oundations of Physics, Vol. 26, No. 9, 1996

The Theoretical Apparatus of Semantic Realism: A New Language for Classical and Quantum Physics

Claudio Garola[1] **and Luigi Solombrino**[1]

Received March 17, 1995; revised February 23, 1996

The standard interpretation of quantum physics (QP) and some recent generalizations of this theory rest on the adoption of a verificationist theory of truth and meaning, while most proposals for modifying and interpreting QP in a "realistic" way attribute an ontological status to theoretical physical entities (ontological realism). *Both terms of this dichotomy are criticizable, and many quantum paradoxes can be attributed to it. We discuss a new viewpoint in this paper* (semantic realism, *or briefly SR*), *which applies both to classical physics (CP) and to QP, and is characterized by the attempt of giving up verificationism without adopting ontological realism. As a first step, we construct a formalized* observative *language L endowed with a correspondence truth theory. Then, we state a set of axioms by means of L which hold both in CP and in QP, and construct a further language L_c which can express both* testable *and* theoretical *properties of a given physical system. The concepts of* meaning *and* testability *do not collapse in L and L_c, hence we can distinguish between semantic and pragmatic compatibility of physical properties and define the concepts of testability and conjoint testability of statements of L and L_c. In this context a new metatheoretical principle (MGP) is stated, which limits the validity of empirical physical laws. By applying SR (in particular, MGP) to QP, one can interpret quantum logic as a theory of testability in QP, show that QP is semantically incomplete, and invalidate the widespread claim that contextuality is unavoidable in QP. Furthermore, SR introduces some changes in the conventional interpretation of ideal measurements and Heisenberg's uncertainty principle.*

1. INTRODUCTION

We propound a general scheme for physical theories in the present paper (*semantic realism*, or, briefly, SR) that produces, in particular, a new interpretation of quantum physics (QP). Our proposal has already been

[1] Dipartimento di Fisica dell'Università and INFN, Via Arnesano, 73100 Lecce, Italy.

25 26 9-1

partially discussed and applied by one of us in various papers[1–9]; here, we provide a unified and generalized treatment of it that allows us to show that a SR approach to QP can solve a number of conceptual problems and cope successfully with some old and more recent quantum paradoxes.[2]

Our *incipit* above could have already annoyed the reader concerned with the research on the foundations of QP. Indeed, the interpretations of this theory have multiplied beyond any reasonable limit in the last decades, and one could look with suspect at the birth of a new interpretation. Thus, we feel the need of justifying thoroughly our boldness, and we devote most of this Introduction to this aim.

We retain that the proliferation of interpretations and approaches to QP has a deep root: the unsolved problem of the epistemological conception that must support QP. Indeed, notwithstanding the efforts of the fathers of QP and of a great number of following researchers, a universal agreement on this subject has not yet been attained, and it is apparent that different epistemological conceptions lead to different interpretations of physical theories. If one then wonders about the reasons of this lack of agreement, one can find them mainly in the aforesaid quantum paradoxes (in particular, in the *EPR-like paradoxes* that appear in quantum physics whenever the thought experiment invented in 1935 by Einstein, Podolsky, and Rosen[10] is analyzed), which show that there are predictions of the theory that conflict with some intuitive conditions, as the requirement of *locality* of physical theories or the expectation that in every theory one can talk about the properties of a physical system in a given state independently of the observations that one intends to perform on the system. Of course, one can ignore this problem and give up all requirements that conflict with his favorite interpretation of QP; but this procedure opens the way to a number of serious philosophical problems on the nature itself of physics and science.

The contrasting interpretations of QP can be summarily grouped in two classes. From one side, we find the attempts of providing a "realistic" interpretation, or modification, of QP (for instance, by introducing hidden variables), most of which attribute an ontological status to physical entities that appear in the theory (as *wave function*, *electron*, *quark*, etc.), especially whenever they can be represented by some intuitive model: we call this

[2] The word "paradox" is often used rather loosely in the literature. We denote by this term here a result in the theory which is counterintuitive or, more rigorously, which contradicts some epistemological requirements regarding the theory; a paradox must then be distinguished from an "antinomy," which is an internal contradiction of the theory.[2]

tendency *ontological realism* here. From the other side, we find the interpretations of QP that refuse every kind of "metaphysical" assumptions (like the Copenhagen, or *standard*, interpretation), most of which adopt more or less explicitly a verificationist theory of truth and meaning:[3] this entails that no meaning can be attributed to physical statements that cannot be tested, even if they are syntactically correct, and that the truth value of a statement may change when changing the context in which the verification occurs. Now, the premises on which ontological realism is based are epistemologically untenable: indeed, this position underestimates the fundamental distinction between *theoretical* and *observative* terms in a theory, attributing "reality" to entities that are refined constructions of our mind, which usually change with the evolution of research and are different even in different but empirically equivalent contexts.[15] On the other side, verificationism collapses the *pragmatic* notion of *epistemic accessibility* (here, briefly, *testability*) with the *semantic* notion of *truth*. This collapse can be seen as harmless, or even desirable, from a physicists viewpoint. Yet, it can be severely criticized,[17-20] for example by noticing that the act of verification tests the truth of the statement which is being verified, so that the concept of truth is presupposed by the concept of verification (it follows that the metalinguistic concepts of truth and epistemic accessibility do not coincide, but the latter presupposes the former). More important, the identification of testability and truth constitutes the deep root of a number of difficulties and paradoxes in QP. In particular, it engenders the problem of *nonobjectivity* of quantum measurements.[21] Furthermore, it may lead one to ignore (or to deny explicitly) the basic difference between logical laws

[3] In some of his papers Bohr seems to accept explicitly a verificationist position.[11, 12] Other authors accept verificationism together with some forms of actuality for the microobjects, as Heisenberg,[13] who asserts that a transition from the possible to the actual occurs during the interaction of the physical object with the measuring device; the distinction between possibility and actuality is crucial in the modal interpretation forwarded by Van Fraassen[14]. In any case, many authors consider "metaphysical" the introduction of theoretical terms denoting theoretical entities in the language of physics. This may lead to radical consequences, as the reduction of microphysics to a theory of measurement, since the physical properties of a microscopical (hence theoretical) physical system are not retained to be inherent to the object itself, which should be ruled out when rigorously speaking, but rather to the measurement process. Such a viewpoint limits in our opinion the explanatory power and the fruitfulness of physical theories. It is therefore important to note that, according to some well-known analysis of scientific theories,[15, 16] theoretical terms necessarily appear in the language of any physical theory which is not purely phenomenological and their introduction does not imply in principle any ontological engagement on the "actual existence" of the corresponding theoretical entities, so that it cannot be charged with being "metaphysical."

(that should hold independently of a specific interpretation, rule our inference processes, and provide *a priori* rationality criteria for physical theories) and physical laws (the validity of which depends on the interpretation); this occurs, for instance, in some quantum logical approaches to QP,[3, 4] as witnessed by the paper itself that started this kind of approaches.[22]

Because of the above arguments, we retain that a way out of quantum paradoxes cannot be found by means of elaborate physical models or involved "physical" reasonings, at least as long as these are conceived within the epistemological frameworks that we have just criticized. Thus, one must decidedly afford the task of working out a new epistemological position, which can be applied to QP and allow him to escape the dichotomy between ontological realism and verificationism. But this should be done preserving some fundamental features of both these positions, which one cannot give up without encountering serious difficulties. In particular, it is important to maintain the distinction between logical and physical laws, which is inherent in the realistic approaches, but also to preserve some operational features that characterize verificationism. We offer SR in this paper as a general scheme attempting to fulfill all these contrasting demands.

Basically (see Sec. 2), SR consists in the construction of a formal language L for physical theories which formalizes an *observative* language of these (to be distinguished from the *general* language, the formalization of which would be long and difficult and is not needed for our purposes, see Remark 2.3). L is a first-order predicate calculus with monadic predicates only, and the leaving of verificationism in favor of a more "realistic" viewpoint is realized by choosing a correspondence theory of truth for L: to be precise, we endow L with a family of Tarskian interpretations,[4] and the labels of the family are interpreted on *laboratories*, i.e., space-time domains in the actual world (these take the place of the *possible worlds* that appear in standard Kripkean semantics). This implies that all (*atomic* or *complex*) statements of L have, when interpreted, a truth value, and that the logic of L is classical: hence, a SR approach to QP is

[4] We remind that the Tarskian theory of truth is modeled on an abstract set theory and is not involved with ontological assumptions on the elements, or subsets, that appear in its models, so that it can be considered *ontologically* neutral.[23] Intuitively, one can say that recognizing that a sentence is true in a given semantic context does not require the acceptance of some ontologically existing underlying reality which is faithfully described by the sentence itself. On the other side, Tarski's theory is not *semantically* neutral,[24] since it admits a notion of truth (and falsity) which goes beyond verifiability (and falsifiability), and it is obviously compatible with a realistic attitude. Thus, the choice of this theory of truth meets our demands above and justifies the name that we have chosen for our approach.

essentially different from the approaches which assume that QP stands on the adoption of a nonstandard logic,[25] as modern quantum logic (QL) or historical Reichenbach's[26] three-valued logic. But a number of operational requirements are maintained in SR: for instance, we assume that all primitive predicates of L can be interpreted either on classes of physically equivalent preparations of a physical system (*states*), or on classes of physically equivalent dichotomic registering devices (*effects*), and that the set of effects contains a subset of idealized effects (*exact effects*) which can be identified with the set of all *testable physical properties* of the system. We can thus classify as *testable* all atomic statements in L, in the sense that their truth values either are known or can be tested, but not all complex statements, since a test of the truth value of these is generally possible only if all properties that appear in them are *conjointly* testable; the concepts of testability and truth are thus distinguished in SR. Moreover, no ontological status is attributed to the theoretical entities that appear in a physical theory (which can be defined by means of suitable subsets of characterizing properties[11]).

As a consequence of the choice of a correspondence theory of truth, all testable properties of a *physical object* (i.e., an individual sample of a given physical system obtained by means of an act of preparation and possibly identified with it) can be attributed or not to the object independently of the observations that one may decide to perform on the system (we loosely say that a *realism of properties* is substituted to the *realism of entities* that characterizes ontological realism). As a consequence of our operational requirements, SR is a purely semantic form of realism, since it is *ontologically neutral*: this means that it is compatible with different philosophical positions (in particular, with instrumentalism or strict realism), hence with different ontologies, so that we do not need to make a choice among these in order to work out our general scheme.

It is apparent from our above description that the fundamental assumptions of SR are not consistent with the standard interpretation of QP, which is based, as we have seen, on the adoption of a verificationist truth theory according to which a statement attributing two or more noncompatible properties to a physical object is meaningless, hence has no truth value. Therefore, the consistency of SR with the mathematical apparatus and the observative content of QP (which we do not intend to question at all) is not granted, and one must treat this point with care when he attempts to provide a SR approach to QP. But an experienced reader could now immediately classify an attempt of this kind as vain, quoting the basic "no-go" Bell–Kochen–Specker (or Bell-KS[27–29]) and Bell[29–34] theorems. Indeed, these theorems ought to prove that QP necessarily is a *contextual* and *nonlocal* theory[5]: should this be true, it would entail that the adoption of a verificationist (or, at least, contextual) truth theory is unavoidable in QP, since it is imposed by results that are internal to the theory rather than being an *a priori* choice, as in the canonical formulation of QP. Thus, the adoption of a correspondence truth theory would be inherently inconsistent with QP, and it would be impossible to adopt an SR viewpoint in this theory.

The above objection is rather thwarting. Indeed, it is well known that the contextuality of QP seems to introduce some mysterious conspiracy of nature (in particular, with regard to marginal distributions of the values of physical observables[(29)]). Furthermore, the occurrence of contextuality, even in the case of compound quantum systems whose elements are far apart (*locality*), sounds paradoxical and leaves in many physicists a feeling of uneasiness, which is explicitly witnessed, for instance, by Sakurai.[(35)]

However, SR can cope successfully with this challenge. In order to reach this goal the original version of SR,[(1)] which applied to noncompound physical systems only, must be suitably generalized and refined. The basic ideas for this generalization have been sketched by one of us in a series of disconnected papers.[(6–9)] We provide here the first integrated discussion of this subject, also stating a number of new results that follow from our present treatment.

The first idea (see Sec. 3) follows from noticing that it is arbitrary to assume *a priori* that the partially ordered set of all testable properties is an orthocomplemented lattice, as usual in CP and QP (we recall that the poset of testable properties is represented by the lattice of orthogonal projections in standard Hilbert space quantum theory, here briefly called HSQT). Rather, one can assume that this poset can be suitably *completed* so as to obtain a lattice, but at the possible expense of introducing *theoretical* properties that are not directly testable. This implies that a new language L_c must be introduced which contains predicates interpreted on these properties (see Sec. 4). When considering the special cases of CP and QP one finds that theoretical properties do not occur in CP, while they appear in QP whenever compound physical systems are described, and we identify them with the properties that are represented by one-dimensional

[5] Here *contextual* means that the value of an observable belonging to a set of observables that are measured on a physical system in a given state may depend on the choice of the set, so that it cannot be thought of as prefixed; equivalently, there are statements attributing physical properties to a physical object which cannot be thought of as true or false independently of the choices of the observer, since their truth values depend on the set of measurements that one decides to perform. *Nonlocal* means that contextuality occurs even if properties are measured that belong to different and spatially separated subsystems of a given physical system.

projections associated to *entangled* states in HSQT (this unusual assumption is physically justified in Remarks 3.2 and 3.3).

The second idea rests on a critical analysis of the concepts of compatibility of physical properties (see Secs. 5 and 6) and of testability of statements of L and L_c (see Sec. 7). Indeed this analysis allows us to distinguish between *theoretical* and *empirical* laws of a theory (see Sec. 8), and to realize that, if one wants to be consistent with an operational viewpoint, a statement expressing an empirical physical law (which has in any case a truth value according to SR) cannot be asserted to be true (it could be false) in physical contexts that the theory itself defines as not epistemically accessible. This remark is formalized by stating a new general principle (MGP principle) that limits the validity of empirical physical laws to the set of laboratories in which the theory does not prohibit that one can get information that confirms this validity.

By using the ideas sketched above in the special case of QP one can invalidate the proofs of the Bell–KS and Bell theorems.[6,8,36] This invalidation means that QP is not necessarily a contextual and nonlocal theory, so that verificationism is not an obliged choice. We can thus simultaneously conclude that SR is not inconsistent with QP and that it allows one to avoid some crucial quantum paradoxes.

By adopting a SR approach to QP, one can find a number of interesting results. For instance, one obtains that, contrary to a widespread belief, the Bell inequalities do not provide a method for testing experimentally whether QP or locality is valid. Indeed, a Bell inequality turns out to be a theoretical formula that is not epistemically accessible, so that any possible physical experiment actually tests something else (correlations among properties of physical objects in accessible contexts), and obviously yields the results predicted by QP.[36] Moreover, the controversial role of modern QL can be clearly specified. Indeed, QL can be obtained by using the (theory dependent) pragmatic concept of testability in QP for selecting suitable subsets of formulas of L_c, and restricting the logical order to this subset[1,3,4] (see also Remark 7.1). Thus, QL is not seen as a theory of truth in competition with classical logic, but, rather, as a mathematical structure that is embedded (in the sense of order) within L_c, and that formalizes properties of the concept of testability in QP (note that the nondistributive lattice of QL obviously is not a subalgebra of the Boolean Lindenbaum–Tarski algebra of L_c). Furthermore, QP proves to be an incomplete theory in a well-defined technical sense (Sec. 9). Finally, some new perspectives on the role of ideal filters in the quantum theory of measurement and on Heisenberg's uncertainty principle can be attained (Secs. 10 and 11).

We would like to close this Introduction with some remarks.

First, we note that the SR approach to QP can be considered orthodox from various viewpoints. In particular, it admits HSQT as a model and it embodies the standard *minimal interpretation* of QP.[21] In addition, it shows that the canonical interpretation of states as "amounts of information" (Sec. 5) is not trivial nor superficial, as maintained by some authors[37] (our concept of information does not take into account the single physicist's contingent information, so that it does not introduce any kind of "subjectivity" in physics). But if one wants to place the SR approach to QP in the context of recent research on the foundations of QP, the results quoted above show that he can collocate it among the theories that retain that QP is incomplete. This has some important consequences in the quantum theory of measurement according to SR (on which we give some hints only in this paper). In particular, the *objectification* problem,[21] which is typical of the approaches that generalize or modify QP but maintain a verificationist theory of truth, like the theory of positive operator valued measures, does not occur. However, the SR approach is not a conventional hidden variables theory for QP, since truth values are not bound in it by the constraints that are usually imposed on hidden variables,[28, 29] but only by the weaker constraints established by MGP (see Remark 8.2).

Second, we observe that the SR approach to QP differs from the current attempts of getting rid of nonlocality by limiting the justified use of counterfactual definiteness,[38] since it does not restrict the set of valid inferences in the language of QP for reasons depending on the laws of QP itself, thus maintaining a distinction between logical and physical levels[4] which avoids a number of conceptual and epistemological troubles.[39]

Finally, we notice that we have provided only sample references on the topics treated in this paper, the literature on the subject being so wide that it is quite impossible to cover it within the limited space of an article's bibliography.

2. THE LANGUAGE L

As we have anticipated in the Introduction, we will take as a starting point here the approach proposed by one of us in a previous paper,[1] which will be briefly mentioned as G.91 in the following. This approach will be refined and modified here. We therefore dedicate this section to present a formalized language L that constitutes our basic tool in the following. Our treatment will be intuitive and informal; a more rigorous treatment can be carried out following the methods adopted in the paper quoted above. In particular, the symbols used in our formal languages will often be used also as metalinguistic variables running on the symbols themselves, which favors understandability at the expense of exactness.

The language L is a classical first-order predicate calculus extended by means of a family of *statistical quantifiers*, it is endowed with a Tarskian truth theory, and by means of it all statements regarding testable physical properties of samples of a given physical system can be expressed (we use the term *physical system* here as a synonym of *physical entity*).[40, 41]

The construction of L can be schematized as follows.

(i) *Alphabet of L.* The set X of individual variables; two disjoint sets $\mathscr{S}$ and $\mathscr{F}$ of monadic predicates, called (*nouns of*) *states* and (*nouns of*) *effects*, respectively; standard logical connectives $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, and quantifiers $\exists$, $\forall$; a family $\{\pi_r\}_{r\in[0,1]}$ of *statistical quantifiers* (we introduce here a simplified family for the sake of brevity); the auxiliary symbols (,) and /.

(ii) *Formation rules.* The set Ψ of all well-formed formulas (wffs) of L is obtained by means of standard (recursive) formation rules, together with the following rule regarding statistical quantifiers:

$$\text{let} \quad x\in X,\ A(x),\ B(x)\in\Psi,\ r\in[0,1]; \qquad \text{then} \quad (\pi_r x)(A(x)/B(x))\in\Psi$$

(iii) *Semantics.* The following sets and objects are introduced in L: the set I of *laboratories*; for every $i\in I$, the (finite) domain D_i of i; for every $i\in I$, the set $\Sigma_i=\{\sigma_i: X \rightarrowtail D_i\}$ of the interpretations of the (individual) variables; for every $i\in I$, $\sigma_i\in\Sigma_i$ and $x\in X$, the *extension* $\sigma_i(x)\in D_i$ of x; for every $i\in I$ and $S\in\mathscr{S}$, the *extension* $\rho_i(S)\subseteq D_i$ of S; for every $i\in I$ and $F\in\mathscr{F}$ the *extension* $\rho_i(F)\subseteq D_i$ of F; the set $\hat{I}$ of all *statistically relevant* laboratories. Then, a Tarskian truth theory, suitably extended in such a way as to apply to statistical wffs, is assumed on L (see G.91). We do not enter here in the details of this theory, and consider only the essentials of it. Thus, let $i\in I$, let $\sigma_i\in\Sigma_i$, and let $x\in X$, $S\in\mathscr{S}$, F, F_1, F_2,..., $F_n\in\mathscr{F}$. Then the atomic wff S(x) (respectively, F(x)) is said to be true in i iff $\sigma_i(x)\in\rho_i(S)$ (respectively, $\sigma_i(x)\in\rho_i(F)$); the molecular wff $\neg F(x)$ is said to be true in i iff $\sigma_i(x)\in D_i\setminus\rho_i(F)$; the molecular wff $F_1(x)\wedge F_2(x)\wedge\cdots\wedge F_n(x)$ is said to be true in i iff $\sigma_i(x)\in\rho_i(F_1)\cap\rho_i(F_2)\cap\cdots\cap\rho_i(F_n)$. Furthermore, for every finite set Γ let $n(\Gamma)$ denote the number of elements in Γ. We say that the quantified statistical wffs $(\pi_r x)(F(x)/S(x))$ and $(\pi_r x)((F_1(x)\wedge F_2(x)\wedge\cdots\wedge F_n(x))/S(x))$ are true in i iff, respectively, $n(\rho_i(F)\cap\rho_i(S))=r\cdot n(\rho_i(S))$ and $n(\rho_i(F_1)\cap\rho_i(F_2)\cap\cdots\cap\rho_i(F_n)\cap\rho_i(S))=r\cdot n(\rho_i(S))$.

It follows from the above assumptions that the connectives $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$ and the quantifiers $\exists$, $\forall$ are interpreted as *not*, *and*, *or*, *if...then*, *iff* and *exists*, *for every*, respectively, as usual in classical logic; furthermore, a statistical quantifier in a wff of the form $(\pi_r x)(A(x)/B(x))$ is intuitively interpreted as stating that the ratio between the number of elements that make A(x) and B(x) true and the number of elements that make B(x) true is r whenever the latter number is not zero. It also follows that, if an interpretation σ_i of the variables is given in the laboratory i (we assume in the following that σ_i is surjective, so that every $x \in X$ can be considered in i as a noun of a physical object and every physical object in i has at least one noun), every wff of L has a truth value in i.

(iv) *Interpretation.* States and effects are interpreted as equivalence classes of the sets Π and $\mathscr{R}$ of all *preparing devices* and *dichotomic registering devices* associated to a given physical system $\mathfrak{P}$, respectively, as in Ludwig.[42] We do not specify for the moment the equivalence relations on Π and $\mathscr{R}$ underlying this interpretation, but assume that the interpretations of states and effects are bijective, i.e., to every equivalence class of Π (respectively, $\mathscr{R}$) corresponds one and only one state (respectively, effect). Furthermore, every laboratory in I is interpreted as a space-time region in the actual world. For every $i \in I$, the set D_i is interpreted as the set of all individual samples of $\mathfrak{P}$ prepared in i (possibly at different times), or *physical objects*, hence for every $\sigma_i \in \Sigma_i$ and $x \in X$, $\sigma_i(x)$ is a physical object in D_i (in principle, a physical object can be identified with the act of preparing it in order to avoid any ontological commitment, but we do not insist on this procedure here for the sake of simplicity); the extension $\rho_i(S)$ of the state S in i is interpreted as the set of all physical objects that are actually prepared in i by means of devices belonging to the equivalence class S (this extension can be identified with the ensemble of physical objects described by the state S according to the statistical interpretation of QP[37]); the extension $\rho_i(F)$ of the effect F is interpreted as the set of all physical objects in i which *would pass* the test whenever tested with any device belonging to the equivalence class F immediately after their preparation (note that our definitions here guarantee that $\rho_i(S)$ and $\rho_i(F)$ do not depend on the choice of a specific instant in the time domain associated to i; indeed, evolution in time is outside the scopes of the present paper). Finally, $\hat{I}$ is interpreted as the set of laboratories where a large number of physical objects is produced for any desired state and/or effect (the time interval associated to a given $i \in I$ can extend in the future, so that there is no finite limit for the number of physical objects that can be produced in i in this case), and all preparations and registrations are performed with the caution required by the physical theory that is adopted (we will refer from now on to $\hat{I}$ rather than to I; in particular, this will be made when universally quantifying on laboratories, as in the expression "for every laboratory i").

(v) *Preorder relations.* By referring to $\hat{I}$, three preorder relations can be defined on Ψ, as follows.

(v.1) *Logical* preorder $\subset$:
for every A_1, $A_2 \in \Psi$, $A_1 \subset A_2$ iff for every $i \in \hat{I}$, A_2 is true for every interpretation σ_i such that A_1 is true.

The preorder $\subset$ canonically induces on Ψ a *logical* equivalence relation $\equiv$, as follows.

For every A_1, $A_2 \in \Psi$, $A_1 \equiv A_2$ iff $A_1 \subset A_2$ and $A_2 \subset A_1$.

Furthermore, $\subset$ canonically induces a *logical* partial order relation on $\Psi/\equiv$, which we still denote by $\subset$. The theory of truth adopted on L implies that $(\Psi/\equiv, \subset)$ is a Boolean lattice (Lindenbaum–Tarski algebra of L).

Finally, the preorder $\subset$ also induces *logical* preorder and equivalence relations on $\mathscr{F}$, as follows.

For every F_1, $F_2 \in \mathscr{F}$, $F_1 \subset F_2$ iff $F_1(x) \subset F_2(x)$ for some (equivalently, all) $x \in X$
(hence, $F_1 \subset F_2$ iff for every $i \in \hat{I}$, $\rho_i(F_1) \subseteq \rho_i(F_2)$).

For every F_1, $F_2 \in \mathscr{F}$, $F_1 \equiv F_2$ iff $F_1 \subset F_2$ and $F_2 \subset F_1$.

(v.2) *Statistical* preorder $\angle$:
for every A_1, $A_2 \in \Psi$, $A_1 \angle A_2$ iff for every $S \in \mathscr{S}$ and $i \in \hat{I}$, $(\pi_{r_1}x)(A_1/S(x))$ true in i implies $(\pi_{r_2}x)(A_2/S(x))$ true in i, with $r_1 \leqslant r_2$.

The preorder $\angle$ canonically induces on Ψ a *statistical* equivalence relation $\simeq$, as follows.

For every A_1, $A_2 \in \Psi$, $A_1 \simeq A_2$ iff $A_1 \angle A_2$ and $A_2 \angle A_1$.

Furthermore, $\angle$ canonically induces a *statistical* partial order relation on $\Psi/\simeq$, which we still denote by $\angle$.

Finally, the preorder $\angle$ also induces *statistical* preorder and equivalence relations on $\mathscr{F}$, as follows.

For every F_1, $F_2 \in \mathscr{F}$, $F_1 \angle F_2$ iff $F_1(x) \angle F_2(x)$ for some (equivalently, all) $x \in X$
(hence, $F_1 \angle F_2$ iff for every $i \in \hat{I}$ and $S \in \mathscr{S}$, $n(\rho_i(S) \cap \rho_i(F_1)) \leqslant n(\rho_i(S) \cap \rho_i(F_2))$).

For every F_1, $F_2 \in \mathscr{F}$, $F_1 \simeq F_2$ iff $F_1 \angle F_2$ and $F_2 \angle F_1$.

(v.3) *Deterministic* preorder $<$:
for every A_1, $A_2 \in \Psi$, $A_1 < A_2$ iff for every $S \in \mathscr{S}$, $(\forall x)(S(x) \rightarrow A_2)$ is true in every $i \in \hat{I}$ whenever $(\forall x)(S(x) \rightarrow A_1)$ is true in every $i \in \hat{I}$.

The preorder $<$ canonically induces on Ψ a *deterministic* equivalence relation $\approx$, as follows.

For every A_1, $A_2 \in \Psi$, $A_1 \approx A_2$ iff $A_1 < A_2$ and $A_2 < A_1$.

Furthermore, $<$ canonically induces a *deterministic* partial order relation on $\Psi/\approx$, which we still denote by $<$.

Finally, the preorder $<$ also induces *deterministic* preorder and equivalence relations on $\mathscr{F}$, as follows.

For every $F_1, F_2 \in \mathscr{F}$, $F_1 < F_2$ iff $F_1(x) < F_2(x)$ for some (equivalently, all) $x \in X$

(hence $F_1 < F_2$ iff for every $S \in \mathscr{S}$, $\rho_i(S) \subseteq \rho_i(F_2)$ in every $i \in \hat{I}$ whenever $\rho_i(S) \subseteq \rho_i(F_1)$ in every $i \in \hat{I}$).

For every $F_1, F_2 \in \mathscr{F}$, $F_1 \approx F_2$ iff $F_1 < F_2$ and $F_2 < F_1$.

The intuitive meaning of the logical, statistical, and deterministic preorder and equivalence easily follows from the interpretations of connectives and quantifiers provided above. Furthermore, we get:

for every $A_1, A_2 \in \Psi$, $A_1 \subset A_2$ implies $A_1 \angle A_2$ implies $A_1 < A_2$, while the converse implications do not generally hold. Hence,

for every $A_1, A_2 \in \Psi$, $A_1 \equiv A_2$ implies $A_1 \simeq A_2$ implies $A_1 \approx A_2$, but $A_1 \approx A_2$ does not imply $A_1 \simeq A_2$, nor this implies $A_1 \equiv A_2$. Analogously,

for every $F_1, F_2 \in \mathscr{F}$, $F_1 \subset F_2$ implies $F_1 \angle F_2$ implies $F_1 < F_2$, but the converse implications do not generally hold, so that

for every $F_1, F_2 \in \mathscr{F}$, $F_1 \equiv F_2$ implies $F_1 \simeq F_2$ implies $F_1 \approx F_2$, but $F_1 \approx F_2$ does not imply $F_1 \simeq F_2$, nor this implies $F_1 \equiv F_2$.

(vi) *Basic derived definitions.* For every $F \in \mathscr{F}$, we define a *certainly yes domain* $\mathscr{S}_t(F)$ and a *certainly no domain* $\mathscr{S}_f(F)$, as follows:

$$\mathscr{S}_t(F) = \{S \in \mathscr{S} \mid \text{for every } i \in \hat{I}, \rho_i(S) \subseteq \rho_i(F)\}$$

$$\mathscr{S}_f(F) = \{S \in \mathscr{S} \mid \text{for every } i \in \hat{I}, \rho_i(S) \cap \rho_i(F) = \varnothing\}$$

Hence, the mappings $\mathscr{S}_t$ and $\mathscr{S}_f$ will be defined as follows:

$$\mathscr{S}_t : F \in \mathscr{F} \rightarrowtail \mathscr{S}_t(F) \in \mathscr{P}(\mathscr{S})$$

$$\mathscr{S}_f : F \in \mathscr{F} \rightarrowtail \mathscr{S}_f(F) \in \mathscr{P}(\mathscr{S})$$

(it is noteworthy that we immediately obtain from the definition of the order $<$ on $\mathscr{F}$ that for every $F_1, F_2 \in \mathscr{F}$, $F_1 < F_2$ iff $\mathscr{S}_t(F_1) \subseteq \mathscr{S}_t(F_2)$; hence, for every $F_1, F_2 \in \mathscr{F}$, $F_1 \approx F_2$ iff $\mathscr{S}_t(F_1) = \mathscr{S}_t(F_2)$).

Furthermore, for every $S \in \mathscr{S}$ we define a *certainly true domain* $\mathscr{F}_S$ of S in $\mathscr{F}$ and a *certainly false domain* $\mathscr{F}'_S$ of S in $\mathscr{F}$, as follows:

$$\mathscr{F}_S = \{F \in \mathscr{F} \mid \text{for every } i \in \hat{I}, \rho_i(S) \subseteq \rho_i(F)\}$$

$$\mathscr{F}'_S = \{F \in \mathscr{F} \mid \text{for every } i \in \hat{I}, \rho_i(S) \cap \rho_i(F) = \varnothing\}$$

Remark 2.1. Whenever $S \in \mathscr{S}$ and the interpretation σ_i is such that S(x) is true in the laboratory i, we briefly say in the following that *the physical object x is in the state S in i*, leaving implicit the reference to the interpretation σ_i and adopting a terminology that is standard in physics (in many cases the reference to i will also be understood). We note that the (finite) set $\rho_i(S)$ of all physical objects in the state S in i is defined in such a way that all its elements are known independently of any registration procedure.

Remark 2.2. For every effect F, the extension $\rho_i(F)$ in the laboratory i is unique, which implies that all registering devices in the equivalence class on which F is intensionally interpreted must select the same physical objects. Furthermore, the (finite) set $\rho_i(F)$ is defined in such a way that one does not know *a priori* all its elements, and this knowledge can be attained only by means of registration procedures that possibly destroy the set itself. Finally, $\rho_i(\mathscr{F})$ generally is a proper subset of the power set $\mathscr{P}(D_i)$, which means that there are subsets of physical objects in i which are not extensions of some $F \in \mathscr{F}$ (hence, neither $(\rho_i(\mathscr{F}), \subseteq)$ nor the Boolean sublattice $(<\rho_i(\mathscr{F})>, \subseteq)$ of $(\mathscr{P}(D_i), \subseteq)$ generated by $\rho_i(\mathscr{F})$ are generally isomorphic to the Boolean lattice $(\mathscr{P}(D_i), \subseteq)$).

Remark 2.3. We anticipate that the language L can be considered a sublanguage of the higher-order language L* that should be needed in order to express formally all physical laws (L* must admit, in particular, quantification on predicative variables,[2,4] but we will not discuss it here, since its construction would be long and difficult), and we briefly say that L is an *observative* sublanguage of L*.[5] However, it must be noted that not all statements in L are necessarily observative, as we will see in Sec. 8.

3. STATES AND EFFECTS

We introduce in this section some assumptions on states and effects that characterize a class of physical theories that contains both CP and QP. In other words, we introduce a general scheme for physical theories that admits CP and QP (and, in particular, HSQT) as models. Our assumptions generalize the conditions introduced in G.91, which will be assumed here to refer to the case of noncompound systems only, while our present generalization overcomes this restriction.

We begin with the following axioms (see G.91, conditions SB, PR, and CE).

AX 1. (i) Let $S_1, S_2 \in \mathscr{S}$. If, for every $i \in \hat{I}$ and $F \in \mathscr{F}$, $n(\rho_i(S_1) \cap \rho_i(F)) \cdot n(\rho_i(S_2)) = n(\rho_i(S_2) \cap \rho_i(F)) \cdot n(\rho_i(S_1))$, then, $S_1 = S_2$.

(ii) For every $i \in \hat{I}$, $\bigcup_{S \in \mathscr{S}} \rho_i(S) = D_i$, and for every $S_1, S_2 \in \mathscr{S}$, $S_1 \neq S_2$ implies $\rho_i(S_1) \cap \rho_i(S_2) = \varnothing$.

AX 2. For every $F_1, F_2 \in \mathscr{F}$, $F_1 = F_2$ iff $F_1 \equiv F_2$ iff $F_1 \simeq F_2$.

AX 3. For every $F \in \mathscr{F}$, an $F' \in \mathscr{F}$ exists such that, for every $i \in \hat{I}$, $\rho_i(F') = D_i \setminus \rho_i(F)$.

Let us comment briefly on these axioms. AX 1 and AX 2 implicitly define the equivalence classes of preparing and registering devices on which states and effects are intensionally interpreted, respectively. It is then worth noticing that our implicit definitions of states and effects here match the explicit definitions provided in the Ludwig approach,[42] and that the set $\mathscr{R}$ of registering devices introduced here can be considered a proper subset of the broader set of the questions introduced in the Piron approach.[43] In addition, AX 2 also implies that the relations $\subset$ and $\angle$ defined on $\mathscr{F}$ are partial orders, hence we get in particular, by setting $\mathscr{F}(x) = \{F(x) \mid F \in \mathscr{F}\}$ and $[\mathscr{F}(x)] = \{[A(x)]_{\equiv} \mid A(x) \in \mathscr{F}(x)\}$, that $([\mathscr{F}(x)], \subset)$, $(\mathscr{F}(x), \subset)$, $(\mathscr{F}, \subset)$ are order-isomorphic posets. Furthermore, AX 1(ii) implies that in every laboratory i the set of all nonvoid extensions of states is a partition on D_i, which is consistent with our interpretation of states in Sec. 2 (in every laboratory, different states are realized by different preparations, which prepare different, nonintersecting sets of physical objects). Finally, AX 3 states that every $F \in \mathscr{F}$ has a complement $F' \in \mathscr{F}$, which allows us to interpret F' as the class of dichotomic devices obtained by exchanging the yes and no answers in every device of the class on which F is interpreted.

Now, let us refer to the definition of certainly true domain $\mathscr{F}_S$ of a state S in $\mathscr{F}$ supplied at the end of Sec. 2, and for every $i \in \hat{I}$ and $S \in \mathscr{S}$ let us introduce the following set:

$$\hat{\rho}_i(S) = \bigcap_{F \in \mathscr{F}_S} \rho_i(F)$$

Then, one obviously gets:

(i) for every $S \in \mathscr{S}$ and $i \in \hat{I}$, $\rho_i(S) \subseteq \hat{\rho}_i(S)$;

(ii) for every $S \in \mathscr{S}$ and $F \in \mathscr{F}$, if, for every $i \in \hat{I}$, $\rho_i(S) \subseteq \rho_i(F)$, then, for every $i \in \hat{I}$, $\hat{\rho}_i(S) \subseteq \rho_i(F)$.

Furthermore, one can define the subset $\mathscr{S}_P \subseteq \mathscr{S}$ of pure states as follows:

$$\mathscr{S}_P = \{S \in \mathscr{S} \mid \text{for every } S^* \in \mathscr{S}, \rho_i(S^*) \subseteq \hat{\rho}_i(S) \text{ in every } i \in \hat{I} \text{ implies } S^* = S\}$$

Of course, $\mathscr{S}_P$ will be identified with the set of pure states in the standard approaches to CP and QP. By referring to the set of pure states, we can introduce the following axiom (see G.91, condition SY).

AX 4. For every $i \in \hat{I}$ and $S_1, S_2 \in \mathscr{S}_P$,

$$n(\rho_i(S_1) \cap \hat{\rho}_i(S_2)) \cdot n(\rho_i(S_2)) = n(\rho_i(S_2) \cap \hat{\rho}_i(S_1)) \cdot n(\rho_i(S_1))$$

By using the above definitions and axioms we can define a *preclusivity* (nonreflexive and symmetric) relation $\perp\!\!\!\perp$ on $\mathscr{S}_P$, as follows:

for every $S_1, S_2 \in \mathscr{S}_P$, $S_1 \perp\!\!\!\perp S_2$ iff for every $i \in \hat{I}$, $\rho_i(S_1) \cap \hat{\rho}_i(S_2) = \emptyset$ (equivalently, $\rho_i(S_2) \cap \hat{\rho}_i(S_1) = \emptyset$).

The relation $\perp\!\!\!\perp$ defined above can be used in order to introduce a *weak orthocomplementation* $\perp$ on the power set $\mathscr{P}(\mathscr{S}_P)$ by setting:

$$\perp : H \in \mathscr{P}(\mathscr{S}_P) \rightarrowtail H^{\perp} = \{S \in \mathscr{S}_P \mid \text{for every } S^* \in H, S \perp\!\!\!\perp S^*\}$$

Furthermore, $\perp$ can be used in order to define a *closure* operation on $\mathscr{P}(\mathscr{S}_P)$, as follows:

$$\perp\perp : H \in \mathscr{P}(\mathscr{S}_P) \rightarrowtail (H^{\perp})^{\perp} \in P(\mathscr{S}_P)$$

We say that a subset $H \in \mathscr{P}(\mathscr{S}_P)$ is closed whenever $H = H^{\perp\perp}$. Then, it follows from known general mathematical results (see G.91, Sec. 2.3) that the set $\mathscr{L}$ of all closed subsets of $\mathscr{S}_P$, partially ordered by set inclusion $\subseteq$, is a complete lattice $(\mathscr{L}, \subseteq)$, with minimal element $\emptyset$ and maximal element $\mathscr{S}_P$, orthocomplemented by the restriction of $\perp$ to $\mathscr{L}$ itself. Therefore, we introduce the new symbols $\bigcap$ and $\bigcup$ in order to denote meet and join in $(\mathscr{L}, \subseteq)$, respectively, in the following.

By using the above definitions and the mappings $\mathscr{S}_t$, $\mathscr{S}_f$ introduced at the end of Sec. 2, we can select a subset $\mathscr{F}_e \subseteq \mathscr{F}$ that is basic both in CP and in QP (see G.91, condition OE):

$$\mathscr{F}_e = \{F \in \mathscr{F} \mid \mathscr{S}_t(F) \in \mathscr{L}, \mathscr{S}_f(F) = \mathscr{S}_t^{\perp}(F)\}$$

The subset $\mathscr{F}_e$ of $\mathscr{F}$ will be called the set of (nouns of) *exact effects* and its elements will be interpreted as equivalence classes of *idealized* dichotomic registering devices, which exactly test whether the value of a given physical observable lies in a given Borel subset of the real line or, briefly, which exactly test whether a given *testable physical property* holds. Therefore, we assume that $\mathscr{F}_e$ can be bijectively mapped on the set of all testable physical properties, and the two sets will be identified in the following.

We can now state a further axiom (see G.91, condition EM).

AX 5. For every $F_1, F_2 \in \mathscr{F}_e$, $\mathscr{S}_t(F_1) \subseteq \mathscr{S}_t(F_2)$ implies $F_1 \subset F_2$.

It follows from AX 5 (see Sec. 2(vi)) that $F_1 < F_2$ implies $F_1 \subset F_2$; hence, we get (see Sec. 2(v)):

for every $F_1, F_2 \in \mathscr{F}_e$, $F_1 < F_2$ iff $F_1 \angle F_2$ iff $F_1 \subset F_2$,

for every $F_1, F_2 \in \mathscr{F}_e$, $F_1 \approx F_2$ iff $F_1 \simeq F_2$ iff $F_1 \equiv F_2$ iff $F_1 = F_2$ (the last equivalence follows by using AX 2).

Furthermore, the restriction of $\mathscr{S}_t$ to $\mathscr{F}_c$ (still denoted by $\mathscr{S}_t$, by abuse of language),

$$\mathscr{S}_t \colon \mathrm{F} \in \mathscr{F}_c \rightarrowtail \mathscr{S}_t(\mathrm{F}) \in \mathscr{L},$$

maps bijectively $\mathscr{F}_c$ onto $\mathscr{S}_t(\mathscr{F}_c)$, and preserves the order. We assume that $\mathscr{S}_t(\mathscr{F}_c)$ has the following further property.

AX 6. The poset $(\mathscr{S}_t(\mathscr{F}_c), \subseteq)$ is dense in $(\mathscr{L}, \subseteq)$

(the word *dense* means here that every $a \in \mathscr{L}$ is the greatest lower bound of at least one subset of $\mathscr{S}_t(\mathscr{F}_c)$ and the least upper bound of at least one subset of $\mathscr{S}_t(\mathscr{F}_c)$).

It follows from AX 6 that $\mathscr{S}_t$ is a minimal embedding of $\mathscr{F}_c$ into a complete lattice and that it coincides (up to lattice isomorphisms) with the *normal embedding* of $\mathscr{F}_c$ into the complete lattice of all its closed ideals.[44] Furthermore, let $a \in \mathscr{L}$. By setting $a^{\uparrow} = \{b \in \mathscr{S}_t(\mathscr{F}_c) \mid a < b\}$, and $a^{\downarrow} = \{b \in \mathscr{S}_t(\mathscr{F}_c) \mid b < a\}$, we also get (ibidem):

$$a = \bigcap_{b \in a^{\uparrow}} b = \bigcup_{b \in a^{\downarrow}} b.$$

Taking into account the lattice $(\mathscr{L}, \subseteq)$, the set $\mathscr{F}_c$ will be extended by means of a set $\mathscr{G}_c$ of monadic predicates such that $\mathscr{F}_c \cap \mathscr{G}_c = \emptyset$ and that the mapping $\mathscr{S}_t$ can be extended to $\mathscr{F}_c \cup \mathscr{G}_c$ so as to map bijectively this set onto $\mathscr{L}$ (by abuse of language, we still denote by $\mathscr{S}_t$ the extended mapping in the following). Thus, we are endowed with a new set of predicates $\mathscr{E}_c = \mathscr{F}_c \cup \mathscr{G}_c$ which contains a subset $\mathscr{G}_c$ of predicates that are not interpreted as equivalence classes of idealized dichotomic registering devices. We say that every predicate $\mathrm{E} \in \mathscr{E}_c$ still denotes a physical property of a given physical system, but this property is *testable* iff $\mathrm{E} \in \mathscr{F}_c$, while it is *non-testable* (or *theoretical*) iff $\mathrm{E} \in \mathscr{G}_c$ (of course $\mathscr{G}_c$ can be void, as it occurs, for instance, in CP, see Remark 3.2).

The set $\mathscr{E}_c$ can be canonically ordered by considering the bijective mapping $\mathscr{S}_t$ which maps $\mathscr{E}_c$ onto the lattice $(\mathscr{L}, \subseteq)$. The restriction to $\mathscr{F}_c$ of the order induced on $\mathscr{E}_c$ by $\mathscr{S}_t$ obviously coincides with the order $<$ (which however also coincides on $\mathscr{F}_c$ with $\subset$ and $\angle$ because of AX 5), hence we denote this order by $<$ in the following. Thus, $(\mathscr{E}_c, <)$ is a complete orthocomplemented lattice (by abuse of language, we still denote by $\perp$, $\bigcap$, and $\bigcup$ orthocomplementation, meet, and join, respectively, in $(\mathscr{E}_c, <)$), which is such that $\mathscr{F}_c$ is dense in $(\mathscr{E}_c, <)$, so that:

$$\text{for every} \quad \mathrm{E} \in \mathscr{E}_c,\ \mathrm{E} = \bigcap_{\mathrm{F} \in \mathscr{F}_c, \mathrm{E} < \mathrm{F}} \mathrm{F} = \bigcup_{\mathrm{F} \in \mathscr{F}_c, \mathrm{F} < \mathrm{E}} \mathrm{F}.$$

It is important to observe that, for every $F \in \mathscr{F}_e$, $F^{\perp} = F' \in \mathscr{F}_e$ because of AX 3 and AX 5 (indeed, F' is such that $\mathscr{S}_t(F') = \mathscr{S}_t^{\perp}(F) = \mathscr{S}_t(F^{\perp})$ and $\mathscr{S}_t^{\perp}(F') = \mathscr{S}_t(F) = \mathscr{S}_t^{\perp}(F^{\perp})$, hence $F' \in \mathscr{F}_e$ and $F' = F^{\perp}$, being $\mathscr{S}_t$ bijective).

Furthermore, it can be proved that there are mathematical properties (as atomicity or orthomodularity) that, whenever assumed on $(\mathscr{F}_e, \subset)$, also hold on $(\mathscr{E}_e, <)$.[45] This suggests to introduce the following further axiom on $(\mathscr{L}, \subseteq)$.

AX 7. The lattice $(\mathscr{L}, \subseteq)$ is atomic, and $\{\{S\} \mid S \in \mathscr{S}_P\}$ is the set of its atoms.

The atomicity of $(\mathscr{L}, \subseteq)$ stated by Axiom 7 implies the atomicity of $(\mathscr{E}_e, <)$, being $\mathscr{S}_t$ bijective and order preserving. Moreover, we can distinguish between *first type* and *second type* pure states by introducing, for every $S \in \mathscr{S}_P$, the *support* $E_S = \mathscr{S}_t^{-1}(\{S\})$ of S (obviously, E_S is an atom of $(\mathscr{E}_e, <)$ because of AX 7), and by setting:

for every $S \in \mathscr{S}_P$, S is a first type state iff $E_S \in \mathscr{F}_e$, a second type state iff $E_S \in \mathscr{G}_e$.

Finally, we state the following further axiom (which assures, in particular, that for every first type state S, E_S is the smallest effect in $\mathscr{F}$ whose extension contains the extension of S in every laboratory).

AX 8. For every $S \in \mathscr{S}_P$, S is a first type state iff an effect $F_S \in \mathscr{F}$ exists such that, for every $i \in \hat{I}$, $\hat{\rho}_i(S) = \rho_i(F_S)$, and $F_S = E_S = \mathscr{S}_t^{-1}(\{S\})$ in this case.

We have thus concluded the presentation of our axioms on states and effects. As we have already seen at the beginning of this section, they generalize the axioms stated in G.91, where an extensive discussion was made on the basis of the interpretation of L reported in Sec. 2. Therefore, we will limit ourselves here to introducing a number of remarks aimed to point out the main novelties in our generalized approach.

Remark 3.1. The set of axioms in this section is not complete in several senses. For instance, it does not allow us to distinguish between CP and QP. Furthermore, it dispenses from mixed states from the very beginning, (see in particular AX 4), while these could actually be recovered in our framework (see G.91, conditions FI and MS, etc.), but at the expense of unnecessary complication for our purposes in this paper. However, it is important to observe that the poset of all positive trace one operators on the Hilbert space of a physical system in HSQT is a model of $(\mathscr{E}, <)$.

We also notice that our set of axioms implies that $(\mathscr{E}_e, <)$ is a complete, orthocomplemented, atomic lattice, which admits as a model the lattice of all orthogonal projections on the Hilbert space of the system in HSQT (hence we will use the symbols $\bigcap$, $\bigcup$, $\perp$ in what follows even in order to denote meet, join, and orthocomplementation, respectively, in this lattice). Then, one may wonder on the links between ($\mathscr{E}_c$, $<$) and the lattices that appear in several known approaches to QP,[42,43,46,47] but a detailed discussion of this topic would be rather lengthy. Therefore, we limit ourselves here to refer to G.91 and to some earlier papers,[48–51] and to identify ($\mathscr{E}_c$, $<$) in the case of noncompound quantum systems (up to lattice isomorphisms) with the Mackey[46] lattice of questions, or with the Jauch[47] and Piron[43] lattice of propositions. Because of this identification, we can also assume that ($\mathscr{E}_c$, $<$) is distributive in classical physics (CP), weakly modular, and satisfying the covering law in QP.

Finally, we stress that the elements of the set $\mathscr{F}_c \subseteq \mathscr{E}_c$ of all exact effects are interpreted on idealized dichotomic devices. In some cases (e.g., observables with continuous spectra) it may be impossible *in principle* to construct a device that realizes an exact effect. This can be intuitively explained in terms of *efficiency* of the devices[49]: indeed, if a device must be characterized by a continuous efficiency, it can never register exactly whether the value of a given observable lies within a prefixed interval Δ of the continuous part of the spectrum.

Remark 3.2. Let us consider the main novelty in this section, that is, the introduction of the set $\mathscr{S}_c$ of theoretical properties. It is apparent that, whenever $\mathscr{S}_c$ is void, all pure states are first type states, and our axioms immediately reduce to axioms already forwarded in G.91. Therefore, it seems important to provide an intuitive physical justification for the introduction of $\mathscr{S}_c$ in the general case. To this end let us observe that the standard scheme for an elementary experiment on a physical system consists in considering a physical object x in a state S (that is, prepared by means of a device in the equivalence class denoted by S) in a laboratory i and measuring whether x has a given property F_1. Usually, the choice of F_1 prohibits in QP that another property F_2 be also measured if F_2 is not compatible with F_1. Furthermore, repeated measurements of this kind on a set of identically prepared physical objects can be performed in order to obtain statistical frequencies to be compared with predicted probabilities.

Whenever a set F_1, F_2,..., F_n of compatible properties is considered, a different kind of experiments can be conceived. Indeed, one can prepare sets of physical objects in the state S and test the (perfect or statistical) correlations existing among the properties F_1, F_2,..., F_n by means of repeated measurements (a particular case that interests us here is provided by a physical system x composed of n subsystems, each property F_i referring to a different subsystem x_i of x). Our point is that this kind of experiments actually tests a (second order, n-adic) property G of F_1, F_2,..., F_n, or *correlation property*, not a property of an individual sample x of the given system. Expressing a correlation property either requires the enlargement of L by means of second-order predicates, or the use of quantified wffs of L; in any case, it cannot be stated by means of an atomic statement of L (note that the attribution of a second-order property to a physical object x would violate a syntactical rule in type theory). Now, only a correlation property can be used in HSQT in order to characterize an entangled state S of a compound physical system, distinguishing it from a mixed state.[37] Hence, we cannot associate a testable physical property of the first order (interpreted as an equivalence class of idealized dichotomic devices) to S. But S is associated to a (one-dimensional) projection in HSQT, which represents the physical first order property E_S according to the standard interpretation of QP. Our arguments then suggest that E_S must be considered a theoretical, not a testable property.

The above discussion provides a physical justification for the introduction of theoretical properties in our approach. Of course, one expects that no theoretical property appears in CP, or, equivalently, that $\mathcal{S}_c = \varnothing$ in this theory, even if compound physical systems are considered.

Remark 3.3. The impossibility of associating (first-order) testable properties to second type states of a compound quantum physical system which is made up of *separate* quantum subsystems has already been recognized by Aerts[40] (we observe that Aerts deduces it as a theorem following from a set of axioms regarding *questions*, while it is assumed to be a definition of second type states in our context). But Aerts concludes that this proves the nonseparability of systems resulting from the composition of different quantum systems (which are described by tensor products in HSQT). On the contrary, we explicitly admit, as we have seen in Remark 3.2, that (i) second type states are associated to theoretical properties, hence to projections in HSQT that do not represent testable physical properties, and that (ii) the correlations between properties of subsystems of a given physical system (that occur even in CP) are second-order properties that cannot be represented by projections in HSQT. This prevents us[36] from accepting quantum nonseparability in the sense established by the Bell theorem (according to which the correlations themselves depend on what is observed, differently from classical correlations, which are prefixed in a given state of the system and do not depend on the observer).

It is interesting to note that our above point (i) has some further relevant consequences. In particular, not all projections that are strictly contained in a projection that appears in the spectral decomposition of a Hermitian operator which represents a physical observable necessarily correspond to physical apparatuses in the case of compound systems, which implies that a complete observation can be impossible. In addition, if we still want to interpret states as equivalence classes of preparations, as we have done in Sec. 2 and intend to maintain in the following for obvious physical reasons, we must accept that a second type (pure) state S cannot be produced by means of standard textbook procedures, that is, by performing an ideal measurement of a suitable observable. Indeed, should this procedure exist, the ideal measurement would characterize an equivalence class $E_S \in \mathscr{F}_c$ of devices that is the support of the state S, which contradicts our assumption that S is a second type state.

Remark 3.4. Let us consider the case $\mathscr{G}_c = \emptyset$, hence $\mathscr{E}_c = \mathscr{F}_c$. By referring to Remark 2.2, we note explicitly that, in every laboratory $i \in \hat{I}$, the Boolean lattice $(\langle \rho_i(\mathscr{F}_c) \rangle, \subseteq)$ generated by all extensions in i of exact effects (which obviously is a sublattice of $(\mathscr{P}(D_i), \subseteq))$ is not necessarily isomorphic to the lattice $(\mathscr{F}_c, <)$, even if $<$ and $\subset$ coincide on $\mathscr{F}_c$; we only expect that $(\mathscr{F}_c, <)$ can be canonically embedded into $(\langle \rho_i(\mathscr{F}_c) \rangle, \subseteq)$, preserving the order. In particular, we have, for every $i \in \hat{I}$ and F, $F_1, F_2 \in \mathscr{F}_c$,

$$\rho_i(F^{\perp}) = D_i \setminus (\rho_i(F)),$$

$$\rho_i(F_1 \bigcap F_2) \subseteq \rho_i(F_1) \cap \rho_i(F_2),$$

$$\rho_i(F_1) \cup \rho_i(F_2) \subseteq \rho_i(F_1 \bigcup F_2).$$

4. THE LANGUAGE L_c

Making reference to Secs. 2 and 3, we now convene that the set $\mathscr{E}_c$ is used in order to construct a new language L_c. To be precise, L_c is obtained by using the same symbols and rules introduced in Sec. 2 when constructing L, with the exception of the set $\mathscr{F}$, which is substituted by $\mathscr{E}_c = \mathscr{F}_c \cup \mathscr{G}_c$. The set of all wffs of L_c will be denoted by Ψ_c.

It is apparent that our new language reduces to a sublanguage of L whenever $\mathscr{G}_c = \emptyset$. On the contrary, L_c is not a sublanguage of L whenever $\mathscr{G}_c \neq \emptyset$. In the latter case the problem arises of the truth values to be attributed to the wffs of L_c, since the extensions of the theoretical properties in a laboratory $i \in \hat{I}$ cannot be interpreted operationally. Therefore, we firstly agree that every predicate $E \in \mathscr{F}_c$ has in every laboratory $i \in \hat{I}$ the same extension attributed to it in the semantics of L. Furthermore, we attribute to every $E \in \mathscr{G}_c$, in every laboratory $i \in \hat{I}$, a conventional extension $\rho_i(E)$, which satisfies the following conditions:

(i) for every $S \in \mathscr{S}$ and $E \in \mathscr{G}_c$, $S \in \mathscr{S}_1(E)$ iff, for every $i \in \hat{I}$, $\rho_i(S) \subseteq \rho_i(E)$;

(ii) for every $i \in \hat{I}$ and $E \in \mathscr{G}_c$, $\bigcup_{F \in \mathscr{F}_c, F < E} \rho_i(F) \subseteq \rho_i(E) \subseteq \bigcap_{F \in \mathscr{F}_c, E < F} \rho_i(F)$;

(iii) for every $i \in \hat{I}$ and $E \in \mathscr{G}_c$, $\rho_i(E^\perp) = D_i \setminus \rho_i(E)$.

The above conditions can be justified as follows. First, $\mathscr{S}_t$ associates a (closed) subset of states to every $E \in \mathscr{E}_c$, which coincides with the certainly-yes domain of E whenever $E \in \mathscr{F}_c$, so that condition (i) can be regarded as a natural extension to theoretical properties of an attribute of testable properties. Second, condition (ii) obviously follows from the expressions of E as meet or join of elements of $\mathscr{F}_c$ provided in Sec. 3. Third, condition (iii) expresses the requirement that the interpretation of $F^\perp$ as the *negation* of F, which follows for every $F \in \mathscr{F}_c$ from the identification of $F^\perp$ with F' in Sec. 3, can be extended to every $E \in \mathscr{E}_c$.

We stress that conditions (i), (ii) and (iii) may be insufficient to determine uniquely the extension $\rho_i(E)$ in a laboratory i of the theoretical property E; it follows that $\rho_i(E)$ is assigned with some degree of arbitrariness. However, the above procedures allow us to assign a conventional extension to every predicate of L_c, hence a conventional truth value will be attributed to every wff of L_c by adopting a Tarskian truth theory, as we have done in L.

The attribution of a truth value to every wff of L_c allows us to introduce a logical preorder relation $\subset$ on Ψ_c, a logical equivalence relation $\equiv$ on Ψ_c, a logical order $\subset$ on $\Psi_c/\equiv$, and a logical order $\subset$ on $\mathscr{E}_c$, by means of the same definitions adopted at the end of Sec. 2, with Ψ_c in place of Ψ and $\mathscr{E}_c$ in place of $\mathscr{F}$. Similarly, we define a statistical preorder relation $\angle$ on Ψ_c, a statistical equivalence relation $\simeq$ on Ψ_c, a statistical order $\angle$ on $\Psi_c/\simeq$, and a statistical order $\angle$ on $\mathscr{E}_c$. Finally, we define a deterministic preorder relation $<$ on Ψ_c, a deterministic equivalence relation $\approx$ on Ψ_c, a deterministic order $<$ on $\Psi_c/\approx$, and a deterministic order $<$ on $\mathscr{E}_c$.

The restrictions of the orders $\subset$, $\angle$, $<$ defined on $\mathscr{E}_c$ to $\mathscr{F}_c$ can be identified with the orders denoted by the same symbols in Sec. 3, which coincide because of Ax 5. When considering $\mathscr{E}_c$, we obtain from AX 5, AX 6, and condition (ii) that for every $E_1, E_2 \in \mathscr{E}_c$, $\mathscr{S}_t(E_1) \subseteq \mathscr{S}_t(E_2)$ implies $E_1 \subset E_2$, so that we get:

for every $E_1, E_2 \in \mathscr{E}_c$, $E_1 < E_2$ iff $E_1 \angle E_2$ iff $E_1 \subset E_2$,

hence,

for every $E_1, E_2 \in \mathscr{E}_c$, $E_1 \approx E_2$ iff $E_1 \simeq E_2$ iff $E_1 \equiv E_2$ iff $E_1 = E_2$.

Finally, let us prove the following propositions.

P 4.1. Let $E_1, E_2 \in \mathscr{E}_c$. Then $(E_1 \bigcap E_2)(x) \approx E_1(x) \wedge E_2(x)$.

Proof. Let us show that $(E_1 \bigcap E_2)(x) < E_1(x) \wedge E_2(x)$. Indeed, let $S \in \mathscr{S}$, and let the wff $(\forall x)(S(x) \rightarrow (E_1 \bigcap E_2)(x))$ be true in every $i \in \hat{I}$. It follows that $S \in \mathscr{S}_i(E_1 \sqcap E_2)$ (if $E_1 \sqcap E_2 \in \mathscr{S}_c$, use condition (i)), hence $S \in \mathscr{S}_i(E_1)$ and $S \in \mathscr{S}_i(E_2)$. This implies that, for every $i \in \hat{I}$, $\rho_i(S) \subseteq \rho_i(E_1)$ and $\rho_i(S) \subseteq \rho_i(E_2)$ (if E_1 and/or E_2 belong to $\mathscr{S}_c$, use again condition (i)), that is, $\rho_i(S) \subseteq \rho_i(E_1) \cap \rho_i(E_2)$. Because of the truth theory adopted in L_c, this means that the wff $(\forall x)(S(x) \to (E_1(x) \wedge E_2(x)))$ is true in every $i \in \hat{I}$, which proves our statement.

The inequality $E_1(x) \wedge E_2(x) < (E_1 \sqcap E_2)(x)$ can be proved by reversing the above arguments. □

P 4.2. Let $x \in X$, $A(x)$, $B(x) \in \Psi_c$, E_A, $E_B \in \mathscr{E}_c$, and let $A(x) \approx B(x)$, $A(x) \approx E_A(x)$, $B(x) \approx E_B(x)$. Then, $E_A = E_B$.

Proof. Since $\approx$ is an equivalence relation, it follows from our assumptions that $E_A(x) \approx E_B(x)$ hence $\mathscr{S}_i(E_A) = \mathscr{S}_i(E_B)$, which implies $E_A \equiv E_B$, so that $E_A = E_B$. □

Remark 4.1. We have seen in Sec. 3 that all elements of $\mathscr{E}_c$ are nouns of physical properties, so that we briefly call them *properties* in this paper, carefully distinguishing between *testable properties* (the elements of $\mathscr{F}_c$) and *theoretical properties* (the elements of $\mathscr{S}_c$). In addition, whenever $E \in \mathscr{E}_c$ and the interpretation σ_i makes $E(x)$ true in the laboratory i, we say that σ_i *is such that the (testable or theoretical) property* E *is true in i for the physical object* x, or, briefly, that x *has the property* E in i, leaving implicit the reference to σ_i. Whenever $E \in \mathscr{E}_c$, $S \in \mathscr{S}$ and $(\pi_r x)(E(x)/S(x))$ is true in a laboratory i, we say that *the physical objects in the state* S *have the property* E *with frequency* r *in i*. Whenever $E_1, E_2, \ldots, E_n \in \mathscr{E}_c$, $S \in \mathscr{S}$, and $(\pi_r x)((E_1(x) \wedge E_2(x) \wedge \cdots \wedge E_n(x))/S(x))$ is true in a laboratory i, we say that *the physical objects in the state* S *have the properties* $E_1, E_2, \ldots, E_n$ *with frequency* r *in i*.

5. SEMANTIC COMPATIBILITY (CONSISTENCY)

Let us note that states and effects appear as (first-order, monadic) predicates in our approach, but there are relevant semantic differences (both intensional and extensional) between these two kinds of predicates. In particular (see AX 1 and AX 2 in Sec. 3), no physical object in a laboratory i can belong to the extensions of two different states (let they be pure states or not), while a physical object usually belongs to the extension of a number of effects (which can be infinite). Hence, different states never can be attributed to a given physical object, while the possibility that a pair E_1, E_2 of different physical properties be conjointly attributed to a given physical object depends on the physical theory $\mathscr{T}$ that one is considering.

There are, however, some important links between states and properties that follow from the axioms stated in Sec. 3 and from the conventions on ρ_i established in Sec. 4. More specifically, we have associated in Sec. 3 a support $E_S \in \mathscr{E}_e$ to every $S \in \mathscr{S}_P$, which is the physical property with minimal extension in $(\mathscr{E}_e, <)$ that is true, in every laboratory $i \in \hat{I}$, for every $x \in X$ such that S(x) is true (the minimality follows from the coincidence of the orders $\subset$ and $<$ on $\mathscr{E}_e$ and must not be confused with the property stated by AX 8 in Sec. 3). Furthermore, the *certainly true domain* of S in $\mathscr{E}_e$

$$\mathscr{E}_S = \{E \in \mathscr{E}_e \mid E_S < E\}$$

(equivalently, $\mathscr{E}_S = \{E \in \mathscr{E}_e \mid \text{ for every } i \in \hat{I},\ \rho_i(S) \subseteq \rho_i(E)\}$), and the *certainly false domain* of S in $\mathscr{E}_e$

$$\mathscr{E}_S^{\perp} = \{E \in \mathscr{E}_e \mid E^{\perp} \in \mathscr{E}_S\}$$

(equivalently, $\mathscr{E}_S^{\perp} = \{E \in \mathscr{E}_e \mid E < E_S^{\perp}\}$, or $\mathscr{E}_S^{\perp} = \{E \in \mathscr{E}_e \mid$ for every $i \in \hat{I}$, $\rho_i(S) \cap \rho_i(E) = \emptyset\}$) are the sets of all properties that are true or false, respectively, in every laboratory $i \in \hat{I}$ for every physical object in the state $S \in \mathscr{S}_P$. It is apparent that both E_S and $\mathscr{E}_S$ are natural choices in order to characterize S. The characterization of any $S \in \mathscr{S}_P$ by means of E_S translates in our context the choice made by Piron,[43] but it is important to note explicitly that this does not mean that S can be identified with E_S from a semantic viewpoint, the extensions $\rho_i(S)$ and $\rho_i(E_S)$ in a laboratory i being generally different (see Remark 5.1). The characterization of S by means of $\mathscr{E}_S$ allows us to recover in our context the standard conception of pure states as maximal "amounts of information" in QP: this provides a new interpretation of pure states, which adjoins to our previous interpretation as equivalence classes of preparations in Sec. 2.

By using the above definitions, a (theory dependent) binary relation C can be introduced on the set $\mathscr{S}_P$ which defines the *semantic compatibility*, or *consistency*, of states (elsewhere[2] C was simply called *compatibility* relation; the new name is needed in our present broadened framework). To be precise, we put:

for every $S_1, S_2 \in \mathscr{S}_P$, $S_1 C S_2$ iff $\mathscr{E}_{S_1} \cap \mathscr{E}_{S_2}^{\perp} = \emptyset = \mathscr{E}_{S_1}^{\perp} \cap \mathscr{E}_{S_2}$.

Then, intuitively, we can say that S_1 and S_2 are in the relation C iff no contradiction occurs between the information embodied in S_1 and the information embodied in S_2.

The relation C proves to be an accessibility relation (it is reflexive and symmetric but not, generally, transitive), and the following statement holds for every $S_1, S_2 \in \mathscr{S}_P$:[2]

$S_1 C S_2$ iff for every $i \in \hat{I}$, $\rho_i(S_1) \cap \rho_i(E_{S_2}) \neq \varnothing$ iff for every $i \in \hat{I}$, $\rho_i(S_2) \cap \rho_i(E_{S_1}) \neq \varnothing$.

It is easy to see that $S_1 C S_2$ iff $S_1 = S_2$ in CP. Coming to QP, one can prove that S_1 is consistent with S_2 iff the vectors $|\psi_1\rangle$ and $|\psi_2\rangle$ that represent S_1 and S_2, respectively, in HSQT, are not orthogonal. Hence, we get:

for every $S_1, S_2 \in \mathscr{S}_P$, $S_1 C S_2$ iff $\langle \psi_1 | \psi_2 \rangle \neq 0$.

Let us turn now to $\mathscr{E}_e$. We have said above that, for every $E_1, E_2 \in \mathscr{E}_e$, the physical theory $\mathscr{T}$ that one is considering establishes whether E_1 and E_2 can or cannot be conjointly true for to a given physical object. In the former case one can say that E_1 and E_2 are consistent in $\mathscr{T}$. This suggests to introduce a binary theory-dependent relation on $\mathscr{E}_e$, which we again denote by C and call *semantic compatibility*, or *consistency*, by setting:

for every $E_1, E_2 \in \mathscr{E}_e$, $E_1 C E_2$ iff an $i \in \hat{I}$ exists such that $\rho_i(E_1) \cap \rho_i(E_2) \neq \varnothing$.

We have thus provided a purely semantic definition of consistency, avoiding any (possibly misleading) resort to a model of $\mathscr{T}$. Furthermore, let us recall from Secs. 3 and 4 that the symbol $\perp$ denotes orthocomplementation in the lattice $(\mathscr{E}_e, <)$ and that, for every laboratory i and for every $E \in \mathscr{E}_e$, $\rho_i(E^\perp) = D_i \setminus \rho_i(E)$. Hence, for every physical theory which satisfies our axioms in Sec. 3, we get:

for every $E_1, E_2 \in \mathscr{E}_e$, $E_1 \not{C} E_2$ iff for every $i \in \hat{I}$, $\rho_i(E_1) \cap \rho_i(E_2) = \varnothing$ iff for every $i \in \hat{I}$, $\rho_i(E_1) \subseteq D_i \setminus \rho_i(E_2)$ iff for every $i \in \hat{I}$, $\rho_i(E_1) \subseteq \rho_i(E_2^\perp)$ iff $E_1 < E_2^\perp$ iff $E_2 < E_1^\perp$.

This characterization of consistency intuitively fits with the interpretation of the mapping $\perp$ as a negation in $\mathscr{E}_e$. More important, it allows us to establish a link between the consistency relation on $\mathscr{S}_P$ and the consistency relation on $\mathscr{E}_e$. Indeed, we get, for every $S_1, S_2 \in \mathscr{S}_P$, $E_1 \in \mathscr{E}_{S_1}$, $E_2 \in \mathscr{E}_{S_2}$, that E_1 and E_2 are not consistent iff $E_1 < E_2^\perp$, which implies $E_1 < E_{S_2}^\perp$, hence $E_{S_1} < E_{S_2}^\perp$; this last inequality holds iff $\mathscr{E}_{S_1} \cap \mathscr{E}_{S_2}^\perp \neq \varnothing$, or equivalently iff $S_1 \not{C} S_2$ so that $S_1 C S_2$ implies $E_1 C E_2$. Intuitively, this means that, whenever the properties E_1 and E_2 belong to the certainly yes domains of the states S_1 and S_2, respectively, they are consistent whenever S_1 and S_2 are consistent. In addition, the implications can be reversed whenever $E_1 = E_{S_1}$ and $E_2 = E_{S_2}$, so that we get:

$$S_1 C S_2 \quad \text{iff} \quad E_{S_1} C E_{S_2},$$

the interpretation of which is immediate.

Remark 5.1. It can be proved in CP (see G.91) that, for every laboratory i and pure state S, $\rho_i(E_S)$ contains all physical objects in $\rho_i(S)$ and a suitable percentage of physical objects for every nonpure state which admits a decomposition in terms of pure states where S appears, but it does not contain any physical object that belongs to the extension in i of a pure state different from S. This means that a physical object x which is in the pure state S^* has the property E_S iff $S^* = S$. In QP, on the contrary, the physical object x could have the property E_S even if it is in a pure state S^* different from S. Hence, when ignoring mixtures, pure states can be identified with their supports from a semantic viewpoint in CP, while they cannot in QP; this prohibits any identification between states and physical properties in QP, as we have already observed above (we retain that an erroneous identification of this kind is the deep root of some old quantum paradoxes in the literature[36]).

Remark 5.2. For every laboratory i the following equation can be proven to hold by inserting some further physical assumptions regarding nonpure states (see in particular G.91, condition MS) in our present context:

$$\bigcup_{S \in \mathscr{S}_p} \rho_i(E_S) = D_i.$$

It follows that, for every laboratory i and for every physical object x, at least one pure state S exists such that x has the property E_S in i; this exhibits, in particular, the semantic basis on which an "ignorance interpretation"[14] of nonpure states can be embodied in our approach.

6. PRAGMATIC COMPATIBILITY

The standard notion of compatibility of observables has an outstanding importance in QP. From a SR viewpoint, it is a *pragmatic* notion, which must be distinguished from the notion of semantic compatibility, or consistency, introduced in Sec. 5. Indeed, it can be translated in our present context by saying that the (testable) properties $F_1, F_2 \in \mathscr{F}_c$ are compatible according to the standard notion iff one can establish whether they both are true for a physical object x by means of a suitable measurement.

Of course, F_1 and F_2 are always compatible, according to the above notion, in CP, while they can be noncompatible in QP. Let us briefly resume the reasons of this difference between the two theories. In CP, one can always establish whether both F_1 and F_2 are true by performing a measurement that consists of the simultaneous measurements of F_1 and F_2, or, equivalently, of the measurements of F_1 and F_2 in sequence, provided that the first measurement does not influence the second (which is always possible, since it is assumed in CP that the disturbance, i.e., the change of state, induced by a measuring apparatus can be reduced below any prefixed limit). On the contrary, one is obliged in QP to measure F_1 and F_2 in sequence, and the disturbance induced by the measuring apparatuses cannot be reduced at will. It follows that the results of the measurements may depend on their order, since the second measurement applies to a physical object that has been disturbed by the first measurement. If this occurs, F_1 and F_2 are not compatible, for one cannot construct an apparatus testing whether F_1 and F_2 are simultaneously true. If this does not occur, i.e. if in every physical situation (state) sequential ideal measurements of F_1 and F_2 lead to results that are independent of the order of the measurements, an (ideal) apparatus testing F_1 and F_2 in sequence defines a testable property F, which is true for a given physical object x iff F_1 and F_2 are simultaneously true for x. In the latter case F_1 and F_2 are compatible (we shall see when discussing ideal measurements in Sec. 9 that, whenever an ideal test of F on a physical object x yields a positive answer, the properties F_1 and F_2 can be attributed to x both before and after the measurement). This leads us to formalize the standard notion of compatibility in our generalized framework as follows.

Let $F_1, F_2 \in \mathscr{F}_c$. We say that F_1 and F_2 are *pragmatically compatible*, or *conjointly testable*, and write $F_1 \mathscr{K} F_2$, iff a property $F \in \mathscr{F}_c$ exists such that, for every laboratory $i \in \hat{I}$, $\rho_i(F) = \rho_i(F_1) \cap \rho_i(F_2)$.

Since the set $(\mathscr{E}_c, <)$ is a lattice (Secs. 3 and 4), the following statement can be easily proved which characterizes pragmatic compatibility.

For every $F_1, F_2 \in \mathscr{F}_c$, $F_1 \mathscr{K} F_2$ iff the meet $F = F_1 \sqcap F_2$ belongs to $\mathscr{F}_c$ and, for every laboratory $i \in \hat{I}$, $\rho_i(F) = \rho_i(F_1) \cap \rho_i(F_2)$.

The above statement implies, in particular, that, for every first type state $S \in \mathscr{S}_1$ and $F \in \mathscr{F}_c$, the support E_S of S is pragmatically compatible with F iff $F \in \mathscr{E}_S \cup \mathscr{E}_S^{\perp}$. More important, it allows us to explore the links between our definition of pragmatic compatibility and some current definitions of compatibility in the literature. To this end, let us add the following axiom to the axioms listed in Sec. 3.

AX 9. For every $F_1, F_2 \in \mathscr{F}_c$, $F_1 \mathscr{K} F_2$ iff $F_1 \mathscr{K} F_2^{\perp}$ iff $F_1^{\perp} \mathscr{K} F_2$ iff $F_1^{\perp} \mathscr{K} F_2^{\perp}$.

AX 9 rests on the obvious physical remark that, for every $F \in \mathscr{F}_c$, F and $F^{\perp}$ can be tested by means of the same dichotomic device in every laboratory i, since $\rho_i(F^{\perp}) = D_i \setminus \rho_i(F)$, as we have seen in Sec. 3. Moreover it entails, because of the above characterization of pragmatic compatibility, that, whenever $F_1 \mathscr{K} F_2$, ρ_i is a lattice isomorphism of the sublattice of $(\mathscr{E}_c, <)$ generated by F_1 and F_2 onto the Boolean sublattice of $(\mathscr{P}(D_i), \subseteq)$ generated by $\rho_i(F_1)$ and $\rho_i(F_2)$. Hence we conclude, in particular, that every pair of pragmatically compatible properties generates a Boolean sublattice of $(\mathscr{E}_e, <)$. This result is relevant, since a relation of compatibility is introduced in $(\mathscr{E}_e, <)$ (to be precise, in a lattice isomorphic to $(\mathscr{E}_e, <)$, see Sec. 3) in some approaches to QP by saying that the properties F_1, $F_2 \in \mathscr{F}_e$ are compatible iff they generate a Boolean sublattice of $(\mathscr{E}_e, <)$.[43]

Finally, we note that our definition of pragmatic compatibility can be generalized as follows.

Let $F_1, F_2, \ldots, F_n \in \mathscr{F}_e$. We say that $F_1, F_2, \ldots, F_n$ are *pragmatically compatible*, or *conjointly testable*, iff a property $F \in \mathscr{F}_e$ exists such that, for every laboratory $i \in \hat{I}$, $\rho_i(F) = \rho_i(F_1) \cap \rho_i(F_2) \cap \cdots \cap \rho_i(F_n)$.

Resting on the above generalization, we add the following further axiom to the axioms listed in Sec. 3.

AX 10. Let $F_1, F_2, \ldots, F_n \in \mathscr{F}_e$. Then, $F_1, F_2, \ldots, F_n$ are pragmatically compatible whenever they are pairwise pragmatically compatible, that is, $F_j \mathscr{K} F_k$ for every $j, k \in \{1, 2, \ldots, n\}$.

Stating AX 10 can be justified by observing that it holds both in QP and in CP, which we want to be models for our general SR scheme.

Remark 6.1. We recall that the properties F_1 and F_2 are assumed to be compatible in HSQT iff they are represented by commuting projections, hence the relation of compatibility is defined on the whole $\mathscr{E}_e$. In our general scheme, of which HSQT is a model (Sec. 3), we have preferred to define pragmatic compatibility on the set $\mathscr{F}_e$ of testable properties only, since the enlargement of our formal definition to $\mathscr{E}_e$ would let $\mathscr{K}$ depend on the (partially) arbitrary choice of the extensions, in every laboratory, of the theoretical properties; furthermore, $\mathscr{K}$ would have no direct operational interpretation when referring to theoretical properties. Of course, this discrepancy is connected with the refinements in the interpretation of HSQT suggested by the general theory and underlined in Remark 3.2.

7. TESTABILITY AND CONJOINT TESTABILITY ON Ψ_e

Let us now shift from the set $\mathscr{F}_e$ to the set Ψ_e of all wffs of L_e. Then, the notions of testability and pragmatic compatibility (conjoint testability) can be canonically extended to Ψ_e, as follows.

Let $A \in \Psi_e$. We say that A is *testable* whenever in every laboratory $i \in \hat{I}$ the truth value of A for every interpretation σ_i can be determined by means of suitable measurements.

By recalling our interpretation of L_e in Sec. 4, we immediately pick out the following basic sets of testable wffs of L_e (see G.91):

(i) for every $x \in X$, the set $\mathscr{F}_c(x) = \{F(x) | F \in \mathscr{F}_c\}$;

(ii) for every $S \in \mathscr{S}$, the set Ψ_c^S of all quantified wffs of the form $(\pi_r x)(A(x)/S(x))$, with $A(x)$ a testable nonquantified (or *open*) wff of Ψ_c (note that the wffs of Ψ_c^S are testable since, for every laboratory i, the domain D_i is finite).

Furthermore, we can prove the following *criterion of testability*.

CT. Let $A(x) \in \Psi_c$ be an open wff, which contains the individual variable x only. Then, $A(x)$ is testable iff an atomic wff $F_A(x) \in \mathscr{F}_c(x)$ exists such that $A(x) \equiv F_A(x)$.

Indeed, let us note that, whenever $A(x)$ is logically equivalent to $F_A(x)$, the truth value of $A(x)$ in every laboratory i for every interpretation σ_i can be determined by means of a measurement of the truth value of $F_A(x)$, hence $A(x)$ is testable. Conversely, if $A(x)$ is testable, it defines a *derived* property of the physical object x that is testable, hence it is logically equivalent to some testable property $F_A \in \mathscr{F}_c$, since we have assumed that $\mathscr{F}_c$ is the set of all testable properties of the physical system that we are considering (Sec. 3).

It is apparent that CT can be used in order to single out further sets of testable statements in Ψ_c. For instance, we get that, for every $F \in \mathscr{F}_c$, the molecular wff $\neg F(x)$ is testable; indeed, $\neg F(x)$ is equivalent to the atomic statement $F^\perp(x)$, where $F^\perp \in \mathscr{F}_c$ (see Sec. 2(iii) and Sec. 3).

Let us define now *conjoint testability* in Ψ_c, as follows.

Let $A_1, A_2 \in \Psi_c$. Then, we say that A_1 and A_2 are *conjointly testable* iff A_1 and A_2 are testable and, for every $i \in \hat{I}$ and $\sigma_i \in \Sigma_i$, the truth values of A_1 and A_2 can be determined conjointly by means of suitable measurements.

By using this definition, we can prove the following proposition.

P 7.1. Let $A_1(x), A_2(x), \ldots, A_n(x)$ be open wffs of Ψ_c, and let $F_1, F_2, \ldots, F_n \in \mathscr{F}_c$ be such that $A_1(x) \equiv F_1(x)$, $A_2(x) \equiv F_2(x), \ldots, A_n(x) \equiv F_n(x)$. Then $A_1(x), A_2(x), \ldots, A_n(x)$ are pairwise conjointly testable iff $F_1, F_2, \ldots, F_n$ are pairwise pragmatically compatible, or iff the wff $A(x) = A_1(x) \wedge A_2(x) \wedge \cdots \wedge A_n(x) \equiv F_1(x) \wedge F_2(x) \wedge \cdots \wedge F_n(x)$ is testable.

Proof. The first equivalence follows from the definition of conjoint testability in Ψ_c, from CT, and from our interpretation of the formal definition of pragmatic compatibility in Sec. 6. In order to prove the second equivalence, let us consider the wff $A(x)$. Because of CT, $A(x)$ is testable iff it is logically equivalent to an atomic wff $F_A(x)$ of $\mathscr{F}_c(x)$. But we have seen in Sec. 2 that $F_1(x) \wedge F_2(x) \wedge \cdots \wedge F_n(x)$, hence $A(x)$, is true in a laboratory i iff $\sigma_i(x) \in \rho_i(F_1) \cap \rho_i(F_2) \cap \cdots \cap \rho_i(F_n)$. Therefore, $A(x)$ is testable iff a property $F_A \in \mathscr{F}_c$ exists such that for every laboratory i, $\rho_i(F_A) = \rho_i(F_1) \cap \rho_i(F_2) \cap \cdots \cap \rho_i(F_n)$. By using the generalized definition of pragmatic compatibility on $\mathscr{F}_c$ introduced at the end of Sec. 6, we get that A(x) is testable iff $F_1, F_2, \ldots, F_n$ are pragmatically compatible, hence iff they are pairwise pragmatically compatible, because of AX 10. □

As a corollary of P 7. 1, we note that, because of the characterization of pragmatic compatibility supplied in Sec. 6, the property F_A that appears in the above proof whenever A(x) is testable is given by $F_A = F_1 \sqcap F_2 \sqcap \cdots \sqcap F_n$.

Remark 7.1. It is important in what follows to make an explicit recognition of some properties of the basic set $\mathscr{F}_c(x)$ of testable wffs of L_c introduced above. Therefore, we note that the restriction of the pre-orders $\subset$, $\angle$, and $<$ to $\mathscr{F}_c(x)$ are partial orders, which coincide because of our assumptions on $\mathscr{F}_c$ in Sec. 3 (see in particular AX 5). Of course, $(\mathscr{F}_c(x), \subset)$ is isomorphic to the poset $(\mathscr{F}_c, <)$, which is dense in the lattice $(\mathscr{E}_c, <)$ (Sec. 3), hence $(\mathscr{F}_c(x), \subset)$ is dense in the lattice $(\mathscr{E}_c(x), \subset)$, with $\mathscr{E}_c(x) = \{E(x) | E \in \mathscr{E}_c\}$. This latter lattice is isomorphic to the further lattice $([\mathscr{E}_c(x)]_\equiv, \subset)$, with $[\mathscr{E}_c(x)]_\equiv = \{[E(x)]_\equiv \mid E \in \mathscr{E}_c\}$. In QP, both $(\mathscr{E}_c(x), \subset)$ and $([\mathscr{E}_c(x)]_\equiv, \subset)$ can be identified with standard QL, which proves that quantum logics can be obtained by using the (theory dependent) pragmatic concept of testability for selecting suitable subsets of wffs in L_c, as we have anticipated in the Introduction (see also G.91, and Refs. 3 and 4). The non-Boolean character of QL is then originated by the fact that $([\mathscr{E}_c(x)]_\equiv, \subset)$ is a subposet but not a sublattice of the Boolean lattice $(\Psi_c/\equiv, \subset)$ in QP (while it is a sublattice of $(\Psi_c/\equiv, \subset)$ in CP).

Let us denote the lattice operations in $(\mathscr{E}_c(x), \subset)$ by the same symbols that we have introduced in Sec. 3 in order to denote the corresponding operations in $(\mathscr{E}_c, <)$. Because of the truth theory assumed in L_c, every wff in $\mathscr{E}_c(x)$ has a truth value in every laboratory i when x is interpreted. But the connective $\sqcap$ cannot be identified with the classical conjunction $\wedge$, and $\sqcup$ cannot be identified with the disjunction $\vee$. In particular, $\sqcap$ and $\sqcup$ are not true-functional in QP,[4] which means that, for example, the truth value of the join $E_1(x) \sqcap E_2(x) \equiv (E_1 \sqcap E_2)(x)$ (where $E_1, E_2 \in \mathscr{E}_c$) generally cannot be deduced from the truth values of $E_1(x)$ and $E_2(x)$ only. Furthermore, if we embed canonically $\mathscr{E}_c(x)$ in Ψ_c and regard $\perp$, $\sqcap$, $\sqcup$ as connectives in Ψ_c defined on the subset $\mathscr{E}_c(x)$ of Ψ_c, we get, for every E, $E_1, E_2 \in \mathscr{E}_c$ (see Sec. 3, Remark 3.4. and Sec. 4. conditions (i), (ii), and (iii)):

$$\neg E(x) \equiv E^\perp(x),$$

$$E_1(x) \sqcap E_2(x) \equiv (E_1 \sqcap E_2)(x) \subset E_1(x) \wedge E_2(x),$$

$$E_1(x) \wedge E_2(x) \subset (E_1 \sqcup E_2)(x) \equiv E_1(x) \sqcup E_2(x),$$

By using P 7.1 and its corollary, we see that, for every $F_1, F_2 \in \mathscr{F}_c$, $F_1(x) \bigcap F_2(x) \equiv F_1(x) \wedge F_2(x)$ iff $F_1(x) \wedge F_2(x)$ is testable or, equivalently, iff F_1 and F_2 are pragmatically compatible. Analogously, $F_1(x) \bigcup F_2(x) \equiv F_1(x) \vee F_2(x)$ iff F_1 and F_2 are pragmatically compatible. Indeed, let $F_1 \mathscr{K} F_2$. Then, $F_1^\perp \mathscr{K} F_2^\perp$ (Sec. 6), hence, for every $i \in \hat{I}$, $\rho_i(F_1^\perp \bigcap F_2^\perp) = \rho_i(F_1^\perp) \cap \rho_i(F_2^\perp) = D_i \setminus \rho_i(F_1) \cap D_i \setminus \rho_i(F_2) = D_i \setminus (\rho_i(F_1) \cup \rho_i(F_2))$. It follows that $\rho_i((F_1^\perp \bigcap F_2^\perp)^\perp) = \rho_i(F_1 \bigcup F_2) = \rho_i(F_1) \cup \rho_i(F_2)$, so that $F_1(x) \bigcup F_2(x) \subset F_1(x) \vee F_2(x)$. Therefore, $F_1(x) \bigcup F_2(x) \equiv F_1(x) \vee F_2(x)$, as stated. Conversely, one easily gets that this last equivalence implies $F_1 \mathscr{K} F_2$.

8. THE METATHEORETICAL GENERALIZED PRINCIPLE

We recall that, according to a standard epistemological conception (*received viewpoint*[19]), one must distinguish between *theoretical* and *empirical* physical laws. Whenever the language of physics is suitably formalized by means of a general language L*, where second-order predicates, predicative variables, and quantification on predicative variables occur (see Remark 2.3), the theoretical laws are expressed by sentences of L* that contain primitive or derived *theoretical terms*, are only partially interpreted (hence cannot be directly tested), and have no truth value, or a conventional truth value only. On the contrary, empirical laws can be expressed by means of an observative part (not necessarily a sublanguage as we shall see in the following) of L*, can be formally deduced from theoretical laws, and have a truth value, since they are empirically interpreted.

Let us accept from now on the above viewpoint. One may then wonder about the role played in L* by the formalized languages L and L_c introduced in Sects. 2 and 4, respectively. Let us firstly consider L. Then, we see that all atomic wffs of L are constructed by means of predicates that are interpreted operationally, hence we say that they express observative statements. But the complex wffs of L may be testable or not (Sect. 7), so that we cannot say that all statements in L are observative. We, however, classify L here as an observative sublanguage of L, as we have anticipated in Remark 2.3, since it is generated as a sublanguage of L* by a set of first-order testable wffs (to be precise, L should be classified as an *observatively minimal* sublanguage of L*). Now, let us consider L_c. Then, we see that it contains both observative and theoretical atomic wffs. Therefore, L_c can still be considered a sublanguage of L*, yet not observative in the sense specified above.

It follows from our analysis on L and L_c that both empirical and theoretical laws can be expressed by means of the sublanguages L and L_c (though general theoretical laws require L*). We retain that the awareness that some physical laws expressed in L_c are theoretical, not empirical, is basic for explaining EPR-like paradoxes and, more generally, some difficulties encountered in the quantum theory of compound physical systems. Therefore let us discuss this subject more deeply.

Let us firstly explore the form taken by those physical laws that can be expressed by means of L_c. Bearing in mind our interpretation of L and L_c in Sects. 2 and 4, respectively, and recalling that we have not introduced time in our scheme since we do not intend to deal with evolution in time in this paper, we can agree that a typical sample of physical law is a compound sentence which establishes, whenever we consider all physical objects such that a given sentence of L_c is true, the percentage of objects for which another sentence of L_c is also true (more complex forms are not excluded, but do not interest us here). A particular case of this kind of physical law is formalized in L_c by the wff:

$$V = (\pi_r x)(A(x)/S(x)),$$

with $r \in [0, 1]$, $S \in \mathscr{S}$ and $A(x)$ an open molecular wff of L_c where only predicates denoting physical properties occur.

Let us comment on this canonical form.

First, we note that, whenever $r = 1$, one gets (see G.91):

$$V = (\pi_1 x)(A(x)/S(x)) \equiv (\forall x)(S(x) \to A(x)).$$

which provides a sample of deterministic law (which can obviously occur even in QP).

Second, we note that many wffs usually exist in Ψ_c that are logically equivalent to V. In particular, whenever a wff $B(x) \in \Psi_c$ exists such that $B(x) \equiv A(x)$, we get $V \equiv (\pi_r x)(B(x)/S(x))$. Because of the definition of statistical equivalence $\simeq$ in Sec. 2(v), we get $V \equiv (\pi_r x)(B(x)/S(x))$ also if $B(x)$ satisfies the condition $B(x) \simeq A(x)$, which is weaker than $B(x) \equiv A(x)$. In addition, whenever $r = 1$ and $B(x)$ satisfies the condition $B(x) \approx A(x)$, which is weaker than $B(x) \simeq A(x)$, we get, because of the definition of deterministic equivalence in Sec. 2(v), $V \equiv (\forall x)(S(x) \to B(x))$.

Let us now introduce the basic distinction between *empirical* and *theoretical* laws.

We say that V expresses an empirical physical law whenever $A(x)$ is testable in the sense specified in Sec. 7, that is, because of CT, whenever a testable property $F_A \in \mathscr{F}_c$ exists such that $A(x) \equiv F_A(x)$. In this case $V \equiv (\pi_r x)(F_A(x)/S(x))$, and the truth value of V can be determined empirically in every laboratory $i \in \hat{I}$ by means of any registering device in the class denoted by F_A: an empirical physical law can be *directly tested*.

We say that V expresses a theoretical physical law whenever A(x) is not testable, that is, whenever no $F \in \mathscr{F}_c$ exists such that $A(x) \equiv F(x)$. Now, it is apparent that the truth value of such a law in a laboratory i, though defined in our approach, cannot be directly tested. Furthermore, the truth value of A(x) is partially conventional, because of the conventions introduced for logical connectives and/or because of the possible presence in A(x) of theoretical predicates, the extensions of which have a certain degree of arbitrariness (see Sec. 4). Therefore, we must attribute to V a role which is basically different from the role attributed to empirical physical laws. Following a standard epistemological viewpoint,[15, 16] we agree to consider V as a formal expression, which is acceptable as a theoretical physical law (independently of its partially conventional truth value) if every empirical physical law that can be deduced from it (and from suitable premises) by means of standard procedures in classical logic turns out to be true when directly tested in a laboratory $i \in \hat{I}$.

The deduction of empirical physical laws from a theoretical law V can be rather complicate. In some cases it can be made easier by substituting A(x) (which is nontestable, since V is theoretical) with a suitable wff B(x) which is logically, or statistically, or deterministically equivalent to A(x). For instance, let $A(x) \simeq F(x)$, with $F \in \mathscr{F}_c$. Then, we immediately obtain from V (which cannot be directly tested) the empirical physical law $(\pi_r x)(F(x)/S(x))$. Analogously, let $r = 1$ and $A(x) \approx F(x)$, with $F \in \mathscr{F}_c$. Then, we immediately obtain from V the empirical physical law $(\forall x)(S(x) \rightarrow F(x))$.

We must now discuss a crucial problem in our approach, that is, the problem of the *truth mode* that is to be attributed to empirical laws deduced from theoretical laws of the kind considered above. Indeed, one of us has attributed to empirical laws a truth mode in G.91 which makes explicit the classical epistemological viewpoint that is universally adopted when dealing with this kind of laws, i.e., has assumed that an empirical law must be true in every laboratory $i \in \hat{I}$ (*metatheoretical classical principle*, or briefly, MCP). But our point here is that this perspective does not take into due account the existence of physical theories, as QP, where a nontrivial relation of pragmatic compatibility is defined on the set of physical properties. Let us consider the problems that occur in this case.

We start from the obvious remark that the statement of testable physical predictions is one of the relevant aims of any physical theory. In order to obtain these predictions physicists usually adopt rather complex inference procedures, the basic step of which can be schematized as follows. An empirical physical law of the form $(\pi_r x)(A(x)/S(x))$, possibly deduced from the general theoretical apparatus of the theory, is introduced, together with a *boundary condition* S(x) and a set $\{A_1, A_2, ..., A_n\}$ of testable wffs of

L_c or *premises*. Then, a testable physical prediction A_{n+1} is deduced which adds to the set of premises for a further step of the same type. The wffs $A_1, A_2, \ldots, A_n$ can be considered predictions following from previous inference steps, or measurement results, or simply assumptions (we note that an elementary testable physical prediction regarding a physical object x is expressed in L_c by a testable atomic wff F(x) which attributes a testable property $F \in \mathscr{F}_c$ to x, while an elementary testable statistical prediction is expressed by a statistical wff $(\pi_r x)(F(x)/S(x))$, where $S \in \mathscr{S}$ and $F \in \mathscr{F}_c$).

Whenever a nontrivial relation of pragmatic compatibility is defined on $\mathscr{F}_c$, the premises $A_1, A_2, \ldots, A_n$ could be chosen so that they are not conjointly testable. A physical situation of this kind is not epistemically accessible, in the sense that one cannot empirically test whether it occurs, and no prediction can be verified. Therefore, it is inconsistent with an operational philosophy to assert that an empirical law is true in a laboratory where such a situation occurs, hence that it can be used in order to obtain further physical predictions. Thus, it seems appropriate to look for a generalization of MCP which reduces to MCP whenever a physical theory (as CP) is considered in which the relation of pragmatic compatibility is trivial but yields a new and subtler characterization of the truth mode of empirical physical laws whenever a nontrivial relation of pragmatic compatibility occurs.

A generalization of this kind has been proposed by one of the authors in some previous papers[6-9] and has been called *metatheoretical generalized principle* (MGP). We express it here as follows.

MGP. Let $V \in \Psi_c$ express a theoretical physical law, let $x \in X$, $S \in \mathscr{S}$, $A(x) \in \Psi_c$, let A(x) be testable, and let the wff $V_A = (\pi_r x)(A(x)/S(x))$ express an empirical physical law deduced from V. Then, V_A can be asserted to be true in every laboratory $i \in \hat{I}$ where a set of conjointly testable premises is assumed.

Let us comment briefly on MGP. It is apparent that this new principle does not modify any empirical quantum prediction, but it establishes a kind of restricted availability of empirical physical laws, since it implies that an empirical law that can be formally deduced from a theoretical law V cannot be asserted to be true in a given laboratory $i \in \hat{I}$ if one assumes in *i* premises that are not conjointly testable (indeed, V could be true as well as false in *i*). This restricted availability may seem disconcerting, since the classical viewpoint is deeply rooted in our usual way of thinking, but it does not contradict a minimal realistic viewpoint (*realism of properties* in the Introduction), and it is based on full acceptance of the operational philosophy of QP together with a correspondence truth theory for L_c: it only prohibits applying that form of ontological realism according to which all theoretical entities in our theories are "real" (i.e., must correspond to natural entities which actually exist), to a theory endowed with a nontrivial relation of pragmatic compatibility. One can say that MGP limits the physicist's "presumption of omniscience," and opens the way to a more flexible conception of physical theories.

It is obvious that MGP satisfies our above condition of reducing to MCP whenever it is applied to a theory, as CP, in which all properties are pragmatically compatible. But the (usually implicit) adoption of MCP in place of MGP in QP can be viewed at as the deep root of many quantum paradoxes.[36]

Before closing this section, we briefly study some particular cases, which help to grasp the deep meaning of MGP.

(i) Let $V \in \Psi_c$ express a theoretical physical law, and let $V_1 = (\forall x)(S(x) \rightarrow F_1(x))$, $V_2 = (\forall x)(S(x) \rightarrow F_2(x))$,..., $V_n = (\forall x)(S(x) \rightarrow F_n(x))$, with $F_1, F_2, \ldots, F_n \in \mathscr{F}_c$, be empirical physical laws that can be deduced from V. Let $i \in \hat{I}$ and let us choose in i a subset $\{V_{i_1}, V_{i_2}, \ldots, V_{i_m}\}$ of $\{V_1, V_2, \ldots, V_n\}$. Then, MGP does not allow us to assert that $V_{i_1}, V_{i_2}, \ldots, V_{i_m}$ are true in i, except whenever $F_{i_1}, F_{i_2}, \ldots F_{i_m}$ are pragmatically compatible. Indeed, let the physical object x be in the state S in i. By choosing in i a set of premises (that can be void) and deducing $V_{i_1}, V_{i_2}, \ldots, V_{i_m}$ from V in sequence, we see that in every step the deduction of an empirical law, say V_{i_k}, yields a premise $F_{i_k}(x)$ that must be adjoined to the set of premises in the further step. This increase can transform a set of pragmatically compatible premises into a set of premises that are not pragmatically compatible, which prohibits one to deduce any further empirical law in $\{V_{i_1}, V_{i_2}, \ldots, V_{i_m}\}$ as a true wff in i.

Let us apply this result in a special case. Let $V = (\forall x)(S(x) \rightarrow (F_1 \cap F_2 \cap \cdots \cap F_n)(x)) = (\forall x)(S(x) \rightarrow E(x))$ be a theoretical physical law in QP, with $E = F_1 \cap F_2 \cap \cdots \cap F_n \in \mathscr{G}_c$ and $F_1, F_2, \ldots, F_n \in \mathscr{F}_c$. Because of proposition P 4.1 (the generalization of which to the case of $n \geqslant 3$ testable physical properties is obvious), we get:

$$V \equiv (\forall x)(S(x) \rightarrow F_1(x) \wedge F_2(x) \wedge \cdots \wedge F_n(x))$$

(we also note that E is unique, in the sense that $V \equiv (\forall x)(S(x) \rightarrow E_1(x))$ implies $E_1 = E$ because of proposition P 4.2). Then, the above empirical laws $V_1, V_2, \ldots, V_n$ can be deduced from V. Let $i \in \hat{I}$, let us choose in i the empty set of premises and a subset $\{V_{i_1}, V_{i_2}, \ldots, V_{i_m}\}$ of $\{V_1, V_2, \ldots, V_n\}$, and let us consider the wffs $U = (\forall x)(S(x) \rightarrow A(x))$ and $W = (\forall x)(S(x) \rightarrow B(x))$, with $A(x) = (F_{i_1} \cap F_{i_2} \cap \cdots \cap F_{i_m})(x)$ and $B(x) = F_{i_1}(x) \wedge F_{i_2}(x) \wedge \cdots \wedge F_{i_m}(x)$. Then U and W express physical laws that can be formally deduced from V. Furthermore, $A(x) \subset B(x)$ (Remark 7.1) and $A(x) \approx B(x)$ (P 4.1), so that $U \equiv W$. Three possibilities occur, which we will consider separately.

First, $F_{i_1}, F_{i_2}, \ldots, F_{i_m}$ are pairwise pragmatically compatible, hence pragmatically compatible (AX 10). Then, MGP implies that $V_{i_1}, V_{i_2}, \ldots, V_{i_m}$ are true in i. Furthermore, the wffs A(x) and B(x) are testable and logically equivalent (Remark 7.1), and U and W are empirical physical laws that are true in i.

Second, $F_{i_1}, F_{i_2}, \ldots, F_{i_m}$ are not pragmatically compatible and $F_{i_1} \cap F_{i_2} \cap \cdots \cap F_{i_m} \in \mathscr{F}_e$. In this case MGP does not imply directly that all $V_{i_1}, V_{i_2}, \ldots, V_{i_m}$ are true in i. However, let us note that A(x) is testable while B(x) is not testable (P 7.1), so that A(x) and B(x) are not logically equivalent, and U and W are an empirical and a theoretical physical law, respectively. Then, MGP entails that U is true in i. Since $U \equiv W$, we can assert that also W is true in i, which implies that $V_{i_1}, V_{i_2}, \ldots, V_{i_m}$ are true in i.

Third, $F_{i_1}, F_{i_2}, \ldots, F_{i_m}$ are not pragmatically compatible, and $F_{i_1} \cap \cdots$ $F_{i_2} \cap \cdots \cap F_{i_m} \in \mathscr{T}_e$. The wffs A(x) and B(x) are not testable, nor they are logically equivalent, and U and W are theoretical physical laws. In this case MGP does not assure that all $V_{i_1}, V_{i_2}, \ldots, V_{i_m}$ are true in i, nor that U or W are true in i.

(ii) The case studied above can be generalized as follows. Let us consider the theoretical physical law $V = (\forall x)(S(x) \rightarrow E(x))$ in QP, with $E \in \mathscr{T}_e$. Since, for every $E \in \mathscr{T}_e$, $E = \bigcap_{F \in \mathscr{F}_e, E < F} F$ (Sec. 3), we obtain:

$$V \equiv (\forall x)\left(S(x) \rightarrow \left(\bigcap_{F \in \mathscr{F}_e, E < F} F\right)(x)\right).$$

Let us assume that proposition P 4.1 can be extended to the meet in the expression of E. Hence, we get:

$$V \equiv (\forall x)\left(S(x) \rightarrow \bigwedge_{F \in \mathscr{F}_e, E < F} F(x)\right).$$

If a laboratory $i \in \hat{I}$ is given, MGP can be applied, and the arguments offered in case (i) can be easily generalized. In particular, let $\mathscr{G}$ be a finite subset of $\{F \in \mathscr{F}_e | E < F\}$. Whenever for every $F_1, F_2 \in \mathscr{G}$, $F_1 \mathscr{K} F_2$, MGP implies that the empirical physical law

$$U = (\forall x)\left(S(x) \rightarrow \bigwedge_{F \in \mathscr{G}} F(x)\right)$$

is true in i. But MGP does not guarantees in general, the truth of U if the pragmatic compatibility of all $F \in \mathscr{G}$ is not assured.

Remark 8.1. Bearing in mind our arguments in Remark 3.3, we can say that a theoretical law of the form $V = (\pi_r x)(A(x)/S(x))$ translates in our language L_c, which is a first-order predicate calculus extended by means of statistical quantifiers, a theoretical law that should otherwise be expressed by means of higher-order predicates and variables and, possibly, modal probability operators[36] of the general language L*. The language L_c is obviously simpler than L* but the main reason for introducing it is the need for explaining the success of HSQT from one side, and to avoid EPR-like paradoxes from the other side.

Remark 8.2. We have already observed at the end of the Introduction that SR is not a conventional hidden variables theory. Our statement of MGP in this section allows us to make our argument more perspicuous. Indeed, it has been pointed out by Kochen and Specker[28] that hidden variables can be invented for every physical theory $\mathscr{T}_1$ if the requirements imposed on their values are sufficiently weak. Thus, the actual problem with hidden variables consists in defining the restrictions that the theory $\mathscr{T}_2$, assumed to be a hidden variables theory for $\mathscr{T}_1$, must satisfy. The constraints proposed by Kochen and Specker[28] seem quite reasonable; this notwithstanding, when $\mathscr{T}_1$ is identified with QP, they lead to the well-known "no-go" Bell and Bell–KS theorems[29] which show that no noncontextual hidden variables theory is possible for QP. This engenders a number of epistemological difficulties which suggests looking deeper into the matter. Thus, one sees that Kochen and Specker's constraints follow from accepting implicitly the general principle MCP, since they formalize the requirement that $\mathscr{T}_2$ preserve the unrestricted validity of the relations among observables established in $\mathscr{T}_1$ (which express theoretical physical laws). If one adopts a SR position, he is then induced to introduce weaker constraints for $\mathscr{T}_2$, that is, the constraints following from the general principle MGP that substitutes MCP: in particular, $\mathscr{T}_2$ could entail empirical physical laws different from those of $\mathscr{T}_1$ in physical situations that are not accessible according to $\mathscr{T}_1$.

The above general arguments explain why the aforesaid "no-go" theorems fail to be true in the specific case of the SR approach to QP,[6, 8, 36] and open the way to noncontextual hidden variables theories for QP. It must, however, be noted that SR, being a general scheme, cannot constitute in itself a detailed hidden variables model for QP (hence, in particular, SR cannot provide a model that gives a "physical explanation" of the possible falsity of the physical laws of QP in a laboratory where premises are assumed that are not conjointly testable according to QP, even if it does not prohibit that such a model exist). Rather one could interpret each property in the SR approach as a hidden variable, taking values 0 and 1, and consider SR as a general, nonconventional (in the sense that it does not assume MCP) hidden variables frame for every physical theory.

9. SEMANTIC INCOMPLETENESS OF QP

The distinction between truth and testability that has been introduced in our approach allows us to distinguish between t-completeness and s-completeness of a physical theory $\mathscr{T}$ with respect to the general language L* by means of which the theory is stated (Remark 2.3). To be precise, a theory will be said to be t-complete (respectively, s-complete) with respect to L* iff the laws of the theory, together with suitable sets of specific assumptions, allow one to predict the truth values of all testable (respectively all) interpreted wffs of L*.[2,4] Of course, s-completeness implies t-completeness, but the converse is not generally true.

The distinction between t-completeness and s-completeness can also be introduced with reference to a sublanguage of L* (note that a theory is t-complete iff it is s-complete when restricting to L; this follows from the fact that all atomic wffs of L are testable and from the adoption of a Tarskian true-functional truth theory in L). In particular, by using the interpretation of pure states as maximal amounts of information discussed in Sec. 5, we conclude that, whenever a theory $\mathscr{T}$ is t-complete with respect to L_c, the knowledge that a statement of the form S(x), with $S \in \mathscr{S}_p$, is true in a laboratory i must allow us to deduce the truth value in i of all (testable) statements of the form F(x), with $x \in X$ and $F \in \mathscr{F}_c$, by using the laws of $\mathscr{T}$ (an interpretation σ_i of the individual variables being implied). Then, one can easily prove, adapting procedures worked out elsewhere[2] to our present generalized context, that QP is not t-complete, hence it is not s-complete with respect to L_c (while CP proves to be complete in both senses).

The above assertion might appear a rather complicated way of restating an obvious consequence of QP; indeed, it essentially means that the knowledge that a physical object is in the state S does not necessarily allow us to know, for any given $F \in \mathscr{F}_c$, whether x has the testable property F. But there is an important novelty in our present perspective: indeed, we assign a truth value to every (atomic or molecular) interpreted statement of L_c, independently of the epistemic accessibility of the truth value itself in QP (this assignment generalizes in our context the "realistic" assumptions introduced by Wigner in his 1970 proof of a Bell inequality[31,36]). On the contrary, a statement attributing a testable physical property F_2 to a physical object x whenever a testable property F_1, not pragmatically compatible with F_2, has been measured on x, or a molecular statement where the meet of nonconjointly testable statements appears, would be considered meaningless, hence having no truth value, by physicists adopting a verificationist theory of truth. Thus, we can classify QP as semantically incomplete, while the orthodox viewpoint considers QP complete, since no meaningless statement is allowed in the language of physics.

The incompleteness of QP and the differences between our interpretation and the orthodox one can be better understood by introducing, for every physical object x and laboratory *i*, the sets $\mathscr{F}^{T}_{ix}$ and $\mathscr{F}^{F}_{ix}$ of all true testable properties and of all false properties of x in *i*, respectively (an interpretation σ_i being implied). Indeed, let us firstly note that the pair $\{\mathscr{F}^{T}_{ix}, \mathscr{F}^{F}_{ix}\}$ is a partition of $\mathscr{F}_e$ because of our semantic model for L_e. Then, for every pure state S, let us recall the definitions of $\mathscr{E}_S$ and $\mathscr{E}^{\perp}_S$ introduced in Sect. 5, and let us put $\mathscr{F}_S = \mathscr{E}_S \cap \mathscr{F}_e$, $\mathscr{F}^{\perp}_S = \mathscr{E}^{\perp}_S \cap \mathscr{F}_e$. It follows that $\mathscr{F}_S$ (respectively, $\mathscr{F}^{\perp}_S$) is the broadest set of testable properties that one can predict to be true (respectively, false) for any physical object x in a laboratory *i*, by using physical laws and the assumption that x is in the state S in *i*. Then, trivially, $\mathscr{F}_S \subseteq \mathscr{F}^{T}_{ix}$ and $\mathscr{F}^{\perp}_S \subseteq \mathscr{F}^{F}_{ix}$, and it can be proved that $\mathscr{F}_S = \mathscr{F}^{T}_{ix}$ and $\mathscr{F}^{\perp}_S = \mathscr{F}^{F}_{ix}$ in CP, and that $\mathscr{F}_S \subset \mathscr{F}^{T}_{ix}$ and $\mathscr{F}^{\perp}_S \subset \mathscr{F}^{F}_{ix}$ in QP.

The strict inclusions formally express the t-incompleteness of QP with respect to L_e, and have some important consequences. First, they imply that a change of state of a physical object does not necessarily modify its testable physical properties in QP (while it does in CP), though it modifies the set of testable properties that can be predicted to be true (this occurs, for instance, whenever an ideal quantum measurement is made, as we shall see in Sec. 10): thus, we explain some features of Bohr's "relational conception of quantum states"[25] in our context, though our viewpoint is different from Bohr's. Second, they imply that different objects in the same state S can be thought of as endowed with different properties (the difference can be detected by means of further measurements), though the properties in $\mathscr{F}_S$ (respectively, $\mathscr{F}^{\perp}_S$) must be true (respectively, false) for them all: even this feature is unacceptable for physicists adopting a verificationist theory of meaning.

10. IDEAL MEASUREMENTS

Let us consider an ideal quantum measurement of an observable A which yields the result a_j on a physical object x in a pure state S in a laboratory *i*, and assume that this result is not certainly true in the state S. Then the property

$$E = \text{the observable A takes value } a_j \text{ on } x$$

is testable, but it does not belong to $\mathscr{E}_S$ and is ascertained to be true for the physical object x by the measuring apparatus, which acts at this stage as an (exact) registering device. By referring to the definition of $\mathscr{F}_{ix}^T$ in Sec. 9, we can say that the measurement shows that $E \in \mathscr{F}_{ix}^T$, which could not have been predicted before the measurement because of the incompleteness of QP. It must be stressed that E is recognized to be true for x at the instant of the measurement, and this recognition has nothing to do with the properties of x after the measurement or, more generally, with the transformation of the state of x during the measurement process. If the measuring apparatus is an *ideal filter*, the state after the measurement can be obtained from S by means of the *projection postulate*, and the apparatus acts at this stage as a preparing device (we do not want to cope here with the widely debated problem of the role and logical status of the projection postulate in QP[52–54]). Then the state after a measurement of this kind is a first type pure state S_j, endowed with a support E_{S_j}, and the property E is true for x even after the measurement. Furthermore it can be easily shown that S_j is consistent with S in the sense specified in Sec. 5, i.e., $S\,C\,S_j$, so that no inconsistency occurs between the information in S and in S_j: this means in particular, that all properties in $\mathscr{E}_S \cup \mathscr{E}_{S_j}$ might be conjointly true for the physical object x both before and after the measurement, though the incompleteness of the theory makes it impossible to know in QP whether such a situation occurs. More generally, we can say that the sets $\mathscr{F}_{ix}^T$ and $\mathscr{F}_{ix}^F$ might remain unchanged during this kind of measurement process even if the state of x changes. If a change occurs in $\mathscr{F}_{ix}^T$ and $\mathscr{F}_{ix}^F$, it can be tested by means of further measurements and it is intuitively ascribed to the interaction of the physical object with the measuring apparatus, but possible changes are limited by the requirement that the property E is true for x even after the measurement (E is such that, for every laboratory *i*, $\rho_i(E_{S_j}) \subseteq \rho_i(E)$; furthermore, whenever A has a discrete, nondegenerate spectrum, E is an atom of $(\mathscr{E}_e, <)$ and coincides with E_{S_j}). In conclusion, the physical object x belongs to $\rho_i(S) \cap \rho_i(E)$ before the measurement, to $\rho_i(S_j) \cap \rho_i(E) = \rho_i(S_j)$ after the measurement, and this displacement corresponds to a change in our knowledge on x, not necessarily to a change of the physical properties that are true for x.

Remark 10.1. Let S_k be the state of x after a measurement of A which yields the result a_k, with $a_k \neq a_j$; then, $S\,C\,S_k$ because of the results reported above, but $S_j \not C\, S_k$ (indeed, S_j and S_k are represented by orthogonal vectors in HSQT). This can occur since C is not transitive in QP, so that $S\,C\,S_j$ and $S\,C\,S_k$ do not imply $S_j\,C\,S_k$, and it can be intuitively interpreted by observing that the information that the observable A takes the value a_j on x is not consistent with the information that A takes value a_k on x. This obviously agrees with the fact that $S_j \not\subset S_k$ iff $E_{S_j} \not\subset E_{S_k}$ because of our arguments in Sec. 5.

A different case occurs if we consider two (pure, first kind, ideal) quantum measurements of two noncommuting observables A and B (with discrete, nondegenerate spectra) that can be (not conjointly) performed on a given physical object x in a pure state S. Indeed, let a, b be possible outcomes of the measurements of A, B respectively, and let S_a, S_b be the corresponding eigenstates. Then, the information in S_a can be consistent with the information in S_b, and this occurs whenever S_a and S_b are represented by nonorthogonal vectors in HSQT. This can be intuitively explained by saying that different questions (measurements) that cannot be answered conjointly could, this notwithstanding, admit answers that can be consistently referred to the same physical object, and obviously agrees with the fact that $S_a \subset S_b$ iff $E_{S_a} \subset E_{S_b}$, again because of our arguments in Sec. 5. However, this intuitive explanation is typical of our approach. Indeed, in the orthodox interpretation of QP, where a verificationist viewpoint is adopted, it would be considered meaningless to ask whether E_{S_a} and E_{S_b} are consistent.

Remark 10.2. It is interesting to consider another kind of measurement which can occur both in CP and in QP. Let a physical object be in a nonpure state T that is a mixture of pure states $S_1, S_2, \ldots$, and let a measurement be performed which refines the information in T, so that after the measurement x is in the pure state S_m which appears in the decomposition of T. One of us has proved elsewhere[2] that, for every m, $T \subset S_m$. In addition whenever S_m and S_n appear in the decomposition of T and $m \neq n$, $S_m \not\subset S_n$ in CP (while $S_m \subset S_n$ in QP whenever S_m and S_n are represented by nonorthogonal vectors in HSQT). It follows in particular that this measurement mimics in the classical case an ideal quantum measurement. However, there is a relevant conceptual difference between the two cases since we have now, for every m, $\mathcal{E}_T \subset \mathcal{E}_{S_m}$ (both in CP and in QP), so that we can say that the measurement provides a refinement of the information, while the strict inclusion $\mathcal{E}_S \subset \mathcal{E}_{S_i}$ is wrong in the ideal quantum measurement considered above. This explains why we cannot think of a measurement performed on a physical object in a mixed state in CP as a faithful model for a quantum measurement on a physical object in a pure state.

11. CONCLUDING REMARKS

The theoretical framework constructed in the previous sections allows us to introduce some general remarks that help one to avoid a number of difficulties when dealing with QP.

First, we observe that the concepts of *preparation* and *registering device*, which are basic in our approach (Sec. 2), can be connected with the notions of *prediction* and *retrodiction*, respectively. Indeed, a preparation π individuates a state, say S, and if π is performed in a laboratory i so that a physical object x is produced, we can say that the statement S(x) is true in i immediately after the preparation, say at time t. Then, S(x) can be taken as an assumption in order to deduce, by using physical laws (Sec. 10), that some properties are true for x in i at t (prediction; if evolution in time is taken into account, one can obviously extend this kind of prediction to any time subsequent to t). A registering device individuates an effect, or better, if suitably chosen, a testable property $F \in \mathscr{F}_c$, and it allows us to say that the statement F(x) is true in i immediately before the measurement (Sec. 10) if it is applied to x in i and yields answer yes. Thus r provides an information that regards the physical object entering the registering device, no matter what its properties may be after the measurement (retrodiction; note that this interpretation is unacceptable according to the standard approach to QP, since this would consider the statement F(x) meaningful, and true, only after the measurement).

Second, we observe that the above remark suggests that a new (theory dependent) binary relation can be introduced on $\mathscr{E}_c$, as follows.

Let $E_1, E_2 \in \mathscr{E}_c$. We say that E_1 and E_2 are *conjointly predictable* and write $E_1 \mathscr{P} E_2$, iff a state $S \in \mathscr{S}_P$ exists such that E_1 and E_2 belong to $\mathscr{E}_S \cup \mathscr{E}_S^\perp$.

The intuitive interpretation of the relation $\mathscr{P}$ is immediate if we refer to the interpretations of $\mathscr{E}_S$ and $\mathscr{E}_S^\perp$ in Sec. 5. Furthermore, it is apparent that, from a verificationist viewpoint, two testable physical properties F_1 and F_2 cannot be conjointly predictable if they are not pragmatically compatible (for, properties that cannot be conjointly tested cannot be retained to be conjointly true for a given physical object, hence they cannot be conjointly attributed to it). But in our general framework the relations of pragmatic compatibility (conjoint testability) and conjoint predictability are disentangled (indeed, the former is defined on $\mathscr{F}_c$ in terms of testability, the latter is defined on $\mathscr{E}_c$ in terms of truth), and some relationships between them can be deduced by using the definitions and results in Sec. 6. In particular, for every $F_1, F_2 \in \mathscr{F}_c$, $F_1 \mathscr{K} F_2$ implies $F_1 \mathscr{P} F_2$, and for every first kind pure state S and $F \in \mathscr{F}_c$, $F \mathscr{K} E_S$ iff $F \mathscr{P} E_S$. Of course, further connections between $\mathscr{K}$ and $\mathscr{P}$ in a specific theory $\mathscr{I}$ can be deduced from the analysis of the measurement process according to $\mathscr{I}$ (for instance, $\mathscr{K}$ and $\mathscr{P}$ obviously coincide in CP).

Third, we note that it is not impossible in our context to attribute simultaneously two properties $F_1, F_2 \in \mathscr{F}_c$ that are not pragmatically compatible nor conjointly predictable to a given physical object. Indeed, let us consider a laboratory i and a physical object x that is prepared in i in such a way that the sentence S(x), with $S \in \mathcal{S}$, is true in i at the instant t: if one can predict that $F_1(x)$ is true at t because of some physical laws and a measurement of F_2 that yields the yes answer is made on x at t, then F_1 and F_2 can be simultaneously attributed to x at t (note that an analogous remark has been made by Cattaneo and Nisticó[55] with reference to QP, but, seemingly, in the framework of the standard von Neumann approach; yet, we retain that the arguments by these authors rest on giving up implicitly the strict verificationist attitude that characterizes the standard interpretation of QP). One can loosely say that the noncompatibility of properties refers to retrodictions (measurements) or to predictions (preparations), not to a combination of these procedures.

The above remarks suggest that, as has already been noticed by other authors,[38] Heisenberg's uncertainty principle in QP can be interpreted as regarding either position and momentum measurements or position and momentum predictions; intuitively, these different interpretations rest on different arguments, since the former follows from our reasonings on pragmatic compatibility at the beginning of Sec. 6, the latter from considering the probability distributions of momentum and position in any possible state.

Finally, we notice that the ideal filters considered in Sec. 10 simultaneously play the role of preparations and registering devices in QP. But our above remarks show that these roles must be carefully distinguished in our approach. Indeed, let us consider two ideal filters that are apt to test whether the testable properties F_1 and F_2 are true for a given physical object, and let F_1 be not pragmatically compatible with F_2. Then, F_1 and F_2 can neither be tested nor predicted conjointly in QP, but the ideal filter testing F_1 can be used as a preparation of a physical object x with property F_1 in a laboratory i for an ideal filter testing F_2 on x; this latter filter then works as a registering device, and if it yields answer yes, F_1 and F_2 are conjointly true for x in i at the instant t immediately following the measurement of F_1 and preceding the measurement of F_2.

REFERENCES

1. C. Garola, "Classical foundations of quantum logic," *Int. J. Theor. Phys.* **30**, 1 (1991).
2. C. Garola, "Semantic incompleteness of quantum physics," *Int. J. Theor. Phys.* **31**, 809 (1992).
3. C. Garola, "Quantum logics seen as quantum testability theories," *Int. J. Theor. Phys.* **31**, 1639 (1992).
4. C. Garola, "Truth versus testability in quantum logic," *Erkenntnis* **37**, 197 (1992).
5. C. Garola, "Semantic incompleteness of quantum physics and EPR-like paradoxes," *Int. J. Theor. Phys.* **32**, 1863 (1993).

6. C. Garola, "Reconciling local realism and quantum physics: a critique to Bell," *Theor. Mat. Fiz.* **99**, 285 (1994).
7. C. Garola, "Criticizing Bell: Local realism and quantum physics reconciled," *Int. J. Theor. Phys.* **34**, 269 (1995).
8. C. Garola, "Questioning nonlocality: an operational critique to Bell's theorem," in *The Foundations of Quantum Mechanics. Historical Analysis and Open Questions*, C. Garola and A. Rossi, eds. (Kluwer Academic, Dordrecht, 1995), p. 273.
9. C. Garola, "Pragmatic versus semantic contextuality in quantum physics," *Int. J. Theor. Phys.* **34**, 1383 (1995).
10. A. Einstein, B. Podolsky, and N. Rosen, "Can quantum mechanical description of reality be considered complete?," *Phys. Rev.* **47**, 777 (1935).
11. N. Bohr, *Atomic Physics and Human Knowledge* (Wiley, New York, 1958).
12. N. Bohr, *Essays 1958 1962 on Atomic Physics and Human Knowledge* (Wiley, New York, 1963).
13. W. Heisenberg, *The Physical Principles of Quantum Theory* (Dover, New York, 1930).
14. B. C. Van Fraassen, "A modal interpretation of quantum mechanics," in *Current Issues in Quantum Logic*, E. G. Beltrametti and B. C. Van Fraassen, eds. (Plenum, New York, 1981).
15. R. B. Braithwaite, *Scientific Explanation* (Cambridge University Press, Cambridge, 1953).
16. C. C. Hempel, *Aspects of Scientific Explanation* (Free Press, New York, 1965).
17. B. Russell, *An Inquiry into Meaning and Truth* (Allen & Unwin, London, 1940).
18. R. Carnap, "Truth and confirmation," in *Readings in Philosophical Analysis*, H. Feigl and W . Sellars, eds. (Appleton-Century-Crofts, New York, 1949).
19. R. Carnap, *Philosophical Foundations of Physics* (Basic Books, New York 1966).
20. K. R. Popper, *Conjectures and Refutations* (Routledge & Kegan Paul, London, 1969).
21. P. Bush, P. J. Lathi, and P. Mittelstaedt, *The Quantum Theory of Measurement* (Springer, Berlin, 1991).
22. G. Birkhoff and J. von Neumann, "The logic of quantum mechanics," *Ann. Math.* **37**, 823 (1936)
23. A. Tarski, "The semantic conception of truth and the foundations of semantics," in *Semantics and the Philosophy of Language*, L. Linsky, ed. (University of Illinois Press, Urbana, 1952).
24. C. Dalla Pozza and C. Garola, "A pragmatic interpretation of intuitionistic propositional logic," *Erkenntnis* **43**, 81 (1995).
25. M. Jammer, *The Philosophy of Quantum Mechanics* (Wiley, New York, 1974).
26. H. Reichenbach, *Philosophic Foundations of Quantum Mechanics* (University of California Press, Los Angeles, 1965).
27. J. S Bell, "On the problem of hidden variables in quantum mechanics," *Rev. Mod. Phys.* **38**, 447 (1966).
28. S. Kochen and E. P. Specker, "The problem of hidden variables in quantum mechanics," *J. Math. Mech.* **17**, 59 (1967).
29. N. D. Mermin, "Hidden variables and the two theorems of John Bell," *Rev. Mod. Phys.* **65**, 803 (1993).
30. J. S. Bell, "On the Einstein Podolsky Rosen paradox," *Physics* **1**, 195 (1964).
31. E. P. Wigner, "On hidden variables and quantum mechanical probabilities," *Am. J. Phys.* **38**, 1005 (1970).
32. F. Selleri, "History of the Einstein Podolsky Rosen paradox," in *Quantum Mechanics Versus Local Realism*, F. Selleri, ed. (Plenum, New York, 1988), p. 1.
33. D. M. Greenberger, M. A. Horne, A. Shimony, and A. Zeilinger, "Bell's theorem without inequalities," *Am. J. Phys.* **58**, 1131 (1990).

34. R. K. Clifton, M. L. G. Redhead, and J. M. Butterfield, "Generalization of the Greenberger–Horne–Zeilinger algebraic proof of nonlocality," *Found. Phys.* **21**, 149 (1991).
35. J. J. Sakurai, *Modern Quatum Mechanics* (Benjamin, Reading, Massachusetts, 1985).
36. C. Garola and L. Solombrino, "Semantic realism versus EPR-like paradoxes: the Furry, Bohm–Aharonov, and Bell paradoxes," *Found. Phys.* **26**, 1329 (1996).
37. B. D'Espagnat, *Conceptual Foundations of Quantum Mechanics* (Benjamin, Reading, Massachusetts, 1976).
38. W. M. De Muynck, W. De Baere, and H. Martens, "Interpretation of quantum mechanics, joint measurements of incompatible observables, and counterfactual definiteness," *Found. Phys* **24**, 1589 (1994).
39. H. P. Stapp, "Comments on 'Interpretation of quantum mechanics, joint measurement of incompatible observables and counterfactual definiteness'," *Found. Phys.* **24**, 1665 (1995).
40. D. Aerts, "Description of many physical entities without the paradoxes encountered in quantum mechanics," *Found. Phys.* **12**, 1131 (1982).
41. D. Foulis, C. Piron, C. Randall, "Realism, operationalism and quantum mechanics," *Found. Phys.* **13**, 813 (1983).
42. G. Ludwig, *Foundations of Quantum Mechanics I* (Springer, New York, 1983).
43. C. Piron, *Foundations of Quantum Physics* (Benjamin, Reading, Massachusetts, 1976).
44. C. Garola, "Embedding of posets into lattices in quantum logic," *Int. J. Theor. Phys.* **24**, 423 (1985).
45. K. Bugajska and S. Bugajski, "The lattice structure of quantum logics," *Ann. Inst. Henri Poincaré* **XIX**, 333 (1973).
46. G. M. Mackey, *The Mathematical Foundations of Quantum Mechanics* (Benjamin, New York, 1963).
47. J. M. Jauch, *Foundations of Quantum Mechanics* (Addison Wesley, Reading, Massachusetts, 1968).
48. C. Garola "Propositions and orthocomplementation in quantum logic," *Int. J. Theor. Phys.* **19**, 369 (1980).
49. C. Garola and L. Solombrino, "Yes-no experiments and ordered structures in quantum physics," *Nuovo Cimento* **77B**, 87 (1983).
50. G. Cattaneo, C. Dalla Pozza, C. Garola, and G. Nisticò, "On the logical foundations of the Jauch–Piron approach to quantum physics," *Int. J. Theor. Phys.* **27**, 1313 (1988).
51. G. Cattaneo, C. Garola, and C. Nisticò, "Preparation-effects versus question-preparation structures," *J. Phys. Ess.* **2**, 197 (1989).
52. D. Dieks, "Quantum mechanics without the projection postulate and its realistic interpretation," *Found. Phys.* **19**, 1397 (1989).
53. K. Gottfried, "Does quantum mechanics carry the seeds of its own destruction?," *Phys. Worlds* **4** (10), 34 (1991).
54. J. Bub, "Quantum mechanics without the projection postulate," *Found. Phys.* **22**, 737 (1992).
55. G. Cattaneo and G. Nisticò, "Interpretative remarks in quantum mechanics," in *The Foundations of Quantum Mechanics. Historical Analysis and Open Questions*, C. Garola and A. Rossi, eds. (Kluwer Academic, Dordrecht, 1995), p. 127.