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NOTE ON EXCITATION THEORIES

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Offner's demonstration of the excitation time equivalence, for all currents, of the theories of Hill and Rashevsky is here extended to a more general case which includes both. It is found that any system of this general type may be replaced by an equivalent one of the Rashevsky type, thus effecting considerable simplification in the mathematical detail.

It has been shown by Offner[1] that the excitation equations of Rashevsky [2]

$$\left.\begin{aligned} \frac{de}{dt} &= K\,I(t) - k(e - e_0) \\ \frac{di}{dt} &= M\,I(t) - m(i - i_0) \end{aligned}\right\} \quad \text{- - - - - - -} \quad (1)$$

and those of Hill[3]

$$\left.\begin{aligned} \frac{de'}{dt} &= K'I(t) - k'(e' - e_0') \\ \frac{di'}{dt} &= M'(e' - e_0') - m'(i' - i_0') \end{aligned}\right\} \quad \text{- - - - -} \quad (2)$$

will give the same values for the excitation time (at which $e = i$) for all $I(t)$, if the constants be related by

$$\left.\begin{aligned} k' &= k \\ m' &= m \\ K' &= r(K - M) \\ M' &= M(k - m)/(K - M) \end{aligned}\right\} \quad \text{- - - - - - -} \quad (3)$$

The zero subscripts denote values at $t = 0$, and $r = \dfrac{(e_0' - i_0')}{(e_0 - i_0)}$.

Both (1) and (2) are included under

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(1) Unpublished work of Mr. E. A. Offner, Dept. of Physiology, University of Chicago.

(2) Rashevsky, N., 1933. *Protoplasma*, V. 20, 42.

(3) Hill, A. V., 1936. *Proc. Roy. Soc. London*, B, V. 119, 305. Here given in different notation.

$$\left.\begin{aligned} \frac{de}{dt} &= k_{11}(e - e_0) + k_{12}(i - i_0) + a\,I \\ \frac{di}{dt} &= k_{21}(e - e_0) + k_{22}(i - i_0) + a\,b\,I\,, \end{aligned}\right\} \quad \text{- - - - -} \quad (4)$$

and in view of the above result it seems of interest to consider this more general case.

Solution of Equations: The solution of a system of simultaneous linear differential equations with constant coefficients can always be reduced to quadratures. This may be done in a number of ways, but perhaps the most convenient is in a notation due to Bartky.[4]

The system

$$\left.\begin{aligned} \frac{dx_1}{dt} &= k_{11}x_1 + \cdots + k_{1n}x_n + \varphi_1(t) \\ &\vdots \\ \frac{dx_n}{dt} &= k_{n1}x_1 + \cdots + k_{nn}x_n + \varphi_n(t) \end{aligned}\right\} \quad \text{- - - - -} \quad (5)$$

or, in vector-matrix notation,

$$\frac{dX}{dt} = K\,X + \varphi(t) \ ,$$

has the general solution

$$X = e^{Kt}X_0 + e^{Kt}\int_0^t e^{-Kt}\,\varphi\,dt \ ; \qquad \text{- - - -} \quad (6)$$

where X_0 is the vector of initial values, and e^{Kt} is a matrix which, for a 2×2 matrix K, is defined as follows:

$$\begin{aligned} e^{Kt} &= \frac{1}{\lambda_1 - \lambda_2}\,e^{\lambda_1 t}(K - \lambda_2 E) + \frac{1}{\lambda_2 - \lambda_1}\,e^{\lambda_2 t}(K - \lambda_1 E) \text{ if } \lambda_1 \neq \lambda_2\ , \\ e^{Kt} &= e^{\lambda t}[K - (\lambda - 1)E] \text{ if } \lambda_1 = \lambda_2 = \lambda\ ; \end{aligned} \qquad (7)$$

where λ_1, λ_2 are the latent roots of K, and E is the identity matrix.

For the case $n = 2$, $\lambda_1 \neq \lambda_2$, the vector integral in (6) has components Ψ_1, Ψ_2 given by

$$\left.\begin{aligned} (\lambda_1 - \lambda_2)\Psi_1 &= (k_{11} - \lambda_2)I_{11} + k_{12}I_{21} - (k_{11} - \lambda_1)I_{12} - k_{12}I_{22} \\ (\lambda_1 - \lambda_2)\Psi_2 &= k_{21}I_{11} + (k_{22} - \lambda_2)I_{21} - k_{21}I_{12} - (k_{22} - \lambda_1)I_{22} \end{aligned}\right\} \text{-}(8)$$

[4] MacMillan, W. D., *Dynamics of Rigid Bodies*, pp. 413-434.

where

$$I_{ij}(t) = \int_0^t \varphi_i e^{-\lambda_j t} dt \quad - - - - - - - \quad (9)$$

The first component of $\vartheta = e^{Kt}\Psi$ is

$$\vartheta_1 = \left| \begin{array}{l} \{(k_{11}-\lambda_2)^2 + k_{12}k_{21}\} F_{111} \\ + k_{12}(k_{11}+k_{22}-2\lambda_2) F_{121} \\ -\{(k_{11}-\lambda_2)(k_{11}-\lambda_1) + k_{12}k_{21}\}(F_{211}+F_{112}) \\ - k_{12}(k_{11}+k_{22}-\lambda_1-\lambda_2)(F_{221}+F_{122}) \\ + \{(k_{11}-\lambda_1)^2 + k_{12}k_{21}\} F_{212} \\ + k_{12}(k_{11}+k_{22}-2\lambda_1) F_{222} \end{array} \right| \cdot \frac{1}{(\lambda_1-\lambda_2)^2} \quad - - - \quad (10)$$

where

$$F_{nij} = e^{\lambda_n t} I_{ij} \, . \quad - - - - - - - - \quad (11)$$

Using the fact that λ_1, λ_2 satisfy

$$(k_{11}-\lambda)(k_{22}-\lambda) = k_{12}k_{21} \quad (\text{so that } \lambda_1+\lambda_2 = k_{11}+k_{22}) \, ,$$

this simplifies to

$$\vartheta_1 = \{(k_{11}-\lambda_2)F_{111} + k_{12}F_{121} - (k_{11}-\lambda_1)F_{212} - k_{12}F_{222}\}/(\lambda_1-\lambda_2) \quad - - - - \quad (12)$$

The second component, obtained by interchanging the subscripts, is

$$\vartheta_2 = \{k_{21}F_{111} + (k_{22}-\lambda_2)F_{121} - k_{21}F_{212} - (k_{22}-\lambda_1) F_{222}\}/(\lambda_1-\lambda_2) \, . \quad - - - - - - - - \quad (13)$$

Application to Excitation Theory: In (4) take $x_1 = e - e_0$, $x_2 = i - i_0$. Since now $\varphi_2 = b\,\varphi_1$ it follows from (9) and (11) that $F_{n2j} = bF_{n1j}$, and (12), (13) reduce to

$$\vartheta_1 = \{(k_{11}-\lambda_2+bk_{12})F_{111} - (k_{11}-\lambda_1+bk_{12})F_{212}\}/(\lambda_1-\lambda_2) \quad - - - - - - - \quad (14)$$

$$\vartheta_2 = [\{k_{21}+b(k_{22}-\lambda_2)\}F_{111} - \{k_{21}+b(k_{22}-\lambda_1)\}F_{212}]/(\lambda_1-\lambda_2) \, . \quad - - - - - - - - \quad (15)$$

Since the initial conditions are $x_1 = x_2 = 0$ at $t = 0$, the first term on the right of (6) is absent; hence x_1 and x_2 are given by (14) and (15) respectively.

The excitation time is to be determined by setting $e = i$; that is, $x_1 - x_2 = i_0 - e_0$, or

$$\{k_{11}-\lambda_2-k_{21}+b(k_{12}-k_{22}+\lambda_2)\}\, a\, J_{11} - \{k_{11}-\lambda_1-k_{21}+b(k_{12}-k_{22}+\lambda_1)\}\, a\, J_{22} = (\lambda_1-\lambda_2)(i_0-e_0) \quad - - - - - - - - - \quad (16)$$

where

$$J_{ij}(t) = e^{\lambda_i t} \int_0^t I(t)\, e^{-\lambda_j t} dt \ .$$

Denoting the respective brackets by A and B, this may be written

$$a\, A\, J_{11} - a\, B\, J_{22} = (\lambda_1 - \lambda_2)\,(i_0 - e_0) \ .$$

In connection with (16) consider the corresponding equation obtained with different (primed) constants in (4). If the corresponding λ_1 are equal, then $J_{ij}' \equiv J_{ij}$. The two equations will then be identical if

$$\left.\begin{array}{l} r\, a\, A \equiv a'\, A' \\ r\, a\, B \equiv a'\, B' \end{array}\right\} \quad \text{- - - - - -} \quad (17)$$

Subtracting, $r\, a(A - B) = a'(A' - B')$ or $r\, a(1 - b) = a'(1 - b')$. Using this in (17) gives

$$r\, a\{k_{11} - k_{21} + b\,(k_{12} - k_{22})\} = a'\{k'_{11} - k'_{21} + b'\,(k'_{12} - k'_{22})\} \ ; \quad \text{- - - - - - -} \quad (18)$$

which, with

$$\left.\begin{array}{r} \lambda_1 = \lambda_1' \\ \lambda_2 = \lambda_2' \\ a'(1 - b') = ra(1 - b) \end{array}\right\} \quad \text{- - - - - - -} \quad (19)$$

constitutes four relations on the six quantities k_{ij}, a, b, sufficient to make (16) identical for the primed and unprimed systems. These relations correspond to the equations (3) in the special case there considered, but are less restrictive since the latent roots in (19) may be numbered in either order. Thus Offner's case has the solution

$$\left.\begin{array}{l} k' = m \\ m' = k \\ K' = r(K - M) \\ M' = K(k - m)/(K - M) \end{array}\right\} \quad \text{- - - - - - -} \quad (20)$$

as well as that, (3), given by him.

The above results show that any system of type (4) can be replaced, at least insofar as regards excitation time, by an equivalent system of type (1) (or (2)). Thus the fit to observation can not be improved by passing to the more complicated case (4); but, by allowing for a greater variety of mechanism, this may provide a more flexible physical interpretation of the values of the constants in (1).