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ON THE RELIABILITY OF A WEIGHTED COMPOSITE*

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A general formula for the reliability of a weighted composite has been derived by which that reliability can be estimated from a knowledge of the weights whatever their source, reliabilities, dispersions, and intercorrelations of the components. The Spearman-Brown formula has been shown to be a special case of the general statement. The effect of the internal consistency or intercorrelation of the components has been investigated and the conditions defining the set of weights yielding maximum reliability shown to be that the weight of a component is proportional to the sum of its intercorrelations with the remaining components and inversely proportional to its error variance.

Although the usual purpose of weighting the component parts of a test battery in combining them to form a composite total score is to increase the validity of the composite, there may be times when it is desired to assign weights in such a way as to produce the most reliable composite. In the absence of a measurable external criterion, for instance, the maximum reliability of the composite is certainly as legitimate a basis for assigning weights as is the intuitive assignment of small whole numbers. In any event, even if the weights are assigned in such a way as to provide a least-squares estimate of some quantitative criterion, the investigation of the effect of the weighting of the components on the reliability of the composite is a problem of considerable, even if secondary, importance.

The present paper develops a generalized formula for the reliability of a weighted composite, investigates certain special cases of unusual interest, and derives the conditions of maximum reliability.†

The Generalized Formula for the Reliability of a Composite

The generalized statement of the reliability of a composite is a function of the reliabilities, the dispersions, and the intercorrelations of the components. It is here derived and stated in both explicit and in parametric form. Let us suppose a composite variable y, consisting

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† Certain of the deductions set forth in this paper are implicit in Kelley, T. L., *Statistical Method,* Chapter IX, and others in Richardson, M. W. "The Combination of Measures," in Horst, Paul, *The Prediction of Personal Adjustment,* Social Science Research Council, 1941.

of the weighted sum of n component scores, $x_1, x_2, \cdots, x_j, \cdots, x_n$, so that for any individual, i,

$$y_i = W_1 x_{i1} + W_2 x_{i2} + \cdots + W_j x_{ij} + \cdots + W_n x_{in}, \tag{1}$$

where both y_i and x_i are deviation scores.

Let us represent the variance due to error of the j-th component by its squared standard error, given by

$$\varepsilon^2{}_j = \sigma^2{}_j\ (1 - r_{jj}), \tag{2}$$

where r_{jj} is the reliability of the component x_j. Now, since the error variance of each x_j is multiplied, in its contribution to the error variance of the composite, by W_j and since by definition the error variances of the components are uncorrelated, we may write the error variance of y *as*

$$\varepsilon^2{}_y = \sum_j^n W^2{}_j\, \varepsilon^2{}_j = \sum_j^n W^2{}_j\, \sigma^2{}_j (1 - r_{jj}) = \sum_j^n W^2{}_j \sigma^2{}_j - \sum_j^n W^2{}_j \sigma^2{}_j r_{jj}. \tag{3}$$

Now the total variance of the composite y is given by the expression

$$\begin{aligned} \sigma^2{}_y &= W^2{}_1 \sigma^2{}_1 + W^2{}_2 \sigma_2{}^2 + \cdots + W^2{}_j \sigma^2{}_j + \cdots + W^2{}_n \sigma^2{}_n \\ &\quad + \cdots + 2 r_{1j} W_1 W_j \sigma_1 \sigma_j + \cdots + 2 r_{jk} W_j W_k \sigma_j \sigma_k \\ &= \sum_j^n W^2{}_j \sigma^2{}_j + 2 \sum_j^n \sum_k^n W_j W_k \sigma_j \sigma_k r_{jk}. \qquad (j < k) \end{aligned} \tag{4}$$

But the reliability of y may be expressed as a function of the ratio of the error variance to the total variance, thus

$$\begin{aligned} r_{yy} &= 1 - \frac{\varepsilon^2{}_y}{\sigma^2{}_y} \\ &= 1 - \frac{\sum W^2{}_j \sigma^2{}_j - \sum W^2{}_j \sigma^2{}_j r_{jj}}{\sum_j^n W^2{}_j \sigma^2{}_j + 2 \sum_{j=1}^{n-1} \sum_{k=2}^{n} W_j W_k \sigma_j \sigma_k r_{jk}}. \qquad (j < k) \end{aligned} \tag{5}$$

This, then, is the basic formula by which the reliability of the weighted composite may be estimated from a knowledge of the dispersions, the weights, the reliabilities, and the intercorrelations of the components. The W_j may be the weights obtained from a least-squares regression equation, or the small whole numbers assigned on some arbitrary or intuitive basis to increase the "validity" with which the composite will predict some vaguely defined, unmeasured "criterion." In either case the reliability of the resulting composite may be estimated from equation (5).

It is convenient, and in accord with common practice, to simplify equation (5) without loss of generality by the reduction of the x_j to standard scores and the introduction of the parametric equation.

$$w_j = W_j \sigma_j \tag{6}$$

so that (5) becomes

$$r_{vv} = 1 - \frac{\sum_{j=1}^{n} w^2_j - \sum_{j=1}^{n} w^2_j r_{jj}}{\sum_{j=1}^{n} w^2_j + 2 \sum_{j=1}^{n-1} \sum_{k=2}^{n} w_j w_k r_{jk}}, \tag{7}$$

remembering that $j < k$. Equation (7) may also be written as (8)

$$r_{vv} = \frac{\sum_{j}^{n} w^2_j r_{jj} + 2 \sum_{j=1}^{n-1} \sum_{k=2}^{n} w_j w_k r_{jk}}{\sum w^2_j + 2 \sum_{j=1}^{n-1} \sum_{k=2}^{n} w_j w_k r_{jk}}. \tag{8}$$

Equation (8) is of especial interest, since it can readily be reduced to matric form. Let us define a row vector, W_j, on the range j, consisting of the n weights, w_j; let the matrix of intercorrelations of the components be r, where r_{jj} is, as defined above, the reliability coefficient of the test j; let us also define a second matrix R so that each element $R_{jk} = r_{jk}$, but the diagonal elements, $R_{jj} = 1 \neq r_{jj}$. Then the numerator of (8) is seen to be $W_j \, r \, W'_j$ and the denominator is seen similarly to be WRW', the expressions for numerator and denominator differing only in the diagonal elements of the matrix of intercorrelations, since these are reliability coefficients in the numerator term and total variances in the denominator term. Equation (8) written in matric notation is thus seen to be:

$$r_{vv} = W \, r \, W' (W \, R \, W')^{-1}. \tag{9}$$

Special Cases and General Interpretations. Before determining the values of w_j which make r_{vv} a maximum, it may be of interest to investigate certain special cases and general properties of the formulation. The most obvious conclusion is, of course, that r_{vv} is unity if, and only if, *every* r_{jj} for which w_j is not zero is also unity—a conclusion of some consequence for the practice of attempting to increase reliability solely by adding items. Similarly, if every r_{jj} is zero, with its corollary that every r_{jk} is zero, then it follows that r_{W} is zero—not a very profound conclusion.

Concerning the values of the variables, w_j, r_{jk} and r_{jj}, we may make certain assumptions and investigate the effect of these special conditions. This has been done systematically for equal and unequal weights, equal and unequal reliability coefficients, and for r_{jk} equal to zero, unity, and unity when corrected for attenuation, i.e., $r_{jk} = \sqrt{r_{jj} r_{kk}}$. Only those cases of particular interest are reported here.

The Spearman-Brown Formula. A special case of particular interest is the Spearman-Brown formula. If not only the dispersions are equal [or equated as in (6)], but the weights of the components are equal, and set for convenience equal to unity, and if, in addition, the reliabilities are also equal, equation (7) becomes

$$r_{W} = 1 - \frac{n - n\,r_{jj}}{n + 2\sum_{j=1}^{n-1} \sum_{k=2}^{n} r_{jk}}. \tag{10}$$

But remembering that $j < k$, and letting $\bar{r}_{jk}$ be the average value of r_{jk},

$$2\sum_{j=1}^{n-1} \sum_{k=2}^{n} r_{jk} = n(n-1)\bar{r}_{jk}. \tag{11}$$

and making this substitution in equation (10), we have

$$r_{W} = 1 - \frac{n(1 - r_{jj})}{n + n(n-1)\bar{r}_{jk}} = 1 - \frac{1 - r_{jj}}{1 + (n-1)\bar{r}_{jk}}. \tag{12}$$

Equation (12) not only constitutes a lemma convenient in the derivation to follow, but contains certain conclusions of importance, to which reference will shortly be made.

If the intercorrelations of the components are equal and, when corrected for attenuation, are equal to unity so that each component is measuring the same variable, i.e.,

$$\frac{r_{jk}}{\sqrt{r_{jj}\,r_{kk}}} = 1 \tag{13}$$

and $r_{jj} = r_{kk}$, $\underline{r_{jk}} = r_{jj}$, then equation (12) becomes

$$r_{W} = 1 - \frac{1 - r_{jj}}{1 + (n-1)r_{jj}} = \frac{n\,r_{jj}}{1 + (n-1)r_{jj}}. \tag{14}$$

This is the familiar Spearman-Brown formula for estimating test reliability, seen here to be a special case of the reliability of a weighted composite as given by equation (5) under the conditions that

$$W_j = W_k,\ \sigma_j = \sigma_k,\ r_{jj} = r_{kk},\ \text{and}\ \frac{r_{jk}}{\sqrt{r_{jj}\, r_{kk}}} = 1.$$

Returning to equation (12), it can be seen that the reliability of a composite of equally weighted standard scores increases with increasing average intercorrelation of the components. This conclusion has been stated before by Richardson and is implicit in the Richardson-Kuder treatment of reliability. The significance of its restatement here is that the same conclusion appears again as a lemma in the development of the Spearman-Brown formula, which has been considered by Kelley* as a measure of reliability of a different nature from the Richardson-Kuder reliability. The effect of the intercorrelations of the components is investigated further below.

Effect of Interrelationships of the Components. Certain effects of correlated vs. uncorrelated components on the reliability of the resulting composite can be deduced as special cases from equation (8), repeated here for ready reference.

$$r_{ww} = \frac{\sum_{j}^{n} w^2_j r_{jj} + 2 \sum_{j=1}^{n-1} \sum_{k=2}^{n} w_j w_k r_{jk}}{\sum w^2_j + 2 \sum_{j=1}^{n-1} \sum_{k=2}^{n} w_j w_k r_{jk}}. \tag{8}$$

If the components are mutually uncorrelated, so that for any pair of components, j and k, $r_{jk} = 0$, equation (8) reduces to

$$r_{ww} = \frac{\sum w^2_j r_{jj}}{\sum w^2_j}. \tag{15}$$

This is equivalent to the statement that, *for mutually uncorrelated components, the reliability of the weighted composite is a weighted mean of the reliabilities* the weight of each being w^2_j. For equal weights, this reduces to the mean reliability of the components, a value generally less than that of the most reliable of the sub-tests. In view of the tendency among some workers in the field to seek uncorrelated components for prediction, due to the dependence on the multiple regression concept,† this conclusion is of more than passing interest.

* Kelley, T. L., The Reliability Coefficient, *Psychometrika*, 1942, 7, 75-83.

† On this, cf. Richardson, M. W. "The combination of measures." In Horst, Paul, *The Prediction of Personal Adjustment*, Social Science Research Council, 1941, especially pp. 392 and 397.

The frequent loose statement of the Spearman-Brown formula, ignoring the assumptions following equation (14), as "The reliability of a test increases as its length increases," together with a loose application of the principle of multiple regression and its corollary that "Other things equal, $R_{y \cdot jk \cdots n}$ is a maximum if the intercorrelations, r_{jk} are a minimum," has led to the uncritical combination of unrelated measures to increase "validity" for some vague and ill-defined criterion, without regard for the effect on the reliability of the measure.

When, on the other hand, the intercorrelations are all equal to a given value, say $\bar{r}_{jk}$, equation (8) becomes

$$r_w = \frac{\sum w^2_j r_{jj} + 2\,\bar{r}_{jk} \sum \sum w_j w_k}{\sum w^2_j + 2\,\bar{r}_{jk} \sum \sum w_j w_k}. \qquad (j < k) \qquad (16)$$

If, moreover, the reliabilities are also equal, this becomes

$$r_w = \frac{r_{jj} \sum w^2_j + 2\,\bar{r}_{jk} \sum \sum w_j w_k}{\sum w^2_j + 2\,\bar{r}_{jk} \sum \sum w_j w_k}. \qquad (17)$$

If, however, the weights are equal, but the reliability unrestricted, equation (16) becomes (remembering $j < k$ and letting the mean of the n reliabilities be $\bar{r}_{jj}$),

$$r_w = \frac{\bar{r}_{jj} + (n-1)\,\bar{r}_{jk}}{1 + (n-1)\,\bar{r}_{jk}}. \qquad (18)$$

The third possibility with regard to the intercorrelations of the components is that, when corrected for attenuation, they are equal to unity as in

$$r_{jk} = \sqrt{r_{jj}\, r_{kk}}\,. \qquad (13)$$

Under this condition, the expression for the reliability of a composite becomes of especial interest only if the reliabilities are equal. Then $r_{jk} = r_{jj}$ and equation (8) may be written

$$r_w = r_{jj} \frac{\sum w^2_j + 2 \sum \sum w_j w_k}{\sum w^2_j + 2\, r_{jj} \sum \sum w_j w_k}. \qquad (19)$$

Since $r_{jj} \leq 1$, the denominator of the fraction is always less than the numerator and, hence, for equal reliabilities of tests all measuring the same factor, r_w is greater than r_{jj}. If, as we have seen earlier, the weights are also equal, r_w is given by the Spearman-Brown formula.

Weighting for maximum reliability. Given the algebraic statement of the reliability of the composite in terms of r_{jj}, r_{jk}, and w_j, in which the first two sets of values are given by the data, it would appear simple to determine that set of weights yielding maximum reliability. The general approach is, of course, obvious—obtain the set of partial derivatives of equation (7) with respect to each w_j in turn, and solve the resulting set of homogeneous equations simultaneously, arbitrarily equating one of the w_j to unity. The actual solution for the general case, however, is far from simple.

Differentiating equation (7) with respect to each weight, w_p, in turn yields:

$$\frac{\partial r_{W}}{\partial w_p} = -\frac{(2w_p - 2w_p r_{pp})(\sum w^2_j + 2\sum\sum w_j w_k r_{jk})}{(\sum w^2_j + 2\sum\sum w_j w_k r_{jk})^2} - \frac{(2w_p + 2\sum w_k r_{pk})(\sum w^2_j - \sum w^2_j r_{jj})}{(\sum w^2_j + 2\sum\sum w_j w_k r_{jk})^2}. \qquad (k \neq p) \tag{20}$$

If $\partial r_{W}/\partial w_p = 0$, then the numerators of the two terms above are equal, thus:

$$w_p(1 - r_{pp})(\sum w^2_j + 2\sum\sum w_j w_k r_{jk}) = (w_p + \sum w_k r_{pk})(\sum w^2_j - \sum w^2_j r_{jj}). \tag{21}$$

Equation (21) defines a set of n homogeneous simultaneous equations, since p may take each value from one through n. These equations can be solved only by imposing a further limitation.

It is convenient to define the weight of one variable, say the q-th component, as unity. Dividing each of the remaining $(n-1)$ equations of (21) by the equation resulting from equating $\frac{\partial r_{W}}{\partial w_p}$ to zero gives

$$\frac{w_p(1 - r_{pp})}{w_q(1 - r_{qq})} = \frac{w_p + \sum_k^{n-1} w_k r_{pk}}{w_q + \sum_k^{n-1} w_k r_{qk}}, \qquad \begin{matrix}(k \neq p) \\ (k \neq q)\end{matrix} \tag{22}$$

or, since $w_q = 1$,

$$w_p \frac{(1 - r_{pp})}{(1 - r_{qq})} = \frac{w_p + \sum_k^{n-1} w_k r_{pk}}{1 + \sum_k^{n-1} w_k r_{qk}}. \tag{23}$$

Clearing of fractions yields

$$w_p(1 - r_{pp})\underline{(1 + \sum w_k r_{qk})} = w_p(1 - r_{qq}) + (1 - r_{qq})\sum w_k r_{pk}. \tag{24}$$

Subtracting $w_p(1 - r_{qq})$ from both sides yields

$$w_p[(1 - r_{pp})(1 + \sum w_k r_{qk}) - (1 - r_{qq})] = (1 - r_{qq}) \sum w_k r_{pk} . \quad (25)$$

Expanding the bracketed term gives

$$w_p[1 - r_{pp} + \sum w_k r_{qk}(1 - r_{pp}) - 1 + r_{qq}] = (1 - r_{qq}) \sum w_k r_{pk} . \quad (26)$$

Solving explicitly for w_p,

$$w_p = \frac{\sum w_k r_{pk}(1 - r_{qq})}{(\sum w_k r_{qk})(1 - r_{pp}) + r_{qq} - r_{pp}} . \qquad \begin{matrix}(k \neq p) \\ (k \neq q)\end{matrix} \quad (27)$$

From equation (27) we can state that, for maximum reliability of the composite, the weight of a component is directly proportional to the weighted sum of its intercorrelations with the remaining components, and inversely proportional to its error variance.

The explicit statement of equation (27) is given below for the case of two variables, where variable 2 is taken as the q-th component.

$$\begin{aligned} w_1 &= \frac{w_2 r_{12}(1 - r_{22})}{w_2 r_{12}(1 - r_{11}) + r_{22} r_{11}} \\ w_2 &\equiv 1 , \end{aligned} \quad (28)$$

whence

$$w_1 = \frac{r_{12}(1 - r_{22})}{r_{12}(1 - r_{11}) + r_{22} - r_{11}} .$$

An exact solution to the equations for more than two variables is not apparent to the writer, although an iterative approximation should yield results. The difficulties of computation, however, render equation (27) of theoretical rather than practical value.