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Symmetric Periodic Noncollision Solutions for N-Body-Type Problems

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Abstract. Using the calculus of variations in the large, especially computing the category of the symmetric configuration space of symmetric N-body-type problems, we prove the existence of infinitely many symmetric noncollision periodic solutions about the symmetric and nonautonomous N-body-type problems under the assumptions that the symmetric potentials satisfy the strong force condition of Gordon.

§1. Introduction

Calculus of variations in the large was used to study periodic solutions for N-body-type problems in the last few years. In this paper, we will consider a class of solutions of the following system of ordinary differential equations

$$m_i \ddot{x}_i (t) + \nabla_{x_i} V(t, x_1(t), \cdots, x_N(t)) = 0, \quad x_i(t) \in R^k, \quad i = 1, \cdots, N, \tag{1}$$

where $m_i > 0$ for all i, and V satisfies the following conditions:

(V1) $V(t, x_1, \cdots, x_N) = \frac{1}{2} \sum_{0 \le i \ne j \le N} V_{ij}(t, x_i - x_j)$,

(V2) $V_{ij} \in C^2(R \times (R^k - \{0\}); R)$, for all $1 \le i \ne j \le N$,

(V3) $V_{ij}(t, \xi) \to -\infty$ uniformly on t as $|\xi| \to 0$, for all $1 \le i \ne j \le N$,

(V4) $V(t, x_1, \cdots, x_N) \le 0$, for all $t \in R$, $(x_1, \cdots, x_N) \in (R^k - \{0\})^N$ and

(V5) the strong force condition (see [7]) holds for V_{ij}, i.e., there exist a function $U \in C^1(R^k - \{0\}, R)$ and a neighborhood N of 0 in R^k such that

$$\begin{cases} \lim_{\xi \to 0} U(\xi) = -\infty, \\ -V_{ij}(t, \xi) \ge |\nabla U(\xi)|^2, \quad \forall t, \xi \in N - \{0\}. \end{cases}$$

We will say that a function $X(t) = (x_1(t), \cdots, x_N(t)) \in C^2(R, (R^k)^N)$ is a T-periodic noncollision solution of (1) if $X(t)$ is a T periodic solution of (1) and $x_i(t) \ne x_j(t)$ for all $i \ne j$, and $t \in R$.

The following symmetric assumption is motivated by the Keplero N-body problem and the symmetry introduced by Bessi and Coti Zelati in [1].

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(V6) there exists an element g of finite order s in $SO(k)$ which has no fixed point other than the origin (i.e., 1 is not an eigenvalue of g), such that

$$V(t, x_1, \cdots, x_N) = V(t + T/s, gx_1, \cdots, gx_N).$$

Note that the hypothesis that $g \in SO(k)$ has no fixed point other than the origin will force k to be an even integer.

If the potential V satisfies (V1)–(V6), and $X(t)$ is a T-periodic noncollision solution of (1) and satisfies $X(t+T/s) = (x_1(t+T/s), \cdots, x_N(t+T/s)) = (gx_1(t), \cdots, gx_N(t)) = gX(t)$, we will say that $X(t)$ is a g-symmetric T-periodic noncollision solution. Throughout this paper we will always assume that the potential V satisfies (V1)–(V6), and a g-symmetric T-periodic noncollision solution will be simply called a (g, T) noncollision solution.

The main result in this paper is the following theorem.

Theorem 1.2. *Suppose that V satisfies* (V1)–(V6) *and T is any positive real number. The system of ordinary differential equations* (1) *has infinitely many (g, T) noncollision solutions.*

Let k be an even integer. Then $g = -id \in SO(k)$ and it has no fixed point other than the origin. The above theorem gives an affirmative answer to half of the question proposed by Coti Zelati in [2] (in the case where k is even).

This paper is organized in three sections. §2 contains some basic facts about the functional corresponding to the system of ordinary differential equations (1), in §3 we will discuss the category of the g-symmetric free loop space, and the proof of our main result will be given in §4.

§2. The Functional f

Let $g \in SO(k)$ be an element of finite order s and have no fixed point other than the origin. We introduce spaces

$$E_g^N = \{(x_1, \cdots, x_N) | x_i \in H^1(R/TZ; R^k), x_i(t + T/s) = g(x_i(t)), \forall t, i\},$$

$$\triangle_g^N = \{(x_1, \cdots, x_N) | x_i \in H^1(R/TZ; R^k), x_i(t + T/s) = g(x_i(t)), \forall t, i \text{ and } x_i(t) \neq x_j(t), \quad \forall t, i \neq j\}.$$

where $H^1(R/TZ; R^k)$ is the metric completion of smooth T-periodic function for the norm $\|x\|_{H^1} = \left(\int_0^T |x(t)|^2 + |\dot{x}(t)|^2 dt\right)^{1/2}$, and the functional $f : \triangle_g^N \to R$

$$f(x_1, \cdots, x_N) = \sum_{i=1}^N \frac{m_i}{2} \int_0^T |\dot{x}_i(t)|^2 dt - \int_0^T V(t, x_1(t), \cdots, x_N(t)) dt.$$

Clearly, E_g^N is a closed subspace of $H^1(R/TZ; (R^k)^N)$, so it is a Hilbert space, and $\triangle_g^N$ is an open subset of E_g^N.

Using a standard argument (see for instance [3]), it is easy to prove the following lemma.

Lemma 2.1. *Suppose* (*V1*)–(*V6*) *hold. Then the critical points of f in $\triangle_g^N$ are (g, T) noncollision solutions of* (*1*).

Lemma 2.2. *Suppose $g \in SO(k)$ is an element of finite order s and it has no fixed point other than the origin. Then $\int_0^T X(t)dt = 0$ for all $X(t) = (x_1(t), \cdots, x_N(t)) \in E_g^N$.*

Proof. Since 1 is not an eigenvalue of $g, id + g + \cdots + g^{s-1} = 0$. So we have

$$\int_0^T X(t)dt = \sum_{l=0}^{s-1} \int_{lT/s}^{(l+1)T/s} X(t)dt = \int_0^{T/s} \sum_{l=0}^{s-1} g^l X(t)dt = 0.$$

By Wirtinger's inequality we know that there is a constant $C > 0$ such that $\|X(t)\|_{H^1} \leq C\|\dot{X}(t)\|_{L^2}$ for all $X(t) \in E_g^N$.

Corollary 2.3. *For any real number K, the set $F_K = \{X \in \triangle_g^N | f(X) \leq K\}$ is precompact in both C^0 topology and weak topology of E_g^N.*

Proof. By (V4), $\|\dot{X}(t)\|_{L^2} \leq 2(\min_i\{m_i\})^{-1} f(X)$, so the set F_K is bounded in H^1. Then applying the Sobolev embedding theorem and Alaoglu's theorem, we complete the proof.

This corollary also implies that f is coercive.

The closed subset $\Gamma_g^N = E_g^N - \triangle_g^N$ of E_g^N will be called the collision set, and a standard argument can be applied to show that the strong force assumption (V5) implies that $f(X) \to \infty$ when X approaches the collision set Γ_g^N. More precisely, we have the following lemma.

Lemma 2.4. *Let $\{X^n\}$ be a sequence in $\triangle_g^N$ and $X^n \to X \in \Gamma_g^N$ in both C^0 topology and weak topology of E_g^N. Then $f(X^n) \to \infty$.*

Proof. Let the limit $X = (x_1, \cdots, x_N) \in \Gamma_g^N$, which means that there exist a $t_0 \in [0, T]$ and an $i_0 \neq j_0$ such that $x_{i_0}(t_0) = x_{j_0}(t_0)$. By the hypothesis, $x_{i_0}^n(t) - x_{j_0}^n(t) \to x_{i_0}(t) - x_{j_0}(t)$ uniformly and then $x_{i_0}^n(t_0) - x_{j_0}^n(t_0) \to 0$. In the case where $x_{i_0}(t) - x_{j_0}(t) \equiv 0$, then $V_{i_0 j_0}(t, x_{i_0}^n(t) - x_{j_0}^n(t)) \to -\infty$ uniformly by (V3), and so does V; hence $f(X^n) \to \infty$.

Now we assume that $x_{i_0}(t) - x_{j_0}(t) \not\equiv 0$. Suppose $x_{i_0}(\tau) - x_{j_0}(\tau) \not\equiv 0$, say $\tau > t_0$. Then

$$\begin{aligned}
&U(x_{i_0}^n(\tau) - x_{j_0}^n(\tau)) - U(x_{i_0}^n(t_0) - x_{j_0}^n(t_0)) = \int_{t_0}^{\tau} \frac{d}{dt} U(x_{i_0}^n(t) - x_{j_0}^n(t))dt \\
&= \int_{t_0}^{\tau} \nabla U(x_{i_0}^n(t) - x_{j_0}^n(t))(\dot{x}_{i_0}^n(t) - \dot{x}_{j_0}^n(t))dt \\
&\leq \left(\int_{t_0}^{\tau} |\nabla U(x_{i_0}^n(t) - x_{j_0}^n(t))|^2 dt\right)^{1/2} \|\dot{x}_{i_0}^n(t) - \dot{x}_{j_0}^n(t)\|_{L^2}.
\end{aligned}$$

By (V5), $U(x_{i_0}^n(\tau) - x_{j_0}^n(\tau)) - U(x_{i_0}^n(t_0) - x_{j_0}^n(t_0)) \to \infty$, and $\|\dot{x}_{i_0}^n(t) - \dot{x}_{j_0}^n(t)\|_{L^2}$ is bounded since X^n converges weakly in H^1. It follows that

$$\int_{t_0}^{\tau} |\nabla U(x_{i_0}^n(t) - x_{j_0}^n(t))|^2 dt \to \infty.$$

By (V5) again,

$$\begin{aligned}
f(X^n) &= \sum_{i=1}^{N} \frac{m_i}{2} \int_0^T |\dot{x}_i^n(t)|^2 dt - \int_0^T V(t, x_1^n(t), \cdots, x_N^n(t))dt \\
&\geq -\int_{t_0}^{\tau} V(t, x_1^n(t), \cdots, x_N^n(t))dt \\
&\geq \int_{t_0}^{\tau} |\nabla U(x_{i_0}^n(t) - x_{j_0}^n(t))|^2 dt \to \infty.
\end{aligned}$$

This completes the proof of the lemma.

Lemma 2.5. *The functional f satisfies the Palais-Smale condition on $\triangle_g^N$, i.e., any sequence $X^n \in \triangle_g^N$ satisfying $f(X^n) \to \lambda$ and $f'(X^n) \to 0$ is precompact in H^1.*

Proof. Let $\{X^n\}$ be a Palais-Smale sequence in $\triangle_g^N$. Corollary 2.3 says that X^n has a convergent subsequence, still called X^n, which converges to an element $X \in E_g^N$ weakly in H^1 and strongly in C^0. By Lemma 2.4, $X \in \triangle_g^N$. The rest of the proof is to show that X^n converges to X strongly, and it suffices to show that $\lim \|X^n\|_{H^1} = \|X\|_{H^1}$. We know that $\lim \|X^n\|_{L^2} = \|X\|_{L^2}$ since $X^n \to X$ uniformly, so we only need to prove that $\lim \|\dot{X}^n\|_{L^2} = \|\dot{X}\|_{L^2}$.

Since X^n converges to X strongly in C^0, then $\nabla_{x_i} V(t, X^n)(x_i - x_i^n)$ converges to zero uniformly, $f'(X^n) \to 0$ in H^{-1} and $x_i - x_i^n$ is bounded in H^1. Then we have

$$m_i \|\dot{x}_i\|_{L^2}^2 - m_i \lim \|\dot{x}_i^n\|_{L^2}^2 = \lim \int_0^T m_i \dot{x}_i^n (\dot{x}_i - \dot{x}_i^n) dt$$

$$= \lim \left\{ \langle f'_{x_i}(X^n), x_i - x_i^n \rangle + \int_0^T \nabla_{x_i} V(t, X^n)(x_i - x_i^n) dt \right\} = 0.$$

Therefore, $\lim \|\dot{x}_i^n\|_{L^2} = \|\dot{x}_i\|_{L^2}$ for all i, and hence $\lim \|\dot{X}^n\|_{L^2} = \|\dot{X}\|_{L^2}$.

§3. The Category of Free Loop Spaces

In this section, we will discuss the category of a class of free loop spaces. Our topological setting is as follows. Let M be a manifold and $g \in \mathrm{HOMEO}_0(M)$ be an element of finite order s where $\mathrm{HOMEO}_0(M)$ consists of all homeomorphisms of M which are isotopic to id_M. The g-symmetric free loop space of M is

$$\triangle_g(M) = \{\omega \in C^0(R/TZ; M) | \omega(t + T/s) = g(\omega(t)), \forall t\}.$$

We will study the category of $\Lambda_g(M)$.

Let $p, q \in M$ be any two points. The path space $\Omega^{p,q}(M)$ is defined as follows:

$$\Omega^{p,q}(M) = \{\omega \in C^0([0,1]; M) | \omega(0) = p, \omega(1) = q\}.$$

If $p = q, \Omega^{p,p}(M)$ will usually be denoted by $\Omega^p(M)$ and called a loop space.

Lemma 3.1. *Let M be a connected manifold. Then $\Omega^{p,q}(M)$ is homeomorphic to $\Omega^{p',q'}(M)$ for any four points p, q, p' and $q' \in M$.*

Proof. Since M is connected, it is homogeneous, namely, for any two points $p, p' \in M$, we can find an isotopy $F_u : M \to M$ with compact support such that $F_0 = id_M$ and $F_1(p) = p'$.

Let F_u and F'_u be two such isotopies that $F_0 = F'_0 = id_M$ and $F_1(p) = p'$, $F'_1(q) = q'$. For any $\omega \in \Omega^{p,q}(M)$, define $H(\omega) \in \Omega^{p',q'}(M)$ as follows:

$$\begin{aligned} H : \Omega^{p,q}(M) &\to \Omega^{p',q'}(M) \\ \omega(t) &\mapsto F'_t(F_{1-t}(\omega(t))). \end{aligned}$$

Clearly this H is continuous and its inverse is $H^{-1}(\omega'(t)) = F_{1-t}^{-1}(F'^{-1}_t(\omega'(t)))$. So H is a homeomorphism.

This is a classical result due to Serre[4] that the real cohomology of $\Omega^p(M)$ has nontrivial cup products of arbitrary high length provided M is an admissible manifold (i.e., a simply connected manifold with finitely generated real cohomology $H^*(M)$ and for some $i > 0, H^i(M) \neq 0$).

Now we rewrite $\Lambda_g(M)$ as $\{\omega \in C^0([0, T/s]; M) | \omega(T/s) = g(\omega(0))\}$. Define $\pi : \Lambda_g(M) \to M$ by $\pi(\omega) = \omega(0)$. For any $p \in M, \pi^{-1}(p) = \Omega^{p,g(p)}(M)$. By Lemma 3.1, we know that all these spaces are homeomorphic, so we have a fiberation

$$\Omega^{p,g(p)}(M) \xrightarrow{i} \Lambda_g(M) \xrightarrow{\pi} M.$$

The following theorem is due to Fadell and Husseini (see [5]).

Theorem 3.2. *Let $F \xrightarrow{i} E \xrightarrow{\pi} B$ be a fiberation which admits of a section $\sigma : B \to E$. Then for any subset $Q \subset F$,*

$$cat_F Q \leq cat_E Q.$$

In particular, $catF \leq catE$.

Theorem 3.3. *Using the notation above, we have, for any subset $Q \subset \Omega^p(M)$,*

$$cat_{\Omega^p(M)} Q \leq cat_{\Lambda_g(M)} Q.$$

In particular, if M is admissible, then $cat\Lambda_g(M) = \infty$.

Proof. We will show that the fiberation

$$\Omega^{p,g(p)}(M) \xrightarrow{i} \Lambda_g(M) \xrightarrow{\pi} M,$$

admits of a section $\sigma : M \to \Lambda_g(M)$. If it can be done, applying Theorem 3.2 to the fiberation, we get the result required.

To show that the fiberation $\Omega^{p,g(p)}(M) \xrightarrow{i} \Lambda_g(M) \xrightarrow{\pi} M$ admits of a section, we note that $g \in \mathrm{HOMEO}_0(M)$ and pick an isotopy $h : [0,1] \to \mathrm{HOMEO}_0(M)$ such that $h_0 = id_M$ and $h_1 = g$. Then $(\sigma(p))(t) = h_t(p)$ is a section. We complete the proof of the theorem.

We are interested in the N-th configuration space of R^k.

$$F_N(R^k) = \{(x_1, \cdots, x_N) | x_i \in R^k, x_i \neq x_j, \forall i \neq j\}.$$

If $k > 2, N > 1, F_N(R^k)$ is admissible (see [5]). Each element $g \in SO(k)$ gives a diffeomorphism of R^k, which sends distinct points to distinct points. Therefore $SO(k)$ extends to an action on the configuration spaces. The space Δ_g^N we introduced in §2 is the space of g-symmetric H^1-loops in $F_N(R^k)$, and it has the same homotopy type as the space $\Lambda_g(F_N(R^k))$. Applying Theorem 3.3 to this case, we have

Corollary 3.4. *If $k > 2, N > 1$ and $g \in SO(k)$ is an element of finite order s, then Δ_g^N has an infinite category.*

Remark. In the case $k = 2, N > 1$, the fundamental group of the configuration space $F_N(R^2)$ is the braid group B_{N-1}. The connected components in the g-symmetric free loop space are in one-to-one correspondence with the conjugacy classes in some extension of B_{N-1} by a finite cyclic group, which is an infinite set. So $\mathrm{Cat}\Lambda_g(F_N(R^2)) = \infty$.

§4. The Main Result

The following is the abstract critical point theorem which we will use in the proof of our main result, and a proof of this theorem can be found in [6].

Theorem 4.1. *Let $\triangle$ be an open subset in a Banach space and f a functional on $\triangle$, such that*

1. *$Cat\triangle = \infty$,*
2. *For any sequence $\{q_n\} \subset \triangle$ and $q_n \to q \in \partial\triangle$, we will have $f(q_n) \to \infty$,*
3. *For any $K \in R$, $Cat_{\triangle}(\{q \in \triangle | f(q) \le K\}) < \infty$, and*
4. *There exists a $\lambda_0 \in R$ such that the Palais-Smale condition holds on the set $\{q \in \triangle | f(q) \ge \lambda_0\}$.*

Then f possesses an unbounded sequence of critical values.

Our main theorem is the following

Theorem 4.2. *Suppose that V satisfies* (V1)–(V6) *and T is any positive real number. Then the system of ordinary differential equations* (1) *has infinitely many (g, T) noncollision solutions.*

To prove this theorem, we need one more lemma.

Lemma 4.3. *For any constant $K \ge 0$, the set $D_K = \{X \in \triangle_g^N | \ \|\dot{X}\|_{L^2} \le K\}$ is of finite category in $\triangle_g^N$, i.e., $Cat_{\triangle_g^N}(D_K) < \infty$.*

Proof. D_{K+1} is a neighborhood of D_K in $\triangle_g^N$; so to prove the lemma, it suffices to find a homotopy $H : D_K \times I \to \triangle_g^N$ for all K, such that H_0 is the inclusion and $H_1(D_K)$ is precompact in $\triangle_g^N$.

Obviously the function $p(X) = \min\limits_{t, i \ne j} |x_i(t) - x_j(t)|$ has a positive low bound p on D_K. Pick a $\delta \in (0, T)$ such that $\sqrt{2\delta} K < p$ and define a function

$$\varphi(T) = \begin{cases} \delta^{-1}, & \text{if } t \in [0, \delta]; \\ 0, & \text{otherwise.} \end{cases}$$

Now we can define the homotopy

$$H(X, u) = (1 - u)X + uX * \varphi, \quad \forall X \in D_K, \quad u \in I,$$

where the convolution $X * \varphi = \left(\int_0^T x_1(t - s)\varphi(s)ds, \cdots, \int_0^T x_N(t - s)\varphi(s)ds \right)$. Clearly H_0 is an inclusion and $H_1(D_K)$ is paracompact since H_1 is a convolution operator and it is compact. We only need to prove that $H(D_K \times I) \subset \triangle_g^N$. Supposing this is not the case, then there is a $u_0 \in (0, 1], X \in D_K, i_0 \ne j_0$ and $t_0 \in [0, T]$, such that $(1 - u_0)(x_{i_0}(t_0) - x_{j_0}(t_0)) + u_0((x_{i_0} - x_{j_0}) * \varphi)(t_0) = 0$. It follows that

$$((x_{i_0} - x_{j_0}) * \varphi)(t_0) = (u_0^{-1} - 1)(x_{i_0}(t_0) - x_{j_0}(t_0)),$$

then

$$|(x_{i_0}(t_0) - x_{j_0}(t_0)) - ((x_{i_0} - x_{j_0}) * \varphi)(t_0)| = u_0^{-1} |x_{i_0}(t_0) - x_{j_0}(t_0)| \ge p.$$

On the other hand, for any $X \in D_K, i \neq j$ and $t \in [0,T]$, we have

$$
\begin{aligned}
&|(x_i(t) - x_j(t)) - ((x_i - x_j) * \varphi)(t)| \\
&\leq \delta^{-1} \int_0^\delta |(x_i(t) - x_j(t)) - (x_i(t-s) - x_j(t-s))| ds \\
&\leq \sup_{0 \leq s \leq \delta} |(x_i(t) - x_j(t)) - (x_i(t-s) - x_j(t-s))| \\
&\leq \sqrt{\delta} \|\dot{x}_i - \dot{x}_j\|_{L^2} \leq \sqrt{\delta}(\|\dot{x}_i\|_{L^2} + \|\dot{x}_j\|_{L^2}) \leq \sqrt{2\delta}\|\dot{X}\|_{L^2} < p.
\end{aligned}
$$

This is a contradiction.

Proof of Theorem 4.2. Now Lemmas 2.4, 2.5, 4.3 and Corollary 3.4 say that the functional $f : \triangle_g^N \to R$ satisfies all conditions in Theorem 4.1, and then Theorem 4.1 and Lemma 2.1 imply that the equations (1) has infinitely many (g,T) noncollision solutions.

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