely, a, b, c, d, e, and p, and since the rank of the test correlations is five, it follows that these six parameters are linearly dependent. In fact, the parameter p is now a linear combination of the other five parameters. We can express these relations by a set of parameters such as A, B, C, D, E, and p, in which p is a linear combination of the five primary parameters. The five primaries are parameters descriptive of the first-order domain, and the parameter or factor p is descriptive of the second-order domain, which is here of unit rank. The second-order parameter is a linear combination of the five primaries that are defined by the original test correlations. If some degree of consistency can be found for these parameters for different groups of subjects, then all of these parameters should represent some aspects of the underlying physical and mental functions.

Consider next a set of correlated primaries A, B, C, D, and E in which the parameter p appears in the first order as in the following example:

$$A = f(p, a),$$
$$B = f(p, b),$$
$$C = f(p, c),$$
$$D = f(p, d),$$
$$E = f(p).$$

The rank of the reduced correlation matrix of the tests would now be five. The five primaries listed above would be correlated and of unit rank. The second-order factor p would be determined from the correlations of the primaries. In this case the communality for the primary factor E would be near unity, thus showing that its total variance is common to the second-order factor p, hence the primary factor E of the first order and the factor p of the second order would be identical. The presence, or absence, of the primary E could be determined by including, or excluding, a few tests in the battery. We see, therefore, that the appearance of a factor in the first order or in the second order may depend on the battery of measurements taken; hence a factor should not be considered as intrinsically different because it appears in the second order. This circumstance can be determined by the selection of the test battery.

On the other hand, a parameter which always appears in the measurements in association with some other function would not appear as in the primary E and it would be discovered experimentally nearly always in the second order. Such a limitation could be introduced by the physical nature of the attribute which the factor represents, so that in such a case the second order would represent something fundamentally different from that of the first order. A single factor study is not likely to reveal whether a second order parameter is fundamentally different from the parameters of first order or whether the differentiation is caused merely by the selection of the test battery.

In the following example we have another combination of primaries,

$$A = f(p, a),$$
$$B = f(p, b),$$
$$C = f(p, c),$$
$$D = f(p, d),$$
$$E = f(e).$$

In this example the reduced rank of the test correlations would again be five. The correlations of the primaries would show unit rank for A, B, C, and D. The factor E would be orthogonal to the rest of the system so that its row and column would have side correlations of zero. The correlations of primaries would not be of unit rank if we consider the whole table of order 5×5 but it would be of unit rank if we consider only the 4 × 4 table for *A, B, C,* and *D*. Relations of this kind can be found by inspection of the correlations of the primaries and they may be indicative of the underlying order in the domain that is being investigated.

The principles of a second-order domain have been discussed here in terms of the simple case in which that domain is of unit rank so that there is only one general second-order factor. It should be evident that the organization of the second-order parameters can be of any rank and complexity. For example, the rank may be higher than one, and the second-order factors may extend to all of the primaries or only to some of them. The possibility of third-order and higher-order factors must be recognized but their experimental identification is of increasing difficulty the higher the order because of the instability of such a superstructure on practically feasible experimental data. The number of second-order factors that can be determined from a given number of linearly independent primary factors follows the same restrictive relations that govern the number of primary factors that can be determined from a given number of tests. Thus, for example, it is not to be expected that three second-order factors will be determinate from only five primary factors for the same reason that three primary factors cannot be determined from five tests. Furthermore, it is entirely possible in the same data for the first-order domain to give clear interpretation of a set of primary factors and for the second-order domain to be indeterminate or ambiguous.

2. *The Box Example*

In order to illustrate the nature of first- and second-order factors, we shall make use of populations of simple objects or geometrical figures and their measurable properties instead of dealing with these factors merely as logical abstractions. We have used a population of rectangular boxes and their measurable attributes to illustrate the principles of correlated primary factors and we can use them also for the present discussion.*

A random collection of rectangular boxes was represented by the three measurements length (x), width (y), and height (z). A list of measurements was prepared which could be made on each box, such as the diagonal of the front face, the area of the top surface, the length of a vertical edge, and so on. Each of these measurements represented a test score and each box represented an individual member of the statistical population. The correlations between the measurements were computed and analyzed factorially as if we did not know

* Thurstone, L. L. Current issues in factor analysis. *Psychol. Bull.*, 1940, 37, p. 222.

anything about the exact nature of each measurement, which was treated as a test score of unknown factorial composition. As has been shown previously, the analysis revealed three factors in the correlations for the particular set of measurements used. The configuration showed a complete simple structure and a set of primary vectors was determined by the configuration. These three primaries represented the three basic parameters in terms of which all the test measurements had been expressed.

The three primary vectors were separated by acute angles whose cosines represented the correlations between the three basic parameters that were used in setting up the box example. These three correlations could be assembled into a small correlation matrix of order 3×3. The physical interpretation of the positive correlations was that large boxes tend to have all of their dimensions larger than small boxes. In other words, if one of the dimensions of a box shape is, say, six feet, the other dimensions of the box are not likely to be of the order of, say, two or three inches. The table of correlations of the three primary factors X, Y, Z, could be represented by a single common factor. This factor would be a second-order factor. It would, no doubt, be interpreted as a size factor in the box example. If this second-order size factor were denoted s, we should have four parameters for describing the box shapes, namely, the three dimensions, x, y, and z and the size factor s. These four parameters or factors would be linearly dependent because the rank of the correlation matrix of the tests was three.

In the case of the box example, a size factor or parameter could be determined in the first-order if desired. For this purpose we could use the first centroid axis, the major principal axis, or the volume vector, all of which can be easily defined in the first-order system of test vectors. The four parameters so chosen would also be linearly dependent. If we wanted to use only three linearly independent parameters including a size factor, that could be done in the first order by choosing, say, the two ratios $x/y = r_1$ and $x/z = r_2$ as well as the volume vector v. These three factors would be linearly independent but they would be correlated. The latitude with which we can choose simplifying parameters for the box example is determined in part by the fact that three factors can nearly always be represented by a common factor whereas this is not the case when the rank is higher than three.

3. *Diagrammatic Representation*

The relations between the first-order and the second-order domains can be represented diagrammatically as shown in *Figures 1,* *2*, and *3*. In *Figure 1* we have a set of eight tests whose correlations are accounted for by five primary factors, *A*, *B*, *C*, *D*, and *E*, which are uncorrelated. The factor *A*, for example, is present in the common factor variances of tests 1, 2, and 4. The primary factor *E* is present in the common factor variances of all the tests, and hence *E* would be called a general factor for the particular battery. Since it is orthogonal to all the other primary factors it may be called an *orthogonal general factor of the first order*. In order to determine the nature of the factor *E* it would be necessary to study it in different test batteries so that one could predict with certainty when the factor would be present and when it would be absent from a test. Since the primary factors are here represented as uncorrelated, the matrix of correlations of the primary factors would be an identity matrix and there would be no immediate provocation to investigate a second-order domain.

In *Figure 2* we have represented a set of tests and five primary factors *A*, *B*, *C*, *D*, and *E*. (We are not here concerned as to whether the particular number of tests represented in this diagram is adequate for the determination of five primary factors. The purpose of these diagrams is merely to show the nature of the relations between the two domains.) The rank of the correlation matrix of the tests would here be five, which corresponds to the number of linearly independent primary factors. In the present case we should find that the primary factors are themselves correlated. The matrix of correlations of these primaries would be of order 5×5 and it would be of unit rank. The correlations between the primary factors could therefore be accounted for by a single general second-order factor that is denoted *G*. If both the first-order and second-order factors were to be used for the description of the tests and the relations, we should have six parameters which would be linearly dependent because the rank of the correlations of the tests is only five. In fact, the saturation of each test with the second-order factor *G* would be a linear combination of the saturations of the test with the five primaries of the first order. None of the primary factors are general factors in this figure.

In *Figure 3* we have a more complex relation in that the correlation matrix for the primary factors would be of rank two. One of the second-order factors is here shown to be common to all but one of the primary factors, one of the second-order factors is a factorial doublet in that it represents additional correlation between the primaries *B* and *D*, and the primary factor *A* is orthogonal to the rest of the primaries so that it does not participate in the second-order domain. This diagram is drawn merely to illustrate the variations in complexity that may be found in factorial studies.

The two types of general factor here shown in *Figures 1* and *2* have some interesting differences. The general factor E of *Figure 1* is independent of the other primary factors while the general factor G in *Figure 2* is present in all of the other factors. Hence we must conclude that a second-order general factor is a part of, and must participate in, the definition of the other factors while the orthogonal general factor E of *Figure 1* is, by definition, independent of the other primary factors. It is evident, therefore, that a general second-order factor is likely to be of more fundamental significance for the domain in question than a general orthogonal first-order factor. An orthogonal general factor of the first order might operate in a test without any group factor whereas a second-order general factor would operate, ordinarily, through the mechanism of some function that could be identified as a group factor, a primary factor, or a special ability.

The factor patterns corresponding to the relations shown diagrammatically in these figures are given in *Tables 1, 2* and *3*. *Table 1* shows the factor pattern for *Figure 1*. Here the orthogonal general factor E is identified by the fact that all entries of its column are filled. *Table 2* shows the factor pattern for *Figure 2*. Here it is seen by the factor pattern that a group such as tests 1, 4, 5, and 7 have no primary factor in common and that hence their correlations would be determined only by the second-order general factor G. The determinant of the correlations for these four tests (the tetrad difference) would therefore vanish. The second-order factor matrix is also shown in this table with only one factor G to correspond to this example.

The question might be raised whether both types of general factor could be present in the same battery. That seems possible. In that case a simple structure could define the primary factors A, B, C, and D but not E in the particular battery of *Figure 1*. This factor could be assumed arbitrarily to be orthogonal to the other factors, but then the line GE of *Figure 2* would be erased to correspond to the fact that E is orthogonal to the other factors. One or more second-order general factors could be found in the correlated primaries. If the correlations of A, B, C, and D were of unit rank, another alternative would be to set E in such a relation to the other primary vectors as to maintain the unit rank with the second-order general factor. It might then be found that the vector E has non-vanishing projections on all the test vectors, in which case both types of general factor would be assumed to be a possible set of explanatory parameters for the battery in question. It must be remembered that these various locations of the reference frame for the explanatory parameters in both the first-order and the second-order domains have validity only in so far as they are suggestive of fruitful scientific interpretation. If this is not the purpose, then the factorial resolution might as well remain in the arbitrary orthogonal factors produced by factoring the given test correlations—or, better still, by not doing the factoring at all.

It might be asked how the correlations of a test battery can be resolved into a second-order domain of unit rank which is lower than the rank of the test correlations. The transitions can be regarded geometrically. The unit test vectors usually define a space of as many dimensions as there are tests. When the reduced correlation matrix is considered, its rank is frequently lower than its order. Hence the reduction from the number of tests n to the number of primary factors r represents a reduction from the total variance of the tests to the common factor variance. The complete correlation matrix for the primary factors represents a set of r unit vectors in as many dimensions, the dimensionality of the common factor space. The reduced form of this matrix for the example of *Figure 2* would have unit rank because the side correlations are determined only by that which the primaries have in common, namely, the second-order general factor.

4. *Group Factors and Primary Factors*

In *Figure 4* we have a diagrammatic representation of a different kind of resolution of factors in the second-order domain and their relation to the primary factors. In this example the rank of the correlation matrix of tests is assumed to be five as represented by as many primary factors A, B, C, D, and E. Let it be assumed that the correlation matrix for these five primaries is of unit rank. The general second-order factor G then accounts for the observed correlations of the primary factors. If the five linearly independent primary unit vectors and the second-order unit vector G are to be represented in the same space, the dimensionality of this space must be six. It is possible to locate in this augmented space another set of unit vectors a, b, c, d, and e which are mutually orthogonal and which are also orthogonal to the unit vector G. Then we have the orthogonal reference frame G, a, b, c, d, and e which defines the six dimensions of the first- and second-order factors but not the test space. The five linearly independent primary factors define a five-dimensional space corresponding to the rank of the test correlations, and this space is a part of the total six-dimensional space of this representation.

The unit vector a is a linear combination of the unit vector G and the primary vector A. The relation is similar for the other primary vectors. The primary vectors A, B, C, D, and E are correlated and of unit rank whereas the vectors, a, b, c, d, and e are arbitrarily set orthogonal to each other. In general, if the rank of the test correlations is r, and if the correlations of the primaries are of unit rank, then the primaries define a unit vector G for a general second-order factor in an augmented space of dimensionality $(r + 1)$ and also a set of r mutually orthogonal unit vectors each of which is in the plane of the second-order general factor and one of the primaries. These vectors are arbitrarily set orthogonal to the general second-order factor and they are called *group factors*. In *Figure 4* the primary factors are denoted A, B, C, D, and E and the group factors are denoted a, b, c, d, and e. With this resolution we have $(r + 1)$ linearly dependent factors which represent the test correlations of rank r. This type of resolution is preferred by some students who use the reference frame G, a, b, c, d, and e because it is orthogonal rather than the frame G, A, B, C, D, and E which is oblique.

5. *A General Second-Order Factor*

The algebraic and computational relations between the first-order and the second-order domains will be shown for the case of a single general second-order factor because of the interest of this case for the psychological controversies of the past forty years about Spearman's general intellective factor. The algebraic and computational relations to be shown can be generalized to second-order domains of higher than unit rank. It must be remembered, however, that the restriction of our discussion to unit rank for the second-order domain does not in any way imply that such low rank is always to be expected. The methods of analysis can be readily extended to a second order of higher rank when the data indicate a determinate second-order configuration. In any case, the second-order rank should be considerably lower than the rank r of the first-order factors in order to justify interpretation.

The primary vectors constitute a set of r linearly independent unit vectors that define a space of dimensionality equal to the rank of the test correlations. In order to represent a general second-order factor as a unit vector in the same configuration it is necessary to augment the dimensionality to $(r + 1)$ dimensions. A second-order domain of rank two would thus require an augmented space of dimensionality $(r + 2)$. The projections of the test vectors on these additional vectors in the augmented space can, however, be expressed as linear combinations of the test projections on the primary vectors or on any set of r linearly independent vectors in the common factor space. The procedures for determining these saturations will be shown without writing explicitly the $(r + 1)$ co-ordinates of the second-order unit vectors in the augmented space.

The present discussion is confined to factorial data that satisfy two conditions, namely 1) that a complete simple structure is revealed in the test configuration and 2) that the second-order correlation matrix is of unit rank. These methods can be adapted to the analysis of less than r primary factors and the methods can be adapted to higher second-order rank.

One of two objectives will be assumed, namely (1) to determine the projections (saturations) of the tests on the second-order factor in addition to the projections on the primary reference vectors or (2) to determine the projections on the second-order factor and also on the orthogonal group factors. It will be convenient to discuss the algebraic relations under four cases because of the different computational routes that may be chosen. These four cases are:

Case 1. Transformation from **F** *to* **V** *including the column vector* **G**

This transformation is shown in rectangular notation in *Table 4* for the equation

$$\boldsymbol{F}_{jm}\,\boldsymbol{\Psi}_{mp} = \boldsymbol{V}_{jp}\,, \tag{1}$$

in which the matrix $\boldsymbol{V}_{jp}$ has an extra column for the second-order factor G with elements v_{jg}, which may also be denoted r_{jg} because these are the correlations between the tests j and the general factor G. The transformation matrix $\boldsymbol{\Psi}_{mp}$ is identical with Λ_{mp} except for the added column G with elements ψ_{mg} which are to be determined. Consider the matrix $\boldsymbol{T}$ as an extension of the factor matrix $\boldsymbol{F}$. The rows of $\boldsymbol{T}$ give the direction cosines of the primary vectors $\boldsymbol{T}_t$ with elements t_{tm}. The same transformation gives

$$\boldsymbol{T}_{tm}\,\boldsymbol{\Psi}_{\underline{mp}} = \boldsymbol{V}_{tp}\,, \tag{2}$$

which is the diagonal matrix $\boldsymbol{D}$ except for the first column. Applying the transformation $\boldsymbol{\Psi}_{mg}$ we have

$$\boldsymbol{T}_{tm}\,\boldsymbol{\Psi}_{mg} = r_{tg}\,, \tag{3}$$

where r_{tg} is the first column of $\boldsymbol{V}_t$ and its elements are the correlations of the primary factors with the general factor G. These are known from the factoring of the unit-rank correlation matrix for the primaries. Then

$$\boldsymbol{\Psi}_{\underline{mg}} = \boldsymbol{T}^{-1}{}_{tm}\,r_{tg}\,, \tag{4}$$

and, since $\boldsymbol{T} = \boldsymbol{D}\,\Lambda^{-1}$, we have

$$\boldsymbol{\Psi}_{mg} = \Lambda\,\boldsymbol{D}^{-1}\,r_{tg}\,, \tag{5}$$

from which the first column of Ψ_{mg} can be computed. Hence the column G of the augmented oblique factor matrix V becomes known.

Case 2. Transformation from **F** *to* **U** *including group factors and general factor* G

Here the computation starts again with the orthogonal factor matrix F and the objective is to determine the saturations of the tests j with the r group factors and the second-order general factor G. This transformation is also shown in *Table 4* in rectangular notation by the equation

$$F_{jm}\, \Psi_{mw} = U_{jw}\,, \tag{6}$$

where U is the factor matrix showing the projections of tests j on group factors and general factor G. These $(r + 1)$ mutually orthogonal factors will be designated by the subscript w. The first column of this matrix is again the column of correlations r_{jg}. If the same transformation is applied to the matrix T for the primary vectors, we have

$$T_{tm}\, \Psi_{mw} = U_{tw}\,, \tag{7}$$

which is also a diagonal matrix except for the first column which contains the correlations r_{tg} between the primary factors and the second-order general factor G. These saturations can be determined from the unit-rank correlation matrix $TT' = R_t$ for the primary factors.* Consider the first row of U_t. The two entries in this row show the direction cosines of T_A in terms of the orthogonal frame G, a, b, and c. The primary vector T_A is a linear combination of the two orthogonal unit vectors G and a. Hence, when r_{1g} is known, we have

$$r^2_{1g} + u^2_{12} = 1\,, \tag{8}$$

or

$$r^2_{Ag} + u^2_{Aa} = 1\,, \tag{9}$$

so that the element u_{Aa} is known. The other diagonal elements of U_t are determined in the same way so that, for example,

$$r^2_{Bg} + u^2_{Bb} = 1\,. \tag{10}$$

When the matrix U_t is known, we have, by (7),

$$\Psi_{mw} = T^{-1}\, U_t\,, \tag{11}$$

* Elsewhere we have denoted this matrix R_{pq} but we are here using the subscript t for the primary vectors T_t and reserving the subscripts p and q for the primary reference vectors A, B, and C. Hence the correlations of the primary factors are here denoted R_t instead of R_{pq}.

or

$$\Psi_{mw} = \Lambda D^{-1} U_t \tag{12}$$

so that the transformation Ψ_{mw} is known. The saturations of tests j on the second-order general factor G and the group factors w can then be computed.

The transformation matrix Ψ_{mw} represents a rigid rotation from one orthogonal frame to another orthogonal frame, and hence this transformation matrix must be orthogonal by rows. A fourth row could be added to Ψ_{mw} for a fourth orthogonal unit vector *IV* with cell entries which normalize each column. Then we should have an orthogonal matrix of order 4×4.

Case 3. Transformation from V *to* U *including the group factors and column vector* G.

Here it is assumed that the computations are to be made from the oblique factor matrix V. In *Table 5* we have the transformation equation in rectangular notation, namely,

$$V_{jp}\, \Psi_{pw} = U_{jw}\,, \tag{13}$$

which gives the saturations of the tests j on the group factors and on the general factor. If the factor matrix V is extended to include the primary vectors T_t we have the diagonal matrix D. Applying the same transformation to D we have

$$D_{tp}\, \Psi_{pw} = U_t \tag{14}$$

so that

$$\Psi_{pw} = D^{-1}\, U_t\,. \tag{15}$$

When the elements of U_t have been determined as for *Case 2*, the transformation Ψ_{pw} can be written by merely adjusting the rows of U_t by the multipliers of $D^{-1}{}_{tp}$. The transformation Ψ_{pw} is then known.

Case 4. Transformation from V *to column vector* G

This is the simplest case and perhaps the most useful as regards the second-order domain. The matrix V is known in determining the simple structure and the primaries. The saturations of the tests j on the second-order general factor G are of interest and these can be determined as linear combinations of the columns of V. Here we have the transformation shown in *Table 5*, namely,

$$V_{jp}\, \Psi_{pg} = r_{jg}\,. \tag{16}$$

Applying the same transformation to D_{tp}, we have

$$D_{tp}\, \Psi_{pg} = r_{tg}\,. \tag{17}$$

The elements of the column vector r_{tg} are known from the correlation matrix $TT' = R_t$ of the primaries. Then

$$\Psi_{pg} = D^{-1}{}_{tp} r_{tg} , \tag{18}$$

and hence the column vector Ψ_{pg} is known. In computing, it is only necessary to multiply the elements r_{tg} by the corresponding diagonal elements of $D^{-1}{}_{tp}$ to determine Ψ_{pg}. The desired column vector r_{jg} can then be determined.

6. *A Trapezoid Population*

In previous studies of factorial theory it has been found useful to illustrate the principles by means of a population of simple physical objects or geometrical figures. The box population was used to illustrate three correlated factors and their physical interpretation. In the present case we want four factors in the first-order domain which by their correlations of unit rank determine a general second-order factor. The correlations of three variables can nearly always be accounted for by a single factor and hence it seems better to choose a four-dimensional system in which the existence of a second-order general factor is more clearly indicated by the unit rank of the correlations of four primary factors. For the present physical illustration we have chosen a population of trapezoids whose shapes are determined by four primary parameters or factors.

The measurements on the trapezoids are indicated in *Figure 5.* The base line is bisected and the length of each half is denoted by the parameter c. An ordinate is erected at this midpoint and its length is h. This ordinate divides the top section into two parts which are denoted a and b as shown. These four parameters, a, b, c, and h, completely determine the figure. The test battery was represented by sixteen measurements which are drawn in the figure. The parameters a, b, c, and h are given code numbers 1, 2, 3, and 4, respectively. Variables (12) and (13) are the two areas as shown. The sum of (12) and (13) equals the total area of the trapezoid. In general, each of these measurements is a function of two or three of the parameters but not of all four of them and hence we should expect a simple structure in this set of measurements. There is a rather general impression that a simple structure is necessarily confined to the positive manifold. In order to offset this impression we included here three additional measures which extend the simple structure beyond the positive manifold. The three additional measures are as follows:

$$14 = (1)\ /\ (2) = a/b$$
$$15 = (2)\ /\ (3) = b/c$$
$$16 = (1)\ /\ (3) = a/c$$

These three measures will necessarily introduce negative saturations on some of the basic factors.

In *Table 6* we have a list of dimensions for a set of thirty-two trapezoids. These will constitute the trapzeoid population. Each figure was drawn to scale on cross-section paper and then the sixteen measurements were made on each figure. These constituted the test scores for the present example. In setting up the dimensions of *Table 6* the numbers were not distributed entirely at random. To do so would tend to make the correlations between the four basic parameters a, b, c, and h approach zero and this would lead to an orthogonal simple structure in which there would be no provocation to investigate a second-order domain. The manner in which the generating conditions of the objects determine the factorial results will be discussed in a later section. *Table 6* was so constructed that, in addition to the four basic parameters, there was also a size factor which functioned as a second-order parameter in determining correlation between the four primary factors in generating the figures.

The product-moment correlations between the sixteen measurements for the thirty-two objects were computed and these are listed in *Table 7*. This correlation matrix was factored by the group centroid method and the resulting factor matrix F is shown in *Table 8*. The fourth-factor residuals are listed in *Table 9*, which indicates that the residuals are vanishingly small. Applying the rotational methods to the configuration, we found the transformation matrix Λ of *Table 10*, which produced the oblique factor matrix V of *Table 11*. In this matrix we are now concerned with all but the last column. When pairs of columns of the factor matrix V are plotted we have the configuration shown in the diagrams of *Figure 6*, in which a simple structure is clearly indicated. The cosine of the angle between the reference vectors is indicated on each diagram of *Figure 6*. These cosines were obtained from the relation $C = \Lambda'\Lambda$ as shown in *Table 12*.

So far in the analysis we have found that four primary factors account for the correlations and this corresponds to the fact that we used four parameters in setting up the trapezoid figures. The four primary factors are correlated as indicated by the obliqueness of the reference axes in the diagrams of *Figure 6*. The next step is to determine the correlations between the primary factors that correspond to the primary reference axes. For this purpose the inverse of the matrix C is computed as shown in *Table 12*. From the diagonal values of this matrix are found the numerical values of the diagonal matrix D, which is also shown in *Table 12*. The inverse of this diagonal matrix is also listed. These numerical values are merely the reciprocals of the entries in D.

In *Table 13* we have the correlation matrix R_t showing the correlations between the primary factors. These are the cosines of the angles between the primary vectors. It can be seen by inspection that this matrix is close to unit rank, which indicates that a single general second-order factor can be postulated to account for the correlations between the primary factors. The saturation of each primary factor with this second-order general factor was determined by one of special formulas for unit rank and the saturations are listed in the column vector r_{tg}. The interpretation is, for example, that the primary factor A has a correlation of .71 with the second-order general factor G. The closeness of the correlation matrix to unit rank is shown by the small side correlations in the residual matrix of *Table 13*. The diagonal values of the residual matrix show that part of the total variance of each primary factor which it does not share with the general second-order factor. If the diagonals of this matrix vanished completely, then the primaries would have their total variance in common and the original reduced correlation matrix for the tests would have been of unit rank.

The saturation of each test with the second-order general factor was determined as a linear combination of the columns of the oblique factor matrix V of *Table 11*. The transformation of equation (18) was used and the numerical values of Ψ_{pg} were listed in *Table 13*. Column G of *Table 11* was then computed by equation (16).

The second-order general factor G can be interpreted in this example as a size factor and it also indicates that in generating the thirty-two figures the four parameters a, b, c, and h were not allowed to take entirely independent values. In other words, the extreme forms of figures either did not occur or else they were used only occasionally. If the four parameters had been allowed to take entirely independent values, then there would have been an appreciable number of figures in which one of these parameters had an unusually small value while some other parameter had some unusually large value. This interpretation of the second-order general factor leads to a consideration of what we shall call *generating parameters*. The present geometrical example illustrates the type of factorial organization that is represented diagrammatically in *Figure 2*. The problem of interpreting the four primary factors can be solved in this case without investigating the second-order domain. But if the correlations between the primary factors show unexpectedly low rank, then this fact can be utilized factorially in gaining further insight into the conditions under which the objects were generated. The four primary factors here identified by the simple structure were the four parameters that were used in setting up the problem.

7. *Generating Parameters*

In addition to the principle of simple structure for the description of each individual object, we may consider an extension of this principle to the problem of describing the manner in which the measured objects were generated. Other things being equal, we should prefer a set of descriptive parameters that give some indication of the conditions that were operative in producing the objects. To the extent that a factor analysis can throw some light on the conditions that were responsible for producing the objects and their measurable characteristics in addition to the description of each individual object, both by some simplifying set of parameters representing causative factors, the factorial methods become even more useful as tools in scientific work.

The numerical values of the trapezoid parameters in *Table 6* defined thirty-two figures of various shapes. The method of constructing the table of four measures for each figure determined whether one or more second-order factors would be present and also whether each of the primaries would be equally or differently represented in the second-order factor. The factorial result could be altered indefinitely by the manner in which the objects were generated in constructing *Table 6*. Since it is the object of factor analysis to reveal the underlying order in the domain, it is an essential part of the numerical example to show that there is a relation between the generating principles and the factorial results.

The first column of the table contains the three linear measurements 1, 2, and 3. Suppose that these were inserted in the column entirely at random. Assume that each column was similarly constructed by distributing a set of measurements entirely at random. Then we should expect zero correlations between the four primaries T_A, T_B, T_C, and T_H. The correlation matrix for the four primary factors would be an identity matrix and it would not be factored because the primary factors would be statistically independent. There would be no second-order factor present.

If, for each one of these thirty-two figures with uncorrelated primaries, we should draw another one similar in proportions but with twice the area and another one with similar proportions but three times the area, then we should have a set of ninety-six figures consisting of three sets that have similar shapes but different sizes. If this new set of ninety-six figures were analyzed factorially with the same battery of sixteen measurements, we should find the same primary factors but they would be correlated. Furthermore, the correlations of the primary factors would all be the same, so that we should have a correlation matrix for the primary factors with uniform side correlations. The reduced correlation matrix would have unit rank and all of the four primaries would have the same saturation on the second-order general factor. This would be a situation with a second-order general factor which has a uniform effect on all of the primaries. Here again, the factorial result would be determined by the manner in which the objects were generated.

Suppose that a group of persons were asked to draw some trapezoids of arbitrary shapes and that these trapezoids were assembled as a population of figures to be measured and analyzed factorially. Then we should almost certainly introduce a second-order size factor because our subjects would probably unwittingly draw the figures so that the several dimensions of each figure would be at least roughly of the same general order of magnitude. Some of the subjects might draw trapezoids of the general size of, say, five or six inches while other subjects might draw figures only one or two inches across. Very few would produce trapezoids that are one or two inches wide and ten inches tall. In other words, since some subjects would draw big figures and others small ones and since they would probably produce very few extreme figures, there would be strong correlation between the primary factors and these in turn could be analyzed factorially into secondary factors. In this situation the rank of the correlations of the primary factors would probably not be exactly one but the inference could certainly be drawn from the factorial result that secondary factors were operative to produce some big figures and some small ones in addition to the primary parameters that define the individual figures.

The interpretation of the second-order factor as a size factor in the trapezoid example should be distinguished from the size factor that could be chosen as a parameter in the first-order domain. If one of the measurements had been the total area of the trapezoid, it would have been represented by a test vector in the middle of the configuration since it would be affected by all four of the generating parameters that were used and which appeared in the simple structure. The total area test vector could be normalized to a unit vector and it could be used as one of the parameters for describing the trapezoids. It would not be identical with the second-order size factor but they would be closely related. Whether a size factor appears as a first-order factor or as a second-order factor depends on the restrictive conditions under which the figures or objects are produced or selected and also on the selection of measurements for the test battery. It is interesting to note that here the results would indicate either that the thirty-two trapezoids had been systematically selected by some restrictive conditions or else that the objects themselves had been generated under some restrictive conditions.

When the factorial results are clear in both the first-order and second-order domains, inferences can sometimes be drawn concerning the generating conditions that produced the individual parts of the objects. Such inferences can be the basis for formulating hypotheses that can be investigated further either by factorial methods or by more directly controlled experiments.

8. *Incidental Parameters*

So far we have considered the primary factors determined by a simple structure as representing parameters that can be given some scientific interpretation in terms of concepts that are fundamental for the domain in question. In using the simple structure solution which leads sometimes to the second-order domain, we have tried to avoid using arbitrary parameters whose only merit is that they serve in the condensation of numerical data. We have tried to find in the primary factors a set of parameters that not only describe the individual measurements but which also reveal something about the underlying order in the domain. In looking for meaningful parameters of this kind it would be an error to assume that all of the factors have significance that transcends the particular experiment or the particular group of subjects. It would be strange indeed if factor analysis were immune from the distracting circumstances of the particular occasion. The experimenter must try to distinguish that which is invariant and which transcends the particular experimental arrangement or the particular experimental group of subjects from that which is local and incidental to the particular occasion. In factor analysis we are not relieved of this difficult task any more than in other forms of scientific experimentation. In order to focalize attention to this circumstance it might be well to distinguish the primary factors which represent the invariants for which we are really looking from those primary factors which, though genuine as regards the explanation of the test variances, are local and of significance only for the experimental group or the particular occasion. *Primary factors which characterize only a particular experimental group or a particular situation may be called incidental factors to distinguish them from the invariants which are normally the object of scientific experimentation.* Incidental factors may appear in the first-order or in the second-order domain.

A few examples will serve to illustrate the manner in which incidental factors may appear as primaries in factorial analysis. In addition to the primary factors that would be found in different groups of subjects, we might find primary factors that are unique for the particular occasion. Suppose that an exceptionally good examiner who is skilled in obtaining good rapport with the subjects should give a part of the test battery to a part of the experimental group. A primary factor might appear for this group of tests and the investigator might be at a loss to explain it because he would be thinking about the nature of the tests and he would try to find something common in the psychological nature of these tests. It might not occur to him that this is the very group of tests that were administered by the experienced examiner. Such a factor would probably be left without interpretation in the final results or the interpretation might be one that would not be sustained in a subsequent experiment with different subjects and different examiners. Incidental factors are almost certainly present in every study. Hence the investigator should feel free to leave without interpretation those primary factors which do not lend themselves to rather clear scientific interpretation. Even then the interpretation should be at first in the nature of a hypothesis to be sustained if possible by subsequent factorial studies. The fact that all of the variances are not adequately accounted for in the interpretation has led some students to conclude that the whole result should be discarded, but such is not the case. It is quite possible to make an important discovery concerning the primary factors that are operative in an experiment even though the major part of the common factor variances remains unexplained. It is assumed, of course, that such a finding could be sustained by the construction of new tests with prediction as to how they should behave factorially in new groups of differently selected subjects.

In one factorial study it was found that a primary factor was common to a set of tests that were given by the projector method with individual timing for each response. The interpretation of such a factor was uncertain. Some psychological function might be involved in the projector tests which was absent from the other tests, but the explanation might also be that some motivational condition was common to the projector tests that was absent from the other tests and which would be of only incidental significance as far as the major purposes were concerned.

Suppose that one of the examiners misunderstands the time limits for a set of tests and that he gives the shorter time limits to a part of the group of subjects for some of the tests. A factor might appear under certain circumstances that would be incidental and of no fundamental significance, but the primary factors that are significant might still be revealed. An unexpected interruption in a school examination such as fire drill, a street parade, or the expectancy of an important school event may act to introduce incidental factors.

One of the most important sources of incidental factors is to be found in the selective conditions. If a group of subjects is selected because of qualification in a composite of two or more tests, the unique variances of such selective tests combine to form one or more incidental common factors which would have remained a part of the unique variance if the selective conditions had not been imposed. The correlations between the factors are determined in large part by the selective conditions. If a group of subjects is selected because of certain test qualifications, it is to be expected that the primary factors will show correlations between factors that are different from the correlations between the same factors in an unselected population. It must not be assumed that the factors are different just because they correlate differently in different populations. This effect is well known with physical measurements, height and weight with intelligence, for example, whose intercorrelations are determined in large part by the selective conditions. These changes do not affect the identity of the factors. An incidental factor which is introduced by conditions of selection may be trivial or it may be of significance, depending on the nature of the unique variances which are introduced into the common factors by the selective conditions.

It should be remarked that in a well planned factorial experiment the incidental factors are usually of secondary importance in comparison with the variance that is assignable to the principal primary factors for which an experiment was planned. When one or more primary factors have relatively small variance and do not seem to lend themselves to clear interpretation, they should be reported without interpretation. Some reader of such a report may find a fruitful hypothesis for it, or the factor may be of only incidental significance.

These few examples will serve to call attention to the fact that not all the primary factors can be expected to have meaning in the fundamental sense of representing functional unities whose identity transcends the particular group of subjects and the experimental conditions of any particular occasion. It does not follow that incidental factors are in any sense artifacts. They may represent genuine factors that were operating to produce the observed individual differences but their significance may not extend beyond the particular occasion. In that sense they are irrelevant to the purposes of the experimenter even though they are valid as factors which can sometimes be identified.

An interesting application of second-order factors is an attempt to reconcile three theories of intelligence, namely, Spearman's theory of a general intellective factor, Godfrey Thomson's sampling theory with what he calls "sub-pools," and our own theory of correlated multiple factors which are interpreted as distinguishable cognitive functions. The tetrad differences vanish when there are no primary factors common to the four tests of each tetrad, the correlations being determined only by the general second-order factors. This application of second-order factor theory will be the subject of a subsequent paper.

TABLE 1
Orthogonal Factors

	A	*B*	*C*	*D*	*E*
1	x				x
2	x	x			x
3			x		x
4	x	x			x
5				x	x
6			x	x	x
7			x		x
8				x	x

TABLE 2
Correlated Factors

	A	*B*	*C*	*D*	*E*
1	x				
2	x	x	x		
3	x				
4		x			
5			x		x
6		x	x	x	
7				x	
8				x	x
9					x

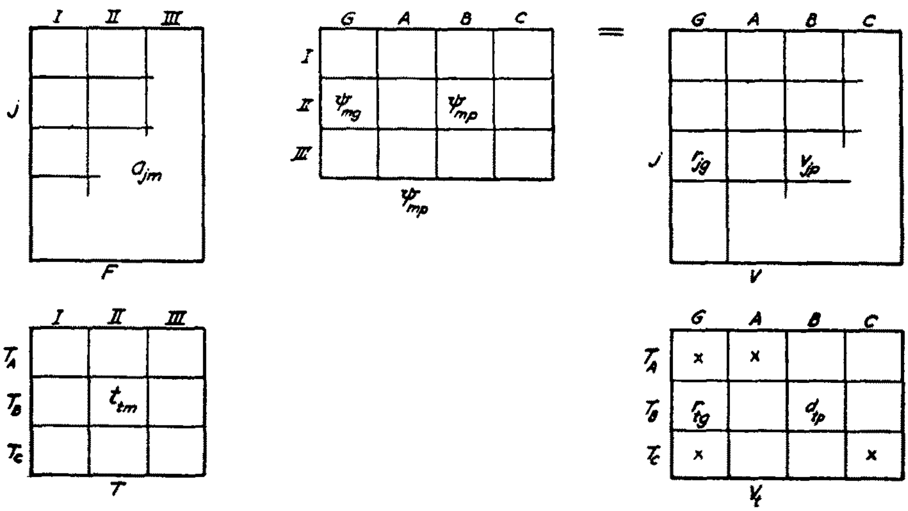
	G
A	x
B	x
C	x
D	x
E	x

TABLE 3
Second-Order Domain
of Figure 3

	P	*Q*
A		
B	x	x
C	x	
D	x	x
E	x	
F	x	
G	x	

TABLE 4

Case 1. Transformation from $\boldsymbol{F}$ to $\boldsymbol{V}$ including the column vector G



Case 2. Transformation from $\boldsymbol{F}$ to $\boldsymbol{U}$ including the group factors and the column vector G

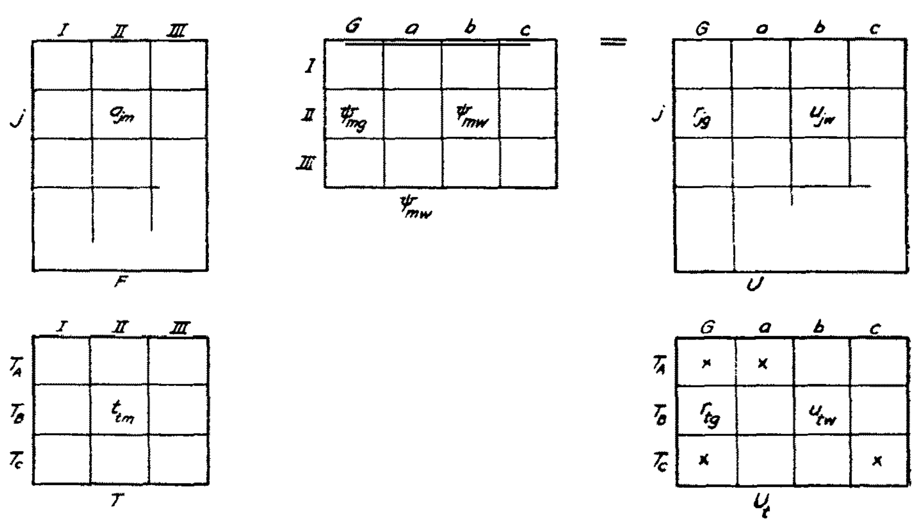
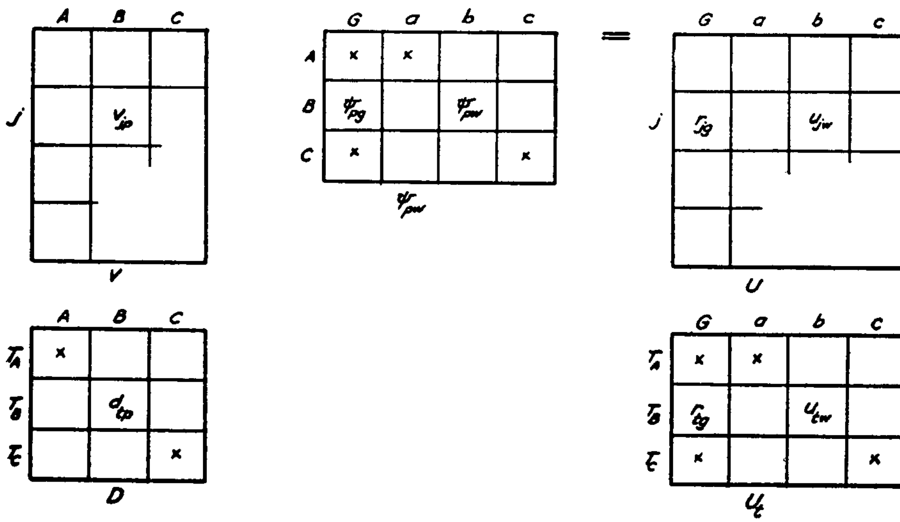


TABLE 5

Case 3. Transformation from V to U including the group factor and the column vector G



Case 4. Transformation from V to column vector G

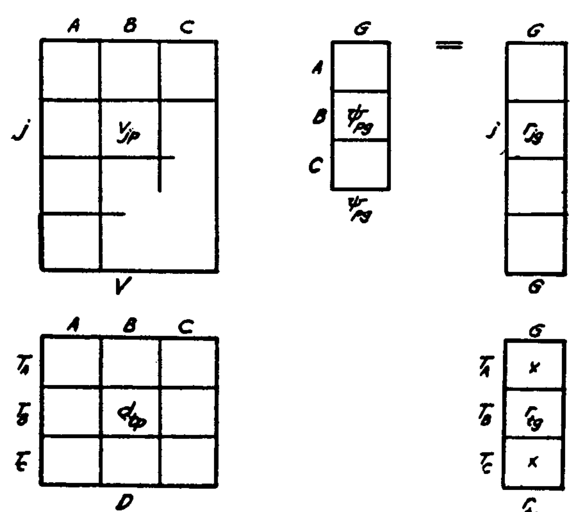


TABLE 6
Trapezoid Parameters

	a	*b*	*c*	*h*
1	1	2	1	2
2	1	2	1	4
3	1	2	3	2
4	1	2	3	4
5	1	3	1	2
6	1	3	1	4
7	1	3	3	2
8	1	3	3	4
9	2	2	1	2
10	2	2	1	4
11	2	2	3	2
12	2	2	3	4
13	2	3	1	2
14	2	3	1	4
15	2	3	3	2
16	2	3	3	4
17	2	3	3	3
18	2	3	3	5
19	2	3	5	3
20	2	3	5	5
21	2	4	3	3
22	2	4	3	5
23	2	4	5	3
24	2	4	5	5
25	3	3	3	3
26	3	3	3	5
27	3	3	5	3
28	3	3	5	5
29	3	4	3	3
30	3	4	3	5
31	3	4	5	3
32	3	4	5	5

TABLE 7

Correlation Matrix

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	1.00	.50	.50	.32	.29	.58	.72	.49	.58	.45	.31	.66	.53	.76	-.35	.11
2	.50	1.00	.50	.32	.36	.42	.57	.49	.74	.67	.33	.54	.64	-.16	-.14	-.23
3	.50	.50	1.00	.32	.52	.42	.88	.82	.90	.45	.30	.78	.75	.19	-.84	-.72
4	.32	.32	.32	1.00	.95	.96	.65	.80	.61	.91	.98	.78	.82	.12	-.22	-.15
5	.29	.36	.52	.95	1.00	.90	.75	.90	.75	.90	.94	.84	.89	.05	-.37	-.31
6	.58	.42	.42	.96	.90	1.00	.78	.83	.70	.92	.94	.86	.86	.34	-.29	-.09
7	.72	.57	.88	.65	.75	.78	1.00	.95	.95	.74	.64	.95	.91	.39	-.69	-.46
8	.49	.49	.82	.80	.90	.83	.95	1.00	.93	.83	.78	.94	.95	.19	-.64	-.52
9	.58	.74	.90	.61	.75	.70	.95	.93	1.00	.79	.60	.90	.93	.11	-.64	-.57
10	.45	.67	.45	.91	.90	.92	.74	.83	.79	1.00	.90	.83	.90	.01	-.22	-.21
11	.31	.33	.30	.98	.94	.94	.64	.78	.60	.90	1.00	.77	.80	.11	-.12	-.09
12	.66	.54	.78	.78	.84	.86	.95	.94	.90	.83	.77	1.00	.97	.34	-.59	-.39
13	.53	.64	.75	.82	.89	.86	.91	.95	.93	.90	.80	.97	1.00	.12	-.52	-.44
14	.76	-.16	.19	.12	.05	.34	.39	.19	.11	.01	.11	.34	.12	1.00	-.28	-.34
15	-.35	-.14	-.84	-.22	-.37	-.29	-.69	-.64	-.64	-.22	-.12	-.59	-.52	-.28	1.00	-.76
16	.11	-.23	-.72	-.15	-.31	-.09	-.46	-.52	-.57	-.21	-.09	-.39	-.44	.34	.76	1.00

TABLE 8

Orthogonal Factor Matrix F

	I	II	III	IV
1	.57	.44	.63	.16
2	.59	—.03	—.01	.59
3	.79	—.46	.38	.03
4	.81	.35	—.42	—.25
5	.88	.13	—.38	—.24
6	.87	.46	—.14	—.12
7	.96	—.02	.30	.00
8	.98	—.07	—.02	—.10
9	.95	—.19	.10	.27
10	.88	.25	—.36	.17
11	.78	.39	—.44	—.14
12	.97	.09	.10	—.05
13	.98	.01	—.09	.06
14	.21	.47	.65	—.38
15	—.60	.50	—.44	.37
16	—.46	.77	.02	.08

TABLE 9

Distribution of Residuals

Dev.	f
.00	50
.01	94
.02	46
.03	30
.04	8
.05	2
.06	2
.07	0
.08	4
.09	2
.10	2

$N = 240.$

TABLE 10

Transformation Matrix Λ

	A	B	C	H
I	.07	.12	.39	.53
II	.70	—.01	—.81	.35
III	.71	—.32	.28	—.64
IV	—.01	.94	—.34	—.44

TABLE 11

Oblique Factor Matrix V

	A	B	C	H	G
1	.79	.01	—.01	—.02	.68
2	.01	.63	.05	.05	.63
3	.00	.01	.78	.00	.73
4	.01	—.01	.00	.93	.46
5	—.11	.00	.21	.86	.52
6	.28	.03	—.03	.76	.62
7	.27	.02	.47	.31	.84
8	.01	.03	.47	.55	.74
9	.00	.34	.46	.25	.85
10	—.02	.38	—.02	.71	.64
11	.02	.10	—.09	.91	.47
12	.20	.04	.35	.50	.78
13	.01	.20	.33	.55	.76
14	.81	—.54	.01	.03	.27
15	—.01	.41	—.89	—.02	—.49
16	.52	.01	—.82	—.02	—.31

TABLE 12

Matrix $C = \Lambda'\Lambda$

	A	B	C	H
A	1.00	—.24	—.34	—.17
B	—.24	1.00	—.35	—.15
C	—.34	—.35	1.00	—.11
H	—.17	—.15	—.11	1.00

Matrix D_{tp}

	A	B	C	H
T_A	.808			
T_B		.808		
T_C			.786	
T_H				.912

Matrix C^{-1}

	A	B	C	H
A	1.53	.73	.83	.46
B	.73	1.53	.84	.44
C	.83	.84	1.62	.45
H	.46	.44	.45	1.19

Matrix $D^{-1}{}_{tp}$

	T_A	T_B	T_C	T_H
A	1.237			
B		1.237		
C			1.273	
H				1.091

TABLE 13

Correlation Matrix $R_t = D_{tp} C^{-1}_{pq} D_{pt}$

	T_A	T_B	T_C	T_H
T_A	1.00	.48	.53	.34
T_B	.48	1.00	.53	.33
T_C	.53	.53	1.00	.32
T_H	.34	.33	.32	1.00

Column Vector r_{tg}

	r_{tg}
T_A	.71
T_B	.70
T_C	.73
T_H	.45

Residuals $= R_t - r_{tg} r'_{tg}$

	T_A	T_B	T_C	T_H
T_A	.50	—.02	.01	.02
T_B	—.02	.51	.02	.01
T_C	.01	.02	.47	—.01
T_H	.02	.01	—.01	.80

Column Vector $\Psi_{pg} = D^{-1}_{tp} r_{tg}$

	G
A	.878
B	.866
C	.929
H	.491

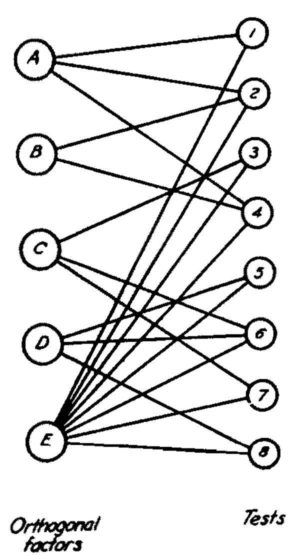


FIGURE 1

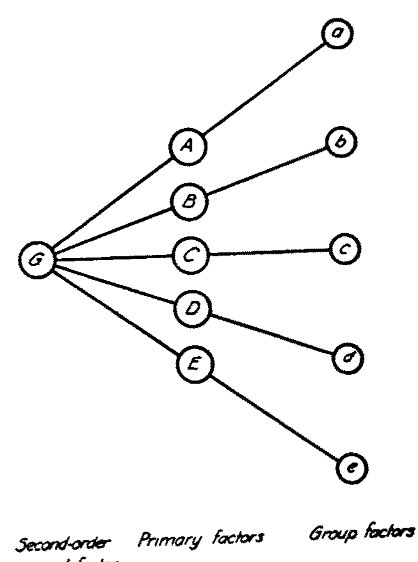


FIGURE 2

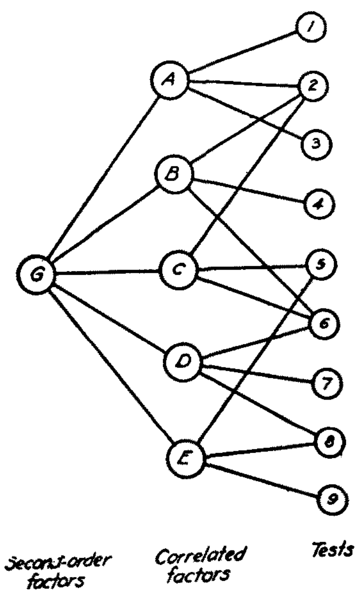


FIGURE 3

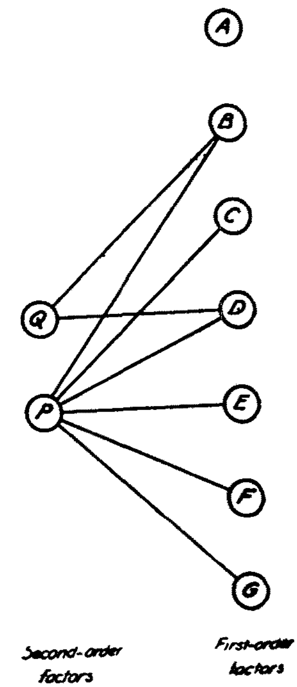
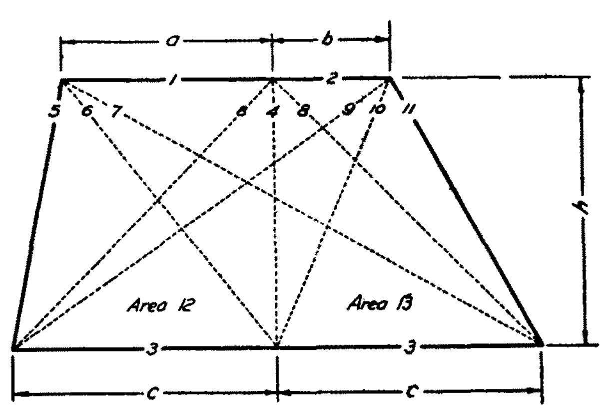


FIGURE 4



$14 = \frac{a}{b}$, $15 = \frac{b}{c}$, $16 = \frac{a}{c}$

FIGURE 5

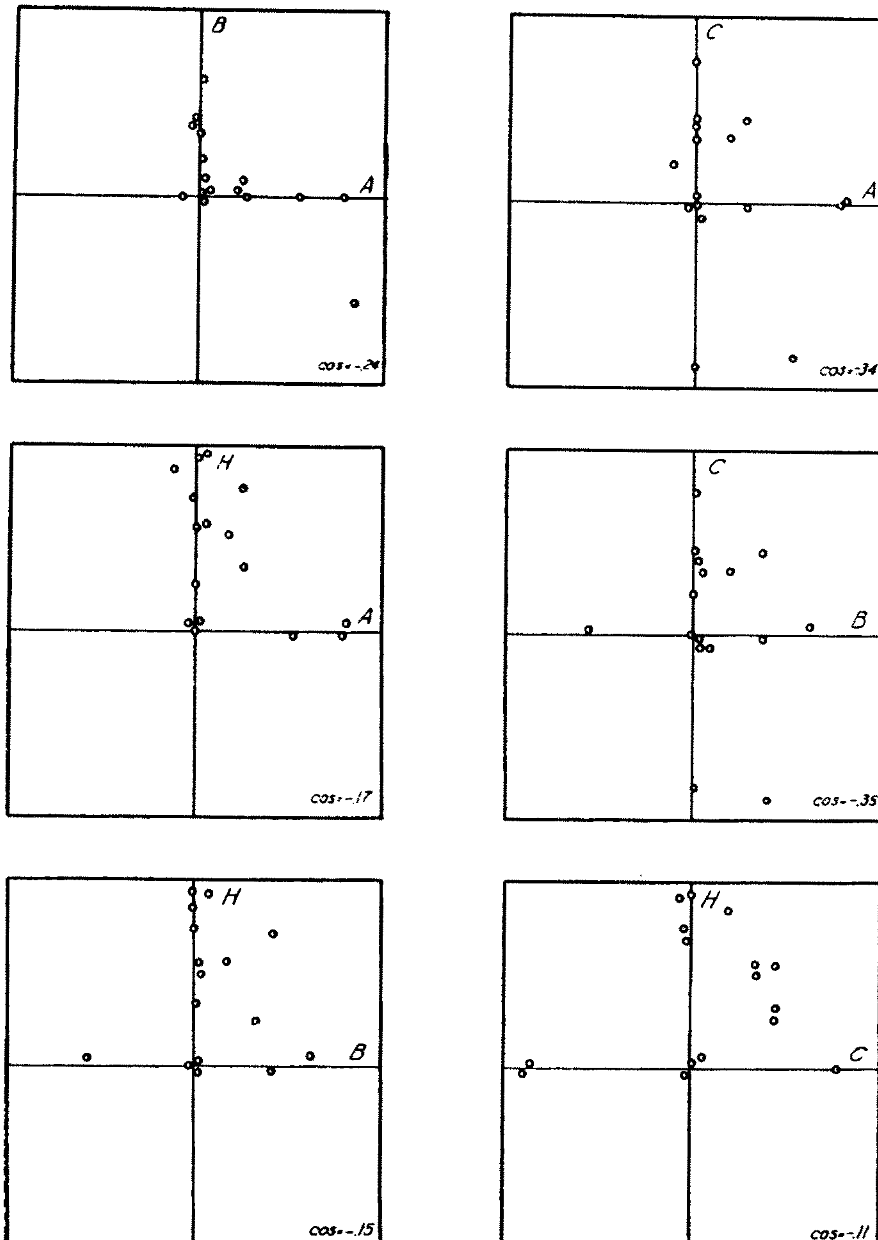


FIGURE 6