AN EXAMPLE OF AN UNORDERED GROUP WITH STRICTLY ISOLATED IDENTITY ELEMENT

V. V. Bludov

UDC 519.46

The subgroup $H \subset G$ is said to be strictly isolated in the group if $gx_1^{-1}gx_1 \ldots x_n^{-1}gx_n \in H$, $g \in G$, $x_i \in G$, $i=1,\ldots,n$, implies that $g \in H$. We know that the strict isolation of the identity element is a necessary condition for a group to be ordered. For nilpotent, bigrade, solvable groups, and also for the extension of an Abelian group by a nilpotent group and for a group with nilpotent commutator group, the strict isolation of the identity element is also a sufficient criterion for the group to be ordered [1].

The following questions were posed at the First All-Union Symposium on Group Theory in 1965: Are there groups with strictly isolated identity elements which are not ordered groups, and can a linearly ordered Abelian strictly isolated normal subgroup of a group with a strictly isolated identity element be such that its order is preserved under internal isomorphisms of the whole group [1, 2]?

The example constructed in this article gives a positive answer to the first question and a negative one to the second.

Let $F_1 = \{x_1, x_2\}$, $F_2 = \{y_1, y_2\}$ be free groups with two generators and $F = F_1 \times F_2$ their direct product. For elements of F_1, in the same way as for words v of the alphabet $\langle x_1, x_2, x_1^{-1}, x_2^{-1} \rangle$, we introduce the following numerical characteristics:

$\ell(v)$ is the length of the word v. The length of the unit e (the empty word) is assumed to be zero.

$$m(v) = \begin{cases} \ell(v) - 1, & \text{if } v \doteq x_i v', \ i=1,2; \\ \ell(v), & \text{if } v \doteq x_i^{-1} v, \ i=1,2; \\ 0, & \text{if } v = e. \end{cases}$$

Here the symbol $\doteq$ denotes the graphic equality of words and is only used between words in reduced (abbreviated) form. If now $f \in F$ and $f = vu$, where $v \in F_1$, $u \in F_2$, we put $\ell(f) = \ell(v)$, $m(f) = m(v)$. Let Z denote the set of integers, $Z_+(F)$ the subset of elements of an integer ring over F with strictly positive coefficients. The notations $Z_+(F_1)$ and $Z_+(F_2)$ are defined similarly. If $\sigma \in Z_+(F)$, $\sigma = n_1 f_1 + \ldots + n_k f_k$, we put $\ell(\sigma) = \max \ell(f_i)$, $m(\sigma) = \max m(f_i)$, $i = 1, \ldots, k$. In F_2 we introduce a linear order relation $\leq$ so that for the generators we have $y_1 > e$, $y_2 > e$.

Now we consider the free Abelian group M with the following basis elements: a_α, b_α^β, $\alpha \in F_2$, $\beta \in F_1$. We construct the semi-direct product of M and F, using an additive representation for the operation in M and a multiplicative representation for the action of F on M. We specify the operation of F on M by the following relations:

$$a_\alpha \cdot x_i = -a_\alpha + (-1)^i (b_\alpha^e + b_{y_i\alpha}^e); \tag{1}$$

$$a_\alpha \cdot y_i = a_{\alpha y_i}; \tag{2}$$

Translated from Algebra i Logika, Vol. 11, No. 6, pp. 619-632, November-December, 1972. Original article submitted October 18, 1972.

$$b_\alpha^\beta \cdot x_i = b_\alpha^{\beta x_i}\ ; \tag{3}$$
$$b_\alpha^\beta \cdot y_i = b_{\alpha y_i}^\beta, \quad i=1,2. \tag{4}$$

We denote the group thus constructed by G. Let A denote the subgroup of M generated by the elements a_α, and let B denote the subgroup generated by the elements b_α^β. If now $c\in M$ and $\sigma\in Z_+(F)$, $\sigma=n_1f_1+\ldots+n_kf_k$, then $c\sigma$ denotes $n_1c\cdot f_1+\ldots+n_kc\cdot f_k$.

To prove that the identity element of G is strictly isolated we need some auxiliary propositions.

LEMMA 1. Let $a\in A$, $a\neq 0$, $\sigma\in Z_+(F_2)$. Then $a\sigma\in A$, $a\sigma\neq 0$.

Proof. It follows directly from (2) that $a\sigma\in A$. Now let $a=n_1a_{\alpha_1}+\ldots+n_ka_{\alpha_k}$, and, for the sake of definiteness, $\alpha_1>\alpha_i$, $i=2,\ldots,k$, $n_1\neq 0$, $\sigma=p_1u_1+\ldots+p_Nu_N$, $u_1>u_j$ for $j=2,\ldots,N$. Then

$$a\sigma = n_1a_{\alpha_1}\cdot p_1u_1 + n_1a_{\alpha_1}\cdot\sum_{j=2}^N p_ju_j + \sum_{i=2}^k n_ia_{\alpha_i}\cdot\sigma =$$
$$= n_1p_1a_{\alpha_1u_1} + \sum_{j=2}^N n_1p_ja_{\alpha_1u_j} + \sum_{i=2}^k\sum_{j=1}^N n_ip_ja_{\alpha_iu_j}\ .$$

Since $p_1n_1\neq 0$ and $\alpha_1u_1>\alpha_1u_j$ for $j=2,\ldots,N$, and $\alpha_1u_1>\alpha_iu_j$ for $i=2,\ldots,k$, $j=1,\ldots,N$, we have $a\sigma\neq 0$ and the lemma is proved.

LEMMA 2. Let $a\in A$, $\sigma\in Z_+(F)$ and suppose the expansion of the expression $a\sigma$ in the basis elements contains the basis element b_α^β with nonzero coefficient. Then for b_α^β we must have $\ell(\beta)\leq m(\sigma)$.

Proof. Let $\sigma=p_1f_1+\ldots+p_Nf_N$. For arbitrary $j=1,\ldots,N$ consider $a\cdot p_jf_j=p_ja\cdot u_jv_j=a'v_j$, where $u_j\in F_2$, $v_j\in F_1$, $u_jv_j=f_j$, $a'\in A$ by Lemma 1. The proof is by induction on the length of v_j. The lemma is true for $\ell(v_j)=0$. Suppose it is true for v_j such that $\ell(v_j)<L$, $L\geq 1$ and if now $\ell(v_j)=L$, we write v_j as $v_j\equiv wv_j'$ so that $\ell(w)=1$; then $\ell(v_j')<L$. Putting $a'=\sum_{i=1}^k n_ia_{\alpha_i}$, we obtain, by (1),

$$a'\cdot v_j = (a'w)\cdot v_j' = -a'v_j' + (-1)^\varepsilon\sum_{i=1}^k n_i\,(b_{\alpha_i}^{\beta v_j'} + b_{y_\varepsilon\alpha_i}^{\beta v_j'}),$$

where $\beta=w$, if $w=x_\varepsilon^{-1}$ and $\beta=e$ if $w=x_\varepsilon$ $(\varepsilon=1,2)$, from which $\ell(\beta v_j')\leq m(wv_j')=m(v_j)\leq m(\sigma)$. For $a'v_j'$ we have $\ell(v_j')<L$ and the induction hypothesis comes into force. Thus, since $a\cdot p_jf_j$ is arbitrary, the lemma is proved.

LEMMA 3. Let $a\in A$, $\sigma\in Z_+(F)$ and suppose the expansion of $a\sigma$ in the basis elements contains the basis element b_α^β with nonzero coefficient and such that $\ell(\beta)=m(\sigma)$.

Then 1) if $\beta\equiv x_\varepsilon\beta'$ $(\varepsilon=1,2)$, we have $\sigma=v\sigma_1+\sigma_2$, where $v\in F_1$, $v=x_\delta\beta$ $(\delta=1,2)$, $\sigma_1\in Z_+(F_2)$, $\sigma_2\in Z_+(F)$;

2) if $\beta\equiv x_\varepsilon^{-1}\beta'$ $(\varepsilon=1,2)$, we have $\sigma=v\sigma_1+\sigma_2$, where $v\in F_1$, $v=\beta$ or $v=x_\delta\beta$ $(\delta=1,2,\ \delta\neq\varepsilon)$, $\sigma_1\in Z_+(F_2)$, $\sigma_2\in Z_+(F)$.

Proof. Let $a=n_1a_{\alpha_1}+\ldots+n_Na_{\alpha_N}$, $\sigma=p_1f_1+\ldots+p_Mf_M$; then

$$a\sigma = (n_1a_{\alpha_1}+\ldots+n_Na_{\alpha_N})(p_1f_1+\ldots+p_Mf_M) = \sum_{i=1}^N\sum_{j=1}^M n_ip_ja_{\alpha_i}\cdot f_j\ .$$

Consider an arbitrary term of the sum $n_ip_ja_{\alpha_i}f_j=n_ip_ja_{\alpha_i}\cdot u_jv_j=a'v_j$, where $v_j\in F_1$, $u_j\in F_2$, $u_jv_j=f_j$. By Lemma 1, $a'\in A$, we put $a'=n_1'a_{\gamma_1}+\ldots+n_k'a_{\gamma_k}$. If $\ell(v_j)=0$, the lemma holds, otherwise we write v_j as $v_j\equiv wv'$, where $\ell(w)=1$.

Consider b_α^β such that $\ell(\beta)=m(\sigma)$ and $\beta=x_\varepsilon\beta'(\beta=x_\varepsilon^{-1}\beta')$, $\varepsilon=1,2$, and assume that $v_j\neq x_\delta\beta$ ($v_j\neq\beta$ and $v_j\neq x_\delta\beta$, $\delta\neq\varepsilon$), $\delta=1,2$; we can show that in this case b_α^β does not occur in the expansion of $a'v_j$ in basis elements.

By (1) we have

$$a'v_j=(a'w)v_j'=-a'v'+(-1)^\xi\sum_{i=1}^{K}n_i'(b_{\gamma_i}^{\omega v'}+b_{y_\xi\gamma_i}^{\omega v'}),$$

where $\omega=w$ if $w=x_\xi^{-1}$ and $\omega=e$, if $w=x_\xi$, $\xi=1,2$. Since $\omega v'\neq\beta$, b_α^β does not occur in the second term of the above sum. For the first term of $a'v'$ we have: either $m(v')<m(\sigma)=\ell(\beta)$ and then b_α^β does not occur in the expansion of $a'v'$ in the basis elements, by Lemma 2; or $m(v')=m(\sigma)$, but since $\ell(v')<\ell(\sigma)$, we have $v'=x_\delta^{-1}v''$, $\delta=1,2$, and again, by (1), we obtain

$$a'v'=-a'v''+(-1)^\delta\sum_{i=1}^{K}n_i'(b_{\gamma_i}^{v'}+b_{y_\delta\gamma_i}^{v'}).$$

Again, b_α^β does not occur in the second term since $\beta\neq v'$ and for the first term we have $m(v'')<m(v')=m(\sigma)=\ell(\beta)$, from which, by Lemma 2, b_α^β does not occur in the expansion of $a'v''$ in basis elements. In view of the above contradiction and the arbitrary choice of the term $n_jp_ja_{\alpha_i}f_j$, the lemma is proved.

LEMMA 4. Let $a\in A$, $a=n_1a_{\alpha_1}+\ldots+n_Na_{\alpha_N}$ and, for the sake of definiteness, $\alpha_1>\alpha_i$, $i=2,\ldots,N$, $n_1\neq 0$. Let

$$\sigma=\sum_{j=1}^{M}p_jv_1u_j+\sigma_2,\ p_j\in Z,\ p_j>0,\ v_1\in F_1,\ m(v_1)=m(\sigma)=$$
$$=\ell(\sigma),\ v_1=x_\varepsilon^{-1}v_1',\ \varepsilon=1,2,\ u_j\in F_2$$

and, for the sake of definiteness, $u_1>u_j$, $j=2,\ldots,M$, $\sigma_2\in Z_+(F)$, where v_1 occurs in the expansion of σ_2 in terms of the basis. Then the expansion of $a\sigma$ in terms of the basis contains the element $b_{y_\varepsilon\alpha_1u_1}^{v_1}$ with coefficient $(-1)^\varepsilon n_1p_1\neq 0$, and for all other basis elements in $b_\alpha^{v_1}$ of this expansion we have $\alpha<y_\varepsilon\alpha_1u_1$.

Proof. Put $\sigma_1=\sum_{j=2}^{M}p_jv_1u_j$, $a_1=\sum_{i=2}^{N}n_ia_{\alpha_i}$; then $a\sigma=n_1p_1a_{\alpha_1}v_1u_1+a_1p_1v_1u_1+a\sigma_1+a\sigma_2$. Since, $v_1=x_\varepsilon^{-1}v_1'$, using (1)-(4), we obtain

$$a\sigma=-n_1p_1a_{\alpha_1}v_1'u_1+(-1)^\varepsilon n_1p_1b_{\alpha_1u_1}^{v_1}+(-1)^\varepsilon n_1p_1b_{y_\varepsilon\alpha_1u_1}^{v_1}-$$
$$-av_1'\sum_{j=2}^{M}p_ju_j+(-1)^\varepsilon\sum_{i=1}^{N}\sum_{j=2}^{M}n_ir_jb_{\alpha_iu_j}^{v_1}+(-1)^\varepsilon\sum_{i=1}^{N}\sum_{j=2}^{M}n_ip_jb_{y_\varepsilon\alpha_iu_j}^{v_1}+a\sigma_2.$$

Consider the first and fourth terms:

$$-n_1p_1a_{\alpha_1}v_1'u_1-av_1'\sum_{j=2}^{M}p_ju_j=a'v_1'.$$

By Lemma 1, $a'\in A$ and since $m(v_1')<m(v_1)=\ell(v_1)$, by Lemma 2, b_α^β does not occur in the expansion of $a'v_1'$ in terms of the basis. Consider the last term $a\sigma_2$; using Lemma 3 we find that $b_\alpha^{v_1}$ can occur in the expansion of $a\sigma_2$ in terms of the basis, only if either v_1 or $x_\delta v_1$ ($\delta=1,2$, $\delta\neq\varepsilon$) occurs in the expansion of σ_2, but the first is impossible, by hypothesis, and the second is impossible since $\ell(x_\delta v_1)=\ell(v_1)+1>\ell(\sigma)\geq\ell(\sigma_2)$.

Consider the second, fifth, and sixth terms:

$$(-1)^\varepsilon\left[\sum_{i=1}^{N}\sum_{j=1}^{M}n_ip_jb_{\alpha_iu_j}^{v_1}+\sum_{i=1}^{N}\sum_{j=2}^{M}n_ip_jb_{y_\varepsilon\alpha_iu_j}^{v_1}\right],$$

since $\alpha_i u_j \leq \alpha_1 u_1 < Y_\varepsilon \alpha_1 u_1$ for all $i=1,\dots,N$; $j=1,\dots,M$ and $Y_\varepsilon \alpha_i u_j \leq Y_\varepsilon \alpha_1 u_j < Y_\varepsilon \alpha_1 u_1$ for all $i=1,\dots,N$; $j=2,\dots,M$, we find that $b^{v_1}_{Y_\varepsilon \alpha_1 u_1}$ does not occur in these sums and for all other $b^{v_1}_\alpha$ from this expression we have $\alpha < Y_\varepsilon \alpha_1 u_1$.

Thus, $b^{v_1}_{Y_\varepsilon \alpha_1 u_1}$ occurs only in the third term $(-1)^\varepsilon n_1 p_1 b^{v_1}_{Y_\varepsilon \alpha_1 u_1}$ with coefficient $(-1)^\varepsilon n_1 p_1 \neq 0$ and the lemma is proved.

LEMMA 5. Let

$$b = \sum_{i=1}^{N} n_i b^{\beta_1}_{\alpha_i} + b_2$$

and, for the sake of definiteness, $\alpha_1 > \alpha_i$, $i=2,\dots,N$, $n_1 \neq 0$. Suppose for all b^β_α in the expansion of b_2 in terms of the basis we have $\beta \neq \beta_1$, and $\ell(\beta) \leq \ell(\beta_1)$. Let $\sigma \in Z_+(F)$, $\sigma = \sum_{j=1}^{M} p_j v_1 u_j + \sigma_2$, $p_j \in Z$, $p_j > 0$, $v_1 \in F_1$, $u_j \in F_2$, and for the sake of definiteness, $u_1 > u_j$, $j=2,\dots,M$. Let v_1 not occur in the expansion of σ_2 in terms of the basis $\ell(v_1) \geq m(\sigma)$, $\ell(\beta_1 v_1) = \ell(\beta_1) + \ell(v_1)$ and for all v in the expansion of σ in terms of the basis let $\ell(\beta_1 v_1) \geq \ell(\beta_i v_j)$ for all $j=1,\dots,N$. Then the element $b^{\beta_1 v_1}_{\alpha_1 u_1}$ occurs and only with coefficient $n_1 p_1 \neq 0$, in the expansion of $b\sigma$ in terms of the basis and for all other $b^{\beta_1 v_1}_\alpha$ in this expansion we have $\alpha < \alpha_1 u_1$.

Proof. Put

$$\sigma_1 = \sum_{j=2}^{M} p_j v_1 u_j \ , \quad b_1 = \sum_{i=2}^{N} n_i b^{\beta_1}_{\alpha_i} \ ,$$

then $b\sigma = n_1 p_1 b^{\beta_1 v_1}_{\alpha_1 u_1} + n_1 b^{\beta_1}_{\alpha_1}(\sigma_1 + \sigma_2) + b_1(p_1 v_1 u_1 + \sigma_1) + b_2(p_1 u_1 v_1 + \sigma_1) + (b_1 + b_2)\sigma_2$. We transform this using the earlier expressions

$$b\sigma = n_1 p_1 b^{\beta_1 v_1}_{\alpha_1 u_1} + n_1 \sum_{j=2}^{M} p_j b^{\beta_1 v_1}_{\alpha_1 u_j} + \sum_{i=2}^{N} \sum_{j=1}^{M} n_i p_j b^{\beta_1 v_1}_{\alpha_i u_j} + b_2 \cdot \sum_{j=1}^{M} p_j v_1 u_j + b_2 \sigma_2 + \sum_{i=1}^{N} b^{\beta_1}_{\alpha_i} \cdot \sigma_2 ,$$

from which we see that $b^{\beta_1 v_1}_{\alpha_1 u_1}$ occurs in the first term with coefficient $n_1 p_1 \neq 0$. We can show that in the expansion of the remaining terms in elements of the basis this element does not occur and for all other $b^{\beta_1 v_1}_\alpha$, we have $\alpha < \alpha_1 u_1$.

Consider the fourth term

$$b_2 \cdot \sum_{j=1}^{M} p_j v_1 u_j ;$$

the basis elements in the expansion of this expression have the form $b^{\beta v_1}_{\alpha u_j}$, where b^β_α is in the expansion of b_2 in the basis elements and since, by hypothesis, for all such b^β_α we have $\beta \neq \beta_1$, then $b^{\beta_1 v_1}_\alpha$ does not occur in the expansion of the fourth term.

Similarly, for the sixth term $\sum_{i=1}^{N} b^{\beta_1}_{\alpha_i} \cdot \sigma_2$ the basis elements have the form $b^{\beta_1 v}$, where $v \neq v_1$ by hypothesis and this means that $b^{\beta_1 v_1}_\alpha$ does not occur in the expansion of this expression in terms of the basis.

Consider the fifth term $b_2\sigma_2$ and assume that $b^{\beta_1 v_1}_\alpha$ occurs in its expansion in terms of the basis. This means that there can be found a b_2 in the expansion of the element $b^{\beta_t}_\alpha$ in terms of the basis and a σ_2 in the expansion of v_s such that $\beta_t v_s = \beta_1 v_1$. Since $\ell(\beta_t v_s) \leq \ell(\beta_t) + \ell(v_s)$, then either: $\ell(\beta_t v_s) < \ell(\beta_t) + \ell(v_s)$ and then $\ell(\beta_t v_s) \leq \ell(\beta_t) + \ell(v_s) - 2 < \ell(\beta_t) + \ell(v_s) - 1 \leq \ell(\beta_t) + m(v_s) \leq \ell(\beta_1) + m(\sigma) \leq \ell(\beta_1) + \ell(v_1) -$

$\ell(\beta_1 v_1)$, from which the equation $\beta_t v_s = \beta_1 v_1$ is impossible; or $\ell(\beta_t v_s) = \ell(\beta_t) + \ell(v_s)$, and in this case, we use the fact that $\ell(\beta_t) \leq \ell(\beta_1)$. When $\ell(\beta_t) = \ell(\beta_1)$, we find that $\beta_t = \beta_1$, which is impossible. Thus, $\ell(\beta_t) < \ell(\beta_1)$, but then $\ell(v_s) > \ell(v_1)$; however, $\ell(v_1) \geq m(6) \geq m(v_s) \geq \ell(v_s) - 1$ and this means that $\ell(v_1) = \ell(v_s) - 1 = m(v_s)$, so from the second equation we find that $v_s = x_\delta v_s'$ $(\delta = 1, 2)$. By hypothesis $\ell(\beta_1 v_1) \geq \ell(\beta_1 v_s)$, and since $\ell(v_s) > \ell(v_1)$, we have $\ell(\beta_1 v_s) < \ell(\beta_1) + \ell(v_s)$, which means that there is a contraction when β_1 and v_s are multiplied, i.e., $\beta_1 = \beta_1' x_\delta^{-1}$ (with the same δ as in $v_s = x_\delta v_s'$). Now we see that in the $\ell(v_1) + 1$-th place on the right in the element $\beta_1 v_1$ there can be found an element x_δ^{-1} and in the $\ell(v_1)+1$-th place on the right in the element $\beta_t v_s$ there can be found an x_δ which shows that the equation $\beta_1 v_1 = \beta_t v_s$ is impossible and this means that $b_\alpha^{\beta_1 v_1}$ does not occur in the expansion of $b_2 \sigma_2$ in terms of the basis.

Consider the second and third terms:

$$n_1 \sum_{j=2}^{M} p_j b_{\alpha_1 u_j}^{\beta_1 v_1} + \sum_{i=2}^{N} \sum_{j=1}^{M} p_j b_{\alpha_i u_j}^{\beta_1 v_1} .$$

In this case $\alpha_1 u_j < \alpha_1 u_1$ for $j = 2, \ldots, M$ $\alpha_i u_j < \alpha_i u_1 < \alpha_1 u_1$ for $i = 2, \ldots, N$, $j = 1, \ldots, M$, and for all $b_\alpha^{\beta_1 v_1}$ in this expansion $\alpha < \alpha_1 u_1$. Thus the lemma is proved.

LEMMA 6. Let $b_1 \in B$, $b_1 = \sum_{i=1}^{N} n_i (b_{\alpha_i}^{\beta} + b_{y_\varepsilon \alpha_i}^{\beta})$, $\varepsilon = 1, 2$, $b_2 \in B$, $b_2 = \sum_{j=1}^{M} m_j (b_{\gamma_j}^{\beta} + b_{y_\delta \gamma_j}^{\beta})$, $\delta = 1, 2$, $\delta \neq \varepsilon$ and $b_1 + b_2 = 0$. Then

$$\sum_{i=1}^{N} n_i a_{\alpha_i} = 0, \quad \sum_{j=1}^{M} m_j a_{\gamma_j} = 0.$$

Proof. Assume the contrary and suppose, for the sake of definiteness, that $\sum_{i=1}^{N} n_i a_{\alpha_i} \neq 0$; then after cancelling like terms (if necessary), there remain coefficients $n_i \neq 0$ for some $i = 1, \ldots, N$. Assume that $\ell_1 = \max_{n_i \neq 0} [\ell(\alpha_i), \ell(y_\varepsilon \alpha_i)]$. We proceed for the second sum and assume that $\ell_2 = \max_{m_j \neq 0} [\ell(\gamma_j), \ell(y_\delta \gamma_j)]$, while if

$$\sum_{j=1}^{M} m_j a_{\gamma_j} = 0,$$

then $\ell_2 = 0$. Here $\ell(\alpha)$ denotes the length of the word α, just as when $v \in F_1$ $\ell(v)$ denotes the length of the word v.

Let α_s denote one of the indices α_i, $i = 1, \ldots, N$, such that $\ell(\alpha_s) = \ell_1$; then $\alpha_s = y_\varepsilon \alpha'$ or $\alpha_s = y_\varepsilon^{-1} \alpha'$. In the same way, γ_t denotes one of the indices γ_j and $\ell(\gamma_t) = \ell_2$; and in this case $\gamma_t = y_\delta \gamma'$ or $\gamma_t = y_\delta^{-1} \gamma'$.

Since $\varepsilon \neq \delta$, we have $\alpha_s \neq \gamma_t$.

In b_1 we choose a basis element $b_{\alpha_s}^{\beta}$ with nonzero coefficient and if $\ell_1 > \ell_2$, then $b_{\alpha_s}^{\beta}$ does not occur in the expansion of b_2 in terms of the basis and $b_1 + b_2 \neq 0$. This is a contradiction. If $\ell_2 > \ell_1$, then $b_2 \neq 0$ and in the expansion of b_2 there can be found an element $b_{\gamma_t}^{\beta}$ with nonzero coefficient and $\ell(\gamma_t) = \ell_2$ and this element does not occur in the expansion of b_1 in terms of the basis; again $b_1 + b_2 \neq 0$. This contradiction proves the assertion.

THEOREM. There is an unordered group with strictly isolated identity element.

Proof. We choose the group G described above and show that it is unordered and that it has a strictly isolated identity element. From (1)-(4) we obtain relations for the elements $a_e, b_e^e \in M$:

$$a_e + a_e \cdot x_1 = - b_e^e - b_e^e \cdot y_1,$$

$$a_e + a_e \cdot x_2 = b_e^e + b_e^e \cdot y_2.$$

From which, if $a_e > 0$, then $b_e^e > 0$ or if $a_e < 0$, then $b_e^e < 0$. We obtain a contradiction with the first of these relations. If $a_e > 0$, then $b_e^e < 0$, or if $a_e < 0$, we have $b_e^e > 0$. We obtain a contradiction with the second relation. Consequently, G does not have any linear orderings.

Since $G/M \cong F$ and F has a strictly isolated identity element, M is strictly isolated in G. This means that it is sufficient to show that if $c \in M$, $\sigma \in Z_+(F)$ and $c\sigma = 0$, then $c = 0$. Consider two cases:

1) $c = a \in A$. Let $a\sigma = 0$; we write σ as follows: $\sigma = \sigma_1 v_1 + \sigma_2$, where $v_1 \in F_1$, $\ell(v_1) = \ell(\sigma)$, $m(v_1) = m(\sigma)$, $\sigma_1 \in Z_+(F_2)$, $\sigma_2 \in Z_+(F)$ and v_1 does not occur in the expansion of σ_2 in terms of the basis. If $\ell(v_1) = 0$, then $\ell(\sigma) = 0$ and $\sigma \in Z_+(F_2)$; then by Lemma 1, $a\sigma = 0$ implies that $a = 0$. Let $\ell(v_1) > 0$. Put $v_1 = w_1 v_1'$ $\omega_1 = w_1$, if $w_1 = x_\varepsilon^{-1}$ $(\varepsilon = 1,2)$ and $\omega_1 = e$, if $w_1 = x_\varepsilon$, $a_1 = \sum_{i=1}^{N} n_i a_{\alpha_i}$. By (1)-(2), we obtain

$$a\sigma = a\sigma_1 v_1 + a\sigma_2 = a_1 v_1 + a\sigma_2 = -a_1 v_1' + (-1)^{\varepsilon} \sum_{i=1}^{N} n_i (b_{\alpha_i}^{\omega v_1'} + b_{y_\varepsilon \alpha_i}^{\omega v_1'}) + a\sigma_2.$$

If $\omega = x_\varepsilon^{-1}$, then $m(v_1') < \ell(\omega v_1')$ and, by Lemma 2, $b_{\alpha_i}^{\omega v_1'}$ does not occur in the expansion of $a_1 v_1'$ in terms of the basis. By Lemma 3, $b_{\alpha_i}^{\omega v_1'}$ may occur in the expansion of $a\sigma_2$ only when $\sigma_2 = v_2\sigma_3 + \sigma_4$, where $\sigma_3 \in Z_+(F_2)$, $\sigma_4 \in Z_+(F)$ and $v_2 = v_1$ or $v_2 = x_\delta v_1$, $\delta = 1,2$, $\delta \neq \varepsilon$, but the first is impossible since v_1 does not occur in the expansion of σ_2 in terms of the basis by hypothesis, and the second is impossible since $\ell(x_\delta v_1) > \ell(v_1) = \ell(\sigma)$. Hence $a\sigma = 0$ implies that

$$\sum_{i=1}^{N} n_i (b_{\alpha_i}^{\omega v_1'} + b_{y_\varepsilon \alpha_i}^{\omega v_1'}) = 0,$$

from which, by Lemma 6, $a_1 = 0$ and hence $a = 0$. Suppose now that $\omega = e$. If $m(v_1') < \ell(v_1')$, then $\ell(\omega v_1') = \ell(v_1') > m(v_1')$ and, by Lemma 2, $b_{\alpha_i}^{\omega v_1'}$ does not occur in the expansion of $a_1 v_1'$ in terms of the basis. Suppose $b_{\alpha_i}^{\omega v_1'}$ occurs in the expansion of $a\sigma_2$, in terms of the basis. Then, by Lemma 3, $\sigma_2 = v_2\sigma_3 + \sigma_4$, where $v_2 \in F_1$ and either $v_2 = v_1$ or $v_2 = x_\delta v_1'$, $\delta = 1, 2, \delta \neq \varepsilon$, $\sigma_3 \in Z_+(F_2)$, $\sigma_4 \in Z_+(F)$, and suppose v_2 does not occur in the expansion of σ_2 in terms of the basis. Since v_1 does not occur in the expansion of σ_2 in terms of the basis, $v_2 \neq v_1$, consequently, $v_2 = x_\delta v_1'$. We have $a\sigma = a_1 v_1 + a_2 v_2 + a\sigma_4$, where

$$a_2 = a\sigma_3 = \sum_{k=1}^{M} m_k a_{\gamma_k}.$$

Further, from (1)-(2), we obtain

$$a\sigma = -a_1 v_1' - a_2 v_1' + (-1)^{\varepsilon} \sum_{i=1}^{N} n_i (b_{\alpha_i}^{v_1'} + b_{y_\varepsilon \alpha_i}^{v_1'}) + (-1)^{\delta} \sum_{k=1}^{M} m_k (b_{\gamma_k}^{v_1'} + b_{y_\delta \gamma_k}^{v_1'}) + a\sigma_4,$$

and since $b_\alpha^{v_1'}$ does not occur in the expansion of $(a_1 + a_2)v_1' + a\sigma_4$ in terms of the basis, by Lemmas 2 and 3, $a\sigma = 0$ implies that

$$(-1)^{\varepsilon} \sum_{i=1}^{N} n_i (b_{\alpha_i}^{v_1'} + b_{y_\varepsilon \alpha_i}^{v_1'}) + (-1)^{\delta} \sum_{k=1}^{M} m_k (b_{\gamma_k}^{v_1'} + b_{y_\delta \gamma_k}^{v_1'}) = 0,$$

but then, by Lemma 6, $a_1 = 0$ and $a_2 = 0$, and so $a = 0$.

If $\ell(V')=0$, by Lemma 1, $b_{\alpha_i}^{v_1'}$ does not occur in the expansion of $a_1 v_1' = a_1$ and, as in the preceding case, $a\sigma=0$ implies that $a=0$. Finally, let $\ell(v_1') = m(v_1') > 0$.

Then $v_1' = x_\delta^{-1} v_1''$ and $\delta \neq \varepsilon$. By Lemma 3, $b_{\alpha_i}^{v_1'}$ occurs in the expansion of $a\sigma_2$ in terms of the basis if $\sigma_2 = \sigma_3 \cdot v_2 + \sigma_4$, where $v_2 = v_1'$ or $v_2 = x_\varepsilon v_1'$ $(\varepsilon \neq \delta)$; but then $v_2 = v_1$ and this is impossible. Let σ_3 and σ_4 be chosen as in the above case. We have $a\sigma = a_1 v_1 + a_2 v_2 + a\sigma_4$, where $a_2 = a\sigma_1$,

$$a_1 v_1 + a_2 v_2 = (a_2 - a_1) v_1' + (-1)^{\varepsilon} \sum_{i=1}^{N} n_i (b_{\alpha_i}^{v_1'} + b_{y_\varepsilon \alpha_i}^{v_1'}).$$

Put

$$a_2 - a_1 = a_3 - \sum_{k=1}^{M} m_k a_{y_k},$$

then

$$a_1 v_1 + a_2 v_2 = a_3 v_1'' + (-1)^{\delta} \sum_{k=1}^{M} m_k (b_{y_k}^{v_1'} + b_{y_\delta y_k}^{v_1'}) + (-1)^{\varepsilon} \sum_{j=1}^{N} n_j (b_{\alpha_j}^{v_1'} + b_{y_\varepsilon \alpha_j}^{v_1'}),$$

and since $m(v_1'') < \ell(v_1')$, by Lemma 2, $b_\alpha^{v_1'}$ does not occur in the expansion of $a_3 v_1''$ in terms of the basis. Again, using Lemma 6, we find that $a\sigma = 0$ implies that $a=0$. Thus, the case $a \in A$ has been fully considered.

2) Now suppose that $c \in M$, $c = a+b$, $a \in A$, $b \in B$, $\sigma \in Z_+(F)$. Again we assume that $c\sigma = 0$ and show that $c=0$. Since we have already considered the case $c \in A$, we assume that $c \notin A$, i.e., $b \neq 0$.

When $e(\sigma) = 0$, we have $c\sigma \neq 0$ if $c \neq 0$. The proof is by induction. Suppose for σ such that $m(\sigma) + \ell(\sigma) < L$ we have proved that the identity element is strictly isolated. Choose σ such that $m(\sigma) + e(\sigma) = L$, $c\sigma = 0$. Construct c_1 and σ_1 such that $m(\sigma_1) + e(\sigma_1) < L$, $c_1\sigma_1 = 0$ when $c\sigma = 0$ and $c_1 = 0$ if and only if $c = 0$. Thus, we have shown completely that the identity element in the group is strictly isolated.

Thus, let $\sigma \in Z_+(F)$, $\ell(\sigma) > 0$ and let us write σ as follows:

$$\sigma = v_1 \sigma_1 + \sigma_2, \text{ where } v_1 \in F_1,\ m(v_1) = m(\sigma),\ \sigma_1 \in Z_+(F_2),\ \sigma_2 \in Z_+(F).$$

Let $c \in M$, $c = a+b$, $a \in A$, $b \in B$, $b \neq 0$. We write b as

$$b = \sum_{i=1}^{N} n_i b_{\alpha_i}^{\beta_1} + b_2$$

so that $b_{\alpha_i}^{\beta_1}$ does not occur in the expansion of b_2 in terms of the basis, $\ell(\beta_1) \geq \ell(\beta_K)$ for all β_K such that $b_\alpha^{\beta_K}$ occurs in the expansion of the element b, $\alpha_1 > \alpha_i$ for $i = 2, \ldots, N$, $n_1 \neq 0$. We write σ_1 as $\sigma_1 = p_1 u_1 + \ldots + p_M u_M$ so that $u_1 > u_j$ for $j = 2, \ldots, M$, $p_1 > 0$. Further, let v_1 not occur in the expansion of σ_2 in terms of the basis. Consider two cases:

1) $v_1 = x_\varepsilon v_1'$. Then $e(v_1) = \ell(\sigma) - m(\sigma) - 1$. If now $\ell(\beta_1 v_1) = \ell(\beta_1) + \ell(v_1)$, noting that $\ell(\beta_1) \geq \ell(\beta_K)$, $\ell(v_1) \geq \ell(v_S)$ for all β_K such that $b_\alpha^{\beta_K}$ occurs in the expansion of b, for all v_S in the expansion of σ, we can use Lemma 5 and find a basis element $b_\alpha^{\beta_1 v_1}$ in the expansion of $b\sigma$ with nonzero coefficient. Then we have $\ell(\beta_1 v_1) = \ell(\beta_1) + \ell(v_1) = \ell(\beta_1) + m(v_1) + 1 > m(\sigma)$, which means, by Lemma 2, that $b_\alpha^{\beta_1 v_1}$ does not occur in the expansion of $a\sigma$ in terms of the basis, as a result of which $(a+b)\sigma \neq 0$.

This contradiction shows that $\ell(\beta_1 v_1) < \ell(\beta_1) + \ell(v_1)$, i.e., $\beta_1 = \beta_1' x_\varepsilon^{-1}$, which means that $\ell(\beta_1) > 0$.

We can show that for an arbitrary basis element $b_\alpha^{\beta_K}$ in the expansion of the element b such that $\ell(\beta_K) = \ell(\beta_1)$ and for an arbitrary v_S in the expansion of σ such that $\ell(v_S) = \ell(v_1)$, we must have $\ell(\beta_K v_S) < \ell(\beta_K) + \ell(v_S)$ since otherwise all the conditions of Lemma 5 would hold and the expansion of $b\sigma$ would contain the basis element $b_\alpha^{\beta_\kappa v_s}$ with nonzero coefficient. This element does not occur in the expansion of $a\sigma$, by Lemma 2, and then we obtain $(a+b)\sigma \neq 0$. Hence for all such β_κ and v_s, we have $\ell(\beta_\kappa v_s) < \ell(\beta_\kappa) + \ell(v_s)$, which means that $\beta_\kappa = \beta_\kappa' x_\varepsilon^{-1}$, $v_s = x_\varepsilon v_s'$.

Suppose the expansion of σ contains the element v_t such that $\ell(v_t) = \ell(\sigma)-1, \ell(v_t) = m(\sigma)$. For this element, we again must have $v_t = x_\varepsilon v_t'$. Otherwise $\ell(\beta_1 v_t) = \ell(\beta_1) + \ell(v_t)$ and $\ell(\beta_1 v_t) \geq \ell(\beta_i v_t)$ for all β_i in the expansion of b in terms of the basis $b_\alpha^{\beta_i}$ and for all v_t in the expansion of σ. Indeed, $\ell(\beta_1 v_t) = \ell(\beta_1) + \ell(v_1) - 1$. If $\ell(\beta_i) \leq \ell(\beta_1)-1$ or $\ell(v_t) \leq \ell(v_1)-1$, the inequality holds, but if $\ell(\beta_i) = \ell(\beta_1)$ and $\ell(v_t) = \ell(v_1)$, by what has been proved above, $\ell(\beta_i v_t) < \ell(\beta_i) + \ell(v_t)$, from which $\ell(\beta_i v_t) \leq \ell(\beta_i) + \ell(v_t) - 2 < \ell(\beta_1) + \ell(v_1) - 1$ and again we can use Lemma 5. Further, $\ell(\beta_1 v_t) = \ell(\beta_1) + \ell(v_1) - 1 = \ell(\beta_1) + m(\sigma) > m(\sigma)$, since $\ell(\beta_1) > 0$, and we can use Lemma 2 to obtain $(a+b)\sigma \neq 0$.

We choose $c_1 = c x_\varepsilon$, $\sigma_1 = x_\varepsilon^{-1}\sigma$; then $c_1\sigma_1 = c\sigma = 0$, $c_1 = 0$ if and only if $c=0$, $\ell(\sigma_1) \leq m(\sigma) < \ell(\sigma)$, $m(\sigma_1) \leq \ell(\sigma_1) \leq m(\sigma)$, and then $m(\sigma_1) + \ell(\sigma_1) < m(\sigma) + \ell(\sigma) = L$.

Thus, it remains to consider the case 2: $x_1 = x_\varepsilon^{-1} v_1'$. Again, suppose that $\sigma = v_1\sigma_1 + \sigma_2, \sigma_1 \in Z_+(F_2)$; for v_1 we make the following assumption: v_1 does not occur in the expansion of σ_2 in terms of the basis and $\ell(v_1) = m(\sigma)$; but then $m(v_1) = m(\sigma)$.

We can show that $\ell(v_1) = \ell(\sigma)$. If the expansion of σ in terms of the basis contains a v_s such that $\ell(v_s) > \ell(v_1)$, then, since $m(v_s) \leq m(\sigma)$, we have $m(v_s) < \ell(v_s)$ and $v_s = x_\delta v_s'$ $(\delta = 1,2)$, from which, in view of the case considered above, the expansion of σ cannot contain an element v_1 such that $\ell(v_1) = m(\sigma) = \ell(\sigma) - 1$ and $v_1 = x_\varepsilon^{-1} v_1'$.

Suppose in the expansion of σ in terms of the basis there can be found an element v_s such that $\ell(v_s) = \ell(v_1) = m(\sigma)$ and $v_s = x_\delta^{-1} v_s'$. We can show that then $\delta = \varepsilon$.

We have $m(\sigma) = \ell(\sigma)$. Let $c = a + b$ where b is as in the first case; then if $\ell(\beta_1 v_1) = \ell(\beta_1) + \ell(v_1)$ we have $\ell(\beta_1) + \ell(v_1) \geq \ell(\beta_i) + \ell(v_t)$ for all β_i such that $b_\alpha^{\beta_i}$ occurs in the expansion of b in terms of the basis and for all v_s in the expansion of σ. Hence we can use Lemma 5 and find in the expansion of $b\sigma$ an element $b_\alpha^{\beta_1 v_1}$ with nonzero coefficient. By Lemma 2 the element $b_\alpha^{\beta_1 v_1}$ occurs in the expansion of $a\sigma$ provided $\ell(\beta_1 v_1) \leq m(\sigma)$ and this is possible only if $\ell(\beta_1) = 0$.

Now the element b can be written as follows: $b = n_1 b_{\alpha_1}^e + \ldots + n_N b_{\alpha_N}^e$.

The element a can be written as $a = m_1 a_{\gamma_1} + \ldots + m_k a_{\gamma_k}$. We use Lemma 4 and find in the expansion of $a\sigma$ a basis element $b_{\gamma_\varepsilon \gamma_1 u_1}^{v_1}$ with nonzero coefficient. We use Lemma 5 and find in the expansion of $b\sigma$ a basis element $b_{\alpha_1 u_1}^{v_1}$ with nonzero coefficient. Now, in view of these assertions, $(a+b)\sigma = 0$ provided $\gamma_\varepsilon \gamma_1 u_1 = \alpha_1 u_1$ from which $\alpha_1 = \gamma_\varepsilon \gamma_1$.

We write σ in the form $\sigma_3 v_s + \sigma_4$, where $v_s = x_\delta^{-1} v_s'$ and $\ell(v_s) = m(\sigma)$. Again we use Lemmas 4 and 5 to find that $\alpha_1 = \gamma_\delta \gamma_1$ from which $\varepsilon = \delta$.

Now, again let $\ell(v_s) = m(\sigma)$, but $v_s = x_\delta v_s'$ $(\delta = 1,2)$; then, by Lemma 5 the basis element $b_\alpha^{v_s}$ occurs in the expansion of $b\sigma$.

By Lemma 3, the element $b_\alpha^{v_s}$ occurs in the expansion of $a\sigma$, only if the expansion of σ contains an element v_t such that $v_t = x_\xi v_s$ $(\xi = 1,2)$ but this is impossible since then $\ell(v_t) > \ell(\sigma)$.

Consider the remaining case $\ell(\beta_1 v_1) < \ell(\beta_1) + \ell(v_1)$. But then $\beta_1 = \beta' x_\varepsilon$; if the expansion of σ contains an element v_t such that $v_t \neq x_\varepsilon^{-1} v_t'$ and $\ell(v_t) = m(\sigma)$, we have $\ell(\beta_1 v_t) = \ell(\beta_1) + \ell(v_t)$ and $\ell(\beta_1 v_t) \geq \ell(\beta_i v_t)$ for all β_i such that $b_\alpha^{\beta_i}$ is in the expansion of b and v_t is in the expansion of σ. By Lemma 5, $b_\alpha^{\beta_1 v_t}$ is in the expansion of $b\sigma$ with nonzero coefficient, but since $\ell(\beta_1 v_t) > m(\sigma)$, by Lemma 2, $b_\alpha^{\beta_1 v_t}$ does not occur in the expansion of $a\sigma$.

This means that we can put $c_1 = cx_\varepsilon^{-1}$, $\sigma_1 = x_\varepsilon \sigma$; then

$$\ell(\sigma_1) + m(\sigma_1) < \ell(\sigma) + m(\sigma) - L.$$

Consequently, we have proved that the identity element is strictly isolated.

COROLLARY. An Abelian, normal, strictly isolated subgroup of a group with a strictly isolated identity element cannot have any linear orderings which are preserved under internal isomorphisms of the whole group.

The proof follows directly from the construction of the group G.

Note 1. In view of the above corollary the Mal'tsev-Podoeryugin-Riger condition [1] for orderability cannot be weakened.

Note 2. In an over-ordered group strict isolation and infra-invariance of its subgroups is a necessary and sufficient condition for convexity of a subgroup [1]. As our example shows, in the case of an ordered group this condition is no longer sufficient.

LITERATURE CITED

1. A. I. Kokorin and V. M. Kopytov, Linearly Ordered Groups [in Russian], Nauka (1972).
2. Kourovsk Notebook [in Russian], Novosibirsk (1965).