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# Stochastic Dilations of Uniformly Continuous Completely Positive Semigroups★

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**Abstract.** For an arbitrary uniformly continuous completely positive semigroup $(\mathcal{T}_t : t \geqslant 0)$ on the space $B(\mathfrak{h}_0)$ of bounded operators on a Hilbert space $\mathfrak{h}_0$, we construct a family $(U(t) : t \geqslant 0)$ of unitary operators on a Hilbert space $\mathfrak{H}_0 = \mathfrak{h}_0 \otimes \mathfrak{H}$ and a conditional expectation $\mathbb{E}_0$ from $B(\mathfrak{H}_0)$ to $B(\mathfrak{h}_0)$, such that, for arbitrary $t \geqslant 0$, $X \in B(\mathfrak{h}_0)$ $\mathcal{T}_t(X) = \mathbb{E}_0[U(t)X \otimes IU(t)^\dagger]$. The unitary operators $U(t)$ satisfy a stochastic differential equation involving a noncommutative generalisation of infinite dimensional Brownian motion. They do not form a semigroup.

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**Key words.** Completely positive semigroup, operators on Hilbert space, conditional expectation, stochastic differential equation, $\infty$-dimensional Brownian motion, Fock space, Itô product formula, stochastic dilation.

## 1. Introduction

In [2] we constructed a noncommutative extension of the Itô stochastic calculus for operator-valued processes. Using the duality transformation to identify $L^2(w)$, where $w$ is Wiener measure, with the Boson Fock space $\mathfrak{H} = \Gamma(L^2(0, \infty))$, classical Brownian motion is expressed as the sum $A(t) + A^\dagger(t)$ of two mutually noncommuting operator valued processes, which are, respectively, the Fock annihilation and creation operators $A(t) = a(\chi_{[0,t]})$, $A^\dagger(t) = a^\dagger(\chi_{[0,t]})$. The extended Itô product formula for the calculus based on $A$ and $A^\dagger$ is expressed formally by the multiplication table

| | $\mathrm{d}A^\dagger$ | $\mathrm{d}A$ | $\mathrm{d}t$ |
|---|---|---|---|
| $\mathrm{d}A^\dagger$ | 0 | 0 | 0 |
| $\mathrm{d}A$ | $\mathrm{d}t$ | 0 | 0 |
| $\mathrm{d}t$ | 0 | 0 | 0 |

from which the product formula for classical Brownian motion follows as a special case.

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Using this calculus, we showed in [2] that, for given bounded operators $L$ and $\mathscr{H}$ in a Hilbert space $\mathfrak{h}_0$, of which $\mathscr{H}$ is self-adjoint, the stochastic differential equation

$$dU = U(L \otimes dA^\dagger - L^\dagger \otimes dA + (i\mathscr{H} - \tfrac{1}{2}L^\dagger L) \otimes I\, dt), \quad U(0) = I \tag{1.1}$$

has a unique solution which consists of unitary operators in $\mathfrak{H}_0 = \mathfrak{h}_0 \otimes \mathfrak{H}$. Moreover, if $\mathbb{E}_0$ is the vacuum conditional expectation from $B(\mathfrak{H}_0)$ onto $B(\mathfrak{h}_0)$ defined by

$$\langle u, \mathbb{E}_0[T]v \rangle = \langle u \otimes \Psi_0, Tv \otimes \Psi_0 \rangle \quad (T \in B(\mathfrak{h}_0), u, v \in \mathfrak{h}_0)$$

where $\Psi_0$ is the Fock vacuum vector, then the formula

$$\mathscr{T}_t(X) = \mathbb{E}_0[U(t)X \otimes IU(t)^{-1}] \quad (X \in B(\mathfrak{h}_0), t \geqslant 0) \tag{1.2}$$

defines a uniformly continuous semigroup of completely positive maps of which the infinitesimal generator $\mathscr{L}$ is given by

$$\mathscr{L}(X) = i[\mathscr{H}, X] - \tfrac{1}{2}(L^\dagger LX - 2L^\dagger XL + XL^\dagger L) \tag{1.3}$$

Now in [4] it is shown that the general form of the infinitesimal generator of a uniformly continuous semigroup of completely positive maps in $B(\mathfrak{h}_0)$ is

$$\mathscr{L}(X) = i[\mathscr{H}, X] - \tfrac{1}{2}\sum_j (L_j^\dagger LX - 2L_j^\dagger XL_j + XL_j^\dagger L_j) \tag{1.4}$$

where $\mathscr{H} \in B(\mathfrak{h}_0)$ is self-adjoint, and the operators $L_j \in B(\mathfrak{h}_0)$ may be infinite in number, but must be such that $\Sigma_j L_j L_j$ converges strongly. Our purpose in this paper is to construct a stochastic unitary dilation of the semigroup of which (1.4) is the infinitesimal generator, by means of a noncommutative stochastic calculus generalising that of [2].

An intuitive procedure for carrying out this goal would be as follows; introduce independent quantum Brownian motions $A_j$, corresponding to the terms $L_j$ in (1.4), and satisfying the product rules

| | $dA_k^\dagger$ | $dA_k$ | $dt$ |
|---|---|---|---|
| $dA_j^\dagger$ | 0 | 0 | 0 |
| $dA_j$ | $\delta_{jk}\, dt$ | 0 | 0 |
| $dt$ | 0 | 0 | 0 |

and solve the equation

$$dU = U\left(\sum_j L_j \otimes dA_j^\dagger - \sum_j L_j^\dagger \otimes dA_j + \left(i\mathscr{H} - \tfrac{1}{2}\sum_j L_j^\dagger L_j\right) \otimes I\, dt\right),$$
$$U(0) = I. \tag{1.5}$$

However, the operator theoretic difficulties of this approach are formidable when there are infinitely many $L_j$, and an alternative strategy is called for. This is to introduce the single process $A_L(t) = \Sigma_j L_j^\dagger \otimes A_j$ together with its formal adjoint $A_L(t) = \Sigma_j L_j \otimes A_j^\dagger$, for which the Itô rules are

| | $dA_L^\dagger$ | $dA_L$ | $dt$ |
|---|---|---|---|
| $dA_L^\dagger$ | 0 | 0 | 0 |
| $dA_L$ | $\Sigma_j L_j^\dagger L_j \otimes I\,dt$ | 0 | 0 |
| $dt$ | 0 | 0 | 0 |

and (1.5) becomes

$$dU = U\left(dA_L^\dagger - dA_L + \left(i\mathscr{H} - \tfrac{1}{2}\sum_j L_j^\dagger L_j\right)\otimes I\,dt\right),$$
$$U(0) = I.$$

Because adaptedness no longer forces processes to commute with stochastic differentials, the appropriate theory of stochastic integration must now distinguish between the left and the right integral. We turn this complication to advantage by developing the theory of adapted processes and stochastic integrals in such a way that formal adjunction is a symmetry converting the left into the right integral and vice-versa.

In this connection we make constant use of the following extension of the well-known result that an everywhere defined operator in a Hilbert space with a densely defined adjoint is bounded.

THEOREM 1.1. *Let $\mathfrak{h}_0$, $\mathfrak{H}$ be Hilbert spaces and let $\mathfrak{h}_0 \otimes \mathfrak{H}$ be their Hilbert space tensor product. Let $\mathscr{E}$ be a dense subspace of $\mathfrak{H}$ and let $\mathfrak{h}_0 \underline{\otimes}\, \mathscr{E}$ denote the algebraic tensor product. Let $T$ and $T^\dagger$ be mutually adjoint operators with common domain $\mathfrak{h}_0 \underline{\otimes}\, \mathscr{E}$, so that for arbitrary $u, v \in \mathfrak{h}_0$, $\Phi, \Psi \in \mathscr{E}$,*

$$\langle u\otimes\Phi, Tv\otimes\Psi\rangle = \langle T^\dagger u\otimes\Phi, v\otimes\Psi\rangle.$$

*Then for each $\Phi \in \mathscr{E}$, the operators $T_\Phi$, $T_\Phi^\dagger$ given by*

$$T_\Phi u = Tu\otimes\Phi, \qquad T_\Phi^\dagger u = T^\dagger u\otimes\Phi \quad (u \in \mathfrak{h}_0)$$

*are bounded.*

*Proof.* Fix $\Psi \in \mathfrak{h}_0 \underline{\otimes}\, \mathscr{E}$ with $\|\Psi\| \leqslant 1$. The linear map

$$\lambda_\Psi(u) = \langle T^\dagger\Psi, u\otimes\Phi\rangle = \langle\Psi, Tu\otimes\Phi\rangle$$

is bounded on $\mathfrak{h}_0$ since

$$|\langle T^\dagger\Psi, u\otimes\Phi\rangle| \leqslant \|T^\dagger\Psi\|\;\|\Phi\|\;\|u\|.$$

Moreover, the $\lambda_\Psi$, $\Psi \in \mathfrak{h}_0 \underline{\otimes}\, \mathscr{E}$, $\|\Psi\| \leqslant 1$ are pointwise bounded, since

$$|\langle\Psi, Tu\otimes\Phi\rangle| \leqslant \|Tu\otimes\Phi\| \quad \text{for } \|\Psi\| \leqslant 1.$$

Hence, by the uniform boundedness principle, there exists a positive number $M$ such that, for all $\Psi \in \mathfrak{h}_0 \underline{\otimes}\, \mathscr{E}$ with $\|\Psi\| \leqslant 1$,

$$|\langle \Psi, Tu \otimes \Phi \rangle| \leqslant M \|u\|, \quad (u \in \mathfrak{h}_0)$$

and, hence, such that for all $\Psi \in \mathfrak{h}_0 \underline{\otimes} \mathscr{E}$,

$$|\langle \Psi, Tu \otimes \Phi \rangle| \leqslant M \|u\| \; \|\Phi\|, \quad (u \in \mathfrak{h}_0).$$

Since $\mathfrak{h}_0 \underline{\otimes} \mathscr{E}$ is dense in $\mathfrak{h}_0 \otimes \mathfrak{H}$, it follows that

$$\|Tu \otimes \Phi\| \leqslant M \|u\|, \quad (u \in \mathfrak{h}_0)$$

that is $T_\Phi$ is bounded. The argument for $T_\Phi^\dagger$ is similar. □

## 2. Notation and Preliminaries

Let a separable Hilbert space $\mathfrak{h}_0$ and a finite or countably infinite index set $J$ be given, once and for all. We denote by $\mathfrak{h}$ the direct sum $\mathfrak{h} = \oplus_{j \in J} L^2[0, \infty)$. The *Boson Fock space* over $\mathfrak{h}$ may be conveniently characterised as a pair $(\mathfrak{H}, \Psi)$ comprising a Hilbert space $\mathfrak{H}$ and a map $\Psi: \mathfrak{h} \to \mathfrak{H}$ such that $\{\Psi(f) : f \in \mathfrak{h}\}$ is total in $\mathfrak{H}$ and, for all $f, g \in \mathfrak{h}$,

$$\langle \Psi(f), \Psi(g) \rangle = \exp \langle f, g \rangle.$$

$\Psi(f)$ is called the *exponential vector* or coherent state corresponding to $f \in \mathfrak{h}$. The *vacuum vector* is $\Psi_0 = \Psi(0)$. We denote by $\mathscr{E}$ the dense subspace of $\mathfrak{H}$ spanned algebraically by the exponential vectors.

The operator-valued processes which concern us live in the tensor product $\mathfrak{H}_0 = \mathfrak{h}_0 \otimes \mathfrak{H}$ of $\mathfrak{H}$ with the 'initial space' [2] $\mathfrak{h}_0$. If $T$ and $T^\dagger$ are mutually adjoint operators in $\mathfrak{H}_0$ with domains containing $\mathfrak{h}_0 \underline{\otimes} \mathscr{E}$, then for $f \in \mathfrak{h}$ we denote by $\|T\|_f$ and $\|T^\dagger\|_f$ the bounds of the operators on $\mathfrak{h}_0$

$$u \mapsto Tu \otimes \Psi(f), \qquad u \mapsto T^\dagger u \otimes \Psi(f),$$

which are bounded by Theorem 1.1.

We denote by

$$\mathfrak{h} = \mathfrak{h}_t \oplus \mathfrak{h}^t \tag{2.1}$$

the natural decomposition

$$\mathfrak{h} = \bigoplus_{j \in J} L^2[0, \infty) = \left( \bigoplus_{j \in J} L^2[0, t] \right) \oplus \left( \bigoplus_{j \in J} L^2(t, \infty) \right)$$

and for $f \in \mathfrak{h}$ we write $f = (f_t, f^t)$ for its components in these subspaces. Corresponding to the direct sum decomposition (2.1), there is a tensor product decomposition $\mathfrak{H} = \mathfrak{H}_t \otimes \mathfrak{H}^t$ of $\mathfrak{H}$ into the Fock spaces $\mathfrak{H}_t$ and $\mathfrak{H}^t$ over $\mathfrak{h}_t$ and $\mathfrak{h}^t$ respectively, in which for each $f \in \mathfrak{h}$

$$\Psi(f) = \Psi(f_t) \otimes \Psi(f^t).$$

In this decomposition clearly $\mathscr{E} = \mathscr{E}_t \underline{\otimes} \mathscr{E}^t$, where $\mathscr{E}_t$ and $\mathscr{E}^t$ are the spans of the exponential vectors in $\mathfrak{H}_t$ and $\mathfrak{H}^t$, respectively.

Now let $B(\mathfrak{h}_0; J)$ denote the set of $J$-tuples of bounded operators in $\mathfrak{h}_0$, $L = (L_j : j \in J)$ for which $\Sigma_j L_j^\dagger L_j$ converges strongly in $B(\mathfrak{h}_0)$. Then $B(\mathfrak{h}_0; J)$ is a complex vector space under component-wise operations. Furthermore, for $L, M \in B(\mathfrak{h}_0; J)$ the sum $\Sigma_j L_j^\dagger M_j$ converges strongly, as is seen from the polarisation identity. For $f = (f_j : j \in J) \in \mathfrak{h}$ and $0 \leqslant s \leqslant t$, since

$$\sum_j \left(\int_s^t f_j I\right)^\dagger \left(\int_s^t f_j I\right) = \sum_j |\langle f, \chi_{(s,t)}\rangle|^2 I \leqslant (t-s) \|f\|^2 I \tag{2.2}$$

the $J$-tuple $(\int_s^t f_j I) \in B(\mathfrak{h}_0; J)$. Hence, the operators $\Sigma_j \int_s^t f_j L_j^\dagger$ and $\Sigma_j \int_s^t \bar{f}_j L_j$ are well defined in $B(\mathfrak{h}_0)$.

## 3. Processes

DEFINITION 3.1. An *adapted process* is a family of operators $F = (F(t) : t \geqslant 0)$ in $\mathfrak{H}_0$ such that for each $t \geqslant 0$

(a) $D(F(t)) = \mathfrak{h}_0 \underline{\otimes} \mathscr{E}_t \underline{\otimes} \mathfrak{H}^t$.

(b) There is an operator $F^\dagger(t)$ with the same domain adjoint to $F(t)$.

(c) There are operators $F_1(t)$ and $F_1^\dagger(t)$ on $\mathfrak{h}_0 \underline{\otimes} \mathscr{E}_t$ such that $F(t) = F_1(t) \otimes I$, $F^\dagger(t) = F^\dagger(t) = F_1^\dagger(t) \otimes I$.

The *adjoint process* of $F$ is $F^\dagger = (F^\dagger(t) : t > 0)$. A *simple process* is an adapted process of the form

$$F(t) = \sum_{n=0}^{\infty} F_n \chi_{(t_n, t_{n+1})}(t) \quad (t \geqslant 0)$$

for some sequence $0 = t_0 < t_1 < \cdots < t_n \to \infty$. An adapted process is *regular* if there exists a sequence $F^{(n)}$, $n = 1, 2, \ldots$ of simple processes such that, for all $f \in \mathfrak{h}$,

$$\|F(t) - F^{(n)}(t)\|_f, \qquad \|F^\dagger(t) - F^{\dagger(n)}(t)\|_f \xrightarrow[n]{} 0$$

uniformly on compact sets in $(0, \infty)$, and *continuous* if for all $u \in \mathfrak{h}_0$, $f \in \mathfrak{h}$, the maps

$$t \mapsto F(t) u \otimes \Psi(f), \qquad t \mapsto F(t)^\dagger u \otimes \Psi(f) \quad \text{are continuous from } [0, \infty) \text{ to } \mathfrak{H}_0.$$

Then every continuous process is regular. We denote by $\mathscr{A}$, $\mathscr{A}_0$, $\mathscr{A}_r$ and $\mathscr{A}_c$, respectively, the sets of adapted, simple, regular and continuous processes.

Now fix $L \in B(\mathfrak{h}_0, J)$, once and for all.

We define operators $A_L(t)$, $t \geqslant 0$, initially with domain $\mathfrak{h}_0 \underline{\otimes} \mathscr{E}$, by

$$A_L(t) u \otimes \Psi(f) = \left(\sum_j \int_0^t f_j L_j^\dagger u\right) \otimes \Psi(f).$$

Formally,

$$A_L(t) = \sum_j L_j^\dagger \otimes A_j(t),$$

where we make the identification

$$\mathfrak{H} = \Gamma\left(\left\{\bigoplus_{k=1}^{j} L^2[0,\infty)\right\} \oplus \left\{\bigoplus_{k>j} L^2[0,\infty)\right\}\right)$$

$$= \left\{\bigotimes_{k=1}^{j} \Gamma(L^2[0,\infty))\right\} \otimes \Gamma\left(\bigoplus_{k>j} L^2[0,\infty)\right)$$

and set $A_j(t) = \otimes^{j-1}{}_{k=1} I \otimes A(t) \otimes I$.

We wish to establish the existence of an operator $A_L^\dagger(t)$ with the same domain adjoint to $A_L(t)$; formally $A_L^\dagger(t) = \Sigma_j L_j \otimes A_j^\dagger(t)$. We introduce the notation

$$\partial_\Delta^j \Psi(f_1, \ldots, f_j, \ldots) = \frac{d}{d\sigma} \Psi(f_1, \ldots, f_j + \sigma\chi_\Delta, \ldots)|_{\sigma=0}$$

where $\chi_\Delta$ is the indicator function of the finite interval $\Delta \subseteq [0,\infty)$. Then when $J$ is finite, $A_L^\dagger(t)$ is given by the action

$$A_L^\dagger(t) u \otimes \Psi(f) = \sum_j L_j u \otimes \partial_{[0,t]}^j \Psi(f). \tag{3.1}$$

That this sum converges when $J$ is infinite is a corollary of Theorem 3.2. Before stating it we note that, if $F$ is an operator whose domain includes $\mathfrak{h}_0 \underline{\otimes} \mathscr{E}$ such that for each $f \in \mathfrak{h}$, the operator $u \mapsto Fu \otimes \Psi(f)$ is bounded on $\mathfrak{h}_0$, that is $\|F\|_f < \infty$, then for $f, g \in \mathfrak{h}$, $u \in \mathfrak{h}_0$ the sum $\Sigma_j \langle FL_j u \otimes \Psi(f), FL_j u \otimes \Psi(g)\rangle$ converges absolutely. Indeed,

$$\begin{aligned}
&\sum_j |\langle FL_j u \otimes \Psi(f), FL_j u \otimes \Psi(g)\rangle| \\
&\quad \leqslant \left(\sum_j \|FL_j u \otimes \Psi(f)\|^2\right)^{1/2} \left(\sum_j \|FL_j u \otimes \Psi(g)\|^2\right)^{1/2} \\
&\quad \leqslant \|F\|_f \|F\|_g \sum_j \|L_j u\|^2 \\
&\quad = \|F_f\| \ \|F\|_g \left\langle u, \sum_j L_j^\dagger L_j u\right\rangle < \infty.
\end{aligned}$$

THEOREM 3.2. *Let $0 \leqslant s \leqslant t$. Let $F$ and $F^\dagger$ be mutually adjoint operators with domain $\mathfrak{h}_0 \underline{\otimes} \mathscr{E}_s \underline{\otimes} \mathfrak{H}^s$ of form $F_1 \otimes I$ and $F_1^\dagger \otimes I$ where $F_1$ and $F_1^\dagger$ are operators on $\mathfrak{h}_0 \underline{\otimes} \mathscr{E}_s$. Then in the case when $J$ is infinite, the sum*

$$\sum_j FL_j u \otimes \partial_{(s,t]}^j \Psi(f)$$

*converges. Moreover*

(a) *For arbitrary $u \in \mathfrak{h}_0$, $f, g \in \mathfrak{h}$,*

$$\left\langle \sum_j FL_j u \otimes \partial^j_{(s,t]} \Psi(f), \sum_j FL_j u \otimes \partial^j_{(s,t]} \Psi(f) \right\rangle$$

$$= \left\langle F \sum_j \int_s^t \bar{f} L_j u \otimes \Psi(f), F \sum_j \int_s^t \bar{g} L_j u \otimes \Psi(g) \right\rangle +$$

$$+ (t-s) \sum_j \langle FL_j u \otimes \Psi(f), FL_j u \otimes \Psi(g) \rangle, \tag{3.2}$$

(b) *for arbitrary* $u, v \in \mathfrak{h}_0$, $f, g \in \mathfrak{h}$

$$\left\langle \sum_j FL_j u \otimes \partial^j_{(s,t]} \Psi(f), v \otimes \Psi(g) \right\rangle = \left\langle u \otimes \Psi(f), \sum_j \int_s^t g_j L_j^\dagger \otimes IF^\dagger v \otimes \Psi(g) \right\rangle.$$

*Proof.* Assume $J = \mathbb{N}$ and let $\phi_n = \Sigma_{j=1}^n FL_j u \otimes \partial^j_{(s,t]} \Psi(f)$. Then, for $m \geqslant n$,

$$\| \phi_m - \phi_n \|^2$$

$$= \sum_{j,k=n+1}^m \langle FL_j u \otimes \partial^j_{(s,t]} \Psi(f), FL_k u \otimes \partial^k_{(s,t]} \Psi(f) \rangle$$

$$= \sum_{j,k=n+1}^m \langle F_1 L_j u \otimes \Psi(f_s), F_1 L_k u \otimes \Psi(f_s) \rangle \times$$

$$\times \frac{\partial^2}{\partial\sigma\,\partial\tau} \langle \Psi(f_1^s, \dots, f_j^s + \sigma\chi_{(s,t]}, \dots), \Psi(g_1^s, \dots, g_k^s + \tau\chi_{(s,t]}, \dots) \rangle \Big|_{\substack{\sigma=0\\ \tau=0}}$$

$$= \sum_{j,k=n+1}^m \langle F_1 L_j u \otimes \Psi(f_s), F_1 L_k u \otimes \Psi(f_s) \rangle \frac{\partial^2}{\partial\sigma\,\partial\tau} \times$$

$$\times \exp \langle (f_1^s, \dots, f_j^s + \sigma\chi_{(s,t]}, \dots), (g_1^s, \dots, g_k^s + \tau\chi_{(s,t]}, \dots) \rangle \big|_{\substack{\sigma=0\\ \tau=0}}$$

$$= \sum_{j,k=n+1}^n \langle F_1 L_j u \otimes \Psi(f_s), F_1 L_k u \otimes \Psi(f_s) \rangle \times$$

$$\times \left\{ \int_s^t f_j \int_s^t \bar{f}_k + (t-s)\delta_{jk} \right\} \| \Psi(f^s) \|^2$$

$$= \left\{ \left\| F_1 \sum_{j=n+1}^m \int_s^t \bar{f}_j L_j u \otimes \Psi(f_s) \right\|^2 + \right.$$

$$\left. + (t-s) \sum_{j=n+1}^m \| F_1 L_j u \otimes \Psi(f_s) \|^2 \right\} \| \Psi(f^s) \|^2$$

$$= \left\| F \sum_{j=n+1}^m \int_s^t \bar{f}_j L_j u \otimes \Psi(f) \right\|^2 + (t-s) \sum_{j=n+1}^m \| FL_j u \otimes \Psi(f) \|^2$$

$$\leqslant \| F \|_f^2 \left\{ \left\| \sum_{j=n+1}^m \int_s^t \bar{f} L_j u \right\|^2 + (t-s) \sum_{j=n+1}^m \langle u, L_j^\dagger L_j u \rangle \right\} \xrightarrow[m,n]{} 0. \tag{3.3}$$

Hence, $(\Phi_n)$ converges as asserted. A similar calculation to that leading to (3.3) establishes (a). To prove (b) we have (assuming $J = N$ is infinite)

$$\left\langle \sum_j FL_j u \otimes \partial^j_{(s,t]} \Psi(f), v \otimes \Psi(g) \right\rangle$$

$$= \lim_n \sum_{j=1}^n \langle F_1 L_j u \otimes \Psi(f_s), v \otimes \Psi(g_s) \rangle \times$$

$$\times \frac{\mathrm{d}}{\mathrm{d}\sigma} \langle \Psi(f_1^s, \ldots, f_j^s + \sigma\chi_{(s,t]}, \ldots), \Psi(g^s) \rangle |_{\sigma=0}$$

$$= \lim_n \sum_{j=1}^n \langle u \otimes \Psi(f_s), L_j^\dagger \otimes IF_1^\dagger v \otimes \Psi(g_s) \rangle \int_s^t g_j \langle \Psi(f^s), \Psi(g^s) \rangle$$

$$= \lim_n \left\langle u \otimes \Psi(f), \sum_{j=1}^n \int_s^t g_j L_j^\dagger \otimes IF^\dagger v \otimes \Psi(g) \right\rangle$$

$$= \left\langle u \otimes \Psi(f), \sum_j \int_s^t g_j L_j^\dagger \otimes IF^\dagger v \otimes \Psi(g) \right\rangle. \qquad \square$$

Taking $s = 0$, $F = I$ in the theorem, we see that the sum (3.1) converges, and defines an operator $A_L^\dagger(t)$ adjoint to $A_L(t)$ on the domain $\mathfrak{h}_0 \underline{\otimes} \mathscr{E}$.

The operators $A_L(t)$ and $A_L^\dagger(t)$ are clearly of form $A_1 \otimes I$ and $A_1^\dagger \otimes I$, respectively, on $(\mathfrak{h}_0 \underline{\otimes} \mathscr{E}_t) \underline{\otimes} \mathscr{E}^t$. As such they extend naturally to mutually adjoint operators on $(\mathfrak{h}_0 \underline{\otimes} \mathscr{E}_t) \underline{\otimes} \mathfrak{H}^t$, which constitute mutually adjoint adapted processes. Furthermore, these processes are additive, in the sense that for $0 \leqslant s \leqslant t$, $A_L(t) - A_L(s)$ and $A_L^\dagger(t) - A_L^\dagger(s)$ are of form $I \otimes A_2 \otimes I$, $I \otimes A_2^\dagger \otimes I$ on $(\mathfrak{h}_0 \underline{\otimes} \mathscr{E}_s) \underline{\otimes} \mathscr{E}_t^s \underline{\otimes} \mathfrak{H}^t$, where $\mathscr{E}_t^s$ is the span of the exponential vectors in $\mathfrak{H}_t^s = \Gamma(\oplus_{j \in J} L^2(s, t])$. Thus $A_L(t) - A_L(s)$ and $A_L^\dagger(t) - A_L^\dagger(s)$ extend naturally to operators, for which we use the same symbols, on $(\mathfrak{h}_0 \otimes \mathfrak{H}_s) \underline{\otimes} \mathscr{E}_t^s \underline{\otimes} \mathfrak{H}^t$. Then if $F$ and $F^\dagger$ satisfy the hypotheses of Theorem 3.2, the operators $F(A_L(t) - A_L(s))$, $(A_L(t) - A_L(s))F$, and $(A_L^\dagger(t) - A_L^\dagger(s))F$ are well defined on $\mathfrak{h}_0 \underline{\otimes} \mathscr{E}_t \underline{\otimes} \mathfrak{H}^t$. We define the operator $F(A_L^\dagger(t) - A_L^\dagger(s))$ on the same domain using Theorem 3.2 by

$$F(A_L^\dagger(t) - A_L^\dagger(s)) u \otimes \Psi(f) = \sum_j FL_j u \otimes \partial^j_{(s,t]} \Psi(F).$$

Theorem 3.2 (b) shows that the operators $F(A_L^\dagger(t) - A_L^\dagger(s))$ and $(A_L(t) - A_L(s))F^\dagger$ are mutually adjoint, and straightforward calculation shows that the same is true of $F(A_L(t) - A_L(s))$ and $(A^\dagger(t) - A^\dagger(s))F^\dagger$.

We note that the identity (3.2) can be restated as

$$\langle F(A_L^\dagger(t) - A_L^\dagger(s)) u \otimes \Psi(f), F(A_L^\dagger(t) - A_L^\dagger(s)) u \otimes \Psi(g) \rangle$$

$$= \langle F(A_L(t) - A_L(s)) u \otimes \Psi(f), F(A_L(t) - A_L(s)) u \otimes \Psi(g) \rangle +$$

$$+ (t - s) \sum_j \langle FL_j u \otimes \Psi(f), FL_j u \otimes \Psi(g) \rangle. \qquad (3.3)$$

## 4. Stochastic Integrals of Simple Processes

DEFINITION 4.1. Let $F, G, H \in \mathscr{A}_0$ and write

$$F = \sum_{n=0}^{\infty} F_n \chi_{[t_n, t_{n+1})}, \qquad G = \sum_{n=0}^{\infty} G_n \chi_{[t_n, t_{n+1})},$$

$$H = \sum_{n=0}^{\infty} H_n \chi_{[t_n, t_{n+1})} \tag{4.1}$$

where $0 = t_0 < t_1 < \cdots < t_n \xrightarrow[n]{} \infty$. The families of operators $M = (M(t) : t \geqslant 0)$, $N = (N(t) : t \geqslant 0)$ with domains $D(M(t)) = D(N(t)) = \mathfrak{h}_0 \underline{\otimes} \mathscr{E}_t \underline{\otimes} \mathfrak{H}^t$ defined by $M(0) = 0$, $N(0) = 0$,

$$M(t) = M(t_n) + F_n(A_L^\dagger(t) - A_L^\dagger(t_n)) + G_n(A_L(t) - A_L(t_n)) + (t - t_n)H_n$$

$$N(t) = N(t_n) + (A_L^\dagger(t) - A_L^\dagger(t_n))F_n + (A_L(t) - A_L(t_n))G_n + (t - t_n)H_n$$

for $t_n < t \leqslant t_{n+1}$, are called the *right* and *left stochastic integrals* of $(F, G, H)$, and denoted by

$$M(t) = \int_0^t (F\,\mathrm{d}A_L^\dagger + G\,\mathrm{d}A_L + H\,\mathrm{d}\tau),$$

$$N(t) = \int_0^t (\mathrm{d}A_L^\dagger F + \mathrm{d}A_L G + H\,\mathrm{d}\tau).$$

Clearly $M$ and $N$ are adapted processes and

$$\left[\int_0^t (F\,\mathrm{d}A_L^\dagger + G\,\mathrm{d}A_L + H\,\mathrm{d}\tau)\right]^\dagger = \int_0^t (\mathrm{d}A_L^\dagger G^\dagger + \mathrm{d}A_L F^\dagger + H^\dagger\,\mathrm{d}\tau). \tag{4.2}$$

We describe by the differential relations

$$\mathrm{d}M = F\,\mathrm{d}A_L^\dagger + G\,\mathrm{d}A_L + H\,\mathrm{d}t, \qquad \mathrm{d}N = \mathrm{d}A_L^\dagger F + \mathrm{d}A_L G + H\,\mathrm{d}t$$

the situation that, for $t \geqslant 0$,

$$M(t) = M_0 \otimes I + \int_0^t (F\,\mathrm{d}A_L^\dagger + G\,\mathrm{d}A_L + H\,\mathrm{d}t),$$

$$N(t) = N_0 \otimes I + \int_0^t (\mathrm{d}A_L^\dagger F + \mathrm{d}A_L G + H\,\mathrm{d}t)$$

where $M_0, N_0 \in B(\mathfrak{h}_0)$.

THEOREM 4.2. *Let $F, G, H \in \mathscr{A}_0$ and*

$$\mathrm{d}M = F\,\mathrm{d}A_L^\dagger + G\,\mathrm{d}A_L + H\,\mathrm{d}t, \qquad \mathrm{d}N = \mathrm{d}A_L^\dagger F + \mathrm{d}A_L G + H\,\mathrm{d}t.$$

*Then for arbitrary $u, v \in \mathfrak{h}_0$, $f, g \in \mathfrak{h}$ the functions on* $(0, \infty)$

$$t \mapsto \langle u \otimes \Psi(f), M(t) v \otimes \Psi(g)\rangle, \qquad t \mapsto \langle u \otimes \Psi(f), N(t) v \otimes \Psi(g)\rangle$$

*are absolutely continuous, with generalised derivatives*

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle u \otimes \Psi(f), M(t) v \otimes \Psi(g)\rangle \tag{4.3}$$

$$= \left\langle u \otimes \Psi(f), \left[F(t)\sum_j \overline{f_j(t)} L_j \otimes I + G(t)\sum_j g_j(t) L_j^\dagger \otimes I + H(t)\right] v \otimes \Psi(g)\right\rangle,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle u \otimes \Psi(f), N(t) v \otimes \Psi(g)\rangle \tag{4.4}$$

$$= \left\langle u \otimes \Psi(f), \left[\sum_j \overline{f_j(t)} L_j \otimes I F(t) + \sum_j g_j(t) L_j^\dagger \otimes I G(t) + H(t)\right] v \otimes \Psi(g)\right\rangle.$$

*Proof.* We give the proof only for the case of the right integral. Assume $F$, $G$ and $H$ are given by (4.1) and that $t \in (t_n, t_{n+1})$. Then

$$\begin{aligned}
&\langle u \otimes \Psi(f), M(t) v \otimes \Psi(g)\rangle \\
&\quad = \langle u \otimes \Psi(f), M(t_n) v \otimes \Psi(g)\rangle + \langle u \otimes \Psi(f), F_n(A_L^\dagger(t) - A_L^\dagger(t_n)) v \otimes \Psi(g)\rangle + \\
&\qquad + \langle u \otimes \Psi(f), G_n(A_L(t) - A_L(t_n)) v \otimes \Psi(g)\rangle + \\
&\qquad + \langle u \otimes \Psi(f), (t - t_n) H_n v \otimes \Psi(g)\rangle.
\end{aligned} \tag{4.5}$$

The second term can be written as

$$\begin{aligned}
&\langle u \otimes \Psi(f), F_n(A_L^\dagger(t) - A_L^\dagger(t_n)) v \otimes \Psi(g)\rangle \\
&\quad = \langle (A_L(t) - A_L(t_n)) F_n^\dagger u \otimes \Psi(f), v \otimes \Psi(g)\rangle \\
&\quad = \left\langle \sum_j \int_{t_n}^t f_j L_j^\dagger \otimes I F_n^\dagger u \otimes \Psi(f), v \otimes \Psi(g)\right\rangle \\
&\quad = \left\langle F_n^\dagger u \otimes \Psi(f), \sum_j \int_{t_n}^t \overline{f_j} L_j \otimes I v \otimes \Psi(g)\right\rangle \\
&\quad = \sum_j \int_{t_n}^t \langle F_n^\dagger u \otimes \Psi(f), \overline{f(\tau)} L_j \otimes I v \otimes \Psi(g)\rangle\,\mathrm{d}\tau.
\end{aligned} \tag{4.6}$$

Now for arbitrary $\phi \in \mathfrak{H}_0$, by Schwarz's inequality,

$$\begin{aligned}
&\sum_j \int_{t_n}^t |\langle \phi, \overline{f_j(\tau)} L_j v \otimes \Psi(g)\rangle|\,\mathrm{d}\tau \\
&\quad = \sum_j |\langle \chi_{(t_n, t]}, f_j\rangle|\,|\langle \phi, L_j v \otimes \Psi(g)\rangle|
\end{aligned}$$

$$\leqslant (t - t_n)^{1/2} \sum_j \|f_j\| \, \|L_j v\| \, \|\phi\| \, \|\Psi(g)\|$$

$$\leqslant (t - t_n)^{1/2} \left(\sum_j \|f_j\|^2\right)^{1/2} \left(\sum_j \|L_j v\|^2\right)^{1/2} \|\phi\| \, \|\Psi(g)\| < \infty.$$

Hence, by the dominated convergence theorem, we may reverse the order of summation and integration in (4.6) and write the second term in (4.5) as

$$\int_{t_n}^{t} \sum_j \langle F_n^\dagger u \otimes \Psi(f), \overline{f_j(\tau)} L_j \otimes I v \otimes \Psi(g) \rangle \, \mathrm{d}\tau$$

which is manifestly absolutely continuous as a function of $t$, with generalised derivative

$$\sum_j \langle F_n^\dagger u \otimes \Psi(f), \overline{f(t)} L_j \otimes I v \otimes \Psi(g) \rangle$$
$$= \langle u \otimes \Psi(f), F(t) \sum \overline{f_j(t)} L_j \otimes I v \otimes \Psi(g) \rangle$$

since $F(t) = F_n$ for $t \in (t_n, t_{n+1})$. A similar argument shows that the third term in (4.5) is absolutely continuous as function of $t$ with generalised derivative $\langle u \otimes \Psi(f), G(t) \Sigma_j g_j(t) L_j^\dagger \otimes I v \otimes \Psi(g) \rangle$. Since the first term is constant and the fourth differentiable with derivative $\langle u \otimes \Psi(f), H(t) v \otimes \Psi(g) \rangle$, the proof is complete. □

THEOREM 4.3. *Under the hypotheses of Theorem* 4.2, *if* $0 \leqslant s \leqslant t$, $\phi \in \mathfrak{h}_0 \otimes \mathfrak{H}_s$, $f, g \in \mathfrak{h}$, $v \in \mathfrak{h}_0$,

$$\langle \phi \otimes \Psi(f^s), (M(t) - M(s)) v \otimes \Psi(g) \rangle$$
$$= \int_s^t \left\langle \phi \otimes \Psi(f^s), \left[ F(\tau) \sum_j \overline{f_j(\tau)} L_j \otimes I + G(\tau) \sum_j g_j(\tau) L_j^\dagger \otimes I + H(\tau) \right] v \otimes \Psi(g) \right\rangle \mathrm{d}\tau,$$

$$\langle \phi \otimes \Psi(f^s), (N(t) - N(s)) v \otimes \Psi(g) \rangle$$
$$= \int_s^t \left\langle \phi \otimes \Psi(f^s), \left[ \sum_j \overline{f_j(\tau)} L_j \otimes I F(\tau) + \sum_j g_j(\tau) L_j^\dagger \otimes I G(\tau) + H(\tau) \right] v \otimes \Psi(g) \right\rangle \mathrm{d}\tau.$$

*Proof.* We give the proof for the right integral. Assume first that

$$\phi = u \otimes \Psi(f_s^{(1)}) \tag{4.7}$$

for $u \in \mathfrak{h}_0$, $f^{(1)} \in \mathfrak{h}$. We obtain the theorem in this case by replacing $f$ in (4.3) by $f^{(1)} \chi_{[0,s]} + f \chi_{(s,\infty)}$ and integrating from $s$ to $t$. Since vectors of the form (4.7) are total we obtain the general case by passing to limits of finite linear combinations. □

THEOREM 4.4. *Let* $F, G, H \in \mathscr{A}_0$ *and*

$$M(t) = \int_0^t (F \, \mathrm{d}A_L^\dagger + G \, \mathrm{d}A_L + H \, \mathrm{d}\tau),$$

$$N(t) = \int_0^t (\mathrm{d}A_L^\dagger F + \mathrm{d}A_L G + H\,\mathrm{d}\tau).$$

*Then for arbitrary $u \in \mathfrak{h}_0$ and $f, g \in \mathfrak{h}$ the functions*

$$t \mapsto \langle M(t)u \otimes \Psi(f), M(t)u \otimes \Psi(g)\rangle, \qquad t \mapsto \langle N(t)u \otimes \Psi(f), N(t)u \otimes \Psi(g)\rangle$$

*are absolutely continuous, with generalised derivatives*

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\langle M(t)u \otimes \Psi(f), M(t)u \otimes \Psi(g)\rangle \\
&= \Big\langle M(t)u \otimes \Psi(f), \Big[F(t)\sum_j \overline{f_j(t)}L_j \otimes I + \\
&\quad + G(t)\sum_j g_j(t)L_j^\dagger \otimes I + H(t)\Big]u \otimes \Psi(g)\Big\rangle + \\
&\quad + \Big\langle\Big[F(t)\sum_j \overline{g_j(t)}L_j \otimes I + G(t)\sum_j f_j(t)L_j^\dagger \otimes I + H(t)\Big]u \otimes \Psi(f), \\
&\qquad M(t)u \otimes \Psi(g)\Big\rangle + \sum_j \Big\langle F(t)L_j u \otimes \Psi(f), F(t)L_j u \otimes \Psi(g\Big\rangle,
\end{aligned} \tag{4.8}$$

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\langle N(t)u \otimes \Psi(f), N(t)u \otimes \Psi(g)\rangle \\
&= \Big\langle N(t)u \otimes \Psi(f), \Big[\sum_j \overline{f_j(t)}L_j \otimes IF(t) + \\
&\quad + \sum_j g_j(t)L_j^\dagger \otimes IG(t) + H(t)\Big]u \otimes \Psi(g)\Big\rangle + \\
&\quad + \Big\langle\Big[\sum_j \overline{g_j(t)}L_j \otimes IF(t) + \sum_j f_j(t)L_j^\dagger \otimes IG(t) + J(t)\Big]u \otimes \Psi(f), \\
&\quad N(t)v \otimes \Psi(g)\Big\rangle + \sum_j \langle L_j \otimes IF(t)u \otimes \Psi(f), L_j \otimes IF(t)u \otimes \Psi(g)\rangle.
\end{aligned} \tag{4.9}$$

*Proof.* We prove the case (4.8) of the right integral; (4.9) is similar. We assume, $F$, $G$ and $H$ given by (4.1) and $t \in (t_n, t_{n+1})$. Then

$$\begin{aligned}
&\langle M(t)u \otimes \Psi(f), M(t)u \otimes \Psi(g)\rangle \\
&\quad = \langle [M(t_n) + F_n(A_L^\dagger(t) - A_L^\dagger(t_n)) + G_n(A_L(t) - A_L(t_n)) + \\
&\qquad + (t - t_n)H_n]u \otimes \Psi(f), \\
&\qquad [M(t_n) + F_n(A_L^\dagger(t) - A_L^\dagger(t_n)) + G_n(A_L(t) - A_L(t_n)) + \\
&\qquad + (t - t_n)H_n u \otimes \Psi(g)\rangle.
\end{aligned}$$

We replace $A_L(t) - A_L(t_n)$ by its actions $\Sigma_j \int_{t_n}^t f_j L_j^\dagger \otimes I$ on the left and $\Sigma_j \int_{t_n}^t g_j L_j^\dagger \otimes I$ on the right. Similarly, using the commutativity of $A_L(t) - A_L(t_n)$ with $F_n^\dagger$, $M(t_n)$, $F_n$, $G_n$ and $H_n$, as in proof of Theorem 4.2 we replace $A_L^\dagger(t) - A_L^\dagger(t_n)$ by its adjoint actions $\Sigma_j \int_{t_n}^t \bar{g}_j L_j \otimes I$ on the left and $\Sigma_j \int_{t_n}^t \bar{f}_j L_j \otimes I$ on the right. Since $A_L(t) - A_L(t_n)$ fails to commute with $A_L^\dagger(t) - A_L^\dagger(t_n)$, we use (3.3) to deal with the crossterm

$$\langle F_n(A_L^\dagger(t) - A_L^\dagger(t_n))u \otimes \Psi(f), F_n(A_L^\dagger(t) - A_L^\dagger(t_n))u \otimes \Psi(g)\rangle.$$

We obtain in this way

$$\begin{aligned}
&\langle M(t)u \otimes \Psi(f), M(t)u \otimes \Psi(g)\rangle \\
&= \Big\langle \Big[ M(t_n) + F_n \sum_j \int_{t_n}^t \bar{g}_j L_j \otimes I + \\
&\quad + G_n \sum_j \int_{t_n}^t f_j L_j^\dagger \otimes I + (t - t_n)H_n \Big] u \otimes \Psi(f), \\
&\quad \Big[ M(t_n) + F_n \sum_j \int_{t_n}^t \bar{f}_j L_j \otimes I + \\
&\quad + G_n \sum_j \int_{t_n}^t g_j L_j^\dagger \otimes I + (t - t_n)H_n \Big] u \otimes \Psi(g)\rangle + \\
&\quad + (t - t_n) \sum_j \langle F_n L_j u \otimes \Psi(f), F_n L_j u \otimes \Psi(g)\Big\rangle.
\end{aligned}$$

Taking the summations and integrations out of the inner product and reversing their order, as is possible by the argument used in the proof of Theorem 4.2, we see that $\langle M(t)u \otimes \Psi(f), M(t)u \otimes \Psi(g)\rangle$ is absolutely continuous, with generalised derivative

$$\begin{aligned}
&\Big\langle \Big[ M(t_n) + F_n \sum_j \int_{t_n}^t \bar{g}_j L_j \otimes I + G_n \sum_j \int_{t_n}^t f_j L_j^\dagger \otimes I + (t - t_n)H_n \Big] u \otimes \Psi(f), \\
&\Big[ F_n \sum_j \overline{f_j(t)} L_j \otimes I + G_n \sum_j g_j(t) L_j^\dagger \otimes I + H_n \Big] u \otimes \Psi(g) \Big\rangle + \\
&\quad + \Big\langle \Big[ F_n \sum_j \overline{g_j(t)} L_j \otimes I + G_n \sum_j f_j(t) L_j^\dagger \otimes I + H_n \Big] u \otimes \Psi(f), \\
&\Big[ M(t_n) + F_n \sum_j \int_{t_n}^t \bar{f}_j L_j \otimes I + \\
&\quad + G_n \sum_j \int_{t_n}^t g_j L_j^\dagger \otimes I + (t - t_n)H_n \Big] u \otimes \Psi(g \Big\rangle + \\
&\quad + \sum_j \langle F_n L_j u \otimes \Psi(f), F_n L_j u \otimes \Psi(f)\rangle.
\end{aligned}$$

Once more, extracting summations and integrations from the inner product, reversing their order and applying Theorem 3.4, having observed first that

$$\left[F_n \sum_j \overline{f_j(t)} L_j \otimes I + G_n \sum_j g_j(t) L_j^\dagger \otimes I + H_n\right] u \otimes \Psi(g)$$
$$= \phi_1 \otimes \Psi(g^{t_n})$$
$$\left[F_n \sum_j \overline{g_j(t)} L_j \otimes I + G_n \sum_j f_j t) L_j^\dagger \otimes I + H_n\right] u \otimes \Psi(f)$$
$$= \phi_2 \otimes \Psi(f^{t_n})$$

for $\phi_1, \phi_2 \in \mathfrak{h}_0 \otimes \mathfrak{H}_{t_n}$ and that for $\tau \in (t_n, t)$

$$F_n = F(\tau), \qquad G_n = G(\tau), \qquad H_n = H(\tau),$$

we obtain the result. □

## 5. Stochastic Integrals of Regular Processes

In the following we use the identity

$$\left\|\sum_j z_j L_j^\dagger\right\| = \left\|\sum_j \bar{z}_j L_j\right\| \leqslant \left(\sum_j |z_j|^2\right)^{1/2} \left\|\sum_j L_j^\dagger L_j\right\|^{1/2} \tag{5.1}$$

for scalars $z_j$, $j \in J$ with $\Sigma_j |z_j|^2 < \infty$. This holds because for arbitrary $u \in \mathfrak{h}_0$

$$\left\|\sum_j z_j L_j u\right\|^2$$
$$= \sum_{j,k} z_j \bar{z}_k \langle L_j u, L_k u\rangle$$
$$\leqslant \sum_{j,k} |z_j|\,|z_k|\,\|L_j u\|\,\|L_k u\|$$
$$= \left(\sum_j |z_j|\,\|L_j u\|\right)^2$$
$$\leqslant \sum_j |z_j|^2 \sum_j \|L_j u\|^2$$
$$\leqslant \sum_j |z_j|^2 \left\|\sum_j L_j^\dagger L_j\right\| \|u\|^2.$$

Now let $F, G, H \in \mathscr{A}_0$ and $M(t) = \int_0^t (F\,\mathrm{d}A_L^\dagger + G\,\mathrm{d}A_L + H\,\mathrm{d}\tau)$ so that, according to (4.2),

$$M(t)^\dagger = \int_0^t (\mathrm{d}A_L^\dagger G^\dagger + \mathrm{d}A_L F^\dagger + H^\dagger\,\mathrm{d}\tau)$$

From Theorem 4.4 we have, for arbitrary $u \in \mathfrak{h}_0$, $f \in \mathfrak{h}$,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \|M(t)u \otimes \Psi(f)\|^2 \\
&= 2\,\mathrm{Re}\left\langle M(t)u \otimes \Psi(f), \left[F(t)\sum_j \overline{f_j(t)}L_j \otimes I + \right.\right. \\
&\quad \left.\left. + G(t)\sum_j f_j(t)L_j^\dagger \otimes I + H(t)\right]u \otimes \Psi(f)\right\rangle + \\
&\quad + \left\|\sum_j F(t)L_j u \otimes \Psi(f)\right\|^2 \\
&\leqslant 2\|M(t)u \otimes \Psi(f)\|\left[\|F(t)\|_f\left\|\sum_j \overline{f_j(t)}L_j\right\| + \|G(t)\|_f\left\|\sum_j f_j(t)L_j^\dagger\right\| + \|H(t)\|_f\right] \times \\
&\quad \times \|u\| + \|F(t)\|_f^2\left\|\sum_j L_j^\dagger L_j\right\|\|u\|^2 \\
&\leqslant 2\|M(t)u \otimes \Psi(f)\|\left[\left(\sum_j |f_j(t)|^2\right)^{1/2}\left\|\sum_j L_j^\dagger L_j\right\|^{1/2}(\|F(t)\|_f + \|G(t)\|_f) + \|H(t\right. \\
&\quad \times \|u\| + \|F(t)\|_f^2\left\|\sum_j L_j^\dagger L_j\right\|\|u\|^2,
\end{aligned}$$

where we use (5.1). Using the arithmetic-geometric mean inequality gives

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \|M(t)u \otimes \Psi(f)\|^2 \\
&\leqslant 2\|M(t)u \otimes \Psi(f)\|^2\sum_j |f_j(t)|^2 + \left\|\sum_j L_j^\dagger L_j\right\|(\|F(t)\|_f^2 + \|G(t)\|_f^2)\|u\|^2 + \\
&\quad + \|M(t)u \otimes \Psi(f)\|^2 + \|H(t)\|_f^2\|u\|^2 + \|F(t)\|_f^2\left\|\sum_j L_j^\dagger L_j\right\|\|u\|^2 \\
&= \|M(t)u \otimes \Psi(f)\|^2\left\{2\sum_j |f_j(t)|^2 + 1\right\} + \\
&\quad + \left[\left\|\sum_j L_j^\dagger L_j\right\|(2\|F(t)\|_f^2 + \|G(t)\|_f^2) + \|H(t)\|_f^2\right]\|u\|^2.
\end{aligned}$$

Multiplying by the integrating factor $\exp(-2\|f_t\|^2 - t)$ and integrating the differential inequality, we obtain

$$\begin{aligned}
&\|M(t)u \otimes \Psi(f)\|^2 \\
&\leqslant \int_0^t \exp\{2\|f_t\|^2 - 2\|f_\tau\|^2 + t - \tau\} \times \\
&\quad \times \left[\left\|\sum_j L_j^\dagger L_j\right\|(2\|F(\tau)\|_f^2 + G(\tau)\|_f^2) + \|H(\tau)\|_f^2\right]\|u\|^2\,\mathrm{d}\tau. \qquad (5.2)
\end{aligned}$$

The corresponding estimate for the left integral,

$$\| M^\dagger(t)u \otimes \Psi(f) \|^2$$

$$\leqslant \int_0^t \exp\{2\|f_t\|^2 - 2\|f_\tau\|^2 + t - \tau\} \times$$

$$\times \left[ \left\| \sum_j L_j^\dagger L_j \right\| (\|F^\dagger(\tau)\|_f^2 + 2\|G^\dagger(\tau)\|_f^2) + \|H^\dagger(\tau)\|_f^2 \right] \|u\|^2 \,\mathrm{d}\tau \tag{5.3}$$

is proved similarly.

Now let $F$, $G$, $H$ be regular processes, so that these exist simple processes $F^{(n)}$, $G^{(n)}$, $H^{(n)}$, $n = 1, 2, \ldots$ such that for each $f \in \mathfrak{h}$

$$\|F(\tau) - F^{(n)}(\tau)\|_f,\ \|G(\tau) - G^{(n)}(\tau)\|_f,\ \|H(\tau) - H^{(n)}(\tau)\|_f \xrightarrow[n]{} 0$$

and

$$\|F(\tau)^\dagger - F^{(n)}(\tau)^\dagger\|_f,\ \|G(\tau)^\dagger - G^{(n)}(\tau)^\dagger\|_f\ \|H(\tau)^\dagger - H^{(n)}(\tau)^\dagger\|_f \xrightarrow[n]{} 0$$

uniformly on each finite interval. Then if

$$M^{(n)}(t) = \int_0^t (F^{(n)}\,\mathrm{d}A_L^\dagger + G^{(n)}\,\mathrm{d}A_L + H^{(n)}\,\mathrm{d}\tau),$$

the estimates (5.2) and (5.3) show that, for each $u \in \mathfrak{h}_0$, $f \in \mathfrak{h}$ and $t > 0$, the sequences $M^{(n)}(t)u \otimes \Psi(f)$, $M^{(n)}(t)^\dagger u \otimes \Psi(f)$, $n = 1, 2, \ldots$ converge to limits independent of the choice of approximating simple processes. We define the stochastic integrals

$$M(t) = \int_0^t (F\,\mathrm{d}A_L + G\,\mathrm{d}A_L + H\,\mathrm{d}\tau),$$

$$M^\dagger(t) = \int_0^t (\mathrm{d}A_L^\dagger G^\dagger + \mathrm{d}A_L F^\dagger + H^\dagger\,\mathrm{d}\tau)$$

in the first instance on the domain $\mathfrak{h}_0 \underline{\otimes} \mathscr{E}$ by

$$M(t)u \otimes \Psi(f) = \lim_n M^{(n)}(t) \otimes \Psi(f),$$

$$M^\dagger(t)u \otimes \Psi(f) = \lim_n M^{(n)}(t)^\dagger u \otimes \Psi(f).$$

The operators $M(t)$ and $M^\dagger(t)$ inherit from $M^{(n)}(t)$ and $M^{(n)}(t)^\dagger$ the property of being of form $M_1(t) \otimes I$, $M_1(t)^\dagger \otimes I$ on $(\mathfrak{h}_0 \underline{\otimes} \mathscr{E}_t) \underline{\otimes} \mathscr{E}^t$ and, hence, extend naturally to the domain $(\mathfrak{h}_0 \underline{\otimes} \mathscr{E}_t) \underline{\otimes} \mathfrak{H}^t$. $M = (M(t) : t \geqslant 0)$ is then an adapted process of which $M^\dagger = (M^\dagger(t) : t \geqslant 0)$ is the adjoint process.

The estimates (5.2) and (5.3), together with their generalisations obtained by replacing $F$, $G$, $H$ by $F\chi_{(s,\infty)}$, $G\chi_{(s,\infty)}$, $H\chi_{(s,\infty)}$ respectively, namely

$$\|(M(t) - M(s))u \otimes \Psi(f)\|^2$$

$$\leqslant \int_s^t \exp\{2\|f_t\|^2 - 2\|f_\tau\|^2 + t - \tau\} \times$$
$$\times \left[\left\|\sum_j L_j^\dagger L_j\right\| (2\|F(\tau)\|_f^2 + \|G(\tau)\|_f^2) + \|H(\tau)\|_f^2\right] \|u\|^2 \,\mathrm{d}\tau. \tag{5.4}$$

$$\|(M(t)^\dagger - M(s)^\dagger)u \otimes \Psi(f)\|^2$$
$$\leqslant \int_s^t \exp\{2\|f_t\|^2 - 2\|f_\tau\|^2 + t - \tau\} \times$$
$$\times \left[\left\|\sum_j L_j^\dagger L_j\right\| (\|F^\dagger(\tau)\|_f^2 + 2\|G^\dagger(\tau)\|_f^2) + \|H^\dagger(\tau)\|_f^2\right] \|u\|^2 \,\mathrm{d}\tau, \tag{5.5}$$

persist in the transition to the limit, showing that the processes $M$ and $M^\dagger$ are continuous, hence, regular and, in particular, that the maps $\tau \mapsto M(\tau)u \otimes \Psi(f)$, $\tau \mapsto M(\tau)u \otimes \Psi(f)$ are continuous and, hence, bounded on each finite interval $[0, t]$. From this it follows that we may pass to the limit in the integrated forms of (4.3), (4.4), (4.8) and (4.9) and deduce that Theorems 4.2, 4.3 and 4.4 hold for arbitrary $F$, $G$, $H \in \mathscr{A}_r$. We summarise.

THEOREM 5.1. *Theorems* 4.2, 4.3 *and* 4.4 *hold for arbitrary* $F$, $G$, $H \in \mathscr{A}_r$. *Furthermore, if*

$$M(t) = \int_0^t (F\,\mathrm{d}A_L^\dagger + G\,\mathrm{d}A_L + H\,\mathrm{d}\tau)$$

*then*

$$M^\dagger(t) = \int_0^t (\mathrm{d}A_L^\dagger G^\dagger + \mathrm{d}A_L F^\dagger + H^\dagger\,\mathrm{d}\tau)$$

*and the estimates* (5.2) *and* (5.3) *are satisfied.*

## 6. The Unitary Process

Let $\mathscr{H}$ be a bounded self-adjoint operator in $\mathfrak{h}_0$, fixed once and for all.

THEOREM 6.1. *There exists a unique adapted process* $(U(t) : t \geqslant 0)$ *satisfying*

$$\mathrm{d}U = U\left(\mathrm{d}A_L^\dagger - \mathrm{d}A_L + \left(i\mathscr{H} - \tfrac{1}{2}\sum_j L_j^\dagger L_j\right) \otimes I\,\mathrm{d}t\right), \quad U(0) = I. \tag{6.1}$$

*Proof.* We establish existence by iteration. Thus, define $U_0(t) \equiv I$ and, assuming that the regular process $(U_n(t) : t \geqslant 0)$ has been defined, define

$$U_{n+1}(t) = I + \int_0^t U_n(\tau)\left(\mathrm{d}A_L^\dagger - \mathrm{d}A_L + \left(i\mathscr{H} - \frac{1}{2}\sum_j L_j^\dagger L_j\right) \otimes I\,\mathrm{d}\tau\right). \tag{6.2}$$

The process $U_{n+1}$ is then continuous, hence, regular. We write

$$U_{n+1}(t) - U_n(t) = \int_0^t (U_r(\tau) - U_{n-1}(\tau)) \times \times \left( dA_L^\dagger - dA_L + \left( i\mathscr{H} - \tfrac{1}{2} \sum_j L_j^\dagger L_j \right) \otimes I \, d\tau \right)$$

and use the estimate (5.2) to write, for $u \in \mathfrak{h}_0$, $f \in \mathfrak{h}$, $t > 0$

$$\| U_{n+1}(t) - U_n(t))u \otimes \Psi(f) \|^2$$
$$\leqslant \int_0^t \exp\{2 \|f_t\|^2 - 2\|f_\tau\|^2 + t - \tau\}$$
$$\left[ 3 \left\| \sum_j L_j^\dagger L_j \right\| \| U_n(\tau) - U_{n-1}(\tau) \|_f^2 + \right.$$
$$\left. + \left\| (U_n(\tau) - U_{n-1}(\tau)) \left( i\mathscr{H} - \tfrac{1}{2} \sum_j L_j^\dagger L_j \right) \otimes I \right\|_f \right] \|u\|^2 \, d\tau$$
$$\leqslant C \int_0^t \exp\{2\|F_t\|^2 - 2\|F_\tau\|^2 + t - \tau\} \| U_n(\tau) - U_{n-1}(\tau) \|_f^2 \; \|u\|^2 \, d\tau$$

where

$$C = 3 \left\| \sum_j L_j^\dagger L_j \right\| + \left\| i\mathscr{H} - \tfrac{1}{2} \sum_j L_j^\dagger L_j \right\|.$$

Hence

$$\| U_{n+1}(t) - U_n(t) \|_f^2$$
$$\leqslant C \int_0^t \exp\{2\|f_t\|^2 - 2\|f_\tau\|^2 + t - \tau\} \| U_n(\tau) - U_{n-1}(\tau) \|_f^2 \, d\tau. \tag{6.3}$$

By induction on $n$ we obtain that

$$\| U_n(t) - U_{n-1}(t) \|_f^2 \leqslant (n!)^{-1} C^n t^n \exp(2\|f_t\|^2 + t).$$

From this and from the corresponding argument for the adjoint processes based on (5.3) it is clear that, for $u \in \mathfrak{h}_0$ and $f \in \mathfrak{h}$, the limits

$$\begin{aligned} U(t)u \otimes \Psi(f) &= \lim_n U_n(t)u \otimes \Psi(f), \\ U^\dagger(t)u \otimes \Psi(f) &= \lim_n U_n^\dagger(t)u \otimes \Psi(f), \end{aligned} \tag{6.4}$$

exist and define mutually adjoint adapted processes. Moreover the convergence in (6.4) is uniform for $t$ in bounded intervals, enabling us to take strong limits in (6.2) and conclude that $(U(t) : t \geqslant 0)$ satisfies (6.1).

If $(V(t) : t \geqslant 0)$ is a second adapted process satisfying (6.1), then, from the estimate (5.2) we obtain as above

$$\| U(t) - V(t) \|_f^2 \leqslant C \int_0^t \exp\{2\|f_t\|^2 - 2\|f_\tau\|^2 + t - \tau\} \| U(\tau) - V(\tau) \|_f^2 \, d\tau. \tag{6.5}$$

Since $U - V$ is a stochastic integral, the map $\tau \mapsto (U(\tau) - V(\tau))u \otimes \Psi(f)$ is continuous for each $u \in \mathfrak{h}_0, f \in \mathfrak{h}$, and hence bounded on $[0, t]$. Hence, by the uniform boundedness principle there exists $M > 0$ such that, for all $\tau \in [0, t]$,

$$\| U(\tau) - V(\tau) \|_f^2 \leqslant M.$$

But then by iterating (6.5) we find that $\| U(t) - V(t) \|_f = 0$. This being so for all $f \in \mathfrak{h}$ shows that $U = V$. □

The adjoint process $U^\dagger$ to $U$ satisfies

$$\mathrm{d}U^\dagger = \left( -\mathrm{d}A_L^\dagger + \mathrm{d}A_L - \left( i\mathscr{H} + \tfrac{1}{2} \sum_j L_j^\dagger L_j \right) \otimes I \,\mathrm{d}t \right) U^\dagger$$

in view of Theorem 5.1. We apply (4.9) to write, for arbitrary $u \in \mathfrak{h}_0, f, g \in \mathfrak{h}$,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \langle U^\dagger(t)u \otimes \Psi(f), U^\dagger(t)u \otimes \Psi(g) \rangle \\
&= \Big\langle U^\dagger(t)u \otimes \Psi(f), \Big[ -\sum_j \overline{f_j(t)} L_j \otimes I + \\
&\quad + \sum_j g_j(t) L_j^\dagger \otimes I - \Big( i\mathscr{H} + \tfrac{1}{2} \sum_j L_j^\dagger L_j \Big) \otimes I \Big] U^\dagger(t)u \otimes \Psi(g) \Big\rangle + \\
&\quad + \Big\langle \Big[ -\sum_j \overline{g_j(t)} L_j \otimes I + \sum_j f_j(t) L_j^\dagger \otimes I - \\
&\quad - \Big( i\mathscr{H} + \tfrac{1}{2} \sum_j L_j^\dagger L_j \Big) \otimes I \Big] U^\dagger(t)u \otimes \Psi(f), U^\dagger(t)u \otimes \Psi(g) \Big\rangle + \\
&\quad + \sum_j \langle L_j \otimes I U^\dagger(t)u \otimes \Psi(f), L_j \otimes I U^\dagger(t)u \otimes \Psi(g) \rangle \\
&= 0.
\end{aligned}$$

Since $U^\dagger(0) = I$ we conclude that, for all $t \geqslant 0$,

$$\langle U^\dagger(t)u \otimes \Psi(f), U^\dagger(t)u \otimes \Psi(g) \rangle = \langle u \otimes \Psi(f), u \otimes \Psi(g) \rangle.$$

By polarisation we obtain that

$$\langle U^\dagger(t)u \otimes \Psi(f), U^\dagger(t)v \otimes \Psi(g) \rangle = \langle u \otimes \Psi(f), v \otimes \Psi(g) \rangle$$

for arbitrary $u, v \in \mathfrak{h}_0, f, g \in \mathfrak{h}$. Thus $U^\dagger(t)$ is isometric.

THEOREM 6.2. *Each $U(t)$, $t \geqslant 0$, is unitary.*

*Proof.* Since $U(t)$ is the adjoint of an isometry it is bounded. To prove it is unitary we need only prove that its action on a total family of vectors is isometric. For these we choose the vectors $u \otimes \Psi(f)$ where $u \in \mathfrak{h}_0$ is arbitrary and $f = (f_j)$ has only finitely many nonzero components, each of which is piecewise constant. For such vectors $u \otimes \Psi(f)$ and $v \otimes \Psi(g)$ by (4.8)

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle U(t)u \otimes \Psi(f), U(t)v \otimes \Psi(g)\rangle$$

$$= \Big\langle U(t)u \otimes \Psi(f), U(t)\Big[\sum_j \overline{f_j(t)}L_j \otimes I -$$

$$- \sum_j g_j(t)L_j^\dagger \otimes I + \Big(i\mathscr{H} - \tfrac{1}{2}\sum_j L_j^\dagger L_j\Big) \otimes I\Big] v \otimes \Psi(g)\Big\rangle +$$

$$+ \Big\langle U(t)\Big[\sum_j \overline{g_j(t)}L_j \otimes I - \sum_j f_j(t)L_j^\dagger \otimes I +$$

$$+ \Big(i\mathscr{H} - \tfrac{1}{2}\sum_j L_j^\dagger L_j\Big) \otimes I\Big] u \otimes \Psi(f), U(t)v \otimes \Psi(g)\Big\rangle +$$

$$+ \sum_j \langle U(t)L_j u \otimes \Psi(f), U(t)L_j v \otimes \Psi(g)\rangle.$$

It follows that the bounded operator $K_{f,g}(t)$ on $\mathfrak{h}_0$ defined by

$$\langle u, K_{f,g}(t)v\rangle = \langle U(t)u \otimes \Psi(f), U(t)v \otimes \Psi(g)\rangle \ (u, v \in \mathfrak{h}_0)$$

satisfies the weak sense differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t} K_{f,g}(t) = \Big[K_{f,g}, \sum_j \overline{f_j(t)}L_j - \sum_j g_j(t)L_j^\dagger + i\mathscr{H}\Big] -$$

$$- \tfrac{1}{2}\sum_j (L_j^\dagger L_j K_{f,g} - 2L_j^\dagger K_{f,g}L + K_{f,g}L_j^\dagger L_j)$$

with initial condition

$$K_{f,g}(0) = \langle \Psi(f), \Psi(g)\rangle I.$$

Since $K_{f,g} \equiv \langle \Psi(f), \Psi(g)\rangle I$ satisfies this equation we may appeal to the uniqueness theorem for the differential equation

$$\frac{\mathrm{d}K}{\mathrm{d}t} = \mathscr{L}K$$

in the Banach space $B(\mathfrak{h}_0)$, $\mathscr{L}$ being a bounded operator in $B(\mathfrak{h}_0)$, within each interval of constancy of the functions $f_j$ and $g_j$, to conclude that $K_{f,g}(t)$ is indeed equal to $\langle \Psi(f), \Psi(g)\rangle I$ for all $t$. Hence, $U(t)$ is isometric as required. □

THEOREM 6.3. *Let $s \geqslant 0$ and let $T$ be a bounded operator of form $I \otimes T_1 \otimes I$ where $T_1 \in B(\mathfrak{h}_s)$. Then for $t \geqslant s$, $U^\dagger(s)U(t)$ commutes with $T$.*

*Proof.* Define processes $J$ and $K$ by

$$J(t) = \begin{cases} 0 & \text{if } t \leqslant s \\ U^\dagger(s)U(t) & \text{if } t > s. \end{cases} \qquad K(t) = [T, J(t)]$$

These clearly inherit adaptedness and regularity from $U$. Subtracting the corresponding equation with $t$ replaced by $s$ from

$$U(t) = I + \int_0^t U(\tau)\left(\mathrm{d}A_L^\dagger - \mathrm{d}A_L + \left(i\mathscr{H} - \tfrac{1}{2}\sum_j L_j^\dagger L_j\right)\otimes I\,\mathrm{d}\tau\right)$$

and multiplying by $U^\dagger(s)$ gives

$$J(t) = I\chi_{[s,\infty)}(t) + \int_0^t J(\tau)\left(\mathrm{d}A_L^\dagger - \mathrm{d}A_L + \left(i\mathscr{H} - \tfrac{1}{2}\sum_j L_j^\dagger L_j\right)\otimes I\,\mathrm{d}\tau\right).$$

Using Theorem 4.3 we deduce from this that, for $t > s$, $u$, $v \in \mathfrak{h}_0$, $f, g \in \mathfrak{h}$,

$$\begin{aligned}
&\langle u\otimes\Psi(f), K(t)v\otimes\Psi(g)\rangle \\
&= \langle T^\dagger u\otimes\Psi(f), J(t)v\otimes\Psi(g)\rangle - \langle J^\dagger(t)u\otimes\Psi(f), Tu\otimes\Psi(f)\rangle \\
&= \int_s^t \Big\langle T^\dagger u\otimes\Psi(f), J(\tau)\Big(\sum_j \overline{f_j(\tau)}L_j\otimes I - \\
&\quad - \sum_j g_j(\tau)L_j^\dagger\otimes I + \Big(i\mathscr{H} - \tfrac{1}{2}\sum_j L_j^\dagger L_j\Big)\otimes I\Big)v\otimes\Psi(g)\Big\rangle\,\mathrm{d}\tau - \\
&\quad - \int_s^t \Big\langle u\otimes\Psi(f), J(\tau)\Big(\sum_j \overline{f_j(\tau)}L_j\otimes I - \\
&\quad - \sum_j g_j(\tau)L_j^\dagger\otimes I + \Big(i\mathscr{H} - \tfrac{1}{2}\sum_j L_j^\dagger L_j\Big)\otimes I\Big)Tv\otimes\Psi(g)\Big\rangle\,\mathrm{d}\tau \\
&= \int_0^t \Big\langle u\otimes\Psi(f), K(\tau)\Big(\sum_j \overline{f_j(\tau)}L_j\otimes I - \\
&\quad - \sum_j g_j(\tau)L_j^\dagger\otimes I + \Big(i\mathscr{H} - \tfrac{1}{2}\sum_j L_j^\dagger L_j\Big)\otimes I\Big)v\otimes\Psi(g)\Big\rangle\,\mathrm{d}\tau.
\end{aligned}$$

Since this holds trivially for $t < s$, we conclude that $K$ satisfies

$$\mathrm{d}K = K\left(\mathrm{d}A_L^\dagger - \mathrm{d}A_L + \left(i\mathscr{H} - \tfrac{1}{2}\sum_j L_j^\dagger L_j\right)\otimes I\,\mathrm{d}t\right), \quad K(0) = 0.$$

But then adding $K$ to the solution $U$ of (6.1) would yield a different solution $U + K$, contradicting uniqueness unless $K \equiv 0$. □

## 7. The Reduced Semigroup

Let $S$ be a contraction on $\mathfrak{h}$. There is a contraction $\Gamma(S)$ on $\mathfrak{H}$ called the second quantisation [3] of $S$ whose action on exponential vectors is $\Gamma(S)\Psi(f) = \Psi(Sf)$.

We denote by $\gamma(S)$ the operator $I \otimes \Gamma(S)$ in $\mathfrak{H}_0$. From corresponding properties of second quantisations [3] we have

$$\gamma(S_1 S_2) = \gamma(S_1)\gamma(S_2), \qquad \gamma(S^\dagger) = \gamma(S)^\dagger, \quad \gamma(I) = I \tag{7.1}$$

for arbitrary contractions $S_1$, $S_2$, $S$ on $\mathfrak{h}$. Also if $S = S_1 \otimes S_2$ is the direct sum of contractions $S_1$ and $S_2$, $\Gamma(S) = \Gamma(S_1) \otimes \Gamma(S_2)$.

We denote by $S_t$, $t \geqslant 0$, the shift in $\mathfrak{h}$

$$S_t f_j(\tau) = \begin{cases} 0 & \text{if } \tau < t \\ f_j(\tau - t) & \text{if } \tau \geqslant t. \end{cases}$$

$S_t$ is isometric and $S_t S_t^\dagger$ is the projector $E^t$ onto $\mathfrak{h}^t$.

THEOREM 7.1. *For arbitrary* $s$, $t \geqslant 0$

$$U(t) = \gamma(S_s)^\dagger\, U(s)^\dagger\, U(s+t)\gamma(S_s).$$

*Proof.* Fix $s \geqslant 0$ and consider the family of bounded operators

$$V(t) = \gamma(S_s)^\dagger\, U(s)^\dagger\, U(s+t)\gamma(S_s), \quad t \geqslant 0.$$

We prove that this is an adapted process, that is each $V(t)$ is of the form $V(t) = V_1(t) \otimes I$ on $(\mathfrak{h}_0 \otimes \mathfrak{H}_t) \otimes \mathfrak{H}^t$. To do this write $S_s$ as the direct sum $S_s = S_1 \oplus S_2$ of its restrictions $S_1 \colon \mathfrak{h}_t \to \mathfrak{h}_{s+t}$, $S_2 \colon \mathfrak{h}^t \to \mathfrak{h}^{s+t}$. Correspondingly

$$\Gamma(S_s) = \Gamma(S_1) \otimes \Gamma(S_2), \qquad \Gamma(S_s)^\dagger = \Gamma(S_1)^\dagger \otimes \Gamma(S_2)^\dagger,$$

where $\Gamma(S_1)$ maps $\mathfrak{H}_t$ to $\mathfrak{H}_{s+t}$ and $\Gamma(S_2)$ $\mathfrak{H}^t$ to $\mathfrak{H}^{s+t}$. Because $U$ is adapted we can write $U^\dagger(s)U(s+t) = U_1 \otimes I$ for some operator $U_1$ on $\mathfrak{h}_0 \otimes \mathfrak{H}_{s+t}$. Thus,

$$\begin{aligned} V(t) &= \gamma(S_s)^\dagger\, U(s)^\dagger\, U(s+t)\gamma(S_s) \\ &= ((I \otimes \Gamma(S_1)^\dagger) \otimes \Gamma(S_2)^\dagger) U_1 \otimes I((I \otimes \Gamma(S_1)) \otimes \Gamma(S_2)) \\ &= (I \otimes \Gamma(S_1)^\dagger) U_1 (I \otimes \Gamma(S_1)) \otimes \Gamma(S_2)^\dagger\, \Gamma(S_2) \\ &= V_1 \otimes I, \end{aligned}$$

where $V_1 = I \otimes \Gamma(S_1)^\dagger\, U_1 I \otimes \Gamma(S_1)$ is a bounded operator on $\mathfrak{h}_0 \otimes \mathfrak{H}_t$, as required.

The adapted process $V$ inherits regularity from $U$. Let us show that it satisfies the stochastic differential equation (6.1); by the uniqueness of the solution we shall then be able to conclude that $U = V$. Applying Theorem 4.3, in which we take $s = 0$, $\phi = u$ and

$$\mathrm{d}M = V\left(\mathrm{d}A_L^\dagger - \mathrm{d}A_L + \left(i\mathscr{H} - \tfrac{1}{2}\sum_j L_j L_j\right) \otimes I\, \mathrm{d}t\right),$$

for arbitrary $t > 0$, $u$, $v \in \mathfrak{h}_0$, $f$, $g \in \mathfrak{h}$,

$$\left\langle u \otimes \Psi(f), \int_0^t V(\tau)\left(\mathrm{d}A_L^\dagger - \mathrm{d}A_L + \left(i\mathscr{H} - \tfrac{1}{2}\sum_j L_j^\dagger L_j\right) \otimes I\, \mathrm{d}\tau\right) v \otimes \Psi(g)\right\rangle$$

$$= \int_0^t \Big\langle u \otimes \Psi(f), \gamma(S_s^\dagger) U(s)^\dagger U(\tau + s) \gamma(S_s) \times$$

$$\times \Big[ \sum_j \overline{f_j(\tau)} L_j - \sum_j g_j(\tau) L_j^\dagger + i\mathscr{H} - \tfrac{1}{2} \sum_j L_j^\dagger L_j \Big] \otimes I v \otimes \Psi(g) \Big\rangle \,\mathrm{d}\tau$$

$$= \int_0^t \Big\langle U(s) u \otimes \Psi(S_s f), U(\tau + s) \times$$

$$\times \Big[ \sum_j \overline{f_j(\tau)} L_j - \sum_j g_j(\tau) L_j^\dagger + i\mathscr{H} - \tfrac{1}{2} \sum_j L_j^\dagger L_j \Big] \otimes I v \otimes \Psi(S_s g) \Big\rangle \,\mathrm{d}\tau$$

$$= \int_0^t \Big\langle U(s) u \otimes \Psi(S_s f), U(\tau + s) \times$$

$$\times \Big[ \sum_j \overline{S_s f_j(\tau + s)} L_j - \sum_j S_s g_j(\tau + s) L_j^\dagger + i\mathscr{H}_0 - \tfrac{1}{2} \sum_j L_j^\dagger L_j \Big] \otimes I v \otimes \Psi(S_s g) \Big\rangle \,\mathrm{d}\cdot$$

$$= \int_s^{s+t} \Big\langle U(s) u \otimes \Psi(S_s f), U(\tau) \times$$

$$\times \Big[ \sum_j \overline{S_s f_j(\tau)} L_j - \sum_j S_s g_j(\tau) L_j^\dagger + i\mathscr{H}_0 - \tfrac{1}{2} \sum_j L_j^\dagger L_j \Big] \otimes I v \otimes \Psi(S_s g) \Big\rangle \,\mathrm{d}\tau$$

$$= \langle U(s) u \otimes \Psi(S_s f), (U(s + t) - U(s)) v \otimes \Psi(S_s g) \rangle,$$

where we use the adaptedness of $U$ to write $U(s)u \otimes \Psi(S_s f)$ in the form $\phi \otimes \Psi((S_s f)_s)$ for $\phi \in \mathfrak{h}_0 \otimes \mathfrak{H}_s$, so that Theorem 4.3 is applicable again, and (6.1),

$$= \langle u \otimes \Psi(S_s f), (U^\dagger(s) U(s + t) - I) v \otimes \Psi(S_s g) \rangle$$

$$= \langle u \otimes \Psi(f), (\gamma(S_s)^\dagger U^\dagger(s) U(s + t) \gamma(S_s) - I) v \otimes \Psi(g) \rangle$$

$$= \langle u \otimes \Psi(f), (V(t) - I) v \otimes \Psi(g) \rangle,$$

where we use the isometry of $S_s$ and (7.1) to write

$$\gamma(S_s)^\dagger I \gamma(S_s) = \gamma(S_s^\dagger S_s) = \gamma(I) = I.$$

It follows that $V$ satisfies (6.1) and the proof is complete. □

For each $t \geqslant 0$ we define a conditional expectation map $\mathbb{E}_t$ from $B(\mathfrak{H}_0)$ onto $B(\mathfrak{h}_0 \otimes \mathfrak{H}_t)$ ($= B(\mathfrak{h}_0)$ when $t = 0$) as follows. For $T \in B(\mathfrak{H}_0)$, $\mathbb{E}_t(T)$ is the unique bounded operator on $\mathfrak{h}_0 \otimes \mathfrak{H}_t$ such that, for arbitrary $\phi_1, \phi_2 \in \mathfrak{h}_0 \otimes \mathfrak{H}_t$

$$\langle \phi_1, \mathbb{E}_t(T) \phi_2 \rangle = \langle \phi_1 \otimes \Psi_0^t, T \phi_2 \otimes \Psi_0^t \rangle$$

where $\Psi_0^t$ is the vacuum in $\mathfrak{H}^t$. We write $\mathbb{E}^t(T) = \mathbb{E}_t(T) \otimes I$, where $I$ is the identity in $\mathfrak{H}^t$. The maps $\mathbb{E}^t$ have the easily verified properties of conditional expectations

$$\text{(a)} \quad \mathbb{E}^s \mathbb{E}^t = \mathbb{E}^s \quad \text{for } 0 \leqslant s \leqslant t, \qquad (7.2)$$

$$\text{(b)} \quad \mathbb{E}^t(S_1 T S_2) = S_1 \mathbb{E}^t(T) S_2$$

if $S_1$ and $S_2$ are both of form $S \otimes I$ for $S \in B(\mathfrak{h}_0 \otimes \mathfrak{H}_s)$

(c) $\mathbb{E}^t(I) = I$.

We also note

(d) If for $s \geqslant 0$, $T$ commutes with all operators of form $I \otimes S \otimes I$ on $\mathfrak{h}_0 \otimes \mathfrak{H}_s \otimes \mathfrak{H}^s$, then $\mathbb{E}^s(T) = \mathbb{E}^0(T)$.

To prove (d) observe that under the given hypothesis, for $S \in B(\mathfrak{H}_s)$,

$$
\begin{aligned}
(\mathbb{E}_s(T) I \otimes S) \otimes I &= \mathbb{E}^s(T)\,(I \otimes S \otimes I) \\
&= \mathbb{E}^s(TI \otimes S \otimes I) \quad \text{by (b)} \\
&= \mathbb{E}^s(I \otimes S \otimes IT) \\
&= (I \otimes S\mathbb{E}_s(T)) \otimes I
\end{aligned}
$$

reversing the previous steps, hence, $\mathbb{E}_s(T)$ commutes with $I \otimes S$ for all $S \in B(\mathfrak{H}_s)$. But then $\mathbb{E}_s(T)$ is necessarily of form $S_1 \otimes I$ with $S_1 \in B(\mathfrak{h}_0)$, and so by (a), (b) and (c)

$$\mathbb{E}^0(T) = \mathbb{E}^0\mathbb{E}^s(T) = \mathbb{E}^0(S_1 \otimes I) = S_1 \otimes I\mathbb{E}^0(I) = S_1 \otimes I = \mathbb{E}^s(T).$$

Finally we note that, since the second quantisations $\Gamma(T_j)$ map the vacuum to itself,

(e) $\mathbb{E}^0(\gamma(T_1)T\gamma(T_2)) = \mathbb{E}^0(T)$ for arbitrary contractions $T_1$, $T_2$ on $\mathfrak{h}$.

We are ready for our main Theorem.

THEOREM 7.3. *For $t \geqslant 0$ define $\mathcal{T}_t$: $B(\mathfrak{h}_0) \to B(\mathfrak{h}_0)$ by*

$$\mathcal{T}_t(X) = \mathbb{E}_0[U(t)X \otimes IU(t)^\dagger], \quad X \in B(\mathfrak{h}_0).$$

*Then $(\mathcal{T}_t : t \geqslant 0)$ is a uniformly continuous one-parameter semigroup of completely positive maps, whose infinitesimal generator*

$$\mathcal{L} = \left.\frac{\mathrm{d}\mathcal{T}_t}{\mathrm{d}t}\right|_{t=0}$$

*is given by*

$$\mathcal{L}(X) = i[\mathcal{H}, X] - \tfrac{1}{2} \sum_j (L_j^\dagger L_j X - 2L_j^\dagger X L_j + XL_j^\dagger L_j). \tag{7.3}$$

*Proof.* Being the product of a conditional expectation with a unitary conjugation, both of which are necessarily completely positive, $\mathcal{T}_t$ is also completely positive.

To prove the semigroup property, for $s, t \geqslant 0$ and $X \in B(\mathfrak{h}_0)$ write

$$
\begin{aligned}
\mathcal{T}_{s+t}(X) \otimes I &= \mathbb{E}^0[U(s+t)X \otimes IU(s+t)^\dagger] \\
&= \mathbb{E}^0\mathbb{E}^s[U(s)U(s)^\dagger\, U(s+t)X \otimes IU(s+t)^\dagger\, U(s)U(s)^\dagger] \\
&= \mathbb{E}^0[U(s)\mathbb{E}^s\{U(s)^\dagger\, U(s+t)X \otimes IU(s+t)^\dagger\, U(s)\}\, U(s)^\dagger] \\
&= \mathbb{E}^0[U(s)\mathbb{E}^0\{U(s)^\dagger\, U(s+t)X \otimes IU(s+t)^\dagger\, U(s)\}\, U(s)^\dagger]
\end{aligned} \tag{7.4}
$$

using properties (a), (b) and (d) of conditional expectations, respectively, the use of (d) being justified by Theorem 6.3. On the other hand, by Theorem 7.1 and (e),

$$\begin{aligned}\mathcal{T}_t(X)\otimes I &= \mathbb{E}^0[U(t)X\otimes IU(t)^\dagger]\\ &= \mathbb{E}^0[\gamma(S_s)^\dagger\, U(s)^\dagger\, U(s+t)\gamma(S_s)X\otimes I\gamma(S_s)^\dagger\, U^\dagger(s+t)U(s)\gamma(S_s)]\\ &= \mathbb{E}^0[U(s)^\dagger\, U(s+t)\gamma(S_s)\gamma(S_s^\dagger)X\otimes IU^\dagger(s+t)U(s)]\\ &= \mathbb{E}^0[U(s)^\dagger\, U(s+t)\gamma(E^s)X\otimes IU^\dagger(s+t)U(s)].\end{aligned}$$

Now by Theorem 6.3, since $\gamma(E^t) = I\otimes\Gamma(0)\otimes I$ in $\mathfrak{h}_0\otimes\mathfrak{H}_s\otimes\mathfrak{H}^s$, $\gamma(E^t)$ commutes with $U(s)^\dagger\, U(s+t)$.

Hence, using (e) again,

$$\begin{aligned}\mathcal{T}_t(X)\otimes I &= \mathbb{E}^0[\gamma(E^s)U(s)^\dagger\, U(s+t)X\otimes IU^\dagger(s+t)U(s)]\\ &= \mathbb{E}^0[U(s)^\dagger\, U(s+t)X\otimes IU^\dagger(s+t)U(s)].\end{aligned}$$

Substituting in (7.4) we obtain

$$\begin{aligned}\mathcal{T}_{s+t}(X)\otimes I &= \mathbb{E}^0[U(s)(\mathcal{T}_t(X)\otimes I)U(s)^\dagger]\\ &= \mathcal{T}_s[\mathcal{T}_t(X)]\otimes I\end{aligned}$$

and so $\mathcal{T}_{s+t} = \mathcal{T}_s\mathcal{T}_t$.

To complete the proof use the polarised form of (4.9), in which we set $f = g = 0$, to write

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\langle u, \mathcal{T}_t(X)v\rangle\\ &= \frac{\mathrm{d}}{\mathrm{d}t}\langle U^\dagger(t)u\otimes\Psi_0, (X\otimes I)U^\dagger(t)v\otimes\Psi_0\rangle\\ &= \left\langle U^\dagger(t)u\otimes\Psi_0, -X\left(i\mathcal{H}+\tfrac{1}{2}\sum_j L_j^\dagger L_j\right)\otimes IU^\dagger(t)v\otimes\Psi_0\right\rangle +\\ &\quad+\left\langle -\left(i\mathcal{H}+\tfrac{1}{2}\sum_j L_j^\dagger L_j\right)\otimes IU^\dagger(t)u\otimes\Psi_0, U^\dagger(t)v\otimes\Psi_0\right\rangle +\\ &\quad+\sum_j\langle L_j\otimes IU^\dagger(t)u\otimes\Psi_0, XL_j\otimes IU^\dagger(t)v\otimes\Psi_0\rangle\\ &= \langle u, \mathcal{T}_t\mathcal{L}(X)v\rangle,\end{aligned}$$

where $\mathcal{L}$ is given by (7.3). From this it is clear that $\mathcal{T}_t$ is uniformly continuous and has infinitesimal generator $\mathcal{L}$. □

## References

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