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SMALL SUB-RIEMANNIAN BALLS ON R^3

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ABSTRACT. This paper deals with sub-Riemannian metrics on R^3 in the contact case. We study the singularities of the exponential mapping in the neighborhood of its pole. This is in stark contrast with the Riemannian case where this situation never occurs.

1. INTRODUCTION

1. Sub-Riemannian geometry has attracted a lot of interest in the past five years. To give an exhaustive list of references would be beyond the scope of this paper.

Here we will limit ourselves to the dimension 3 and study the conjugate locus of a sub-Riemannian structure in the neighborhood of its pole as well as the wave front sets of small radius.

A sub-Riemannian metric on an open subset M of R^3 is a couple $\Sigma = (\Delta, g)$, of a two-dimensional vector subbundle of TM, and a metric $g : \Delta \to R_+$ on Δ, such that Δ is a contact structure on M. It is well known [17] that this couple defines a metric d on M. The set of the $\Sigma = (\Delta, g)$ will be denoted by $\mathrm{Sub}\, R(M)$.

There are many important differences between Riemannian and sub-Riemannian metrics. One of them is that the conjugate locus and the cut locus of a point x contain x in their adherence. Consequently, even spheres of small radius have singularities.

The classical model for sub-Riemannian metrics is the Heisenberg case: M is the 3-dimensional Heisenberg group, Δ is the unique (up to conjugation) left invariant nonintegrable 2-dimensional distribution, and g is left invariant. This case was analyzed by Brockett [4] and Vershik and Gershkovich [19], among others. This will play a major role in our study, since our approach is to consider any sub-Riemannian metric as a perturbation of the Heisenberg one.

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In the Heisenberg case, the picture of the wave front is well known. Balls are just perfect "apples," and the conjugate locus of a point a is the a-coset with respect to the center of the group.

In the general case, as we will show, the situation is more complicated: the germ of the conjugate locus at the origin is a 2-dimensional surface. The picture of the least singular case has been exhibited for the first time by A. Agrachev in 1994 at the Zürich congress [1].

Here we will give a generic classification of what can happen for the conjugate loci, and their intersection with wave fronts and sub-Riemannian spheres.

2. Statement of our main results. Associated to a sub-Riemannian metric, there is a canonical cylindrical coordinate system (R, φ, r) on T_0^*M (R radius vector, φ angular coordinate, r third cartesian coordinate). The line $R = 0$ is the annihilator of the distribution Δ at zero (see Sec. 4.3 for precise definitions). For $r \neq 0$, we set $\rho = \dfrac{1}{r}$.

As in the Riemannian case, the sub-Riemannian metric (Δ, g) has an exponential mapping. Contrary to what happens in that case, a geodesic is not determined by its initial point and initial velocity, but rather by its initial point and a covector in T_0^*M (see Sec. 4.1). We assume that 0 belongs to M and we denote the exponential mapping of pole zero by $\mathcal{E}(s, \varphi, \rho)$. Precisely, $\mathcal{E}(s, \varphi, \rho)$ is the point at distance s from 0 on the geodesic having the initial point 0 and the initial covector of coordinates $\left(1, \varphi, \dfrac{1}{\rho}\right)$ in T_0^*M. The parameter ρ has a simple differential geometric interpretation: this is the curvature radius of the geodesic with respect to the connection defined in Sec. 3.

The conjugate time mapping $s_c(\varphi, \rho)$ is defined as usual: the graph S of s_c is the set of points at which $\mathcal{E}(s, \varphi, \rho)$ is singular for the first time $s > 0$. The conjugate locus CL is the image of S by $\mathcal{E}$.

When ρ tends to zero, $s_c(\varphi, \rho)$ is equivalent to $2\pi\rho$. Hence CL has its pole 0 in its closure. The precise description of CL near its pole is given in Theorem 1.1.

To state this let us point out a few facts:

(i) The structural group of the metric bundle (Δ, g) can be reduced to $SO(2)$ (since Δ is orientable). This allows us to decompose the tensor fields associated to Δ according to the action of $SO(2)$ on them.

(ii) The metric g on the vector bundle Δ and a choice of orientation on Δ define a volume form vol_g, section of the bundle $\wedge^2\Delta^*$ of 2-skew symmetric covariant tensors on Δ. vol_g defines a complex structure j on Δ by the formula:

For any couple $(u, v) \in \Delta \times_M \Delta$, $\mathrm{vol}_g(u, v) = g\big(j(u), v\big)$. If we change the orientation, j is changed into $-j$.

(iii) We associate to the pair (Δ, g) two symmetric covariant tensor fields Q, V on Δ, of degree 2 and 3 respectively. These tensor fields depend functorially on (Δ, g), and represent differential geometric invariants of (Δ, g). The interpretation of these invariants in terms of the connection canonically associated to (Δ, g) will be given in Sec. 3.

As mentionned in (i), these tensors have canonical decompositions with respect to the action of $SO(2)$:

$$Q = Q_0 + Q_2, \qquad V = V_1 + V_3.$$

The expressions of Q_0, Q_2, V_1, V_3 in terms of j, Q, V can be given as follows:

$$\begin{aligned}
Q_0(v) &= \frac{1}{2}\operatorname{tr}_g Q\, g = \frac{1}{2}\big(Q(v) + Q(j\,v)\big),\\
Q_2(v) &= Q - Q_0 = \frac{1}{2}\big(Q(v) - Q(j\,v)\big),\\
V_1(v) &= \frac{1}{2}V(v) + \frac{1}{8}\big(V(v + j\,v) + V(v - j\,v)\big),\\
V_3(v) &= \frac{1}{2}V(v) - \frac{1}{8}\big(V(v + j\,v) + V(v - j\,v)\big).
\end{aligned}$$

Let us set $\delta(Q) = -\det_g\left(Q - \frac{1}{2}\operatorname{tr}_g Q\, g\right)$.

The exponential mapping $\mathcal{E}(s, \varphi, \rho)$ has an expansion in ρ at $\rho = 0$, of the form

$$\mathcal{E}(s, \varphi, \rho) = \sum_{i=1}^{4} \rho^i\, \mathcal{E}_i\left(\frac{s}{\rho}, \varphi\right) + O(\rho^5), \tag{1.1}$$

where $O(\rho^5)$ is of the form $\rho^5 f\left(\frac{s}{\rho}, \varphi, \rho\right)$ and f and $\mathcal{E}_i$ are smooth functions.

Theorem 1.1. *There exists a linear coordinate system (x, y, w) on T_0M such that $\Delta(0) = \ker w$ and the restrictions of x, y to $\Delta(0)$ satisfy $x \circ j = -y$, $y \circ j = x$. There exists a diffeomorphism ϕ from a neighborhood of zero in T_0M to a neighborhood of zero in M, such that the conjugate time mapping $s_c(v, \rho)$ and the conjugate locus mapping $CL(v, \rho)$ have the* *following expansions in ρ:*

$$\text{(i)} \qquad s_c(v,\rho) = 2\pi\rho - 12\pi\,\rho^3\,Q_0(v) + 20\,\pi\,\rho^4\,\big(V_1(j(v)) - \frac{1}{2}\,V_3(j(v))\big) + O_1(\rho^5);$$

$$\text{(ii)} \qquad \begin{pmatrix} x_c(v,\rho) \\ y_c(v,\rho) \end{pmatrix} = \big(-8\,\pi\,\rho^3\,Q_2(v) - \frac{45}{2}\pi\rho^4\,V_3(j(v))\big)v + \big(-2\,\pi\,\rho^3\,Q_2(v + j(v)) - \frac{15}{2}\pi\rho^4 V_3(v)\big)\,j(v) + O_2(\rho^5);$$

$$w_c(v,\rho) = \pi\,\rho^2 - 3\,\pi\,\rho^4\big(3\,Q_0(v) - 2\,Q_2(v)\big) + O_3(\rho^5).$$

$O_i(\rho^5)$ is a function of the form $\rho^5\,f_i(v,\rho)$, where f_i is a smooth function of ρ and v. v is the unit initial velocity of the geodesic. Moreover, the 3-jet of ϕ (w.r.t. the gradation defined later on in Sec. 2) is unique, and $(x,\,y,\,w)$ is unique up to a rotation of x, y.

Remark 1. Obviously, x_c (resp. y_c, w_c) means $x_c \circ \phi^{-1}$ (resp. $y_c \circ \phi^{-1}$, $w_c \circ \phi^{-1}$).

We will also use the complex notations:

$$\begin{aligned} &z = x + i\,y, \qquad v = e^{i\,\varphi}, \\ &Q_0 = \frac{c_1 + c_2}{2}\,|z|^2, \quad Q_2 = \frac{c_1 - c_2}{2}\,\mathrm{Re}\,(z^2), \\ &V_1 = \mathrm{Re}\,\big(\overline{a}\,z\,|z|^2\big), \quad V_3 = \mathrm{Re}\,(b\,z^3), \\ &\overline{a} = \text{complex conjugate of } a,\ c_1,\,c_2 \in R,\ \ a,\,b \in \mathbb{C}. \end{aligned} \tag{1.2}$$

Then,

$$z_c(\varphi,\rho) = \pi\,\rho^3\,(c_2 - c_1)\,(e^{3\,i\,\varphi} + 3\,e^{-i\,\varphi}) + \frac{15}{2}\pi\,\rho^4\,i\,(b\,e^{4\,i\,\varphi} - 2\,\overline{b}\,e^{-2\,i\,\varphi}) + O_2(\rho^5);$$

$$w_c(\varphi,\rho) = \pi\,\rho^2 - \frac{3}{2}\pi\,\rho^4\,\big(3\,(c_1 + c_2) + 2\,(c_2 - c_1)\,\cos 2\varphi\big) + O_3(\rho^5).$$

Theorem 1.2. (i) *If the differential invariant Q_2 is nonzero, then $\mathcal{E}^3 = \rho\,\mathcal{E}_1 + \rho^2\,\mathcal{E}_2 + \rho^3\,\mathcal{E}_3$ is a sufficient jet for the exponential mapping $\mathcal{E}$, in a neighborhood of $S \cap (0 < \rho < \alpha)$ for α sufficiently small;*

(ii) *if $Q_2 = 0$ but the differential invariant V_3 is nonzero, then $\mathcal{E}^4 = \sum_{i=1}^{4} \rho^i\,\mathcal{E}_i$ is a sufficient jet for the exponential mapping, locally around each point of $S \cap (0 < \rho < \alpha)$ for α sufficiently small.*

As a consequence, the conjugate locus is diffeomorphic to the image of $\mathcal{E}^3$ in case (i) and to the image of $\mathcal{E}^4$ in case (ii), but locally only in this last case. They are shown in Sec. 6, Fig. 2 for case (i), Fig. 11 for case (ii).

In a separate publication, we will deal with the stability in the case (ii) and prove that a global result similar to case (i) of Theorem 1.2 holds.

To prove these two theorems, we construct a normal form for sub-Riemannian metrics. This normal form is interesting in its own right for several reasons, which will appear in the paper.

Theorem 1.3. *Any formal sub-Riemannian metric has an orthonormal basis (F, G) of the following form (setting $e_1 = F(0)$, $e_2 = G(0)$, $e_3 = [F, G](0)$):*

$$(\mathcal{NF}) \quad \begin{cases} F &= (1+y^2\beta)\, e_1 - x\, y\, \beta\, e_2 + \dfrac{y}{2}(1+\gamma)\, e_3 = \\ &= \dfrac{\partial}{\partial x} + y\left(\beta\left(y\dfrac{\partial}{\partial x} - x\dfrac{\partial}{\partial y}\right) + \dfrac{1}{2}(1+\gamma)\dfrac{\partial}{\partial w}\right), \\ G &= (1+x^2\beta)\, e_2 - x\, y\, \beta\, e_1 - \dfrac{x}{2}(1+\gamma)\, e_3 = \\ &= \dfrac{\partial}{\partial y} - x\left(\beta\left(y\dfrac{\partial}{\partial x} - x\dfrac{\partial}{\partial y}\right) + \dfrac{1}{2}(1+\gamma)\dfrac{\partial}{\partial w}\right), \\ & \text{(with an obvious abuse of notations)}, \end{cases}$$

where (x, y, w) are the coordinates on T_0M dual to (e_1, e_2, e_3), β, γ are power series satisfying the conditions

$$\beta(0, 0, w) = \gamma(0, 0, w) = \frac{\partial \gamma}{\partial x}(0, 0, w) = \frac{\partial \gamma}{\partial y}(0, 0, w) = 0.$$

Observe that if $\beta = \gamma = 0$, this is just the orthonormal frame field of the Heisenberg metric.

Corollary 1.4. *In the coordinate system defined by the normal form $\mathcal{NF}$, the values of Q, V at 0, as polynomials in x, y, are related to γ as follows:*

$$\gamma = (Q+V)\,(x, y) + \textit{terms of degree } \geq 3 \textit{ or terms containing } w.$$

Let us now state our generic result.

Theorem 1.5. *For generic sub-Riemannian metrics, $Q_2 \neq 0$ except on a one-dimensional submanifold of M, and on this submanifold, $V_3 \neq 0$.*

This theorem shows that the two situations we have studied in Theorem 1.2, are the only possible generic situations for sub-Riemannian metrics.

3. Organization of the paper. In Sec. 2 and 3, we construct the canonical form and the connection ∇ respectively.

Section 4 is devoted to general facts about the geodesics, the exponential mapping, and to our method of approximation of the conjugate time mapping.

In Corollary 4.6, we state a well-known elementary fact, which simplifies considerably the effective computation of the conjugate time mapping.

This computation and the computation of the conjugate locus are carried out in Sec. 5.

In Sec. 6, we prove our stability results for the exponential mapping (Theorem 1.2, which is stated more precisely in the series of Theorems 6.2 to 6.5).

Pictures of the conjugate locus are shown: Fig. 2 for the first generic situation (i), $Q_2 \neq 0$, and Fig. 11 for the second generic situation (ii), $Q_2 = 0$ but $V_3 \neq 0$.

We show pictures of the bifurcation of the conjugate locus, as we move continuously from situation (i) to situation (ii), in Figs 7 to 11.

In case (i), we show also pictures of the wave fronts of small radius, and their intersection with the conjugate locus (Figs 1, 3, 4, 5, 6).

Section 7, at the end of the paper, is devoted to some complements in the case (ii): we show that, for an open set of values of parameters, the conjugate locus has in fact 6 cuspidal lines (Fig. 12) (and not 3 as in the approximation $\mathcal{E}^4$ of $\mathcal{E}$). Pictures of its intersection with small wave fronts are also shown (Figs 13–16).

2. Normal forms

In this section, we will study the normal forms of orthonormal frames of a *formal* sub-Riemannian metric. Such a frame Σ is a couple (F, G) of two formal vector fields. Since we restrict ourselves to contact distributions, Σ must satisfy the condition: $F(0)$, $G(0)$, $[F, G]\,(0)$ are independent.

A typical point of R^3 is denoted by ξ. We call the canonical coordinates in R^3, (x, y, w), and write z for the couple (x, y). We normalize the couples $\Sigma = (F, G)$ by chosing $dw\big(F(0)\big) = dw\big(G(0)\big) = 0$.

The set of these couples $\Sigma = (F, G)$ will be denoted by $\mathcal{R}$.

2.1. Notations and statement of the problem.

1. $\mathcal{F}$ is the set of all formal power series in the variables x, y, w. We define a gradation on $\mathcal{F}$ as follows: the weights of x, y will be one, that of w, two. Let us denote by $\mathcal{F}_n$ the space of all homogeneous polynomials of degree n with respect to this gradation. Then $\mathcal{F}$ is the completion $\prod\limits_{n\geq 0} \mathcal{F}_n$ of the direct sum $\bigoplus\limits_{n\geq 0} \mathcal{F}_n$ with respect to the valuation defined by this gradation (see [5]).

2. $V\mathcal{F}$ will denote the free $\mathcal{F}$-module of rank 3 of all formal vector fields. It has a basis (e_1, e_2, e_3), where e_1 (resp. e_2, e_3) corresponds to the derivation $\frac{\partial}{\partial x}$ (resp. $\frac{\partial}{\partial y}, \frac{\partial}{\partial w}$). A gradation of $\mathcal{F}$ induces a gradation on the module $V\mathcal{F}$ of derivations of $\mathcal{F}$: e_1, e_2, e_3 have the weights -1, -1, -2 respectively.

Then, $V\mathcal{F}$ is the completion $\prod_{n\geq -2} V\mathcal{F}_n$ of $\bigoplus_{n\geq -2} V\mathcal{F}_n$, where $V\mathcal{F}_n$ is the vector space of all $\sum_{i=1}^{3} a^i e_i$, a^1, $a^2 \in \mathcal{F}_{n+1}$, $a^3 \in \mathcal{F}_{n+2}$.

3. The elements of $V\mathcal{F}$ act on $\mathcal{F}$ as (Lie) derivations as follows:

$$L_{e_1}(f) = \frac{\partial f}{\partial x}, \;\; L_{e_2}(f) = \frac{\partial f}{\partial y}, \;\; L_{e_3}(f) = \frac{\partial f}{\partial w}.$$

The Lie derivation operation is compatible with the gradations of $\mathcal{F}$ and $V\mathcal{F}$:

$$\text{if } V \in V\mathcal{F}_n \text{ and } f \in \mathcal{F}_m, \; L_V(f) \in \mathcal{F}_{n+m}.$$

4. $V\mathcal{F}^0$ denotes the submodule of $V\mathcal{F}$ such that: if $V \in V\mathcal{F}^0$, then $dw\big(V(0)\big) = 0$. Equivalently, $V\mathcal{F}^0 = \prod_{n\geq -1} V\mathcal{F}_n$.

Let us set $(V\mathcal{F})^2 = V\mathcal{F}^0 \times V\mathcal{F}^0$.

5. $V\mathcal{F}$ is also a Lie algebra under the following Lie bracket operation: $X \in V\mathcal{F}$, $Y \in V\mathcal{F}$, $[X, Y] = Z$, where $Z_n = \sum_{i+j=n} [X_i, Y_j]$.

6. Defintion of $\mathcal{D}$. We shall denote by $\mathcal{D}$ the set of formal diffeomorphisms preserving the origin and the plane $w = 0$. This is a group under the natural composition operation, and can be identified in a natural way to a subset of $V\mathcal{F}$ as follows:

Let us denote by $\mathcal{D}_0$ the set of all $\varphi = (\varphi_1^1, \varphi_1^2, \varphi_2^3) \in V\mathcal{F}_0$ such that

$$\det \begin{pmatrix} \dfrac{\partial \varphi_1^1}{\partial x} & \dfrac{\partial \varphi_1^1}{\partial y} \\ \dfrac{\partial \varphi_1^2}{\partial x} & \dfrac{\partial \varphi_1^2}{\partial y} \end{pmatrix} \neq 0 \;\; \text{and} \;\; \frac{\partial \varphi_2^3}{\partial w}(0) \neq 0.$$

Then, $\mathcal{D}$ is the completion

$$\mathcal{D}_0 \bigoplus \prod_{n\geq 1} V\mathcal{F}_n.$$

7. Definition of $\mathcal{R}$, $\mathcal{R}_{-1}$, $\mathcal{R}_H$. $\mathcal{R} \subset (V\mathcal{F}^0)^2$ is the completion

$$\mathcal{R}_{-1} \bigoplus \prod_{n\geq 0} (V\mathcal{F}_n)^2,$$

where

$$\mathcal{R}_{-1} = \left\{ (F_{-1}, G_{-1}) \;\middle|\; \begin{array}{l} F_{-1}, G_{-1} \in V\mathcal{F}_{-1}, \\ F_{-1}(0), G_{-1}(0), [F_{-1}, G_{-1}] \text{ are independent} \end{array} \right\}.$$

Let us denote by $\mathcal{N}_{-1}$ the couple (F_{-1}^0, G_{-1}^0), where

$$(F_{-1}^0,\, G_{-1}^0) = \Big(e_1 + \frac{y}{2}e_3,\, e_2 - \frac{x}{2}e_3\Big),$$

the Heisenberg frame, and set

$$\mathcal{R}_H = \mathcal{N}_{-1} \bigoplus \prod_{n\geq 0} (V\mathcal{F}_n)^2.$$

8. Topologies. We endow $\mathcal{F}$ with the topology of convergence of coefficients, $V\mathcal{F}$ which is isomorphic to $\mathcal{F}^3$ with the product topology, $(V\mathcal{F})^2$ also with the product topology, $\mathcal{R} \subset (V\mathcal{F}^0)^2$ and $\mathcal{D} \subset V\mathcal{F}$ with the induced topologies. $\mathcal{D}$ is a topological group.

9. Group operation and gauge group. Both $\mathcal{D}$ and $\mathcal{F}$ operate on $\mathcal{R}$ as follows:

- Action of $\varphi \in \mathcal{D}$: $\varphi\,.\,(F, G) = \big(\varphi_*(F),\, \varphi_*(G)\big)$;
- Action of $\alpha \in \mathcal{F}$: $\alpha.\,(F, G) = (F\, \cos\alpha + G\, \sin\alpha, -F\, \sin\alpha + G\, \cos\alpha)$.

α is called a *formal gauge transformation.*

This action defines a semi-direct product structure on $\mathcal{D} \times \mathcal{F}$ denoted $\mathcal{D} \propto \mathcal{F}$.

The action of $\mathcal{D} \propto \mathcal{F}$ on $\mathcal{R}$ is continuous.

The semi-direct product of the group of origin-preserving formal diffeomorphisms with the group of formal gauge transformations will be called the *gauge group.*

10. Definition of the normal forms. The following subgroups of $\mathcal{D} \propto \mathcal{F}$ will be needed:

$$\begin{aligned}\mathcal{G}_0^0 =&\big\{(\varphi_0, \alpha_0) \in \mathcal{D}_0 \propto \mathcal{F}_0 \,\big|\, \varphi_1^1 = \cos\alpha_0\, x + \sin\alpha_0\, y,\\ &\qquad\qquad \varphi_1^2 = -\sin\alpha_0\, x + \cos\alpha_0\, y,\ \varphi_2^3 = w\big\},\end{aligned}$$

$$\mathcal{G}_I = \big\{(\varphi, \alpha)\ \in \mathcal{D} \propto \mathcal{F}\ \big|\alpha_0 = 0,\ \varphi_0 = \text{Identity}\big\}.$$

$\mathcal{G}_I$ is a normal subgroup of $\mathcal{D} \propto \mathcal{F}$. Hence, the product set $\mathcal{G}_I\, \mathcal{G}_0^0$ is a group $\mathcal{G}^0$. It is easy to check that the decomposition of a $g \in \mathcal{G}^0$ as a product $g_2\, g_1$, $g_1 \in \mathcal{G}_0^0$, $g_2 \in \mathcal{G}_I$ is unique.

Let us explain how we shall proceed. First, we shall prove that:

$$\left\{\begin{array}{ll}\text{(i)} & \mathcal{R}_H \text{ meets every orbit of the action of } \mathcal{D} \propto \mathcal{F} \text{ on } \mathcal{R},\\ \text{(ii)} & \text{The stabilizer of } \mathcal{R}_H \text{ in } \mathcal{D} \propto \mathcal{F} \text{ is } \mathcal{G}^0,\\ \text{(iii)} & \text{The set of all } g \in \mathcal{D} \propto \mathcal{F} \text{ such that } g.\,\mathcal{R}_H \cap \mathcal{R}_H \neq \emptyset \text{ is just } \mathcal{G}^0.\end{array}\right. \tag{2.1}$$

These remarks reduce the study of the orbits of $\mathcal{D} \propto \mathcal{F}$ in $\mathcal{R}$ to those of $\mathcal{G}^0$ in $\mathcal{R}_H$.

The usual way to study the action of a group is to construct a continuous section of the action. But the action of $\mathcal{G}^0$ in $\mathcal{R}_H$ does not have a continuous section, that is, a section of the canonical projection $\mathcal{R}_H \to \mathcal{R}_H/\mathcal{G}^0$, which is continuous for the topology of coefficients on $\mathcal{R}_H$ and the quotient topology on $\mathcal{R}_H/\mathcal{G}^0$. On the other hand, we shall prove that the action of $\mathcal{G}_I$ on $\mathcal{R}_H$ *possesses such a section.*

Definition 2.1. A *normal form* $\mathcal{NF}$ is an element of such a continuous section $\mathcal{N}$.

11. Reduction to the action of $\mathcal{G}^0$ in $\mathcal{R}_H$. Taking $(F,G) \in \mathcal{R}$, $(\varphi,\alpha) \in \mathcal{D} \propto \mathcal{F}$, we have $(\varphi,\alpha).(F,G) = (F',G')$, where

$$\begin{cases} F'_{-1} = \varphi_{0*}(F_{-1})\cos\alpha_0 + \varphi_{0*}(G_{-1})\sin\alpha_0, \\ G'_{-1} = -\varphi_{0*}(F_{-1})\sin\alpha_0 + \varphi_{0*}(G_{-1})\cos\alpha_0. \end{cases} \tag{2.2}$$

If we take $\alpha_0 = 0$ in (2.2), easy computations left to the reader show that there exists a unique φ_0 such that (F'_{-1}, G'_{-1}) is the Heisenberg frame, because $F_{-1}(0)$, $G_{-1}(0)$, $[F_{-1}, G_{-1}]$ are independent. Hence:

Lemma 2.2. *$\mathcal{R}_{-1}$ is the orbit of (F^0_{-1}, G^0_{-1}) under the action of the group $\mathcal{D}_0 \propto \mathcal{F}_0$. The stabilizer of (F^0_{-1}, G^0_{-1}) is $\mathcal{G}^0_0$.*

The assertions (2.1) follow from this lemma.

2.2. Continuous sections of the action of $\mathcal{G}_I$ on $\mathcal{R}_H$.

1. Let us chose $(\varphi,\alpha) \in \mathcal{G}_I$, with $\varphi = Id + \varphi_{n+1} + \varphi_{n+2} + \dots$, $\alpha = \alpha_{n+1} + \alpha_{n+2} + \dots$, and $(F,G) \in \mathcal{R}_H$, $F = F^0_{-1} + \dots + F_{n-1} + F_n + \dots$, $G = G^0_{-1} + \dots + G_{n-1} + G_n + \dots$.

Set $(\varphi,\alpha).(F,G) = (F',G')$.

Easy computations left to the reader show that:

Lemma 2.3.

$$\begin{cases} 1. & \begin{cases} F'_k = F_k, \\ G'_k = G_k, \end{cases} \quad k \le n-1, \\ 2. & \begin{cases} F'_n - F_n = [\varphi_{n+1}, F^0_{-1}] + \alpha_{n+1}G^0_{-1}, \\ G'_n - G_n = [\varphi_{n+1}, G^0_{-1}] - \alpha_{n+1}F^0_{-1}. \end{cases} \end{cases} \tag{2.3}$$

The relations (2.3) lead to the introduction of the linear operator L_n, $n \geq 0$:

$$L_n : \mathcal{D}_{n+1} \times \mathcal{F}_{n+1} \to (V\mathcal{F}_n)^2,$$

$$L_n(\varphi_{n+1}, \alpha_{n+1}) = \left([\varphi_{n+1}, F^0_{-1}] + \alpha_{n+1}G^0_{-1},\ -\alpha_{n+1}F^0_{-1} + [\varphi_{n+1}, G^0_{-1}]\right).$$

Lemma 2.4.

1. *L_n is injective.*
2. *L_n is not surjective for $n \geq 1$.*

The proof of this lemma is a consequence of the results of the Appendix 9.1.

In order to construct continuous sections of the action of $\mathcal{G}_I$ on $\mathcal{R}_H$, we choose a vector space $\mathcal{N}_n \subset (V\mathcal{F}_n)^2$ which is a complement of the image of L_n for each $n \geq 1$ (the sum of $\mathcal{N}_n$ and $\operatorname{Im} L_n$ is direct and is the whole space). The sequence of choices $(\mathcal{N}_n, n \geq 1)$ determines a continuous section in a unique way.

There are several choices for the $\mathcal{N}_n$. We have made one, which is most convenient because of its simplicity and its *invariance with respect to the action of* $\mathcal{G}_0^0$.

2. Our normal form. $\mathcal{N}_n$ is the vector subspace of $(V\mathcal{F}_n)^2$,

$$\begin{aligned}\mathcal{N}_n = \Big\{ &(y^2 e_1 - xye_2)\beta_{n-1} + \frac{y}{2}\gamma_{n+1}e_3,\ (x^2 e_2 - xye_1)\beta_{n-1} - \frac{x}{2}\gamma_{n+1}e_3, \\ &\beta_{n-1} \in \mathcal{F}_{n-1},\ \gamma_{n+1} \in \mathcal{F}_{n+1}, \\ &\beta_{n-1}(0,0,w) = 0,\ \gamma_{n+1}(0,0,w) = \frac{\partial \gamma_{n+1}}{\partial x}(0,0,w) = \frac{\partial \gamma_{n+1}}{\partial y}(0,0,w) = 0 \Big\}.\end{aligned}$$

Theorem 2.5. *Any couple (F, G) of formal vector fields at 0, such that $F(0), G(0), [F, G](0)$ are independent, is equivalent under the action of the gauge group to one of the following forms:*

$$(\mathcal{NF}) \quad \begin{cases} e_1 + (y^2\, e_1 - x\, y\, e_2)\, \beta(\xi) + \dfrac{y}{2}\, (1 + \gamma(\xi))\, e_3, \\ e_2 + (x^2\, e_2 - x\, y\, e_1)\, \beta(\xi) - \dfrac{x}{2}\, (1 + \gamma(\xi))\, e_3. \end{cases} \tag{2.4}$$

β and γ belong to $\mathcal{F}$ and satisfy the conditions

$$\beta(0,0,w) = 0, \gamma(0,0,w) = 0, \frac{\partial \gamma}{\partial x}(0,0,w) = \frac{\partial \gamma}{\partial y}(0,0,w) = 0.$$

Moreover, the stabilizer of any element $\mathcal{NF}$ under the action of $\mathcal{G}_I$ is trivial, and the section is invariant under the action of $\mathcal{G}_0^0$. The action of $\mathcal{G}_0^0$ is the natural one.

The fact that the section is invariant under the action of $\mathcal{G}_0^0$ is just a matter of trivial computations. That the stabilizers of the elements $\mathcal{NF}$ under the action of $\mathcal{G}_I$ are trivial is a consequence of Lemma 2.4.

This Theorem 2.5 is proved in Appendix 9.1.

2.3. Partial normal forms.

1. If $f \in \mathcal{F}$ (resp. $V \in V\mathcal{F}^0$, $\varphi \in \mathcal{D}$, ...), let us denote by $j^n f$ (resp. $j^n V$, $j^n\varphi, \ldots$) the element $f_0 + f_1 + \ldots + f_n$ (resp. $V_{-1} + \ldots V_n$, $\varphi_0 + \ldots + \varphi_n, \ldots$).

$j^n f \in J^n\mathcal{F}$ (resp. $j^n V \in J^n V\mathcal{F}^0$, $j^n\varphi \in J^n\mathcal{D}, \ldots$).

The action of $\mathcal{D} \propto \mathcal{F}$ on $\mathcal{R}$ induces actions of $J^{n+1}\mathcal{G}_I$ on $J^n\mathcal{R}_H$.

Definition 2.6. We call an element of a section of the action of $J^{i+1}\mathcal{G}_I$ on $J^i\mathcal{R}_H$ a "partial normal form of order i" and denote it by $\mathcal{NF}^i$.

A normal form $\mathcal{NF}$ determines partial normal forms $\mathcal{NF}^i$ of any order i: Any $\Sigma \in \mathcal{R}$ can be written as follows:

$$(\mathcal{NF}^i) \qquad \Sigma = \Sigma_{-1} + \ldots + \Sigma_l + \ldots + \Sigma_i + O^{i+1}, \quad \Sigma_l \in \mathcal{N}_l\,, \tag{2.5}$$

where $O^{i+1} \in (V\mathcal{F})^2$ and O^{i+1} has order $i+1$ in the gradation defined in Sec. 2.1.

For our study in the next sections, we will need mainly the partial normal forms of orders 1 and 2. Since the components β_1, γ_2, γ_3 are fundamental for us, we call them l, Q, V respectively.

$$(\mathcal{NF}^1) \qquad \Sigma = (F^0_{-1}, G^0_{-1}) + (\frac{y}{2} Q\, e_3, -\frac{x}{2} Q\, e_3) + O^2, \tag{2.6}$$

with $Q = Q(z)$, a quadratic form in z.

$$(\mathcal{NF}^2) \begin{cases} \Sigma = \Sigma_{-1} + \Sigma_1 + \Sigma_2 + O^3, \\ \Sigma_{-1} = (F^0_{-1}, G^0_{-1}),\ \Sigma_1 = \left(\frac{y}{2} Q\, e_3, -\frac{x}{2} Q\, e_3\right), \\ \Sigma_2 = \Big((y^2\, e_1 - x\, y\, e_2)\, l(z) + \\ \qquad + \frac{y}{2}\, V(z)\, e_3, (x^2\, e_2 - x\, y\, e_1)\, l(z) - \frac{x}{2}\, V(z)\, e_3\Big), \end{cases} \tag{2.7}$$

where $l(z)$ is linear, $Q(z)$ is quadratic, $V(z)$ is cubic.

Remark 2. Our normal form has three advantages which will simplify our computations of the conjugate locus in the following paragraphs:

(1) it has a high order contact with the Heisenberg canonical form;

(2) the section is invariant under the action of the group $\mathcal{G}^0_0$, as was already noticed;

(3) *the variable w does not appear in the partial normal form $\mathcal{NF}^2$.*

In a certain sense, the coordinates introduced by the normal form play the same role as the so-called "normal coordinates" associated to Riemannian metrics: they are the coordinates in which the expression of the metric is the simplest possible.

2. Action of $\mathcal{G}_0^0$ on partial normal forms. Let us go back, for V, to the notations (1.2) of the introduction. Trivial consequences of Theorem 2.5 are:

Corollary 2.7. *The action of $\mathcal{G}_0^0$ on the partial normal forms $\mathcal{NF}^2$ is as follows. $(\varphi_0, \alpha_0) \in \mathcal{G}_0^0$ transform Q, l, V in the following way:*

$$\begin{aligned} &\text{(i)} && Q(z) \to Q\big(e^{\alpha_0 j}(z)\big);\\ &\text{(ii)} && l(z) \to l\big(e^{\alpha_0 j}(z)\big);\\ &\text{(iii)} && V(z) \to V(e^{\alpha_0 j}(z)),\\ &&& \textit{or,}\ \ a \to e^{-i\alpha_0}\, a,\ \ b \to e^{3\,i\,\alpha_0}\, b. \end{aligned}$$

2.4. Stabilizers. With these complex notations (1.2) of the introduction, $V = V_1 + V_3$, $Q = Q_0 + Q_2$, we get

Corollary 2.8. *The action of $J^2\mathcal{D} \propto \mathcal{F}$ on the open subset*

$$J^1G_1 \subset J^1\mathcal{R}, \quad G_1 = \{\Sigma \,|\, Q_2 \neq 0\},$$

admits the following smooth sections $\overline{\mathcal{N}}^1$.

$\overline{\mathcal{N}}^1$ is the set of all $j^1\Sigma$ such that

$$(\overline{\mathcal{NF}}^1) \qquad j^1\Sigma = (F_{-1}^0, G_{-1}^0) + \left(\frac{y}{2}Q\, e_3, -\frac{x}{2}Q\, e_3\right), \tag{2.8}$$

where $Q(z) = c_1x^2 + c_2y^2$, $c_1 < c_2$.

The stabilizer of any element $\overline{\mathcal{NF}}^1$ is the subgroup $\{(\varphi(0), 0), (\varphi(\pi), \pi)\}$ of $\mathcal{G}_0^0$.

A quasi-section of the action of a topological group G on a set X is a subset of X, which cuts any G-orbit in X such that the stabilizers of its elements are discrete.

Corollary 2.9. *The action of $J^3\mathcal{D} \propto \mathcal{F}$ on the subsets*

$$\begin{aligned} &J^2G_2,\ (\textit{resp.}\ J^2G_3, J^2G_4) \subset J^2\mathcal{R}, && G_2 = \{\Sigma | V_3 \neq 0,\, Q_2 = 0\},\\ &G_3 = \{\Sigma \,|\, V_1 \neq 0,\, Q_2 = 0\}, && G_4 = \{\Sigma \,|\, l \neq 0,\, Q_2 = 0\} \end{aligned}$$

admits the following quasi-section (resp. smooth sections) $\overline{\mathcal{N}}^i$, $i = 2, 3, 4$.

$\overline{\mathcal{N}}^i$, $i = 2,3,4$, *is the set of all* $j^2\Sigma$ *such that*

$$
\begin{aligned}
&(\overline{\mathcal{NF}}^i) \qquad j^2\Sigma = (F^2, G^2),\\
&F^2 = F^0_{-1} + \frac{\mathrm{tr}_g Q}{2}\,\|z\|^2\,\frac{y}{2}\,e_3 + l(z)\,(-x\,y\,e_2 + y^2\,e_1) + \frac{y}{2}V(z)\,e_3,\\
&G^2 = G^0_{-1} - \frac{\mathrm{tr}_g Q}{2}\,\|z\|^2\,\frac{x}{2}\,e_3 + l(z)\,(-x\,y\,e_1 + x^2\,e_2) - \frac{x}{2}V(z)\,e_3,
\end{aligned}
\tag{2.9}
$$

with $\big(V_3(1,0) = 0,\, V_3(0,1) < 0\big)$ $\big($*resp.* $\big(V_1(1,0) = 0,\, V_1(0,1) > 0\big)$, $\big(l(0,1) = 0,\, l(1,0) > 0\big)\big)$.

The stabilizers of all elements $\overline{\mathcal{NF}}^3$, $\overline{\mathcal{NF}}^4$ *are trivial.*

2.5. Codimension of some bad subsets in $\mathcal{R}$.

1. We will consider the two bad subsets of $\mathcal{R}$, B^1 and B^2, that are the complements of G_1 and $G_1 \cup G_2$ respectively (G_1, G_2 just defined in Corollaries 2.8, 2.9).

2. $SJ^n(\mathcal{R})$ denotes the set of standard n-jets of elements (F, G) of $\mathcal{R}$, i.e., $SJ^n(\mathcal{R})$ is the set of couples of standard n-jets of vectors.

Forcing α_0 to be 0, we already know, as a consequence of Lemma 2.2 and Theorem 2.5, that there is a well-defined map $\Pi_{\mathcal{N}}$, putting any (F, G) under normal form, $\Pi_{\mathcal{N}} : \mathcal{R} \to \mathcal{N}_{-1} \bigoplus \prod_{n\geq 0} \mathcal{N}_n$.

$\Pi_{\mathcal{N}}$ induces a map $\Pi^n_{\mathcal{N}} : SJ^{n+2}(\mathcal{R}) \to \mathcal{N}_{-1} \bigoplus \bigoplus_{i=0}^{n} \mathcal{N}_n$.

The following lemma is easy to check (left to the reader).

Lemma 2.10. $\Pi^n_{\mathcal{N}}$ *is an analytic surjective submersion.*

Remark 3. $\Pi^n_{\mathcal{N}}$ is a rational map.

Theorem 2.11. B^1 *and* B^2 *are analytic submanifolds of* $SJ^4(\mathcal{R})$, *of codimension* 2 *and* 4 *respectively.*

Proof. The space $\mathcal{N}_1 \bigoplus \mathcal{N}_2$ is the space $\{Q(z),\, V(z),\, l(z)\}$, where Q, V, l are as in Secs 2.3.1, 2.4.

The set of all quadratic forms with zero discriminant is the set of multiples of $(x^2 + y^2)$, hence has codimension two. Therefore, $\Pi^2_{\mathcal{N}}(B^1)$ has codimension 2. The cubic forms V, such that $V_3 = 0$, form a vector subspace of codimension 2, hence $\Pi^2_{\mathcal{N}}(B^2)$ has codimension 4. Lemma 2.10 implies that B^1 and B^2 are analytic manifolds of codimension 2 and 4 respectively. $\square$

3. Differential-geometric interpretation of our main invariants

In the previous Sec. 2, we were led to consider Q, V, l that appear naturally. Also, V_3 and Q_2 play a special role: They define the bad sets B^1 and B^2 that we will have to consider in order to describe the generic situations for sub-Riemannian metrics. In this section, we will interpret these invariants in terms of classical differential geometry. For this we will associate canonically a linear connection to a given sub-Riemannian metric.

3.1. Canonical connection associated to a sub-Riemannian metric (Δ, g).

1. We consider, the one form ω, unique up to orientation, defined by:
(i) $\Delta = \ker \omega$,
(ii) $d\omega =$ volume form over Δ,
and the unique vector field ν on M, such that

$$i(\nu)(\omega \wedge d\omega) = d\omega\,.$$

This last fact is equivalent to $i(\nu)(d\omega) = 0$, $\omega(\nu) = 1$.

The bundle Δ carries a natural complex structure $j : \Delta \to \Delta$, defined by $d\omega(X, Y) = \langle j\,X,\, Y\rangle_g$. We extend the operator j to TM in a natural way, setting $j\,\nu = 0$, and call the extended operator j again.

Lemma 3.1. (i) *TM (resp. T^*M) is canonically isomorphic to*

$$\Delta \bigoplus_M R\,\nu$$

(resp. $\Delta^* \bigoplus_M \Delta^0$, where Δ^* is the dual of Δ and Δ^0 the annihilator of Δ in T^*M);

(ii) *the structural group of Δ can be reduced to $SO(2)$, that of TM and T^*M to $SO(2) \times Id_R$.*

Proof. (ii) is an immediate consequence of (i). (i) follows from above if we remark that Δ^* can be canonically identified to the annihilator of $R\,\nu$.

2. Distinguished coframe fields. A coframe field $(\omega^1, \omega^2, \omega^3)$ on an open subset $M' \subset M$ is said to be *distinguished* if

$$\text{(i)}\ \ (\omega^1)^2 + (\omega^2)^2\big|_\Delta = g, \qquad \text{(ii)}\ \omega^3 = \omega, \qquad d\omega^3 = \omega^1 \wedge \omega^2.$$

(ii) implies that $\omega^1(\nu) = \omega^2(\nu) = 0$. If (e_1, e_2, e_3) is a frame field of M', dual to $(\omega^1, \omega^2, \omega^3)$, then $e_1, e_2 \in \Delta$, $e_3 = \nu$.

3. Canonical connection. $\underline{TM}$ and $\underline{\Delta}$ will denote the sheaves of sections of TM and Δ respectively.

Theorem 3.2. *There is a unique linear connection such that, if $\nabla : TM \times_M \underline{TM} \to TM$ is the associated covariant derivative, then:*

(i) $\nabla : TM \times_M \underline{\Delta} \to \Delta$, *($\nabla$ has a restriction to the bundle Δ);*
(ii) $\nabla g = 0$ *and* $\nabla \omega = 0$;
(iii) $\nabla_\nu \nu = 0$;
(iv) *the torsion of* ∇ *is* $\nu \otimes d\omega$;
(v) *the bilinear form* $\mathrm{II} : \Delta \times_M \Delta \to R$, $\mathrm{II}(X,Y) = \langle \nabla_X \nu, Y \rangle_g$ *is symmetric.*

Definition 3.3. II is called the second fundamental form.

The proof of Theorem 3.2 is given in Appendix 9.2.

Note that our canonical connection is different from the one considered in [6].

Remark 4. Let g^* be the tensor representing the cometric of g: for $\psi \in T^*M$, $g^*(\psi) = \sup\left\{ \frac{\psi(u)^2}{\langle u, u\rangle_g} \middle| u \in \Delta_q,\ u \neq 0,\ q = \pi_{T^*M}(\psi)\right\}$. Then it is easy to see that $\nabla g^* = 0$.

3.2. Interpretation of the invariants β, γ. Let q be any point in M. Let (F,G) be any germ at q of an orthonormal frame field for (Δ, g). We can apply Theorem 2.5 to the pair $(j_q^\infty F,\ j_q^\infty G)$ of infinite jets of F, G at q and obtain the formal functions β, γ, denoted $\beta_{F,G,q}$, $\gamma_{F,G,q}$ respectively here. The expressions of these functions with respect to the linear coordinates (x,y,w) which are dual to the frame $\big(F(q), G(q), [F,G](q)\big)$ are determined in Theorem 2.5.

For each n, the components of degree n, $\beta_{n,F,G,q}$ and $\gamma_{n,F,G,q}$, of $\beta_{F,G,q}$ and $\gamma_{F,G,q}$ respectively, define tensors belonging to the sub-tensor bundle of $\otimes T^*M$:

$$\otimes\left\{ (S^k\Delta^*)_q \otimes (\Delta_q^0)^{\frac{n-k}{2}} \,\middle|\, 0 \le k \le n,\ k = n \mod 2 \right\},$$

where $S^k\Delta^*$ is the bundle of all symmetric covariant k-tensors on Δ and $(\Delta^0)^{\frac{n-k}{2}}$ is the $\left(\frac{n-k}{2}\right)$th tensor power of Δ^0. We have used the identification of T^*M with $\Delta^* \bigoplus_M \Delta^0$ introduced in Lemma 3.1. Then the definition of the normal form and Theorem 2.5 imply the proposition.

Proposition 3.4.

(i) *The tensors $\beta_{n,F,G,q}$ and $\gamma_{n,F,G,q}$ are independent of the choice of the germ of frame field (F,G) at q;*

(ii) *Denoting them by $B_{n,q}$, $\Gamma_{n,q}$, the correspondences $q \in M \to B_{n,q}$, $\Gamma_{n,q}$ define tensor fields on M which are invariants of the structure (Δ, g).*

Proof. (ii) is an easy consequence of (i) which follows from the fact that if we replace the germ at q of frame field (F, G) by another one, say (F_1, G_1), then the unique mapping $A \in O(3)$, mapping the frame $\big(F(q), G(q), [F, G](q)\big)$ onto $\big(F_1(q), G_1(q), [F_1, G_1](q)\big)$, belongs to $SO(3)$. Then an easy computation shows that $\beta_{F_1,G_1,q} = \beta_{F,G,q} \circ A$ and $\gamma_{F_1,G_1,q} = \gamma_{F,G,q} \circ A$. □

The tensor fields most important to us will be those corresponding to β_1, γ_2, γ_3, which we have denoted by l, Q, V respectively. Note that l, Q, V depend only on (x, y). Hence the corresponding tensor fields, which for simplicity we shall still denote by l, Q, V, belong to the tensor spaces Δ^*, $S^2\Delta^*$, $S^3\Delta^*$ respectively.

3.3. Decomposition of tensor fields. Let J be a subbundle of a tensor bundle $\otimes^{p,q}\Delta$(p-contravariant, q-covariant tensors on Δ). Let $J(0)$ be its typical fiber. Then the structural group $SO(2)$ of Δ operates on $J(0)$. $J(0)$ considered as a $SO(2)$-module decomposes into irreducible components $J(0) = \bigoplus_n \{J_n(0)\,|\, n \in Z,\ n \text{ not necessarily distinct}\}$. The representation of $SO(2)$ on $J_n(0)$ has character χ^n, nth power of the basic character χ : $\chi(e^{i\theta}) = e^{i\theta}$.

Later on, we shall need the following decompositions of $S^2\Delta^*$, $S^3\Delta^*$, $\Delta^* \otimes S^2\Delta^*$, into isotypic components. If N is an $SO(2)$-module, we denote by N_k its isotypic component corresponding to the kth power of the basic character χ, $k \in Z$.

$$S^2\Delta^* = (S^2\Delta^*)_2 \bigoplus (S^2\Delta^*)_0, \qquad S^3\Delta^* = (S^3\Delta^*)_3 \bigoplus (S^3\Delta^*)_1,$$
$$\Delta^* \otimes S^2\Delta^* = \bigoplus_{n=3,\,1} (\Delta^* \otimes S^2\Delta^*)_n.$$

$\Delta^* \otimes \wedge^2\Delta^*$ is irreducible with character χ.

It is easy to see that $(S^2\Delta^*)_0$ can be identified with the line bundle $R\,g$, g the tensor representing the metric, $(S^3\Delta^*)_1$ with the symmetric product $\Delta^* \odot R\,g$. Then $(S^2\Delta^*)_2$ (resp. $(S^3\Delta^*)_3$) is just the orthogonal component of $(S^2\Delta^*)_0$ (resp. $(S^3\Delta^*)_1$) in $S^2\Delta^*$ (resp. $S^3\Delta^*$) with respect to the metric induced by g on $\Delta^* \odot \Delta^*$ (resp. $\Delta^{*\odot 3}$).

If $q \in S^2\Delta^*$, then $q = q_2 + q_0$, $q_n \in (S^2\Delta^*)_n$, $n = 0, 2$, and $q_0 = \dfrac{1}{2}(\operatorname{tr}_g q)g$, where $\operatorname{tr}_g q$ is the trace of q with respect to g and $\operatorname{tr}_g q_2 = 0$. The determinant of q_2 with respect to g is negative or zero. Its opposite is denoted by $\delta(q)$.

Let ω^1, ω^2 be an orthonormal coframe field for (Δ, g), defined on an open subset M' of M such that $\omega^2 \circ j = -\omega^1$. Then $\Omega = \omega^1 + i\,\omega^2 : \Delta \to \mathbb{C}$ is $\mathbb{C}$-linear for the complex structure on Δ defined by j. A tensor $\tau \in S^2\Delta^*_x$, $x \in M'$, belongs to $(S^2\Delta^*)_2$ (resp. $(S^2\Delta^*)_0$) if and only if it can be written as $\tau = \operatorname{Re}\big(\lambda_\tau\,(\Omega)^2\big)$ (resp. $\mu_\tau\,|\Omega|^2$, $\mu_\tau \in R$), and a tensor $\tau \in S^3\Delta^*$ belongs to $(S^3\Delta^*)_3$ (resp. $(S^3\Delta^*)_1$) if and only if $\tau = \mathrm{Re}(b_\tau\,(\Omega)^3)$, $b_\tau \in \mathbb{C}$ (resp. $\mathrm{Re}(\bar{a}_\tau\,\Omega\,|\Omega|^2)$, $a_\tau \in \mathbb{C}$), $\bar{a}_\tau$ is the complex conjugate of a_τ.

In particular, if we go back to the fields l, Q, V introduced in Sec. 2.3.1, we have at the point 0: $Q = Q_2 + Q_0$, where $Q_0 = \frac{1}{2}\,\mathrm{tr}_g\, Q\, g = \frac{c_1 + c_2}{2}\,|dz_0|^2$, $Q_2 = \frac{c_1 - c_2}{2}\,\mathrm{Re}\big((dz_0)^2\big)$. $V = V_3 + V_1$, $V_1 = \mathrm{Re}(\bar{a}\,dz_0\,|dz_0|^2)$, $V_3 = \mathrm{Re}\big(b\,(dz_0)^3\big)$, where $dz_0 = dx_0 + i\,dy_0$, x, y being the coordinates introduced in the normal form $\mathcal{NF}$, c_1, $c_2 \in R$, a, $b \in \mathbb{C}$.

3.4. Evaluation of Q_2, V_1, V_3 in terms of the Gaussian curvature, the second fundamental form and their covariant derivatives. We refer to the notations of our Appendix 9.2.

The Gaussian curvature form is $\Omega_2^1 = d\omega_2^1$. II is a section of $S^2\Delta^*$, $\Omega_2^1\big|_\Delta$ of $\wedge^2\Delta^*$. Hence, the restrictios of $\nabla\mathrm{II}\big|_\Delta$ and $\nabla\Omega_2^1\big|_\Delta$ to $\Delta^{\otimes 3}$ are sections of $\Delta^* \otimes S^2\Delta^*$ and $\Delta^* \otimes \wedge^2\Delta^*$ respectively. Let III_1, III_3 be the components of the restriction $\nabla\mathrm{II}|_\Delta$ in $(\Delta^* \otimes S^2\Delta^*)_1$ and $(\Delta^* \otimes S^2\Delta^*)_3$.

Then:

Theorem 3.5.

$$
\begin{aligned}
&Q_2(u) = \frac{1}{4}\,\mathrm{II}\big(u, j(u)\big),\\
&V_3(u) = \frac{1}{15}\,\mathrm{III}_3\big(j(u), u, u\big),\\
&B_1(u) = l(u) = \mathrm{III}_1\big(j(u), u, u\big) + \frac{1}{6}\,\nabla_u\Omega_2^1\big(u, j(u)\big),\\
&V_1(u) = \frac{1}{10}\,\big(\mathrm{III}_1\big(j(u), u, u\big) + \frac{1}{2}\,\nabla_u\Omega_2^1\big(u, j(u)\big), \text{ for any } u \in \Delta.
\end{aligned}
$$

Also:

$$
\begin{aligned}
&V_3(v) = \frac{1}{30}\,\big(\nabla_v\mathrm{II}(j\,v, v) + \nabla_{j\,v}\mathrm{II}(v, v)\big),\\
&B_1(v) = l(v) = \frac{1}{2}\left(\frac{1}{3}\,\nabla_v\Omega_2^1(v, j\,v) + \nabla_{j\,v}\mathrm{II}(v, v) - \nabla_v\mathrm{II}(j\,v, v)\right),\\
&V_1(v) = \frac{1}{20}\left(\nabla_v\Omega_2^1(v, j\,v) + \nabla_{j\,v}\mathrm{II}(v, v) - \nabla_v\mathrm{II}(j\,v, v)\right),
\end{aligned}
$$

and $\mathcal{K}$ denotes the Gaussian curvature of ∇ $\big(\mathcal{K} = d\omega_2^1(e_1, e_2)\big)$:

$$\mathcal{K} = 6\;\mathrm{tr}_g\, Q.$$

3.5. Equation of geodesics in terms of the connection ∇. In the following sections, we will use the Hamiltonian formalism to compute the geodesics. However, one could use the Lagrangian formalism and the connection ∇.

The following theorem gives the equations of the geodesics in that formalism.

Theorem 3.6. *The geodesics $\lambda(s)$ of a sub-Riemannian metric (s is the arclength) are parametrized by their initial tangent vector $\dot{\lambda}(0)$ and a real parameter r_0. They are the solutions of the following equations*:

$$\begin{cases} \nabla_{\dot{\lambda}}\, \dot{\lambda} = r\, j(\dot{\lambda}), \\ \dot{r} + \mathrm{II}(\dot{\lambda}) = 0, \quad r(0) = r_0. \end{cases}$$

Remark 5. As stated in the introduction, the first equation shows that r is the curvature of the geodesic λ with respect to the connection ∇.

This connection, together with the above equation for geodesics, has been introduced independently by Rumin [16]).

4. Basic facts about geodesics and the conjugate locus

4.1. Geodesics. Let $\Sigma = (\Delta, g)$ be a sub-Riemannian structure on an open set M of R^3, which contains 0. We can define a "cometric" $\mathcal{H} : T^*M \to R$, associated to (Δ, g) as follows: $\mathcal{H}(\psi) = \dfrac{1}{2} \sup \left\{ \dfrac{\psi(v)^2}{\langle v, v\rangle_g} \,\middle|\, v \in \Delta_m \setminus \{0\},\ m = \text{foot of } \psi \right\}$. On each fiber of T^*M, $\mathcal{H}$ is a positive semi-definite quadratic form, the kernel of which is the annihilator Δ^0 of Δ. Finally, if (F, G) is an orthonormal frame field for (Δ, g) defined on an open set O, then on T^*O we have

$$\mathcal{H}(\psi) = \frac{1}{2}\Big((\psi(F))^2 + (\psi(G))^2 \Big).$$

Definition 4.1. A geodesic is a parametrized curve $\lambda : J \to M$, J some interval, which is a projection of a trajectory $\Lambda : J \to T^*M$ of the Hamiltonian vector field $\overrightarrow{\mathcal{H}}$ associated to $\mathcal{H}$.

Since we consider only contact structures, there are no abnormal minimizing curves and the following fact holds:

Proposition 4.2. *Any length-minimizing curve for (Δ, g) is a geodesic.*

It is not true in general that a geodesic curve is minimizing, but for any sufficiently short geodesic this will be the case (see Remark 6, Sec. 4.2 for a precise statement).

As we have said before, a geodesic $\lambda : [0, T[\to M$ is not determined by its initial point $\lambda(0)$ and its initial velocity $\frac{T\lambda}{\partial t}(0)$, but rather by $\lambda(0)$ and the covector $\Lambda(0) \in T^*_{\lambda(0)}M$. For a general sub-Riemannian structure, it can happen that λ has several liftings, but this does not happen in the contact case.

Notations. (1) The unique trajectory Λ of $\overrightarrow{\mathcal{H}}$, of which λ is the projection, will be called the lifting of λ;

(2) in analogy with the initial velocity, we will call the covector $\Lambda(0)$ the initial covector of λ.

4.2. The exponential mapping and the conjugate locus. For $c > 0$, c arbitrary, any minimizing curve is the projection of a trajectory of $\overrightarrow{\mathcal{H}}$ located in the surface $\mathcal{H}^{-1}(c)$. Here, we shall take the level surface $\mathcal{H}^{-1}\left(\frac{1}{2}\right)$ (which is clearly smooth). This corresponds to the trajectories of $\overrightarrow{\mathcal{H}}$, for which *the time parameter s is equal to the arclength.*

The intersection C of $\mathcal{H}^{-1}\left(\frac{1}{2}\right)$ with T_0^*M is a cylinder.

Let us denote by $\mathcal{E} : \widetilde{C} \to M$, the exponential mapping associated to $\mathcal{H}$, $\widetilde{C} \subset R_+ \times C$, $\widetilde{C} = \{(s, \psi) \mid 0 \leq s < e(\psi)\}$. $e(\psi)$ is the positive escape time of the trajectory $\Phi(s, \psi)$ of $\overrightarrow{\mathcal{H}}$ such that $\Phi(0, \psi) = \psi$. $\mathcal{E}(s, \psi) = \pi\left(\Phi(s, \psi)\right)$, π denoting the canonical projection, $\pi : T^*M \to M$.

Definition 4.3. For any $\psi \in C$ the (first) conjugate arclength $s_c(\psi)$ associated to ψ is the positive number $\widehat{s}$, whenever it exists, such that for any $0 < s < \widehat{s}$, $T_{(s,\psi)}\mathcal{E} : T_{(s,\psi)}\widetilde{C} \to T_{\mathcal{E}(s,\psi)}M$ is injective, but $T_{(\widehat{s},\psi)}\mathcal{E}$ is not.

Definition 4.4. The conjugate locus at the source is the graph of $s_c : \mathrm{dom}(s_c) \to R_+$ ($\mathrm{dom}(s_c)$ is the set of all ψ, for which s_c exists).

The *conjugate locus is its image* by the exponential map.

We shall denote the conjugate locus at the source by S and the conjugate locus by CL.

Remark 6. For any $\widehat{s} \in [0, s_c(\psi)[$, the curve $s \in [0, \widehat{s}] \to \mathcal{E}(s, \psi)$ is C^0-locally minimizing.

As we pointed out in the introduction, the main difference from Riemannian geometry is that the point 0 lies in the adherence of CL. The main object of our study in this paper will be the germ of CL at 0.

4.3. A basic property. Let $\mathcal{H}$ be any Hamiltonian homogeneous of degree λ on T^*M. Let k be a regular value of $\mathcal{H}$. $\mathcal{H}^{-1}(k)$ is a smooth hypersurface in T^*M. $\Phi(s,\psi)$ denotes again the trajectory of $\vec{\mathcal{H}}$ such that $\Phi(0,\psi)=\psi$. Let α be the Liouville one-form on T^*M, $\alpha(V_\psi)=\psi(T\pi(V_\psi))$. Let C_s be the open subset of C, where $\Phi(s,.)$ is defined. Set $\Phi_s(\psi)=\Phi(s,\psi)$, $\Phi_s : C_s \to T^*M$.

The following result is well known, but for the convenience of the reader, we shall supply a simple proof.

Lemma 4.5. *Φ_s preserves the Liouville one-form α, restricted to*

$$\mathcal{H}^{-1}(k).$$

Proof. $(\Phi_s)^*\alpha$ denotes the pullback of α.

$$\frac{d}{ds}\big((\Phi_s)^*\alpha\big)=L_{\vec{\mathcal{H}}}\big((\Phi_s)^*\alpha\big),\qquad L_{\vec{\mathcal{H}}}=i_{\vec{\mathcal{H}}}\,d+d\,i_{\vec{\mathcal{H}}}.$$

L and i are the Lie-derivative and contraction operators respectively.

$i_{\vec{\mathcal{H}}}\alpha=\lambda\,\mathcal{H}$. Hence, since $\mathcal{H}$ is constant on $\mathcal{H}^{-1}(k)$,

$$d\,i_{\vec{\mathcal{H}}}\alpha|_{\mathcal{H}^{-1}(k)}=0.$$

Otherwise,

$$i_{\vec{\mathcal{H}}}\,d\alpha=d\mathcal{H}$$

which is also zero on $\mathcal{H}^{-1}(k)$. Hence, $L_{\vec{\mathcal{H}}}\alpha|_{\mathcal{H}^{-1}(k)}=0$. □

The lemma has the following important Corollary 4.6. Let $(x,\,y,\,w)$ be a coordinate system on M such that at 0,

$$g|_{\Delta_0}=(dx)^2+(dy)^2,\quad dx(\nu)=0,\ dy(\nu)=0,\quad dw|_{\Delta_0}=0,\ dw(\nu)=1.$$

(x,y,w) induces a Darboux coordinate system $(x,\,y,\,w,\,p,\,q,\,r)$ on T^*M. In this system $\alpha=p\,dx+q\,dy+r\,dw$.

Let $(R,\,\varphi,\,r_0)$ be the cylindrical coordinate system on the fiber T_0^*M at 0:

$$r_0 \text{ is the restriction of } r \text{ to } T_0^*M,\quad p=R\cos\varphi,\quad q=R\sin\varphi.$$

Then the couple (φ,r_0) is a cylindrical coordinate system on C. (C is the surface $R=1$.)

Set $X_s=\dfrac{T}{\partial s}\mathcal{E}(s,\varphi,r_0)$, $X_{r_0}=\dfrac{T}{\partial r_0}\mathcal{E}(s,\varphi,r_0)$, $X_\varphi=\dfrac{T}{\partial\varphi}\mathcal{E}(s,\varphi,r_0)$. Then, these vectors are the projections on TM of the vectors

$$\vec{\mathcal{H}}\big(\Phi_s(\psi(\varphi,r_0))\big),\qquad T\Phi_s\frac{\partial\psi(\varphi,r_0)}{\partial r_0},\qquad T\Phi_s\frac{\partial\psi(\varphi,r_0)}{\partial\varphi},$$

respectively.

For $s = 0$, α takes the value 0 on the vectors $\frac{T\Phi_s(\varphi, r_0)}{\partial r_0}$, $\frac{T\Phi_s(\varphi, r_0)}{\partial \varphi}$, since they are tangent to $C \subset T_0^*M$. By the invariance of α of Lemma 4.5, $\alpha\Big(\frac{T\Phi_s(\varphi, r_0)}{\partial r_0}\Big) = \alpha\Big(\frac{T\Phi_s(\varphi, r_0)}{\partial \varphi}\Big) = 0$ for all s. Moreover

$$\alpha\big(\vec{\mathcal{H}}(\Phi_s(\varphi, r_0))\big) = 2\mathcal{H}(\Phi_s(\varphi, r_0)) = 1.$$

Hence, the values of the covector $\Phi_s(\varphi, r_0)$ on the vectors X_s, X_φ, X_{r_0} are 1, 0, 0 respectively. Now $dx \wedge dy \wedge \Phi_s(\varphi, r_0) = r\big(\Phi_s(\varphi, r_0)\big)\, dx \wedge dy \wedge dw$.

Corollary 4.6.

$$\begin{aligned}
&dx \wedge dy \wedge dw\,(X_s, X_\varphi, X_{r_0}) = \\
&= \frac{1}{r\big(\Phi_s(\varphi, r_0)\big)}\, dx \wedge dy \wedge \Phi_s(\varphi, r_0)\,(X_s, X_\varphi, X_{r_0}) = \\
&= \frac{1}{r\big(\Phi_s(\varphi, r_0)\big)}\, dx \wedge dy\,(X_\varphi, X_{r_0}).
\end{aligned}$$

Notation. Let us set $\widehat{D} = dx \wedge dy\,(X_\varphi, X_{r_0})$.

4.4. Equations of geodesics in normal form.

1. The most convenient way to write the equations of the geodesics is to introduce the Hamiltonian liftings f, $g : T^*M \to R$, of the vector fields F, G: $f(\psi) = \langle F, \psi\rangle$, $g(\psi) = \langle G, \psi\rangle$. Then the Hamiltonian $\mathcal{H}$ has the expression $\mathcal{H} = \frac{1}{2}(f^2 + g^2)$.

Let $\lambda : I \to M$, $s \to \lambda(s)$, I interval, be a geodesic parametrized by arclength and $\Lambda : I \to T^*M$, $s \to \Lambda(s)$ be its lifting. Let (x, y, w) be any coordinate system on M and p, q, $r : T^*M \to R$ be the dual coordinates. Finally, let $\{\ \}$ denote the Poisson bracket on T^*M associated with its canonical symplectic structure.

The equations determining λ are:

$$\begin{cases}
\text{(i)} & \dfrac{d\lambda(s)}{ds} = f\big(\Lambda(s)\big)\, F\big(\lambda(s)\big) + g\big(\Lambda(s)\big)\, G\big(\lambda(s)\big), \\
\text{(ii)} & \dfrac{df\big(\Lambda(s)\big)}{ds} = g\big(\Lambda(s)\big)\, \{f, g\}\big(\Lambda(s)\big), \\
& \dfrac{dg\big(\Lambda(s)\big)}{ds} = f\big(\Lambda(s)\big)\, \{g, f\}\big(\Lambda(s)\big), \\
\text{(iii)} & \dfrac{dr\big(\Lambda(s)\big)}{ds} = -f\big(\Lambda(s)\big)\dfrac{\partial f\big(\Lambda(s)\big)}{\partial w} - g\big(\Lambda(s)\big)\,\dfrac{\partial g\big(\Lambda(s)\big)}{\partial w}.
\end{cases} \tag{4.1}$$

In fact, if we know a solution $\lambda(s)$, $f(\Lambda(s))$, $g(\Lambda(s))$, $r(\Lambda(s))$ of the equations, then $p(\Lambda(s))$, $q(\Lambda(s))$ can be computed since there are smooth functions A, B, C, $D : M \to R$ such that $p = A\,f + B\,g$, $q = C\,f + D\,g$.

To study equations (4.1), (i), (ii), (iii), we shall use the results of Theorem 2.5, Sec. 2. Let us start with an orthonormal frame field (F, G) of (Δ, g) on M. There exists a formal diffeomorphism φ and a formal function α at 0 such that the formal fields

$$\begin{cases} \widetilde{F} = \varphi_* \left((j^\infty F)\, \cos\alpha + (j^\infty G)\, \sin\alpha\right), \\ \widetilde{G} = \varphi_* \left(-(j^\infty F)\, \sin\alpha + (j^\infty G)\, \cos\alpha\right) \end{cases}$$

are in normal form $\mathcal{N}F$ (with Q diagonal).

By Borel's theorem, φ and α are, respectively, the infinite jets at 0 of a C^∞ diffeomorphism Φ and a C^∞ function A. Then the vector fields

$$\begin{cases} \widehat{F} = \Phi_* \ (F\, \cos A + G\, \sin A), \\ \widehat{G} = \Phi_* \ (-F\, \sin A + G\, \cos A) \end{cases}$$

are C^∞ vector fields having their infinite jets in normal form. Obviously, Φ and A are not unique, but the only thing that will matter later is their infinite jet at 0.

On a neighborhood of 0 in T_0M, Φ will be a diffeomorphism onto a neighborhood of 0 in M. Hence, the functions $x \circ \Phi^{-1}$, $y \circ \Phi^{-1}$, $w \circ \Phi^{-1}$ will form a coordinate system in a neighborhood of 0 in M. Since our study is local, we can assume that the image of Φ is the whole of M.

To simplify the notations, from now on, we identify $x \circ \Phi^{-1}$, $y \circ \Phi^{-1}$, $w \circ \Phi^{-1}$ with x, y, w respectively and we replace the frame field (F, G) by the frame field $(F\, \cos A + G\, \sin A,\ -F\, \sin A + G\, \cos A)$, which defines the same metric.

The gradation on the ring $\mathcal{F}$ of infinite jets of germs at 0 of functions on M induces canonically a gradation on the ring S of infinite jets of germs at 0 of functions on T^*M as follows: the gradation on $\mathcal{F}$ is defined by the following trivial action of R_+^* on R^3: $a \in R_+^*$, $a\,.\,(x,\, y,\, w) = (a\,x,\, a\,y,\, a^2\,w)$. This action lifts canonically to a symplectic action of R_+^* on T^*R^3:

$$a\,.\,(x, y, w, p, q, r) = (a\,x, a\,y, a^2\,w, a^{-1}\,p,\, a^{-1}\,q,\, a^{-2}\,r).$$

It is clear that the Poisson bracket $\{\ \} : S \times S \to S$ is homogeneous for this action, if $u \in S_m$, $v \in S_n$, $\{u, v\} \in S_{n+m}$. ($S = \prod_{n \geq -2} S_n$.)

2. Formal expansions of the variables f, g, $\{f, g\}$. In the gradation just defined on S, we have the expansions

$$
\begin{gathered}
f = f_{-1} + f_1 + f_2 + f_{(3)}, \qquad g = g_{-1} + g_1 + g_2 + g_{(3)}. \\
\theta = \{f, g\} = \theta_{-2} + \theta_0 + \theta_1 + \theta_{(2)} \\
(\text{here if } u = f,\ g \text{ or } \theta,\ u_k \in S_k, \text{ and } u_{(k)} \in S(k) = \prod_{n \geq k} S_n),
\end{gathered} \tag{4.2}
$$

and

$$
\begin{aligned}
&f_{-1} = p + y\frac{r}{2},\ g_{-1} = q - x\frac{r}{2}, && f_1 = y\frac{Q\,r}{2},\ g_1 = -x\frac{Q\,r}{2}, \\
&f_2 = \frac{y}{2}(2\,(y\,p - x\,q)\,l + r\,V), && g_2 = -\frac{x}{2}(2\,(y\,p - x\,q)\,l + r\,V), \\
&\theta_{-2} = \{f_{-1}, g_{-1}\} = r, && \theta_0 = \{f_{-1}, g_1\} + \{f_1, g_{-1}\} = 2\,r\,Q.
\end{aligned}
$$

$$
\begin{gathered}
\theta_1 = \{f_{-1}, g_2\} + \{f_2, g_{-1}\} = 4\,(y\,p - x\,q)\,l + r\left(\frac{5}{2}V + \frac{x^2+y^2}{2}\,l\right), \\
f\,\frac{\partial f}{\partial w} + g\,\frac{\partial g}{\partial w} = \xi = \xi_0 + \xi_{(1)}, \qquad \xi_0 = \frac{y\,f - x\,g}{2}\,r\,\frac{\partial \gamma_4}{\partial w},
\end{gathered}
$$

γ_4 is the component of degree 4 in γ,

$$
\begin{gathered}
f\,\frac{\partial f}{\partial r} + g\,\frac{\partial g}{\partial r} = \eta = \frac{y\,f - x\,g}{2}\,(1+\gamma) = \eta_0 + \eta_1 + \eta_2 + \eta_{(3)}, \\
\eta_0 = \frac{y\,f - x\,g}{2}, \quad \eta_1 = \eta_0\,Q, \quad \eta_2 = \eta_0\,V.
\end{gathered}
$$

3. In our subsequent dealings we shall be constantly computing with expansions and the following lemma will be useful to get estimates of the orders of terms.

Lemma 4.7. *Let $u \in S_n$ and assume that u is linear in p, q, r. Let $x = x_1\,\varepsilon^{\sigma_1} + x_2\,\varepsilon^{\sigma_1+1} + \dots$, $y = y_1\,\varepsilon^{\sigma_1} + y_2\,\varepsilon^{\sigma_1+1} + \dots$, $w = w_1\,\varepsilon^{\sigma_2} + w_2\,\varepsilon^{\sigma_2+1} + \dots$, $p = p_1\,\varepsilon^{\tau_1} + p_2\,\varepsilon^{\tau_1+1} + \dots$, $q = q_1\,\varepsilon^{\tau_1} + q_2\,\varepsilon^{\tau_1+1} + \dots$, $r = r_1\,\varepsilon^{\tau_2} + r_2\,\varepsilon^{\tau_2+1} + \dots$ be formal expansions in a parameter ε with coefficients in some R-algebra A; then $u(x_1\,\varepsilon^{\sigma_1} + \dots, y_1\,\varepsilon^{\sigma_1} + \dots, w_1\,\varepsilon^{\sigma_2} + \dots, p_1\,\varepsilon^{\tau_1} + \dots, q_1\,\varepsilon^{\tau_1} + \dots, r_1\,\varepsilon^{\tau_2} + \dots)$ will start with terms of degree at least*

$$
\begin{aligned}
\min\big(&\min\{P\,\sigma_1 + Q\,\sigma_2 \,/\, P,\ Q \text{ integers},\ P + 2Q = n+1\} + \tau_1, \\
&\min\{P\,\sigma_1 + Q\,\sigma_2 \,/\, P,\ Q \text{ integers},\ P + 2Q = n+2\} + \tau_2\big).
\end{aligned}
$$

The easy proof of this lemma is left to the reader.

It will be convenient to use the following complex notations:

$$
z = x + i\,y, \qquad h = f + i\,g, \qquad \zeta = p + i\,q.
$$

As a first application of the concepts just introduced, we will prove the following proposition which will be useful later on:

Proposition 4.8. *For any compact K of C,* $\inf_K s_c > 0$.

Proof. Let $\lambda : [0,S] \to M$ be a geodesic parametrized by arclength s, such that $\lambda(0) = 0$, and let $\Lambda : [0,S] \to T^*M$ be its lifting. Its initial covector $\Lambda(0)$ is $\cos\varphi\, dx_0 + \sin\varphi\, dy_0 + r_0\, dw_0 \in T_0^*M$. Then

$$z(\lambda(s)) = e^{i\varphi}s + O(s^2), \qquad h(\Lambda(s)) = e^{i\varphi} + O(s),$$
$$w(\lambda(s)) = O(s), \qquad r(\Lambda(s)) = r_0 + O(s).$$

Using expansions (4.2) and Lemma 4.7, we see that $\dfrac{dh(\Lambda(s))}{ds} = -i\,(r_0 + O(s^2))\,h(\Lambda(s))$ and $z(\lambda(s)) = \int_0^s h(\Lambda(s))\,ds + O(s^3)$. Hence, $h(\Lambda(s)) = e^{i\varphi}(1 - i\,r_0\,s + O(s^2))$ and

$$h(\Lambda(s)) = e^{i\varphi}(1 - i\,r_0\,s + O(s^2)) \quad\text{and}\quad z(\lambda(s)) = e^{i\varphi}\Big(s - i\,s^2\,\frac{r_0}{2} + O(s^3)\Big). \tag{4.3}$$

Finally,

$$\frac{dw(\lambda(s))}{ds} = \Big(f\,\frac{\partial f}{\partial r} + g\,\frac{\partial g}{\partial r}\Big)(\Lambda(s)) = \frac{r_0}{4}s^2 + O(s^3).$$

Hence, $w(\lambda(s)) = \dfrac{r_0}{12}s^3 + O(s^4)$. The expansions of $z(\lambda(s))$ and $w(\lambda(s))$ imply the proposition. □

4.5. Equations of geodesics in reduced form with new time and expansions of solutions in terms of $\rho_0 = \dfrac{1}{r_0}$. Since we are interested in the conjugate locus near the origin, Proposition 4.8 shows that we need to consider the values of r_0 near ∞ only. In this section we shall obtain expansions of the geodesics in terms of the parameter $\rho_0 = \dfrac{1}{r_0}$.

To do this, we shall make a change of variables and also reparametrize the geodesics. The change of variable will be

$$(x,y,w,p,q,r) \to \Big(x,y,w,\widehat{p} = \frac{p}{r},\ \widehat{q} = \frac{q}{r}\Big)$$

on the open subset $(T^*M)_r$ of T^*M, where $r \neq 0$. This change is equivalent to replacing the symplectic structure on T^*M by the associated contact structure defined by $\mathcal{H}$.

Let $\widehat{f} = \dfrac{f}{r}$, $\widehat{g} = \dfrac{g}{r}$, $\widehat{h} = \dfrac{h}{r} = \widehat{f} + i\,\widehat{g}$, $\widehat{\theta} = \dfrac{\theta}{r}$, $\widehat{\zeta} = \dfrac{\zeta}{r}$. Note that since f, g, h, θ are linear in p, q, r, then $\widehat{f}$, $\widehat{g}$, $\widehat{h}$, $\widehat{\theta}$ are smooth functions in $(x,y,w,\ \widehat{p},\ \widehat{q})$, affine in $\widehat{p}$, $\widehat{q}$.

Let $\lambda : [0, S] \to M$ be a geodesic parametrized by arclength starting at 0 whose lifting $\Lambda : [0, S] \to T^*M$ lies in $(T^*M)_r$. We make the following change of parameter $s \to t$, *t the new time*,

$$t = t_\lambda(s) = \int_0^s r\big(\Lambda(s)\big)\, ds. \tag{4.4}$$

The inverse change is $t \to s = s_\lambda(t)$. Let $\widehat{\lambda}(t) = \lambda\big(s_\lambda(t)\big)$, $\widehat{\Lambda}(t) = \Lambda(s_\lambda(t))$. Then $\widehat{\lambda}$, $\widehat{\Lambda}$ satisfy the following system of equations:

$$\left\{\begin{array}{ll} \text{(i)} & \dfrac{d\widehat{h}(\Lambda(t))}{dt} = -i\,\widehat{\theta}\big(\widehat{\Lambda}(t)\big)\,\widehat{h}\big(\widehat{\Lambda}(t)\big) + \widehat{\xi}\big(\widehat{\Lambda}(t)\big)\,\widehat{h}\big(\widehat{\Lambda}(t)\big), \\ \text{(ii)} & \dfrac{d\widehat{\lambda}(t)}{dt} = \widehat{f}\big(\widehat{\Lambda}(t)\big)\,F\big(\widehat{\lambda}(t)\big) + \widehat{g}\big(\widehat{\Lambda}(t)\big)\,G\big(\widehat{\lambda}(t)\big), \\ \text{(iii)} & \dfrac{dr\big(\widehat{\Lambda}(t)\big)}{dt} = -r\big(\widehat{\Lambda}(t)\big)\,\widehat{\xi}\big(\widehat{\Lambda}(t)\big), \\ & \text{where } \widehat{\xi} = \widehat{f}\,\dfrac{\partial \widehat{f}}{\partial w} + \widehat{g}\,\dfrac{\partial \widehat{g}}{\partial w}. \end{array}\right. \tag{4.5}$$

The expansions (4.2) give the following expansions:

$$\begin{array}{ll} \widehat{f} = \widehat{f}_1 + \widehat{f}_3 + \widehat{f}_4 + \widehat{f}_{(5)}, & \widehat{g} = \widehat{g}_1 + \widehat{g}_3 + \widehat{g}_4 + \widehat{g}_{(5)}, \\ \widehat{h} = \widehat{h}_1 + \widehat{h}_3 + \widehat{h}_4 + \widehat{h}_{(5)}, & \widehat{\theta} = \widehat{\theta}_0 + \widehat{\theta}_2 + \widehat{\theta}_3 + \widehat{\theta}_{(4)}, \\ \widehat{\xi} = \widehat{\xi}_4 + \widehat{\xi}_{(5)}, & \widehat{\eta} = \widehat{\eta}_2 + \widehat{\eta}_4 + \widehat{\eta}_{(5)}, \end{array}$$

where, if $u = f, g, h, \theta, \xi$ or η and u_n is homogeneous of degree n and $u_{(n)}$ starts with terms of degree at least n:

$$\left\{\begin{array}{l} \widehat{h}_1 = \widehat{\zeta} - i\,\dfrac{z}{2},\ \widehat{h}_3 = -i\,z\,\dfrac{Q(z)}{2},\ \widehat{h}_4 = -i\,\dfrac{z}{2}\left(2\,l(z)\,\operatorname{Im}(\overline{\widehat{\zeta}}\,z) + V(z)\right), \\ \widehat{\theta}_0 = 1,\ \widehat{\theta}_2 = 2\,Q(z),\ \widehat{\theta}_3 = 4\,l(z)\,\operatorname{Im}(\overline{\widehat{\zeta}}\,z) + \dfrac{5}{2}V(z) + \dfrac{|z|^2}{4}l(z), \\ \widehat{\theta}_3 = 4\,l(z)\,\operatorname{Im}(\overline{\widehat{h}}_1\,z) + \dfrac{5}{2}V(z) - 3\dfrac{|z|^2}{4}l(z), \text{ since } \widehat{\zeta} = \widehat{h}_1 + i\,\dfrac{z}{2}, \\ \xi_4 = \dfrac{1}{2}\operatorname{Im}(\overline{h}_1\,z)\,\dfrac{\partial \gamma_4}{\partial w},\ \ \eta_2 = \dfrac{1}{2}\operatorname{Im}(\overline{\widehat{h}}_1\,z), \\ \eta_4 = \dfrac{1}{2}\left(\operatorname{Im}(\overline{\widehat{h}}_1\,z)\,Q(z) + \operatorname{Im}(\overline{\widehat{h}}_3\,z)\right). \end{array}\right. \tag{4.6}$$

Let $\widehat{\lambda} : [0, T] \to M$, $\widehat{\Lambda} : [0, T] \to (T^*M)_r$ be a geodesic and its lifting, such that $\widehat{\lambda}(0) = 0$, $\widehat{\Lambda}(0) = \rho_0 \cos\varphi\, dx_0 + \rho_0 \sin\varphi\, dy_0 + dw_0$. Then,

$\dfrac{d}{dt}z(\widehat{\lambda}(t))\big|_{t=0} = \widehat{h}(\widehat{\Lambda}(0)) = \rho_0\, e^{i\varphi}$. $z(\widehat{\lambda})$, $\widehat{h}(\widehat{\Lambda})$, $w(\widehat{\lambda})$ are smooth functions in t, φ, ρ_0, and we can expand them in powers of ρ_0 :

$$\begin{aligned} z(\widehat{\lambda}) &= \rho_0\, z_1 + \rho_0^2\, z_2 + \rho_0^3\, z_3 + \rho_0^4\, z_4 + \rho_0^5\, z_{(5)}, \\ \widehat{h}(\widehat{\Lambda}) &= \rho_0\, H_1 + \rho_0^2\, H_2 + \rho_0^3\, H_3 + \rho_0^4\, H_4 + \rho_0^5\, H_{(5)}, \end{aligned} \tag{4.7}$$

where z_i, H_i, $1 \le i \le 4$ are smooth functions of t and φ, $z_{(5)}$, $H_{(5)}$ are smooth functions of t, φ, ρ_0. Since $w(\widehat{\lambda}(0)) = 0$ and $\dfrac{d}{dt}w(\widehat{\lambda}(t)) = \eta(\widehat{\Lambda}(t))$, then the relations (4.6) and Lemma 4.7 show that

$$w(\widehat{\lambda}) = \rho_0^2\, w_2 + \rho_0^3\, w_3 + \rho_0^4\, w_4 + \rho_0^5\, w_{(5)}. \tag{4.8}$$

From (4.6) we can also conclude that using Lemma 4.7:

$$\widehat{\zeta}(\Lambda) = \rho_0 e^{i\varphi} + O(\rho_0^2),\ \ \widehat{\theta}(\widehat{\Lambda}) = 1 + 2\rho_0^2\, Q(z_1) + \rho_0^3\, \widehat{\theta}_3(z_1, H_1) + O(\rho_0^4).$$

From this we get the following relations using Lemma 4.7:

$$\begin{cases} \text{(i)} & \dfrac{d}{dt}\widehat{h}(\widehat{\Lambda}(t)) = -i\left(1 + 2\,\rho_0^2\, Q(z_1) + \rho_0^3\,\widehat{\theta}_3(z_1) + O(\rho_0^4)\right)\widehat{h}(\widehat{\Lambda}(t)), \\ \text{(ii)} & \dfrac{d}{dt}w(\widehat{\lambda}(t)) = \dfrac{1}{2}\rho_0^2\, \mathrm{Im}(\overline{H}_1(t)\, z_1(t)) + \\ & \qquad + \dfrac{1}{2}\,\rho_0^4\left(\mathrm{Im}(\overline{H}_1(t)\, z_1(t))\, Q(z_1(t)) + \right. \\ & \qquad \left. + \mathrm{Im}(\overline{H}_3(t)\, z_1(t)) + \mathrm{Im}(\overline{H}_1(t)\, z_3(t))\right) + O(\rho_0^5), \\ \text{(iii)} & \dfrac{d}{dt}z(\widehat{\lambda}(t)) = \widehat{h}(\widehat{\Lambda}(t)) - i\,\rho_0^4\, z_1(t) l(z_1(t))\ \mathrm{Im}(\overline{H}_1(t)\, z_1(t)) + O(\rho_0^5), \\ \text{(iv)} & \dfrac{d}{dt}r(\widehat{\Lambda}(t)) = -\dfrac{1}{2}\rho_0^4\, r(\widehat{\Lambda}(t))\ \ \mathrm{Im}(\overline{H}_1(t)\, z_1(t))\, \dfrac{\partial\gamma_4}{\partial w}(z_1(t), w_2(t)) + \\ & \qquad + O(\rho_0^5). \end{cases}$$

Setting $\rho = \dfrac{1}{r}$, we get

$$\begin{aligned} \text{(v)}\quad \frac{d}{dt}\rho(\widehat{\Lambda}(t)) = \rho(\widehat{\Lambda}(t))\, (\,\rho_0^4\, \frac{1}{2}\ \mathrm{Im}(\overline{H}_1(t)\, z_1(t))\, \frac{\partial\gamma_4}{\partial w}(z_1(t), w_2(t)) + \\ + O(\rho_0^5)). \end{aligned} \tag{4.9}$$

These equations imply the following ones:

$$
\begin{cases}
\frac{d}{dt}H_1(t) = -i\,H_1(t),\ \frac{d}{dt}H_2(t) = -i\,H_2(t),\\
\frac{d}{dt}H_3(t) = -i\,H_3(t) - 2\,i\,H_1(t)\,Q\big(z_1(t)\big),\\
\frac{d}{dt}H_4(t) = -i\,\big(H_4(t) + \widehat{\theta}_3\big(z_1(t), H_1(t)\big)\,H_1(t)\,\big),\\
\frac{d}{dt}z_1(t) = H_1(t),\ \frac{d}{dt}z_2(t) = H_2(t),\ \frac{d}{dt}z_3(t) = H_3(t),\\
\frac{d}{dt}z_4(t) = H_4(t) - i\,z_1(t)\,l\big(z_1(t)\big)\,I\big(\overline{H}_1(t)\,z_1(t)\big),\\
\frac{d}{dt}w_2(t) = \frac{1}{2}\,\mathrm{Im}\big(\overline{H}_1(t)\,z_1(t)\big),\\
\frac{d}{dt}w_4(t) = \frac{1}{2}\big(Q(z_1(t))\,\mathrm{Im}\big(\overline{H}_1(t)\,z_1(t)\big) + \mathrm{Im}\big(\overline{H}_3(t)z_1(t)\big)+\\
\qquad\qquad + \mathrm{Im}\big(\overline{H}_1(t)z_3(t)\big)\big),\\
\text{with initial conditions } z_i(0) = 0,\ 0 \le i \le 4,\ H_i(0) = 0,\ 2 \le i \le 4,\\
\frac{dz_i}{dt}(0) = 0,\ 2 \le i \le 4,\ \frac{dz_1}{dt}(0) = H_1(0) = e^{i\varphi},\ w_2(0) = w_4(0) = 0.
\end{cases}
\tag{4.10}
$$

4.6. Computations. The only informations we will need in the following are the expressions of the functions z_1, w_2 and the values $z_i(2\pi, \varphi)$, $i = 2, 3$, $w_4(2\pi, \varphi)$.

The computations of z_1 are trivial,

$$H_1(t) = e^{i(\varphi - t)}, \qquad z_1(t) = \frac{e^{i\varphi}}{i}\,(1 - e^{-i\,t}). \tag{4.11}$$

In particular, $H_1(2\pi) - H_1(t) = i\,z_1(t)$. The following two remarks make the computations trivial:

(1) Let k be a continuous periodic function of period 2π and let u be a solution of the equation $\frac{du(t)}{dt} = -i\,u(t) - i\,k(t)\,H_1(t)$ such that $u(0) = 0$. Then

$$\int_0^{2\pi} u(t)\,dt = i\int_0^{2\pi} z_1(t)\,k(t)\,dt. \tag{4.12}$$

In fact, $u(t) = -i\,H_1(t)\,\int_0^t k(\tau)\,d\tau$. $u(2\pi) = -i\,H_1(2\pi)\,\int_0^{2\pi} k(t)\,dt$. Integrating the equation for u, we have

$$u(2\pi) = \int_0^{2\pi} \frac{du(t)}{dt}dt = -i\int_0^{2\pi} u(t)\,dt - i\int_0^{2\pi} k(t)\,H_1(t)\,dt.$$

Hence,

$$\int_0^{2\pi} u(t)\,dt = \int_0^{2\pi} \big(H_1(2\pi) - H_1(t)\big)\,k(t)\,dt = i \int_0^{2\pi} z_1(t)\,k(t)\,dt.$$

(2) Let P be a polynomial in z and $\overline{z}$, $P(z,\overline{z}) = \sum a_{m,n}\,z^m\,\overline{z}^n$. Then

$$\int_0^{2\pi} z_1(t)\,P\big(z_1(t),\,\overline{z}_1(t)\big)\,dt = 2\pi \sum a_{m,n}\,\frac{(m+n+1)!}{(m+1)!\,n!}\,\frac{e^{i(m-n+1)\varphi}}{i^{m-n+1}}. \tag{4.13}$$

This is a direct consequence of the following simple computation:

$$\int_0^{2\pi} z_1(t)^m\,\overline{z}_1(t)^n\,dt = \int_0^{2\pi} z_1(t)^{m+n}\,e^{i\,n\,t}\,e^{-2\,i\,n\,\varphi}\,dt = 2\,\pi\,\frac{(m+n)!}{m!\,n!}\,\frac{e^{i(m-n)\varphi}}{i^{m-n}}.$$

Applying (4.12) to the equation for H_3 and the fact that

$$z_3(t) = \int_0^t H_3(\tau)\,d\tau, \quad \big(z_3(0) = 0\big),$$

we get

$$z_3(2\,\pi) = 2\,i \int_0^{2\pi} z_1(t)\,Q\big(z_1(t)\big)\,dt.$$

But $Q(z) = c_1\,x^2 + c_2\,y^2 = \frac{c_1 - c_2}{4}(z^2 + \overline{z}^2) + \frac{c_1 + c_2}{2}\,|z|^2$. Applying (4.13), we get

$$z_3(2\pi, \varphi) = \pi\,(c_2 - c_1)\,(e^{3\,i\,\varphi} + 3e^{-i\,\varphi}) + 6\,\pi\,(c_1 + c_2)\,e^{i\,\varphi}. \tag{4.14}$$

Applying (4.12) to the equation for z_4, we get

$$\begin{aligned} z_4(2\,\pi, \varphi) = i \int_0^{2\pi} \frac{5}{2} V\big(z_1(t)\big)\,z_1(t)\,dt \,+ \\ + 2\,i \int_0^{2\pi} \Big(-\frac{3}{4}\,|z_1(t)|^2 + 2\,\mathrm{Im}\big(\overline{H}_1(t)\,z_1(t)\big) - \\ - \frac{1}{2}\,\mathrm{Im}\big(\overline{H}_1(t)\,z_1(t)\big)\Big)\,l\big(z_1(t)\big)\,z_1(t)\,dt. \end{aligned}$$

But $|z_1(t)|^2 = 2\,(1 - \cos t)$, $\mathrm{Im}\big(\overline{H}_1(t)\,z_1(t)\big) = 1 - \cos t$.

Finally,

$$z_4(2\pi,\,\varphi) = \frac{5\,i}{2}\int_0^{2\pi} z_1(t)\,V\big(z_1(t)\big)\,dt.$$

Since $V(z) = \frac{1}{2} b\,z^3 + \frac{1}{2}\,\overline{b}\,\overline{z}^3 + \frac{1}{2}\overline{a}\,|z|^2\,z + \frac{1}{2}\,a\,|z|^2\,\overline{z}$, applying (4.13), we get

$$z_4(2\,\pi,\varphi) = 5\,\pi\,i\,\left(\frac{1}{2}\,b\,e^{4\,i\,\varphi} - 2\,\overline{b}\,e^{-2\,i\,\varphi} - 2\,\overline{a}\,e^{2\,i\,\varphi} + 3\,a\right). \tag{4.15}$$

Clearly,

$$w_2(t) = \frac{1}{2}\,(t - \sin t), \quad \text{and} \quad w_3(t) = 0; \tag{4.16}$$

$$w_4(2\,\pi) = \frac{1}{2}\int_0^{2\pi}\Big(\mathrm{Im}\big(\overline{H}_1(t)\,z_1(t)\big)\,Q\big(z_1(t)\big) + \mathrm{Im}\big(\overline{H}_3(t)\,z_1(t)\big) + \\ + \mathrm{Im}\big(\overline{H}_1(t)\,z_3(t)\big)\Big)dt,$$

$$w_4(2\,\pi) = \frac{1}{4}\int_0^{2\pi}|z_1(t)|^2\;Q\big(z_1(t)\big)\,dt + \\ + \frac{1}{2}\int_0^{2\pi}\Big(\mathrm{Im}\big(\overline{H}_3(t)\,z_1(t)\big) + \mathrm{Im}\big(\overline{H}_1(t)\,z_3(t)\big)\Big)\,dt.$$

Applying (4.13), we get

$$w_4(2\,\pi) = -3\,\frac{\pi}{2}\,\big(\,2\,(c_2 - c_1)\,\cos 2\varphi + 3\,(c_1 + c_2)\big). \tag{4.17}$$

The preceding considerations allow us to state:

Lemma 4.9. *For any $T > 0$, there exists a compact neighborhood K_T of 0 in R and a smooth mapping $\widehat{\mathcal{E}} : [0,T] \times S^1 \times K_T \to M$, $(t,\, e^{i\,\varphi}, \rho_0) \to \widehat{\mathcal{E}}(t,\varphi,\rho_0)$ such that:* (i) *the curve $t \to \widehat{\mathcal{E}}(t,\varphi,\rho_0)$ is the geodesic parametrized by the new time starting at 0 with initial covector*

$$\rho_0 \cos\varphi\,dx_0 + \rho_0 \sin\varphi\,dy_0 + dw_0$$

(*covector for the lifting parametrized by the new time*).

(ii) $\widehat{\mathcal{E}}$ *has expansions in powers of* ρ_0:

$$\begin{aligned} z(\widehat{\mathcal{E}}) &= \rho_0\, e^{i\,\varphi} \left(\frac{1 - e^{-it}}{i} \right) + \rho_0^3\, z_3 + \rho_0^4\, z_4 + O(\rho_0^5), \\ w(\widehat{\mathcal{E}}) &= \frac{1}{2} \rho_0^2\, (\, t - \sin t\,) + \rho_0^4\, w_4 + O(\rho_0^5). \end{aligned} \tag{4.18}$$

4.7. Relations between the arclength s **and the new time** t. In view of Lemma 4.9, we have:

for any $T > 0$, there exists a smooth function $\nu : [0, T] \times S^1 \times K_T \to R$ such that

$$\rho\big(\widehat{\mathcal{E}}(t, \varphi, \rho_0)\big) = \rho_0\, \exp(\, \rho_0^4\, \nu(t, \varphi, \rho_0)), \text{ for all } (t, \varphi, \rho_0) \in [0, T] \times S^1 \times K_T$$

(see 4.9).

Lemma 4.10. *Using the notations of Lemma* 4.9, *for any* $T > 0$ *there exists a smooth function* $\mu_1 : [0, T] \times S^1 \times K_T \to R$ *such that the length of the geodesic* $\tau \in [0, t] \to \widehat{\mathcal{E}}(\tau, \varphi, \rho_0)$ *for any* $(t, \varphi, \rho_0) \in [0, T] \times S^1 \times K_T$ *is*

$$s(t, \varphi, \rho_0) = \rho_0\, t + \rho_0^5\, \mu_1(t, \varphi, \rho_0). \tag{4.19}$$

The mapping $[0, T] \times S^1 \times K_T \to R_+ \times S^1 \times K_T$, $(t, \varphi, \rho_0) \to (s(t, \varphi, \rho_0), \varphi, \rho_0)$ *is a diffeomorphism onto its image.*

Proof. Inverse function theorem. □

Notation. The inverse diffeomorphism is of the form

$$\big(\delta(s, \varphi, \rho_0),\, \varphi,\, \rho_0\big),\ \delta : s\big([0, T] \times S^1 \times K_T\big) \times S^1 \times K_T \to R_+,$$

$$\delta(s, \varphi, \rho_0) = r_0 s + \rho_0^4\, \mu_2(r_0\, s, \varphi, \rho_0). \tag{4.20}$$

5. Effective approximation of the conjugate time and the conjugate locus

This section is a bit computational but absolutely necessary. All the main formulas are established in this section.

5.1. The method for computing the conjugate time mapping.

1. Let us look for conjugate times in the case of initial conditions $(\varphi, \rho_0) \in C$ such that ρ_0 is small (on a neighborhood of infinity on C).

Our Lemma 4.9 shows that

$$z\big(\widehat{\mathcal{E}}(t, \varphi, \rho_0)\big) = \rho_0\, z_1(t, \varphi) + \rho_0^3\, z_3(t, \varphi) + \rho_0^4\, z_4(t, \varphi) + \rho_0^5\, z_5(t, \varphi, \rho_0). \tag{5.1}$$

By Corollary 4.6, the determinant $\widehat{D}$ we have to compute has the expression

$$\widehat{D}(s,\varphi,\rho_0) = \operatorname{Im}\left(\overline{\frac{\partial(z\circ\mathcal{E})(s,\varphi,\rho_0)}{\partial\rho_0}}\,\frac{\partial(z\circ\mathcal{E})(s,\varphi,\rho_0)}{\partial\varphi}\right). \tag{5.2}$$

Denote by Z the function $z\circ\widehat{\mathcal{E}}$; then $(z\circ\mathcal{E})(s,\varphi,\rho_0) = Z\big(\delta(s,\varphi,\rho_0),\ \varphi,\ \rho_0\big)$. Hence,

$$\widehat{D}(s,\varphi,\rho_0) = \operatorname{Im}\Bigg(\overline{\left(\frac{\partial Z(\delta,\varphi,\rho_0)}{\partial\rho_0} + \frac{\partial Z(\delta,\varphi,\rho_0)}{\partial t}\frac{\partial\delta}{\partial\rho_0}\right)}\cdot \left(\frac{\partial Z(\delta,\varphi,\rho_0)}{\partial\varphi} + \frac{\partial Z(\delta,\varphi,\rho_0)}{\partial t}\frac{\partial\delta}{\partial\varphi}\right)\Bigg).$$

Let D be the composition

$$D(t,\varphi,\rho_0) = \widehat{D}\big(s(t,\varphi,\rho_0),\varphi,\rho_0\big). \tag{5.3}$$

Then we have

Lemma 5.1. *D has an expansion in terms of ρ_0, of the form*

$$\begin{cases} 1. \quad D = \rho_0 D' = \rho_0\big(D_1 + \rho_0^2(D_2 + D_2') + \rho_0^3 D_3 + O(\rho_0^4)\big), \\ 2. \quad D_1 = \operatorname{Im}\left(\overline{\left(z_1 - t\frac{\partial z_1}{\partial t}\right)}\frac{\partial z_1}{\partial\varphi}\right), \\ 3. \quad D_2 = \operatorname{Im}\left(\overline{\left(z_1 - t\frac{\partial z_1}{\partial t}\right)}\frac{\partial z_3}{\partial\varphi}\right), \\ \qquad D_2' = \operatorname{Im}\left(\overline{\left(3\,z_3 - t\frac{\partial z_3}{\partial t}\right)}\frac{\partial z_1}{\partial\varphi}\right), \\ 4. \quad D_3 = \operatorname{Im}\left(\overline{\left(z_1 - t\frac{\partial z_1}{\partial t}\right)}\frac{\partial z_4}{\partial\varphi}\right) + \operatorname{Im}\left(\overline{\left(4\,z_4 - t\frac{\partial z_4}{\partial t}\right)}\frac{\partial z_1}{\partial\varphi}\right). \end{cases} \tag{5.4}$$

Proof. Let us first compute $\dfrac{\partial Z(\delta,\varphi,\rho_0)}{\partial\rho_0} + \dfrac{\partial Z(\delta,\varphi,\rho_0)}{\partial t}\dfrac{\partial\delta}{\partial\rho_0}$.

We know by (4.20) that

$$\frac{\partial\delta}{\partial\rho_0} = -\frac{s}{\rho_0^2} + \rho_0^4\frac{\partial\mu_2}{\partial\rho_0} - \rho_0^2\, s\,\frac{\partial\mu_2}{\partial(r_0 s)}$$

and by (4.19),

$$\frac{\partial\delta}{\partial\rho_0} = -\frac{\delta}{\rho_0} + O(\rho_0^3). \tag{5.5}$$

Therefore, a straightforward computation shows that

$$\frac{\partial Z(\delta,\varphi,\rho_0)}{\partial \rho_0}+\frac{\partial Z(\delta,\varphi,\rho_0)}{\partial t}\frac{\partial \delta}{\partial \rho_0}=\left(z_1-t\frac{\partial z_1}{\partial t}+\rho_0^2\left(3z_3-t\frac{\partial z_3}{\partial t}\right)+\right.$$
$$\left.+\rho_0^3\left(4\,z_4-t\frac{\partial z_4}{\partial t}\right)\right)_{t=\delta}+O(\rho_0^4). \tag{5.6}$$

Now, we compute

$$\frac{\partial Z(\delta,\varphi,\rho_0)}{\partial \varphi}+\frac{\partial Z(\delta,\varphi,\rho_0)}{\partial t}\frac{\partial \delta}{\partial \varphi}.$$

By (4.20),

$$\frac{\partial \delta}{\partial \varphi}=O(\rho_0^4); \tag{5.7}$$

therefore

$$\frac{\partial Z(\delta,\varphi,\rho_0)}{\partial \varphi}+\frac{\partial Z(\delta,\varphi,\rho_0)}{\partial t}\frac{\partial \delta}{\partial \varphi}=\left(\rho_0\frac{\partial z_1}{\partial \varphi}+\rho_0^3\frac{\partial z_3}{\partial \varphi}+\rho_0^4\frac{\partial z_4}{\partial \varphi}\right)_{t=\delta}+O(\rho_0^5). \tag{5.8}$$

All these relations, (5.6), (5.8) in particular, hold for t as large as we need, for ρ_0 small.

Plugging (5.6), (5.8) into the expression (5.3) of the determinant D gives the result. □

2. The equation $D=0$ can be rewritten as

$$D_1+\rho_0^2(D_2+D_2^{'})+\rho_0^3D_3+O(\rho_0^4)=0; \tag{5.9}$$

we can compute D_1: formula (4.11) gives

$$D_1=4\sin\frac{t}{2}\left(\sin\frac{t}{2}-\frac{t}{2}\cos\frac{t}{2}\right). \tag{5.10}$$

The function $\left(\sin\frac{t}{2}-\frac{t}{2}\cos\frac{t}{2}\right)$ does not vanish for $0<t\le 2\pi$.

Hence, in the case of the right invariant metric on the Heisenberg group, for which $D_2=D_2^{'}=D_3=O(\rho_0^4)=0$ in formula (5.9), the conjugate new time is constant and equal to 2π. Since in that case r is constant, we get

$$s_c(\varphi,r_0)=\frac{2\pi}{r_0}.$$

Plugging this value into formula (4.18), we get that $z(2\pi)=0$, since in the Heisenberg case $z_3=z_4=O(\rho_0^5)=0$. Therefore we get the well-known result that in the Heisenberg case the conjugate locus is the w-axis.

Proposition 5.2. *For any compact neighborhood K of the circle $\{r = 0\}$ in C, $\inf_{C \setminus K} t_c > 0$.*

Proof. This is similar to that of Proposition 4.8.

It follows that for any $\varepsilon > 0$, there is a $B > 0$ such that $|\rho_0| < B$ implies $t_c(\varphi, \rho_0) > 2\pi - \varepsilon$. Moreover, one can easily compute $\dfrac{\partial D_1}{\partial t}$ at $t = 2\pi$:

$$\frac{\partial D_1}{\partial t}(2\pi) = -2\pi. \tag{5.11}$$

This shows that one can apply the implicit function theorem to solve Eq. (5.9) around any point (t, φ, ρ_0), with $t = 2\pi$, $\rho_0 = 0$.

There is an open neighborhood $W = T \times U'$ of the subset $\{2\pi\} \times S_1 \times \{0\}$ in $R_+ \times S_1 \times R$, (S_1, the circle) such that the set of solutions of (5.9) in W is the graph of a smooth function $\widehat{t_c} : U' \to T$, such that $\widehat{t_c}(\varphi, 0) = 2\pi$, and the domain $\{(t, \varphi, \rho_0) | \, 0 < t < \widehat{t_c}(\varphi, \rho_0), (\varphi, \rho_0) \in U'\}$ does not contain a solution of (5.9).

These last considerations and Proposition 4.8 allow one to conclude:

Theorem 5.3. *There is an open neighborhood V of 0 in M and an open neighborhood U of infinity in C such that the conjugate time mapping of $\Sigma|_V$, our sub-Riemannian metric restricted to V, has domain U and is the mapping $\widehat{t_c}|_U$, $\widehat{t_c}$ restricted to U.*

Proof. Let us take U' defined just above. U'^c, the complement of U' in C, is compact. By Proposition 4.8, there is $\varepsilon > 0$, such that $s_c(\varphi, r_0) > \varepsilon$ for all $(\varphi, r_0) \in U'^c$. We replace M by a smaller neighborhood V, in such a way that the positive escape time $e(\varphi, \rho_0)$ of any point ψ of U'^c (for the equation of geodesics with arc-length time) is smaller than ε (for instance, look to the expansion (4.3) of geodesics). We just replace U' by the subset U of the (φ, ρ_0) such that $\widehat{s_c}(\varphi, \rho_0) < e(\varphi, \rho_0)$, where $\widehat{s_c}$ is the arc-length time corresponding to $\widehat{t_c}$. U is again an open neighborhood of infinity in C. □

3. A straightforward computation gives the following expansion for $t_c(\varphi, \rho_0)$:

$$t_c(\varphi, \rho_0) = 2\pi + \frac{1}{2\pi}\left(\rho_0^2(D_2(2\pi, \varphi) + D_2'(2\pi, \varphi)) + \rho_0^3 D_3(2\pi, \varphi)\right) + O(\rho_0^4),$$

We note that $z_1(2\pi,\varphi) = 0 = \dfrac{\partial z_1}{\partial\varphi}(2\pi,\varphi)$ by (4.11). Hence, as we already know, $D_1(2\pi) = 0$, but also

$$\begin{aligned} &D_2'(2\pi,\varphi) = 0,\ D_2(2\pi,\varphi) = -2\pi\ \mathrm{Im}\left(\overline{\frac{\partial z_1}{\partial t}}\,\frac{\partial z_3}{\partial\varphi}\right)(2\pi,\varphi),\\ &D_3(2\pi,\varphi) = -2\pi\ \mathrm{Im}\left(\overline{\frac{\partial z_1}{\partial t}}\,\frac{\partial z_4}{\partial\varphi}\right)(2\pi,\varphi). \end{aligned} \tag{5.12}$$

Hence, one has

$$\begin{aligned} &t_c(\varphi,\rho_0) = 2\pi - \rho_0^2\Delta_2 - \rho_0^3\Delta_3 + O(\rho_0^4), \qquad \text{with}\\ &\Delta_2 = \mathrm{Im}\left(\overline{\frac{\partial z_1}{\partial t}}\,\frac{\partial z_3}{\partial\varphi}\right)(2\pi,\varphi), \quad \Delta_3 = \mathrm{Im}\left(\overline{\frac{\partial z_1}{\partial t}}\,\frac{\partial z_4}{\partial\varphi}\right)(2\pi,\varphi). \end{aligned} \tag{5.13}$$

5.2. Computation of the conjugate time and conjugate locus . Let us compute Δ_2 and Δ_3.

Computation of Δ_2: $\Delta_2 = \mathrm{Im}\left(\overline{\dfrac{\partial z_1}{\partial t}(2\pi,\varphi)}\,\dfrac{\partial z_3}{\partial\varphi}(2\pi,\varphi)\right)$.

$\dfrac{\partial z_1}{\partial t}(2\pi,\varphi) = e^{i\varphi}$.

By formula (4.14) :

$$\frac{\partial z_3}{\partial\varphi}(2\pi,\varphi) = 3\,i\,\pi\,(c_2 - c_1)\,(e^{3i\varphi} - e^{-i\varphi}) + 6\,\pi\,i\,(c_1 + c_2)e^{i\varphi};$$

$$\Delta_2 = \mathrm{Im}(3\,i\,\pi\,(c_2 - c_1)\,(e^{2i\varphi} - e^{-2i\varphi}) + 6\,\pi\,i\,(c_1 + c_2)) = 6\pi\,(c_1 + c_2).$$

Computation of Δ_3 : $\Delta_3 = \mathrm{Im}\left(\overline{\dfrac{\partial z_1}{\partial t}(2\pi,\varphi)}\,\dfrac{\partial z_4}{\partial\varphi}(2\pi,\varphi)\right)$.

By formula (4.15):

$$\frac{\partial z_4}{\partial\varphi}(2\pi,\varphi) = 5\,\pi\,i\,\left(2i\,b\,e^{4i\varphi} + 4i\,\bar b\,e^{-2i\varphi} - 4\,i\,\bar a\,e^{2i\varphi}\right);$$

$$\Delta_3 = 5\pi\ \mathrm{Im}(-2be^{3i\varphi} - 4\bar b\,e^{-3i\varphi} + 4\,\bar a\,e^{i\varphi});$$

$$\begin{aligned}\Delta_3 &= 5\pi\left(2\,\mathrm{Im}(be^{3i\varphi}) + 4\ \mathrm{Im}(\bar a\,e^{i\varphi})\right) = 10\pi\,\mathrm{Re}\big(b\,(i\,e^{i\varphi})^3\big) -\\ &\quad - 20\,\pi\ \mathrm{Re}(\bar a\,i\,e^{i\varphi});\end{aligned}$$

$$\Delta_3 = 10\,\pi\,V_3\big(j\,(v)\big) - 20\,\pi\,V_1\big(j(v)\big), \tag{5.14}$$

where v denotes the unit initial velocity of the geodesic.

Hence we have the formula

$$t_c(\varphi,\rho_0) = 2\pi - 6\pi\rho_0^2\,\mathrm{tr}_g\,Q - \pi\,\rho_0^3\left(10\,V_3(j\,(v)) - 20\,V_1\big(j(v)\big)\right) + O(\rho_0^4).$$

To compute the conjugate locus, using the notations of Lemma 4.9, we get

$$z\big(\widehat{\mathcal{E}}\big(t_c(\varphi,\rho_0),\varphi,\rho_0\big)\big) = \rho_0\, z_1(t_c,\varphi) + \rho_0^3\, z_3(t_c,\varphi) + \rho_0^4\, z_4(t_c,\varphi) + O(\rho_0^5);$$
$$z_1(t_c,\varphi) = z_1(2\pi,\varphi) - \frac{\partial z_1}{\partial t}(2\pi,\varphi)(\rho_0^2\,\Delta_2 + \rho_0^3\,\Delta_3) + O(\rho_0^4),$$
$$z_3(t_c,\varphi) = z_3(2\pi,\varphi) + O(\rho_0^2),\;\; z_4(t_c,\varphi) = z_4(2\pi,\varphi) + O(\rho_0^2).$$

Hence,

$$z\big(\widehat{\mathcal{E}}\big(t_c(\varphi,\rho_0),\varphi,\rho_0\big)\big) = \rho_0 z_1(2\pi,\varphi) + \rho_0^3\big(z_3(2\pi,\varphi) - \Delta_2\,\frac{\partial z_1}{\partial t}(2\pi,\varphi)\big) +$$
$$+\rho_0^4\big(z_4(2\pi,\varphi) - \Delta_3\,\frac{\partial z_1}{\partial t}(2\pi,\varphi)\big) + O(\rho_0^5);$$
$$z_1(2\pi,\varphi) = 0.$$

To compute the other terms, we note that

$$\begin{aligned}&z_3(2\pi,\varphi) - \Delta_2\,\frac{\partial z_1}{\partial t}(2\pi,\varphi) =\\ &= e^{i\varphi}\left(e^{-i\varphi}\, z_3(2\pi,\varphi) - \operatorname{Im}\left(\frac{\partial}{\partial\varphi}(e^{-i\varphi}\, z_3(2\pi,\varphi))\right) -\right.\\ &\left.- \operatorname{Im}\left(i\, e^{-i\varphi}\, z_3(2\pi,\varphi)\right)\right) = e^{i\varphi}\left(i\, \operatorname{Im}\left(e^{-i\varphi}\, z_3(2\pi,\varphi)\right) -\right.\\ &\left.- \frac{\partial}{\partial\varphi}\operatorname{Im}\left(e^{-i\varphi}\, z_3(2\pi,\varphi)\right)\right).\end{aligned}$$

Similarly,

$$\begin{aligned}z_4(2\pi,\varphi) - \Delta_3\,\frac{\partial z_1}{\partial t}(2\pi,\varphi) = e^{i\varphi}&\left(i\, \operatorname{Im}\big(e^{-i\varphi}\, z_4(2\pi,\varphi)\big) -\right.\\ &\left.- \frac{\partial}{\partial\varphi}\operatorname{Im}\left(e^{-i\varphi}\, z_4(2\pi,\varphi)\right)\right).\end{aligned}$$

By formula (4.14),

$$e^{-i\varphi}\, z_3(2\pi,\varphi) = \pi\,(c_2 - c_1)\,(e^{2i\varphi} + 3\,e^{-2i\varphi}) + 6\,\pi\,(c_1 + c_2).$$

Hence, $\operatorname{Im}\big(e^{-i\varphi}\, z_3(2\pi,\varphi)\big) = -2\pi\,(c_2 - c_1)\,\sin 2\varphi$;

$$\begin{aligned}z_3(2\pi,\varphi) - \Delta_2\,\frac{\partial z_1}{\partial t}(2\pi,\varphi) &= 2\,\pi\,(c_2 - c_1)\,e^{i\varphi}\,(2\,\cos 2\varphi - i\,\sin 2\,\varphi) =\\ &= 4\,\pi\,(c_2 - c_1)\,(\cos^3\varphi - i\,\sin^3\varphi).\end{aligned}$$

By formula (4.15):

$$e^{-i\varphi}\, z_4(2\pi,\varphi) = 5\,\pi\, i\, \left(\frac{1}{2}\, b\, e^{3\,i\,\varphi} - 2\,\bar{b}\, e^{-3\,i\,\varphi} - 2\,\bar{a}\, e^{i\,\varphi} + 3\, a\, e^{-i\,\varphi}\right);$$

$$\operatorname{Im}\big(e^{-i\varphi}\, z_4(2\pi,\varphi)\big) = 5\pi\left(-\frac{3}{2}\operatorname{Re}(b\, e^{3\,i\,\varphi}) + \operatorname{Re}(\bar{a}\, e^{i\,\varphi})\right);$$

$$\frac{\partial}{\partial\varphi}\operatorname{Im}\big(e^{-i\varphi}\, z_4(2\pi,\varphi)\big) = 5\pi\left(\frac{9}{2}\operatorname{Re}\big(b\,(i\, e^{i\,\varphi})^3\big) + \operatorname{Re}(\bar{a}\, i\, e^{i\,\varphi})\right);$$

$$z_4(2\pi,\varphi) - \Delta_3\,\frac{\partial z_1}{\partial t}(2\pi,\varphi) =$$
$$= 5\pi e^{i\varphi}\left(-\frac{3}{2} i\,\operatorname{Re}(b\, e^{3\,i\,\varphi}) + i\operatorname{Re}(\bar{a}e^{i\varphi}) - \frac{9}{2}\operatorname{Re}\big(b(i\, e^{i\varphi})^3\big) - \operatorname{Re}(\bar{a}ie^{i\varphi})\right).$$

Going back to vector notation and to the notations of Sec. 3.3, we get that

$$z_4(2\pi,\varphi) - \Delta_3\,\frac{\partial z_1}{\partial t}(2\pi,\varphi) =$$
$$= 5\pi\left(-\frac{3}{2}V_3(v)\, j(v) + V_1(v)\, j(v) - V_1\big(j(v)\big)\, v - \frac{9}{2}\, V_3\big(j(v)\big)\, v\right).$$

Computation of $w(t_c,\varphi,\rho_0)$:

$$w(t_c,\varphi,\rho_0) = \rho_0^2\, w_2(t_c,\varphi) + \rho_0^4\, w_4(t_c,\varphi) + O(\rho_0^5) =$$
$$= \rho_0^2\, w_2(2\pi,\varphi) + \rho_0^4\left(w_4(2\,\pi,\varphi) - \frac{\partial w_2}{\partial t}(2\,\pi,\varphi)\,\Delta_2\right) + O(\rho_0^5);$$
$$w_2(2\pi,\varphi) = 2\pi, \qquad \frac{\partial w_2}{\partial t}(2\,\pi,\varphi) = 0,$$
$$w_4(2\pi,\varphi) = -3\frac{\pi}{2}\,\big(2\,(c_2 - c_1)\,\cos 2\varphi + 3\,(c_1 + c_2)\big);$$
$$w(t_c,\varphi,\rho_0) = 2\pi\,\rho_0^2 - 3\frac{\pi}{2}\,\rho_0^4\,\big(2\,(c_2 - c_1)\,\cos 2\varphi + 3\,(c_1 + c_2)\big) + O(\rho_0^5).$$

Finally, let us make a last coordinate change. In vector notations, we set

$$(\widetilde{x},\widetilde{y}) = \widetilde{z} = z - \frac{5}{\pi}\, w^2\,\big(V_1(v)\, j(v) - V_1\big(j(v)\big)\, v\big). \tag{5.15}$$

We get the final expressions for the approximations of the conjugate time mapping (using (4.19)), and the conjugate locus, summarized in the next theorem:

Theorem 5.4. *Making the coordinate change* (5.15), *the following approximating formulas hold for the arc-length conjugate time mapping s_c and* *the conjugate locus CL, provided that we restrict our sub-Riemannian metric to a sufficiently small neighborhood U of the origin* 0 *on M*:

$$\text{(i)}\quad s_c(v,\rho_0) = 2\pi\rho_0 - 6\pi\rho_0^3 \operatorname{tr}_g Q + 20\,\pi\,\rho_0^4\left(V_1(j(v)) - \frac{1}{2}V_3(j(v))\right) + + O_1(\rho_0^5); \tag{5.16}$$

(ii) *CL is the intersection with U of the image of the following mapping ξ_c*:

$$\begin{aligned} \xi_c(v,\rho_0) &= \begin{pmatrix} \widetilde{z}_c(v,\rho_0) \\ w_c(v,\rho_0) \end{pmatrix}, \\ \widetilde{z}_c(v,\rho_0) &= \left(-8\,\pi\,\rho_0^3\,Q_2(v) - \frac{45}{2}\pi\rho_0^4\,V_3(j(v))\right)v+ \\ &\quad +\left(-2\,\pi\,\rho_0^3\,Q_2(v+j(v)) - \frac{15}{2}\pi\rho_0^4 V_3(v)\right)j(v) + O_2(\rho_0^5); \\ w_c(v,\rho_0) &= \pi\,\rho_0^2 - 3\,\pi\,\rho_0^4(3\,Q_0(v) - 2\,Q_2(v)) + O_3(\rho_0^5), \end{aligned} \tag{5.17}$$

v being the unit initial velocity of the geodesics.

In complex notations, (i) and (ii) of Theorem 5.4 give, with $v = e^{i\varphi}$:

$$\begin{cases} s_c = 2\pi\rho_0 - 6\pi\rho_0^3 \operatorname{tr}_g Q + 20\pi\rho_0^4\left(\frac{1}{2}\operatorname{Re}(bie^{3i\varphi}) + \operatorname{Re}(\overline{a}ie^{i\varphi})\right) + O_1(\rho_0^5); \\ z_c(\varphi,\rho_0) = \pi\,\rho_0^3\,(c_2 - c_1)\,(e^{3\,i\,\varphi} + 3\,e^{-i\,\varphi}) + \\ \qquad + \frac{15}{2}\pi\,\rho_0^4\,i\,(b\,e^{4\,i\,\varphi} - 2\,\overline{b}\,e^{-2\,i\,\varphi}) + O_2(\rho_0^5); \\ w_c(\varphi,\rho_0) = \pi\,\rho_0^2 - \frac{3}{2}\pi\,\rho_0^4\,(3\,(c_1+c_2) + 2\,(c_2-c_1)\,\cos 2\varphi) + O_3(\rho_0^5). \end{cases}$$

6. Stability of our approximating formulas for CL

6.1. Genericity.

Proposition 6.1. *There is an open dense subset of* $\operatorname{Sub} R(M)$ *(in the Whitney topology) for which the following holds*:

(i) *condition $Q_2 \neq 0$ holds everywhere on M except on a dimension* 1 *smooth submanifold (could be empty)*;

(ii) *condition $Q_2 \neq 0$ or $Q_2 = 0$ but $V_3 \neq 0$ holds everywhere on M.*

Proof. This follows immediately from our Theorem 2.11 and standard transversality arguments. □

In this paragraph, we will study more precisely the approximation of CL in these two generic situations, (i) and (ii) of Proposition 6.1.

6.2. The case $Q_2 \neq 0$ (generic points for generic elements of $\mathrm{Sub}\,R(M)$). In this section, we restrict ourselves to $w > 0$ in R^3, and $\rho_0 > 0$.

1. Representation of the exponential mapping as a suspension.

Theorem 6.2. *There exist a constant $\eta > 0$, a neighborhood U of $S \cap \{0 < \rho_0 < \eta\}$ in $R_+ \times C$, a neighborhood V of $\mathcal{E}(S \cap \{0 < \rho_0 < \eta\})$ and a coordinate system $(\sigma, \varphi, \varepsilon)$ in U, φ unchanged, a coordinate system $(\tilde{x}, \tilde{y}, \tilde{z})$ in V, such that the expression E in those coordinates of the exponential mapping is a suspension*

$$E(\sigma, \varphi, \varepsilon) = \begin{pmatrix} E_0(\sigma, \varphi) \\ \varepsilon \end{pmatrix} + \begin{pmatrix} O(\varepsilon) \\ 0 \end{pmatrix} = E_{in}(\sigma, \varphi, \varepsilon) + \begin{pmatrix} O(\varepsilon) \\ 0 \end{pmatrix}, \tag{6.1}$$

where $O(\varepsilon)$ has order one at least in ε, and

$$E_0(\sigma, \varphi) = \sigma \frac{\partial z_1}{\partial t}(2\pi, \varphi) + z_3(2\pi, \varphi). \tag{6.2}$$

Proof. Let us go back to the expression (4.7), (4.8) of the exponential mapping, for the new time t:

$$\begin{pmatrix} z(t, \varphi, \rho_0) \\ w(t, \varphi, \rho_0) \end{pmatrix} = \begin{pmatrix} \rho_0\, z_1(t, \varphi) + \rho_0^3\, z_3(t, \varphi) + O_1(\rho_0^4) \\ \rho_0^2\, w_2(t, \varphi) + O_2(\rho_0^4). \end{pmatrix} = \widehat{\mathcal{E}}(t, \varphi, \rho_0). \tag{6.3}$$

To represent this map as a suspension of a map between two-dimensional spaces, we will make simple coordinate changes at the source and at the image of $\widehat{\mathcal{E}}$.

First, observe that we can consider $\widehat{\mathcal{E}}$ as a mapping from $\mathrm{dom}(\widehat{\mathcal{E}}) \cap \{2\pi - \eta < t < 2\pi + \eta,\ 0 < \rho_0 < \eta\}$ into $\{w > 0\}$ for η sufficiently small. We make the following coordinate change at the source:

$$(t, \varphi, \rho_0) \to \Big(\sigma = \frac{t - 2\pi}{\rho_0^2},\ \varphi,\ \rho_0\Big).$$

In the image, first we make the change $(x, y, w) \to \Big(x, y, \varepsilon = \sqrt{\dfrac{w}{\pi}}\Big)$. At the source, we apply $(\sigma, \varphi, \rho_0) \to (\sigma, \varphi, \varepsilon)$, solving the equation $w = \pi\,\varepsilon^2 = \pi\,\rho_0^2 + O(\rho_0^4)$.

Then we notice that for the z-component of $\widehat{\mathcal{E}}$,

$$z(\sigma, \varphi, \varepsilon) = \varepsilon^3 \Big(\sigma \frac{\partial z_1}{\partial t}(2\pi, \varphi) + z_3(2\pi, \varphi)\Big) + O(\varepsilon^4). \tag{6.4}$$

Hence, we can make a second coordinate change in the image:

$$(x, y, \varepsilon) \to \left(\widetilde{x} = \frac{x}{\varepsilon^3}, \widetilde{y} = \frac{y}{\varepsilon^3}, \varepsilon\right). \quad \square$$

2. Stability. To determine CL, that is, the singular set of $\mathcal{E}$, in an appropriate open set U, we shall show that E_{in} in (6.1) is a *sufficient jet* for $\mathcal{E}$ on U. This will show that in a neighborhood of zero, CL is diffeomorphic to the singular locus of E_{in}. In order to prove that E_{in} is a sufficient jet, we will use the fact that E_0 is a "Whitney map" in a neighborhood of S.

We say that a map F between 2-dimensional manifolds is a Whitney map if:

(1) Its singular set S at the source is a smooth curve.
(2) Restricted to its singular set, $F|_S$ is injective and proper.
(3) The image curve $F(S)$ presents only fold points and cusp points as singularities (for these concepts see Whitney [20]).

(6.5)

By well-known facts of singularity theory, Whitney maps are stable, in the sense of Thom and Mather (Whitney [20], or Mather [13]).

Let us consider the map E_0, restricted to a neighborhood

$$S_\alpha = \left\{(\sigma, \varphi) \middle| -6\,\pi\,\mathrm{tr}_g\, Q - \alpha \le \sigma \le -6\,\pi\,\mathrm{tr}_g\, Q + \alpha\right\}$$

of the circle S_0, $S_0 = \left\{(\sigma, \varphi) \middle| \sigma = -6\,\pi\,\mathrm{tr}_g\, Q\right\}$, $\alpha > 0$, sufficiently small.

It is easy to check that for α small enough, the restriction $E_0|_{S_\alpha}$ is a Whitney map. By (4.11), (4.14), we have

$$E_0(\sigma, \varphi) = \sigma \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix} + 2\pi \begin{pmatrix} \cos(\varphi)\left(4c_2 + 2c_1 + \cos(2\varphi)(c_2 - c_1)\right) \\ \sin(\varphi)\left(2c_2 + 4c_1 + \cos(2\varphi)(c_2 - c_1)\right) \end{pmatrix}. \tag{6.6}$$

We already know that the circle S_0 is the singular set of $E_0|_{S_\alpha}$ for α sufficiently small, and it is obvious that the conditions (1), (2), (3) stated above are satisfied. Therefore, for ε sufficiently small, E_0 is R.L. equivalent to our $E_0 + O(\varepsilon)$ in (6.1) and, since $O(\varepsilon)$ is smooth, it is possible to check that the diffeomorphisms appearing in the R.L. equivalence relation between E_0 and $E_0 + O(\varepsilon)$ depend smoothly on ε. Therefore, the following theorem holds.

Theorem 6.3. *If $Q_2 \neq 0$, there is a constant $\varepsilon_0 > 0$, a constant $\alpha > 0$, a neighborhood*

$$S_{\alpha,\varepsilon_0} = \left\{(\sigma, \varphi, \varepsilon) \middle| -6\pi\,\mathrm{tr}_g\, Q - \alpha < \sigma < -6\pi\,\mathrm{tr}_g\, Q + \alpha,\ 0 < \varepsilon < \varepsilon_0\right\}$$

of $S_0 = \{(\sigma,\varphi,\varepsilon) | \sigma = -6\pi \operatorname{tr}_g Q,\ 0 < \varepsilon < \varepsilon_0\}$, *a neighborhood* S' *of* $E_{in}(S_0)$ *in* $M \cap \{w > 0\}$, *and diffeomorphisms* $\Xi : S_{\alpha,\varepsilon_0} \to \overline{S}$, $\Xi' : S' \to \overline{S}'$ *such that:*

(i) Ξ *preserves the foliation* $\varepsilon =$ constant, Ξ' *preserves the foliation* $w =$ constant;

(ii) $E = \Xi' \circ E_{in} \circ \Xi^{-1}$.

The set of singular values of $E_0|_{S_{\alpha,\varepsilon_0}}$ is given by the approximation (5.17) of CL at order ρ_0^3 for $\rho_0 = 1$, i.e.,

$$\begin{pmatrix} x \\ y \end{pmatrix} = 8\pi\sqrt{\delta(Q)} \begin{pmatrix} \cos^3\varphi \\ -\sin^3\varphi \end{pmatrix}.$$

It is a closed curve, without self-intersection, presenting 4 cusps at $\varphi = k\dfrac{\pi}{2}$, $0 \le k \le 3$. Hence, the same holds for sections of CL by planes $w = \varepsilon^2$ for ε small enough. The pictures are shown in the next section.

6.3. The case $Q_2 = 0$ but $V_3 \neq 0$ (generic singular case).

1. In this degenerate situation, we will also conclude on the stability of our approximating formulas for CL, but in the local sense only (that is, for germs at points of S), because of the self-intersections of CL, as we shall see.

When $Q_2 = 0$, using a simple rotation, we can assume that $V_3(1,0) = 0$, $V_3(0,1) = \widetilde{b} > 0$. Our formula (5.17) for the approximation of CL can be rewritten:

$$\begin{cases} \widetilde{z}_c = 15\dfrac{\pi}{2}\widetilde{b}\rho_0^4 \begin{pmatrix} 2\cos(2\varphi) + \cos(4\varphi) \\ -2\sin(2\varphi) + \sin(4\varphi) \end{pmatrix}, \\ w_c = \pi\rho_0^2 - 9\pi\dfrac{\operatorname{tr}_g Q}{2}\rho_0^4. \end{cases} \tag{6.7}$$

Remark 7.

(1) The determinants

$$\Delta_1 = \det\left(\frac{\partial \widetilde{z}_c}{\partial \rho_0}, \frac{\partial \widetilde{z}_c}{\partial \varphi}\right), \qquad \Delta_2 = \det\begin{pmatrix} \dfrac{\partial \widetilde{x}_c}{\partial \rho_0} & \dfrac{\partial \widetilde{x}_c}{\partial \varphi} \\ \dfrac{\partial w_c}{\partial \rho_0} & \dfrac{\partial w_c}{\partial \varphi} \end{pmatrix},$$

$$\Delta_3 = \det\begin{pmatrix} \dfrac{\partial \widetilde{y}_c}{\partial \rho_0} & \dfrac{\partial \widetilde{y}_c}{\partial \varphi} \\ \dfrac{\partial w_c}{\partial \rho_0} & \dfrac{\partial w_c}{\partial \varphi} \end{pmatrix}$$

are, respectively, up to nonzero constant factors,

$$\Delta_1 = \sin^2(3\varphi),\ \Delta_2 = \sin(3\varphi)\cos\varphi,\ \Delta_3 = \sin(3\varphi)\sin\varphi. \tag{6.8}$$

These determinants vanish simultaneously for $\sin(3\varphi) = 0$, that is, $\varphi = k\frac{\pi}{3}$, $0 \le k \le 5$.

(2) The map defined by formula (6.7) has period π in φ.

(3) The image of the mapping (6.7), shown in Fig. 11 in the next section, presents 3 cusps which are double points of this mapping.

To analyze the stability, we will proceed as in the previous section. But there is a crucial difference from the previous case: the approximation E_{in} is not a sufficient jet globally on the singular locus any more, although, locally along the singular locus, it is sufficient everywhere.

Remark 8. The reason for this is that the restriction of E_{in} to its singular locus is invariant under the involution $\rho_0 \to \rho_0$, $\varphi \to \varphi + \pi$, a fact not true for the mapping E.

2. Representation of $\mathcal{E}$ as a suspension.

Theorem 6.4. *If $Q_2 = 0$, there exists a constant $\eta > 0$, a neighborhood U of $S \cap \{0 < \rho_0 < \eta\}$ at the source, a neighborhood V of $\mathcal{E}(S \cap \{0 < \rho_0 < \eta\})$, and a coordinate system $(\sigma, \varphi, \varepsilon)$ in U (φ unchanged), a coordinate system $(\widetilde{x}, \widetilde{y}, \widetilde{w})$ in V, such that in those coordinates, the exponential map E has the suspended form* (6.1), *where*

$$E_0(\sigma, \varphi) = \sigma \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix} + z_4(2\pi, \varphi). \tag{6.9}$$

$z_4(2\pi, \varphi)$ has already been computed (formula (4.15), where we can choose $V_3\,(1,0) = 0$, $V_3\,(0,1) = \widetilde{b} \neq 0$).

Proof of Theorem 6.4.

Our approximation of the exponential mapping is now

$$\widehat{\mathcal{E}}(t, \varphi, \rho_0) = \begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} \rho_0\, z_1(t,\varphi) + \rho_0^3\, z_3(t,\varphi) + \rho_0^4\, z_4(t,\varphi) + O(\rho_0^5) \\ \rho_0^2\, w_2(t,\varphi) + \rho_0^4\, w_4(t,\varphi) + O(\rho_0^5) \end{pmatrix}.$$

We will proceed as in the proof of Theorem 6.2. Our first coordinate change at the source will be $(t, \varphi, \rho_0) \to \left(\sigma = \dfrac{t - 2\pi + 6\pi\, \mathrm{tr}_g\, Q\, \rho_0^2}{\rho_0^3},\ \varphi,\ \rho_0\right)$.

Our first coordinate change at the image will be again

$$(x, y, w) \to \left(x, y, \varepsilon = \sqrt{\frac{w}{\pi}}\right).$$

Our second coordinate change at the source will be the same as in Sec. 6.2.1.

We get for the z-component of $\mathcal{E}$:

$$z(\sigma, \varphi, \varepsilon) = \varepsilon^4\big(\sigma(\cos\varphi, \sin\varphi) + z_4(2\pi, \varphi)\big) + O(\varepsilon^5).$$

The second coordinate change at the image is

$$(x, y, \varepsilon) \to \left(\tilde{x} = \frac{x}{\varepsilon^4}, \tilde{y} = \frac{y}{\varepsilon^4}, \varepsilon\right). \quad \square$$

3. Stability. As before, we consider the map E_0 in a neighborhood S_α of the set

$$S_0 = \left\{(\sigma, \varphi) \,/\, \sigma = 20\,\pi \left(\frac{1}{2}\ \mathrm{Re}(b\, i\, e^{3\,i\,\varphi}) + \mathrm{Re}(\overline{a}\, i\, e^{i\,\varphi})\right)\right\},$$

$$S_\alpha = \{(\sigma + h, \varphi) \mid (\sigma, \varphi) \in S_0,\ |h| < \alpha\}. \tag{6.10}$$

Again, we already know that for α small enough, the singular set at the source of $E_0|_{S_\alpha}$ is exactly the circle S_0 (see (5.13), (5.14)), and the singular locus CL in these coordinates is given by the expression (5.17) for $\rho_0 = 1$.

But the map E_0 is not a Whitney map any more since the property (2) of Whitney maps fails to be true : $E_0|_{S_0}$ is not injective.

Nevertheless, the two other conditions for Whitney maps hold, and the germs of E_0 along S_0 are stable.

As a consequence, we can state the following theorem.

Theorem 6.5. *In the situation where $Q_2 = 0$ but $V_3 \neq 0$, the conjugate locus CL is of the form*

$$\begin{pmatrix} \tilde{z}_c \\ w_c \end{pmatrix} = \begin{pmatrix} 15\frac{\pi}{2}\,\tilde{b}\,\rho_0^4\big(2\cos(2\varphi) + \cos(4\varphi)\big) \\ 15\frac{\pi}{2}\,\tilde{b}\,\rho_0^4\big(-2\sin(2\varphi) + \sin(4\varphi)\big) \\ \pi\rho_0^2 - \frac{9}{2}\pi\,\mathrm{tr}_g\, Q\,\rho_0^4 \end{pmatrix} + O(\rho_0^5).$$

Sections of CL by the level surfaces w = constant, $w \neq 0$ are closed curves, presenting 6 *cusp points (when counted at the source).*

We shall see below that in general there are six distinct cusp points although the approximation at order ρ_0^4 presents only 3 cusp points at the image, due to the fact that it is invariant by the involution of above Remark 8.

6.4. Pictures. In this section we will show a certain number of sub-Riemannian pictures. None of these pictures has required numerical integration. They just required evaluation of our approximating formulas.

1. First, we deal with the case where $Q_2 \neq 0$.

Figure 1. The sub-Riemannian small sphere (wave front). The equations of the approximation are (from (4.18)).

$z(t, \rho_0, \varphi) = \rho_0 z_1(t) + \rho_0^3\, z_3(t, \varphi)$, with

$$z_1(t) = 2\,\sin\frac{t}{2}\left(\cos\left(\varphi - \frac{t}{2}\right), \sin\left(\varphi - \frac{t}{2}\right)\right), \tag{6.11}$$

$z_3(t,\varphi) = \big(x_3(t,\varphi),\, y_3(t,\varphi)\big)$, where

$$
\begin{aligned}
x_3(t,\varphi) = {} & (24c_2 t\cos\varphi + 24\, c_1 t\cos(\varphi - t) + \\
& + 24\, c_2 t\cos(\varphi - t) - 6c_1 t\cos(3\varphi - t) + \\
& + 6c_2 t\cos(3\varphi - t) - 6\, c_1 t\cos(\varphi + t) + \\
& + 6\, c_2 t\cos(\varphi + t) - 42c_1\sin\varphi - \\
& - 18c_2\sin\varphi + 2c_1\sin 3\varphi - \\
& - 2c_2\sin 3\varphi + 6c_1\sin(\varphi - 2t) + \\
& + 6c_2\sin(\varphi - 2t) - 6c_1\sin(3\varphi - 2t) + \\
& + 6c_2\sin(3\varphi - 2t) + 21c_1\sin(\varphi - t) + \\
& + 27c_2\sin(\varphi - t) + c_1\sin(3(\varphi - t)) - \\
& - c_2\sin(3(\varphi - t)) + 3c_1\sin(3\varphi - t) - \\
& - 3c_2\sin(3\varphi - t) + 15c_1\sin(\varphi + t) - \\
& - 15c_2\sin(\varphi + t))/12,
\end{aligned}
\tag{6.12}
$$

$$
\begin{aligned}
y_3(t,\varphi) = {} & (42c_2\cos\varphi + 18\, c_1\cos(\varphi) + 2\, c_2\cos(3\varphi) - \\
& - 2\, c_1\cos(3\varphi) - 6c_1\cos(\varphi 2t) - \\
& - 6c_2\cos(\varphi - 2t) + 6\, c_1\cos(3\varphi - 2t) - \\
& - 6c_2\cos(3\varphi - 2t) - 27c_1\cos(\varphi - t) - \\
& - 21c_2\cos(\varphi - t) - c_1\cos(3(\varphi - t)) + \\
& + c_2\cos(3(\varphi - t)) - 3c_1\cos(3\varphi - t) + \\
& + 3c_2\cos(3\varphi - t) + 15c_1\cos(\varphi + t) - \\
& - 15c_2\cos(\varphi + t) + 24c_1 t\sin\varphi + \\
& + 24c_1 t\sin(\varphi - t) + 24c_2 t\sin(\varphi - t) - \\
& - 6c_1 t\sin(3\varphi - t) + 6c_2 t\sin(3\varphi - t) + \\
& + 6c_1 t\sin(\varphi + t) - 6c_2 t\sin(\varphi + t))/12,
\end{aligned}
$$

$$
w(t,\rho_0,\varphi) = \frac{\rho_0^2}{2}(t - \sin t). \tag{6.13}
$$

Notice that this last formula is the same as in the Heisenberg case, as we have shown.

In these formulas, $\varphi \in [0, 2\pi]$, ρ_0 is small, and $s = \rho_0 t$ is the radius of the sphere (remember that s is the arclength). We obtain the following picture for $c_2 = .85$, $c_1 = -1.52$, $s = \dfrac{1}{160}$.

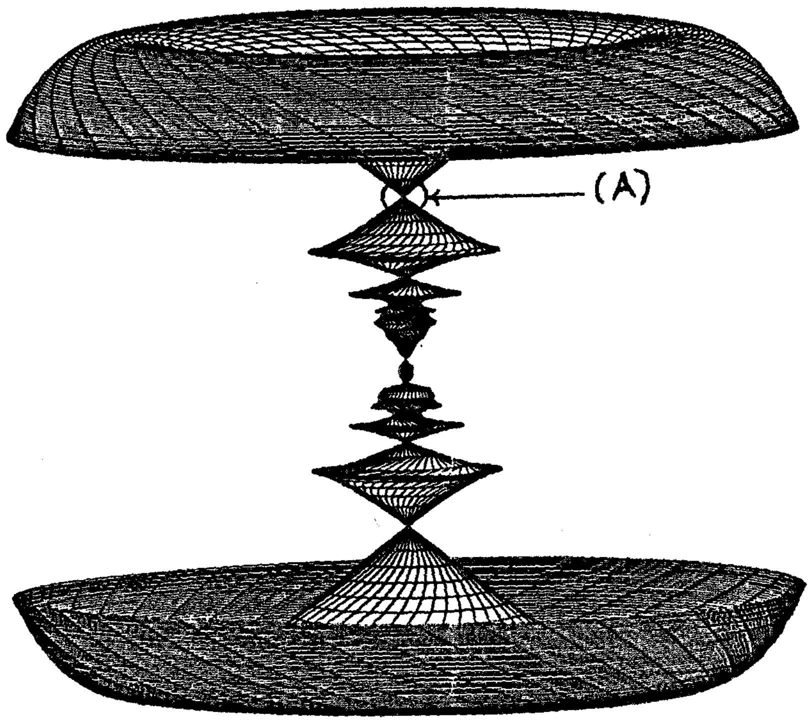

Fig. 1. Generic sub-Riemannian small sphere (wave front).

In Figs. 3 to 6, 13 to 16, we will give a detailed representation of the region marked by (A) in Fig. 1, which corresponds to t close to 2π or s close to $2\pi\rho_0$.

Figure 2: *The generic conjugate locus CL at generic points.*

The expression is that of formula (5.17). We draw the picture for $w > 0$ only.

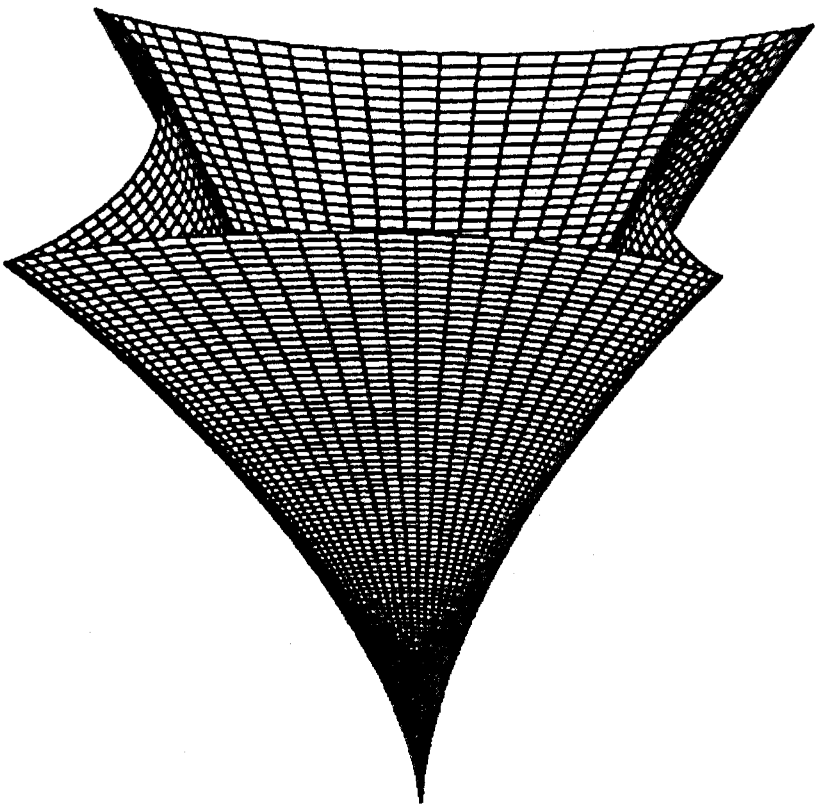

Fig. 2. Generic conjugate locus CL at generic points.

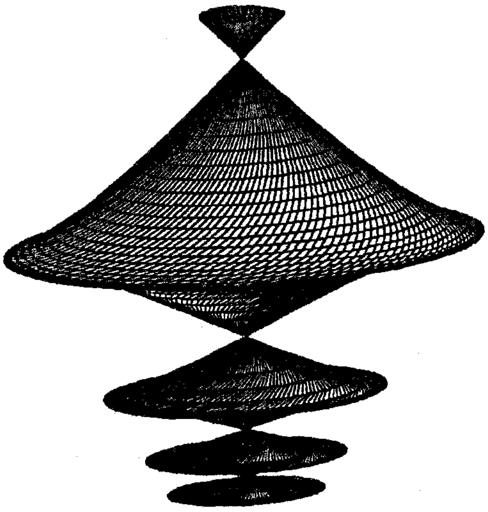

Fig. 3

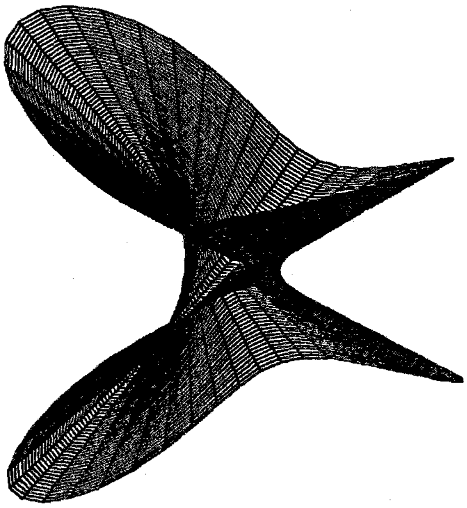

Fig. 4. Region (A).

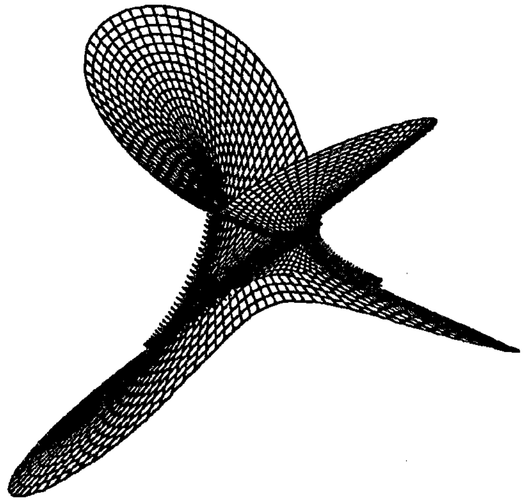

Fig. 5. Region (A).
(Same picture as Fig. 4, but the intersection with the conjugate locus has been marked.)

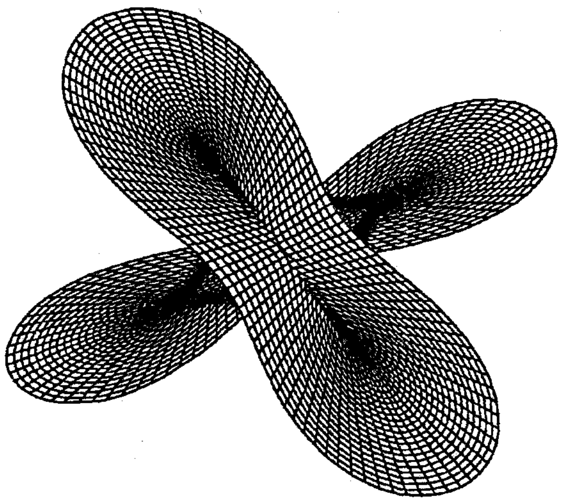

Fig. 6. Region (A).
(Same picture as Fig. 5 from above.)

As one can see, the intersection of the wave front and the conjugate locus is a closed curve, presenting 4 cusps. Moreover, the intersection of the sphere with the cut-locus is a line segment, the endpoints of which are 2 of the 4 cusps.

2. Now we show how the approximation (5.17) of order ρ_0^4 of the conjugate locus CL changes when we start from $c_1 \neq c_2$ (i.e., $Q_2 \neq 0$) and move to $c_1 = c_2$ (i.e., $Q_2 = 0$).

We show a succession of pictures for the following values of c_1, c_2 : $c_1 = -1,\ c_2 = \alpha - 1$, α being specified under each picture.

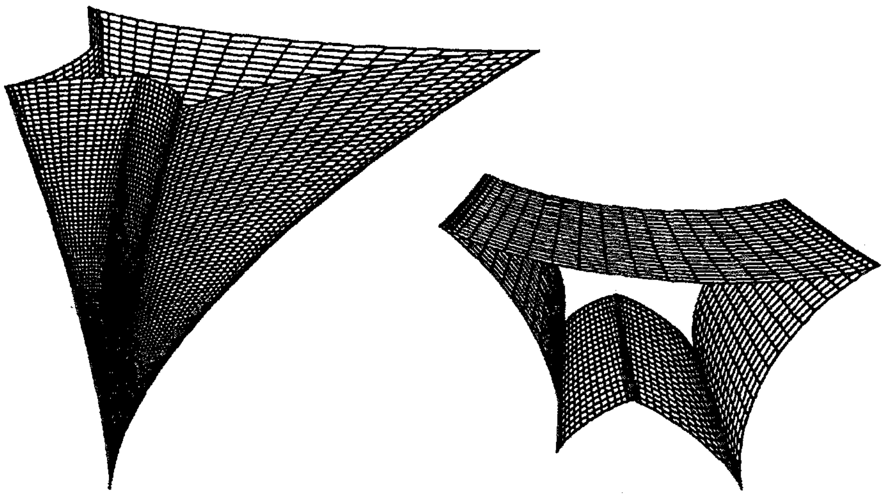

Fig. 7. $\alpha = 0.1$

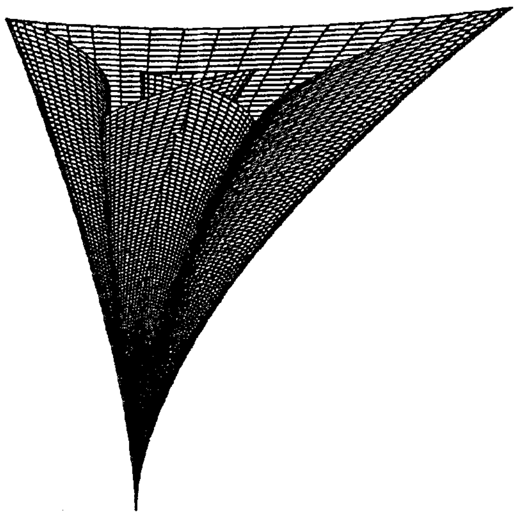

Fig. 8. $\alpha = 0.06$

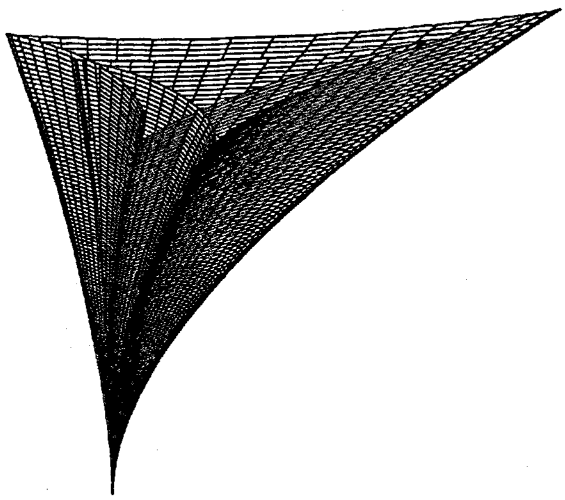

Fig. 9. $\alpha = 0.02$

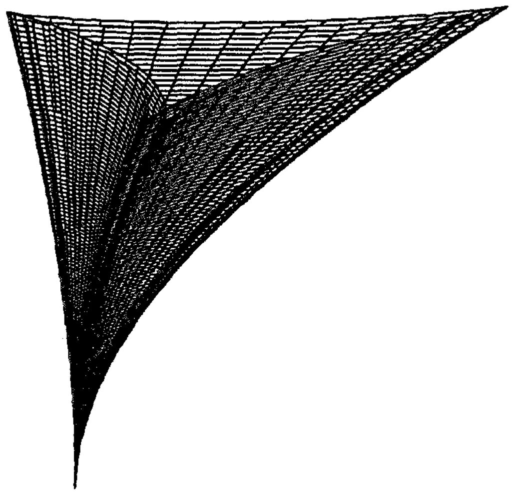

Fig. 10. $\alpha = 0.005$

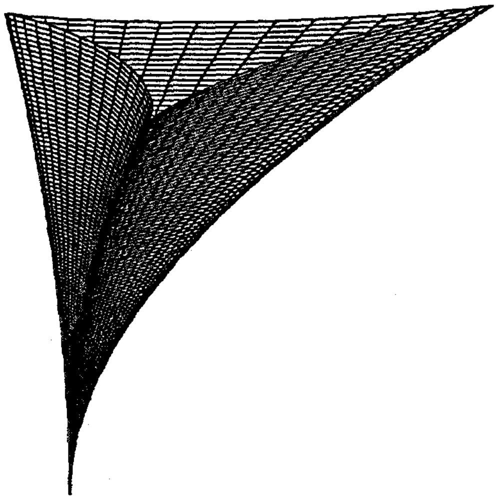

Fig. 11. $\alpha = 0$

Of course, in the figures (Fig. 7 to Fig. 10) the sections $w_c = \varepsilon$ have 4 cusp points for ε sufficiently small. For a larger ε they have 6 of them which coalesce in pairs, to produce the final picture, Fig. 11, with 3 cuspidal lines only.

7. THE GENERIC CASE FOR $Q_2 = 0$

In this section, we give numerical results on the conjugate locus and the wave fronts of order 5 for the following values of the parameters:

$$\begin{gathered} c_1 = -1, \quad c_2 = -1, \quad l_2 = -0.1, \quad l_1 = 0, \\ t_{12} = \frac{-1}{4}, \quad t_{21} = t_{30} = t_{03} = 0, \end{gathered} \tag{7.1}$$

where $V(z) = \sum_{i+j=3} t_{ij}\, x^i\, y^j$.

We use our partial normal form $\mathcal{NF}^3$ of order 3

$$(\mathcal{NF}^3) \quad \Sigma = \begin{pmatrix} \left(1 + y^2\,(l + f_2)\right)\dfrac{\partial}{\partial x} - xy(l + f_2)\dfrac{\partial}{\partial y} + \\ + \dfrac{y}{2}(1 + Q + V + w\,g_2 + h_4)\dfrac{\partial}{\partial w}, \\ \left(1 + x^2\,(l + f_2)\right)\dfrac{\partial}{\partial y} - xy(l + f_2)\dfrac{\partial}{\partial x} - \\ - \dfrac{x}{2}(1 + Q + V + w\,g_2 + h_4)\dfrac{\partial}{\partial w} \end{pmatrix}$$

with the following choices:

$$f_2(x, y) = \frac{2}{5}xy, \quad g_2(x, y) = 0, \quad h_4(x, y) = \frac{1}{2}x^2\,y^2 + \frac{1}{5}x^3\,y.$$

We apply the same method, but take the approximation of order 5 instead of 4 in ρ_0 for conjugate points. We get some complicated formulas that we do not give here. Just evaluating these formulas allows us to draw the following picture:

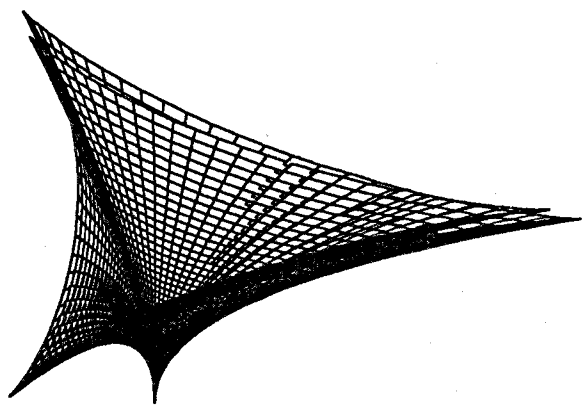

Fig. 12. The conjugate locus CL of order 5, for $Q_2 = 0$.

This picture shows that there are actually 6 cusps, and not 3 (even for w_c arbitrary small).

Now, we show pictures of the typical sub-Riemannian wave front in that case.

The wave front, under several viewpoints, near $t = 2\pi$:

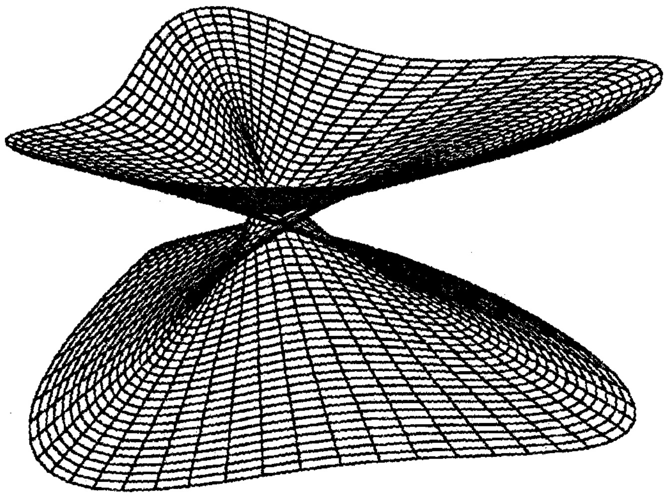

Fig. 13

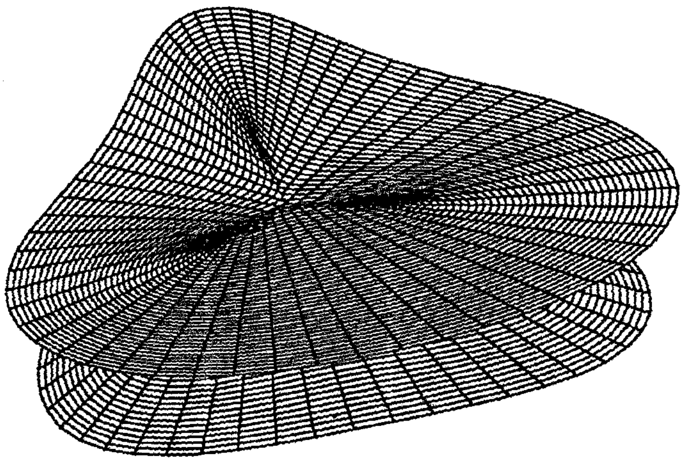

Fig. 14

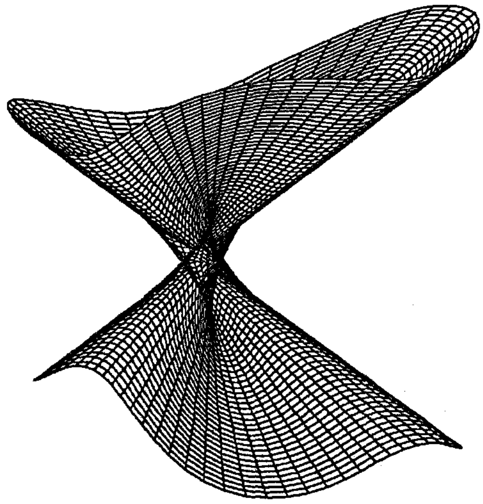

Fig. 15

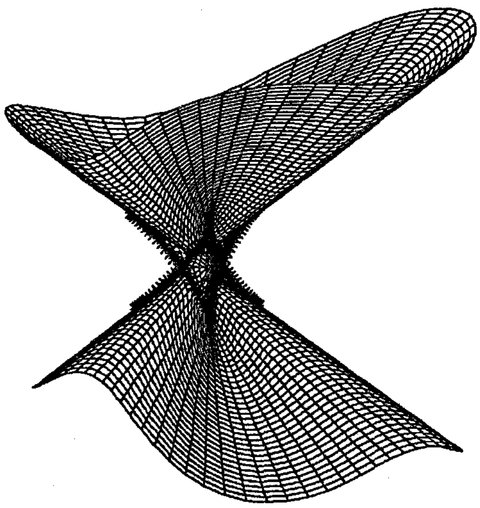

Fig. 16.
Wave front plus intersection with the singular locus marked on it.

Figures 13 to 16 are obtained by evaluating the approximate equation of the geodesics of order 5,

$$\begin{cases} z(t) = \rho_0\, z_1(t,\varphi) + \rho_0^3 z_3(t,\varphi) + \rho_0^4 z_4(t,\varphi) + \rho_0^5 z_5(t,\varphi), \\ w(t) = \rho_0^2\, w_2(t,\varphi) + \rho_0^4\, w_4(t,\varphi) + \rho_0^5\, w_5(t,\varphi); \end{cases} \tag{7.2}$$

$z_1(t,\varphi) = 2\sin\frac{t}{2}\Big(\cos\Big(\varphi - \frac{t}{2}\Big), \sin\Big(\varphi - \frac{t}{2}\Big)$; z_3 is given in formula (6.12). $w_2 = \frac{\rho_0^2}{2}(t - \sin t)$; all the other terms z_4, z_5, w_4, w_5 were computed using Mathematica. We do not give the formulas here because they are too long.

Observe that, in Fig. 16, the intersection (marked) between the wave front and the conjugate locus is a closed curve presenting 6 cusp points. In Fig. 14, we see that the intersection of the sphere and the cut-locus consists of 3 line segments having one endpoint in common. The other 3 endpoints of these segments are 3 of the 6 cusps.

It is an interesting observation that this will also hold, for the approximation at order 4 only in ρ_0, an approximation for which the conjugate locus has only 3 cuspidal lines (Fig. 11).

8. Table of main symbols

M, Σ, $\operatorname{Sub} R(M)$	Sec. 1.1.
Δ, g	Sec. 1.1, 2, Sec. 3.1, 2, 3, Sec. 4.
$\mathcal{E}$, R, φ, r, ρ	Sec. 1.2, Sec. 4.3, 5.
$\beta_{FGq}, \gamma_{FGq}, \beta_{nFGq}, \gamma_{nFGq}$, B_{nq}, Γ_{nq}, Δ^*, Δ^0	Sec. 3.1, 2.
$S^k\Delta^*, S^2\Delta^*$, $(S^2\Delta^*)_2, (S^2\Delta^*)_0$, $(S^3\Delta^*)_3, (S^3\Delta^*)_1, \delta(Q)$	Sec. 3.2, 3.
S, CL	Sec. 1.2, Sec. 4.2.
$\mathcal{E}^i$, $\mathcal{E}_i$	Sec. 1.2.
$F, G, \mathcal{R}$	Sec. 1.2, Sec. 2.
β, γ	Sec. 1.2, Sec. 3.2.
Q, V, Q_2, Q_0, V_3, $V_1, \operatorname{tr}_g Q$, v, l	Sec. 1.2, Sec. 2.3, Sec. 3.2, 3.
ξ, x, y, w, z	Sec. 1.2, Sec. 2, Sec. 4.4.
$\mathcal{F}$, $\mathcal{F}_n, V\mathcal{F}$, $V\mathcal{F}_n$, $V\mathcal{F}^0$	Sec. 2.1.1, 2, 3, 4.
$\mathcal{D}, \mathcal{D}_0$, $\mathcal{R}_{-1}, \mathcal{R}_H$, $F_{-1}, G_{-1}, F^0_{-1}, G^0_{-1}$	Sec. 2.1.6, 7.
$\mathcal{D} \propto \mathcal{F}$	Sec. 2.1.9.
$\mathcal{G}^0_0$, $\alpha_0, \mathcal{G}_I$, $\mathcal{D}_0 \propto \mathcal{F}_0$	Sec. 2.1.10.

L_n, $\mathcal{N}_n$, $\mathcal{NF}$	Sec. 2.2.1, 2.
j^n, J^n, $\mathcal{NF}^i$	Sec. 2.3.1.
j	Sec. 1.2, Sec. 3.1, Sec. 3.3.
$c_1, c_2, \overline{\mathcal{N}}^i, \overline{\mathcal{NF}}^i, G_1, G_2, G_3, G_4$	Sec. 1.2, Sec. 2.4.
$SJ^i, \Pi_{\mathcal{N}}, \Pi_{\mathcal{N}}^n$, B^1, B^2	Sec. 2.5.2.
ω, ν, $\omega^1, \omega^2, \omega^3, e_1, e_2, e_3$	Sec. 3.1.1, 2, 3.
$\underline{TM}$, $\underline{\Delta}$, II, Ω, III_1, III_3, $\mathcal{K}$, ω_2^1, Ω_2^1	Sec. 3.1.3, Sec. 3.2, 3.
ψ, $\mathcal{H}$, $\overrightarrow{\mathcal{H}}$, Λ, λ	Sec. 4.1, Sec. 4.2.
C, $\widetilde{C}$, s_c, α, $C_s, \widehat{D}$, Φ, Φ_s	Sec. 4.3.
p, q, r, f, g, $\{\}$, θ, η, h, ζ, ξ	Sec. 4.4.1, 2, 3.
ρ_0, $\widehat{f}$, $\widehat{g}$, $\widehat{h}$, $\widehat{\theta}$, $\widehat{\zeta}$, $\widehat{\xi}$, $\widehat{\eta}$, $\widehat{p}$, $\widehat{q}$	Sec. 4.4, Sec. 4.5.
s, t	Sec. 4.2, Sec. 4.5.
H_i, z_i, w_i	Sec. 4.4, Sec. 4.5.
a, b	Sec. 1.2, Sec. 4.6.
D, D_1, D_2, D_2', D_3	Sec. 5.1.1.
t_c, Δ_2, Δ_3	Sec. 5.1.3, Sec. 5.2.
δ	Sec. 1.2, Sec. 4.7.
E, E_0, $\widehat{\mathcal{E}}$, E_{in}	Sec. 6.2.1.
Ξ', Ξ, S_{α,ε_0}, S_0, S', $\overline{S}$, $\overline{S}'$	Sec. 6.2.2.
$\widetilde{b}$	Sec. 6.3.1.
f_2, g_2, h_4	Sec. 7.
J	Sec. 9.1.1.

9. Appendices

9.1. Construction of normal forms.

1. In this section, J denotes the 2×2 matrix $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $C(z)$ the 2×2 matrix

$$C(z) = \begin{pmatrix} y^2 & -x\,y \\ -x\,y & x^2 \end{pmatrix} = -J\,z\,z^t\,J.$$

We write $\dfrac{\partial}{\partial z} = \begin{pmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{pmatrix}$.

With these notations, our normal form can be rewritten under the matrix-differential operator form:

$$(\mathcal{NF}) \qquad \left((Id + C(z)\beta(z,w)) \frac{\partial}{\partial z} \right)^t - \frac{z^t\,J}{2}\,(1 + \gamma(z,w))\,\frac{\partial}{\partial w},$$

$$\beta(0,w) = 0 = \gamma(0,w) = \frac{\partial \gamma}{\partial z}(0,w). \tag{9.1}$$

We denote by V_n: The vector subspaces $V_n \subset \mathcal{F}_{n-1} \times \mathcal{F}_{n+1}$, $n \geq 2$,

$$V_n = \left\{ (\beta_{n-1}, \gamma_{n+1}) \,/\, \beta_{n-1}(0,w) = 0 = \gamma_{n+1}(0,w) = \frac{\partial \gamma_{n+1}}{\partial z}(0,w) \right\}.$$

If $n = -1$, 0, we set $V_n = \{0\}$. If $n = 1$, $V_1 \subset \{0\} \times \mathcal{F}_2$,

$$V_1 = \{(0, Q(z)) \,/\, Q \ \text{ quadratic}\}.$$

$N_n : V_n \to (V\mathcal{F}_n)^2$ is the operator

$$N_n(\beta_{n-1}, \gamma_{n+1}) =$$
$$= \Big((y^2 e_1 - x\, y\, e_2)\, \beta_{n-1} + \frac{y}{2} \gamma_{n+1} e_3,\ (x^2 e_2 - x\, y\, e_1)\, \beta_{n-1} - \frac{x}{2} \gamma_{n+1} e_3 \Big).$$

Then $\mathcal{N}_n = \mathrm{Im}(N_n)$.

Our Theorem 2.5 can be seen as a corollary of the following lemma.

Lemma 9.1. *The operator* $M_n = L_n + N_n$,

$$M_n : \mathcal{D}_{n+1} \times \mathcal{F}_{n+1} \times V_n \to (V\mathcal{F}_n)^2, \ \textit{is injective.}$$

Since the dimension of $\mathcal{D}_{n+1} \times \mathcal{F}_{n+1} \times V_n$ is equal to the dimension of $(V\mathcal{F}_n)^2$, this statement is equivalent to: (a) L_n and N_n are injective, (b) $\mathrm{Im}(N_n)$ is a supplement of $\mathrm{Im}(L_n)$, which is exactly what we need.

Observations: $\dim \mathcal{F}_{2p} = (p+1)^2$, $\dim \mathcal{F}_{2p+1} = (p+1)(p+2)$, $\dim V\mathcal{F}_{2p} = (p+2)(3p+4)$, $\dim V\mathcal{F}_{2p-1} = (p+1)(3p+4)$, $\dim V_{2p} = 2p(p+2)$, $\dim V_{2p+1} = (p+1)^2 + (p+2)^2 - 2$, $\dim \mathcal{D}_n = \dim V\mathcal{F}_n$. This shows that $\dim(\mathcal{D}_{n+1} \times \mathcal{F}_{n+1} \times V_n) = \dim(V\mathcal{F}_n)^2$.

Our proof of the fact that $\mathcal{N}$ is a normal form is given just below. In this proof, the heuristic selection rules that we apply for the choice of $\mathcal{N}_n$ will appear clearly.

An alternative proof (but this proof does not put in evidence the selection rules we have chosen) is to check directly Lemma 9.1, i.e., the kernel of M_n is reduced to $\{0\}$.

2. By Lemma 2.3, we have to consider the equations

$$\begin{cases} F'_{n-1} - F_{n-1} = [\varphi_n, F^0_{-1}] + \alpha_n\, G^0_{-1}, \\ G'_{n-1} - G_{n-1} = [\varphi_n, G^0_{-1}] - \alpha_n\, F^0_{-1}. \end{cases}$$

We set

$$\begin{cases} F'_{n-1} - F_{n-1} = A^1_n \dfrac{\partial}{\partial x} + A^2_n \dfrac{\partial}{\partial y} + A^3_{n+1} \dfrac{\partial}{\partial w}, \\ G'_{n-1} - G_{n-1} = B^1_n \dfrac{\partial}{\partial x} + B^2_n \dfrac{\partial}{\partial y} + B^3_{n+1} \dfrac{\partial}{\partial w}; \end{cases} \tag{9.2}$$

$A_n = (A_n^1,\, A_n^2)^t$, $B_n = (B_n^1,\, B_n^2)^t$, $\deg A_k^i = \deg B_k^i = k$;

$$\varphi_n = \varphi_{n+1}^1 \frac{\partial}{\partial x} + \varphi_{n+1}^2 \frac{\partial}{\partial y} + \varphi_{n+2}^3 \frac{\partial}{\partial w},\; \varphi_{n+1}^z = \begin{pmatrix} \varphi_{n+1}^1 \\ \varphi_{n+1}^2 \end{pmatrix},\; \deg \varphi_n^i = n.$$

Any $f \in \mathcal{F}_m$ can be written in a unique way as

$$f = \sum_{\lambda \le m,\, \lambda = m \mod 2} f_\lambda(x,y)\, w^{\frac{m-\lambda}{2}};$$

hence we have

$$A_n = \sum A_\lambda\, w^{\frac{n-\lambda}{2}}, \quad B_n = \sum B_\lambda\, w^{\frac{n-\lambda}{2}}, \quad \alpha_n = \sum \alpha_\lambda\, w^{\frac{n-\lambda}{2}},$$

$$A_{n+1}^3 = \sum A_\mu^3\, w^{\frac{n-\mu+1}{2}}, \quad B_{n+1}^3 = \sum B_\mu^3\, w^{\frac{n-\mu+1}{2}}.$$

We use the following matrix notation:

$$C_\lambda = \begin{pmatrix} A_\lambda^1 & B_\lambda^1 \\ A_\lambda^2 & B_\lambda^2 \end{pmatrix}, \qquad D_\mu = (A_\mu^3,\, B_\mu^3).$$

The equations above are equivalent to

$$(E_{\lambda,n}) d_z\varphi_\lambda^z - \frac{n-\lambda+3}{4}\varphi_{\lambda-2}^z\, z^t\, J - J\,\alpha_{\lambda-1} = C_{\lambda-1},$$
$$0 \le \lambda \le n+1, \quad \lambda = (n+1) \mod 2;$$
$$(E_{\mu,n}^3) d_z\varphi_\mu^3 - \frac{n-\mu+4}{4}\varphi_{\mu-2}^3\, z^t\, J - \frac{z^t}{2}\,\alpha_{\mu-2} + (\varphi_{\mu-1}^z)^t\, \frac{J}{2} = D_{\mu-1},$$
$$0 \le \mu \le n+2, \;\; \mu = n \mod 2.$$

Here $d_z\varphi_\lambda^z$ (resp. $d_z\varphi_\mu^3$) denotes the Jacobian matrix of φ_λ^z (resp. φ_μ^3) w.r.t. z.

3. Elimination of the φ_λ^z's.

Lemma 9.2. *P, Q are homogeneous polynomials in z, of degree n. There does exist a unique system U, V of homogeneous polynomials, $\deg U = n+1$, $\deg V = n-1$, such that*

$$\frac{\partial U}{\partial x} + y\,V = P, \;\; \frac{\partial U}{\partial y} - x\,V = Q.$$

They are given by

$$U = \frac{1}{n+1}(x\,P + y\,Q), \qquad V = \frac{1}{n+1}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right).$$

Therefore, if $\alpha_{\lambda-1}$ is given, the equation $E_{\lambda,n}$ determines uniquely φ^z_λ, $\varphi^z_{\lambda-2}$.

Let us write $\varphi^z_\lambda = E_{\lambda,n}\,\varphi^z_\lambda$, $\varphi^z_{\lambda-2} = E_{\lambda,n}\,\varphi^z_{\lambda-2}$.

But $\varphi^z_{\lambda-2}$ can also be computed using $E_{\lambda-2,n}$. One gets

$$\varphi^z_{\lambda-2} = E_{\lambda-2,n}\,\varphi^z_{\lambda-2}.$$

The system $E_{\lambda,n}$ has a unique solution as soon as the compatibility relations

$$E_{\lambda,n}\,\varphi^z_{\lambda-2} = E_{\lambda-2,n}\,\varphi^z_{\lambda-2} \tag{9.3}$$

are satisfied (here $\varphi^z_\nu = 0$ if $\nu < 0$, or $\nu > n+1$, or $\nu \neq n+1 \mod 2$).

These compatibility relations can be rewritten

$$\begin{aligned}(C_{\lambda-1,n})\quad \frac{\partial}{\partial z}\alpha_{\lambda-1} + \frac{\lambda\,(n-\lambda+3)}{4\,(\lambda-2)}\alpha_{\lambda-3}\,J\,z &= \frac{\partial}{\partial y}A_{\lambda-1} - \frac{\partial}{\partial x}B_{\lambda-1} - \\ &- \frac{\lambda\,(n-\lambda+3)}{4\,(\lambda-2)}\,C_{\lambda-3}\,z, \\ \lambda = (n+1) &\mod 2.\end{aligned}$$

By Lemma 9.2, this equation has a unique solution $\alpha_{\lambda-1} = \widehat{\alpha}_{\lambda-1}$, $\alpha_{\lambda-3} = \widetilde{\alpha}_{\lambda-3}$, and we can state the following lemma.

Lemma 9.3. *The system $E_{\lambda,n}$ has a unique solution $\widehat{\varphi}^z_\lambda$, $\widetilde{\varphi}^z_{\lambda-2}$, $\widehat{\alpha}_{\lambda-1}$, if one requires that $\widetilde{\varphi}^z_{\lambda-2}$ have the following form:*

$$\widetilde{\varphi}^z_{\lambda-2} = \frac{1}{\lambda-2}\,(C_{\lambda-3}\,z + \alpha_{\lambda-3}\,J\,z).$$

Then, $\alpha_{\lambda-3} = \widetilde{\alpha}_{\lambda-3}$, $(\widehat{\alpha}_{\lambda-1}, \widetilde{\alpha}_{\lambda-3})$ is the unique solution of $C_{\lambda-1,n}$. $\widehat{\varphi}^z_\lambda = E_{\lambda,n}\,\varphi^z_\lambda$, with $\alpha_{\lambda-1} = \widehat{\alpha}_{\lambda-1}$.

4. These two Lemmas 9.2 and 9.3 allow us to construct normal forms. For this, we make a combinatorial choice allowing us to solve *in a unique way* some of the equations $E_{\lambda,n}$, $E^3_{\mu,n}$. The other equations, which will not be satisfied, will lead to residual terms that are the coefficients of the formal normal form. We will illustrate this in the case of our normal form $\mathcal{N}$.

Our choice will be the following: for $\widehat{\alpha}_\nu$, we will take the value given by the equation $C_{\nu,n}$, except in the following cases: (i) $n = 0 \mod 2$, $\nu = 0$, (ii) $n = 1 \mod 2$, $\nu = 1$.

Once the α are chosen, we always take $\varphi_\lambda = E_{\lambda,n}\,\varphi_\lambda$ for this choice of $\alpha_{\lambda-1} = \widehat{\alpha}_{\lambda-1}$.

Finally, for φ^3_μ we always choose the solution of $E^3_{\mu,n}$.

In case (i), $n = 0 \mod 2$, α_0 will be the solution $\widetilde{\alpha}_0$ of $C_{2,n}$,

$$\frac{\partial}{\partial z}\alpha_2 + \frac{3\,n}{4}\alpha_0\,J\,z = \frac{\partial A_2}{\partial y} - \frac{\partial B_2}{\partial x} - \frac{3\,n}{4}C_0 z, \tag{9.4}$$

because there is no equation $C_{0,n}$ to determine α_0.

This choice for α_0 and the choice of $\widehat{\alpha}_2$ for α_2 imply that the systems $E_{1,n}$ and $E_{3,n}$ can be solved exactly. Therefore, the residual terms A_0^{res}, B_0^{res}, A_2^{res}, B_2^{res} are zero. Applying Lemma 9.2 again shows that the residual terms $A_1^{3\ \mathrm{res}}$, $B_1^{3\ \mathrm{res}}$ are also zero.

In case (ii), $n = 1 \mod 2$, $\nu = 1$, we do not use $C_{3,n}$ as previously to determine α_1, α_3 ($\alpha_1 = \widetilde{\alpha}_1$, $\alpha_3 = \widehat{\alpha}_3$), but to simplify as far as possible the terms in $\dfrac{\partial}{\partial w}$ in the normal form, we determine α_1 as follows:

We solve $E_{2,n}$ and $E_{1,n}^3$ as functions of α_1, A, B, D. This determines φ_0^z, φ_2^z, φ_1^3 as functions of α_1. This being done, we replace φ_2^z in $E_{3,n}^3$ to get the system

$$(\widetilde{E}_{3,n}^3) \quad T_z\varphi_3^3 - \frac{n+1}{4}\varphi_1^3 z^t J = D_2 + \left(\frac{J}{4} C_1\, z\right)^t + \frac{z^t}{4}\alpha_1.$$

By Lemma 9.2, this system has a unique solution $\varphi_3^3 = \overline{\varphi}_3^3$, $\varphi_1^3 = \overline{\varphi}_1^3$. We express that $\overline{\varphi}_1^3$ is given by $E_{1,n}^3$, $\overline{\varphi}_1^3 = E_{1,n}^3\varphi_1^3$, to get a relation which determines uniquely $\alpha_1 = \overline{\alpha}_1$ and shows that $A_1^{\mathrm{res}} = B_1^{\mathrm{res}} = 0$, $A_0^{3\ \mathrm{res}} = B_0^{3\ \mathrm{res}} = A_2^{3\ \mathrm{res}} = B_2^{3\,res} = 0$.

But, in general, we get that $A_3^{\mathrm{res}} \neq 0, B_3^{\mathrm{res}} \neq 0$.

5. Finally, let us note that the residual terms are given by the relations

$$C_{\lambda-1}^{\mathrm{res}} = T_z\widehat{\varphi}_\lambda^z - \frac{n-\lambda+3}{4}\widehat{\varphi}_{\lambda-2}^z\, z^t J - \widehat{\alpha}_{\lambda-1} J - C_{\lambda-1}. \tag{9.5}$$

Since $\widehat{\varphi}_{\lambda-2}^z = \dfrac{1}{\lambda-2}(C_{\lambda-3}\, z + J\, z\, \widehat{\alpha}_{\lambda-3})$, and since

$$C_{\lambda-1} = T_z\widehat{\varphi}_\lambda^z - \frac{n-\lambda+3}{4}\widetilde{\varphi}_{\lambda-2}^z\, z^t J - \widehat{\alpha}_{\lambda-1} J,$$

where

$$\widetilde{\varphi}_{\lambda-2}^z = \frac{1}{\lambda-2}(C_{\lambda-3}\, z + J\, z\, \widetilde{\alpha}_{\lambda-3}),$$

we have

$$\begin{gathered}
A_{\lambda-1}^{\mathrm{res}} = \frac{n-\lambda+3}{4} y\, (\widehat{\varphi}_{\lambda-2}^z - \widetilde{\varphi}_{\lambda-2}^z),\\
B_{\lambda-1}^{\mathrm{res}} = -\frac{n-\lambda+3}{4} x\, (\widehat{\varphi}_{\lambda-2}^z - \widetilde{\varphi}_{\lambda-2}^z),\\
A_{\lambda-1}^{\mathrm{res}} = \frac{n-\lambda+3}{4\,(\lambda-2)} y\, f_{\lambda-3}\, R, \quad B_{\lambda-1}^{\mathrm{res}} = -\frac{n-\lambda+3}{4\,(\lambda-2)} x\, f_{\lambda-3}\, R;\\
R = y\, e_1 - x\, e_2, \qquad f_{\lambda-3} = (\widehat{\alpha}_{\lambda-3} - \widetilde{\alpha}_{\lambda-3}).
\end{gathered}$$

For φ_μ^3, we take the solution $E_{\mu,n}^3\,\varphi_\mu^3$ of $E_{\mu,n}^3$ and it is easy to give the explicit expression of $A_{\mu-1}^{3\ \mathrm{res}}$, $B_{\mu-1}^{3\ \mathrm{res}}$. Finally, notice that it is also possible to give explicit formulas for A^{res}, B^{res} in terms of A and B.

9.2. Proof of Theorem 3.2. Assume that ∇ does exist. Let $(\omega^1, \omega^2, \omega^3)$ be a distinguished coframe field on a certain open $M' \subset M$.

$\nabla \omega^i = -\sum_{j=1}^3 \omega_j^i \otimes \omega^j, \quad i = 1, 2,$
$\nabla \omega^3 = 0$ (ω_j^i, standard Cartan's forms [9]).
$\frac{1}{2}\nabla g = \left[\nabla \omega^1 \odot \omega^1 + \nabla \omega^2 \odot \omega^2\right]\big|_{\Delta \times_M \Delta}$ ($\odot$ = symmetric tensor product).

$$\left[\nabla \omega^1 \odot \omega^1 + \nabla \omega^2 \odot \omega^2\right] = -\sum_{j=1}^{3} \left(\omega_j^1 \otimes (\omega^j \odot \omega^1) + \omega_j^2 \otimes (\omega^j \odot \omega^2)\right)$$

and

$$0 = \frac{1}{2}\nabla g = -\sum_{j=1}^{2} \left(\omega_j^1 \otimes (\omega^j \odot \omega^1) + \omega_j^2 \otimes (\omega^j \odot \omega^2)\right).$$

Hence,

$$\omega_1^1 = \omega_2^2 = \omega_2^1 + \omega_1^2 = 0. \tag{9.6}$$

Let (e_1, e_2, e_3) be the dual coframe of $(\omega^1, \omega^2, \omega^3)$;

$$\nabla e_1 = \omega_1^2 \otimes e_2, \ \nabla e_2 = \omega_2^1 \otimes e_1, \ \nabla e_3 = \omega_3^1 \otimes e_1 + \omega_3^2 \otimes e_2.$$

Otherwise, $e_3 = \nu|_{M'}$, therefore $\langle \nabla_X \nu, Y \rangle_g = \omega_3^1(X)\,\omega^1(Y) + \omega_3^2(X)\,\omega^2(Y)$.
II being symmetric, one has

$$\omega_3^1 \wedge \omega^1 + \omega_3^2 \wedge \omega^2 = 0 \text{ on } \Delta \times_M \Delta.$$

But, since $\nabla_\nu \nu = 0$, $\omega_3^1(e_3) = \omega_3^2(e_3) = 0$. This shows that

$$\omega_3^1 \wedge \omega^1 + \omega_3^2 \wedge \omega^2 = 0 \text{ on } TM' \times_{M'} TM'. \tag{9.7}$$

Let us compute ∇ in the coframe $(\omega^1, \omega^2, \omega^3)$.
The torsion T of ∇ is $T : TM \times_M TM \to TM$, with

$$\begin{cases} \omega^1 \circ T = d\omega^1 + \omega_2^1 \wedge \omega^2 + \omega_3^1 \wedge \omega^3, \\ \omega^2 \circ T = d\omega^2 + \omega_1^2 \wedge \omega^1 + \omega_3^2 \wedge \omega^3, \\ \omega^3 \circ T = d\omega^3. \end{cases} \tag{9.8}$$

Hence, since

$$T = e_3 \otimes d\omega^3, d\omega^1 + \omega_2^1 \wedge \omega^2 + \omega_3^1 \wedge \omega^3 = d\omega^2 + \omega_1^2 \wedge \omega^1 + \omega_3^2 \wedge \omega^3 = 0.$$

Set

$$\begin{cases} d\omega^i = \frac{1}{2}\Sigma A_{j,k}^i \omega^j \wedge \omega^k, & A_{j,k}^i + A_{k,j}^i = 0, \ i = 1, 2, \\ \omega_j^i = \Sigma B_{k,j}^i \omega^k. \end{cases} \tag{9.9}$$

Because $\omega_3^1(e_3) = \omega_3^2(e_3) = 0$, $B_{3,3}^1 = B_{3,3}^2 = 0$.

By (9.7), $B^1_{2,3} = B^2_{1,3}$. By (9.6), $B^1_{k,2} + B^2_{k,1} = 0$. Therefore, we have the equations

$$\frac{1}{2}\sum_{j,k=1}^{3} A^1_{j,k}\omega^j \wedge \omega^k + \sum_{k=1}^{3} B^1_{k,2}\omega^k \wedge \omega^2 + \sum_{k=1}^{2} B^1_{k,3}\omega^k \wedge \omega^3 = 0,$$

$$\frac{1}{2}\sum_{j,k=1}^{3} A^2_{j,k}\omega^j \wedge \omega^k + \sum_{k=1}^{3} B^2_{k,1}\omega^k \wedge \omega^1 + \sum_{k=1}^{2} B^2_{k,3}\omega^k \wedge \omega^3 = 0,$$

$$A^1_{1,2} + B^1_{1,2} = 0,\ A^1_{2,3} - B^1_{3,2} + B^1_{2,3} = 0,\ A^1_{3,1} - B^1_{1,3} = 0,$$

$$A^2_{1,2} - B^2_{2,1} = 0,\ A^2_{2,3} + B^2_{2,3} = 0,\ A^2_{3,1} - B^2_{1,3} + B^2_{3,1} = 0.$$

Hence, $B^1_{1,2} = -A^1_{1,2}$, $B^1_{1,3} = A^1_{3,1}$, $B^2_{2,1} = A^2_{1,2}$, $B^2_{2,3} = -A^2_{2,3}$;

$$A^2_{3,1} + B^2_{3,1} - B^2_{1,3} = A^2_{3,1} - B^1_{3,2} - B^1_{2,3} = A^1_{2,3} - B^1_{3,2} + B^1_{2,3} = 0.$$

Hence,

$$B^1_{3,2} = -B^2_{3,1} = \frac{1}{2}(A^1_{2,3} + A^2_{3,1}), \qquad B^1_{2,3} = B^2_{1,3} = \frac{1}{2}(A^2_{3,1} - A^1_{2,3}),$$
$$B^1_{2,2} = -B^2_{2,1} = -A^2_{1,2},\ B^2_{1,1} = -B^1_{1,2} = A^1_{1,2}.$$

This determines the ω^i_j and shows that ∇ is unique. In the same way, these formulas determine uniquely a connection with the required properties. $\square$

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