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KAM-STABLE HAMILTONIANS

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ABSTRACT. We present a simple proof of Rüssmann's theorem on invariant tori of analytic perturbations of analytic integrable Hamiltonian systems of the form $dp/dt = 0$, $dq/dt = \partial f(p)/\partial p$, where (p, q) are the action-angle variables. Rüssmann's theorem asserts that if the image of the mapping $p \mapsto \partial f(p)/\partial p$ does not lie in any linear hyperplane passing through the origin, then any sufficiently small Hamiltonian perturbation of this integrable system possesses many invariant tori close to the unperturbed tori $\{p = \text{const}\}$. The main idea of our proof is that we embed the perturbed Hamiltonian in a family of Hamiltonians depending on an external multidimensional parameter. We also show that the Rüssmann condition is necessary (i.e., not only sufficient) for the existence of perturbed tori and give analogs of Rüssmann's theorem for exact symplectic diffeomorphisms, reversible flows, and reversible diffeomorphisms.

1. INTRODUCTION AND THE MAIN RESULT

This paper contributes to the classical problem of invariant tori in nearly integrable Hamiltonian systems. Such tori are the subject of the Kolmogorov–Arnold–Moser (KAM) theory (see, e.g., [1] for a review and extensive bibliography). Consider a Hamiltonian differential equation

$$\frac{dp}{dt} = -\frac{\partial H}{\partial q}, \quad \frac{dq}{dt} = \frac{\partial H}{\partial p}, \quad H = H(p, q), \tag{1}$$

where $p = (p_1, \dots, p_n)$ are action variables ranging in some open domain in $\mathbb{R}^n$ and $q = (q_1, \dots, q_n)$ are the conjugate angle variables varying over the standard n-torus $\mathbb{T}^n = (\mathbb{R}/2\pi\mathbb{Z})^n$. If system (1) is integrable, i.e., its Hamilton function $H = f(p)$ is independent of the angle variables q, then the phase space is foliated into invariant n-tori $\{p = \text{const}\}$, the flow on these tori being linear with frequency vectors $\omega(p) = \partial f(p)/\partial p$ (quasiperiodic for vectors $\omega(p)$ with incommensurable components). The question arises of

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whether nonintegrable Hamiltonian systems with Hamilton functions close to $f(p)$ still admit many invariant tori filled up with quasiperiodic motions. To be more precise, let $S \subset \mathbb{R}^n$ be the closure of a bounded connected open domain and $\mathcal{O}(S) \subset \mathbb{C}^n$ be a complex neighborhood of S. One can assume for simplicity that S is diffeomorphic to a closed n-dimensional ball.

Definition. The holomorphic function $f : \mathcal{O}(S) \to \mathbb{C}$ (real-valued for real values of the argument) is said to be *KAM-stable* if there exist a smaller complex neighborhood $\mathcal{O}_1(S) \subset \mathbb{C}^n$ of S whose closure lies in $\mathcal{O}(S)$ and a complex neighborhood $\mathcal{O}(\mathbb{T}^n) \subset (\mathbb{C}/2\pi\mathbb{Z})^n$ of the standard n-torus $\mathbb{T}^n$ with the following property: for any $\delta > 0$ there is $\varepsilon > 0$ such that for every holomorphic function $g : \mathcal{O}_1(S) \times \mathcal{O}(\mathbb{T}^n) \to \mathbb{C}$ real-valued for real values of the arguments and subject to the inequality $|g| < \varepsilon$ on $\mathcal{O}_1(S) \times \mathcal{O}(\mathbb{T}^n)$, we have the following. System (1) with $H(p,q) = f(p) + g(p,q)$, where $p \in \mathcal{O}_1(S) \bigcap \mathbb{R}^n$ and $q \in \mathbb{T}^n$, possesses analytic invariant n-tori of the form $p = p(q)$ carrying quasiperiodic motions, and the measure of the union of these tori is no less than

$$(1-\delta)\mathrm{meas}_{2n}(S \times \mathbb{T}^n) = (1-\delta)(2\pi)^n \mathrm{meas}_n(S)$$

(here meas_N denotes the N-dimensional Lebesgue measure).

The following conditions guaranteeing the KAM-stability of the unperturbed Hamilton function $f(p)$ are widely used [1]–[8]: the nondegeneracy in the sense of Kolmogorov:

$$\det \frac{\partial \omega}{\partial p} = \det \frac{\partial^2 f}{\partial p^2} \neq 0 \quad \text{in } S, \tag{2}$$

the isoenergetic nondegeneracy:

$$\det \begin{pmatrix} \partial\omega/\partial p & \omega \\ \omega & 0 \end{pmatrix} = \det \begin{pmatrix} \partial^2 f/\partial p^2 & \partial f/\partial p \\ \partial f/\partial p & 0 \end{pmatrix} \neq 0 \quad \text{in } S, \tag{3}$$

the nondegeneracy in the sense of Bruno:

$$\mathrm{rank}\begin{pmatrix} \omega & \dfrac{\partial \omega}{\partial p} \end{pmatrix} = \mathrm{rank}\begin{pmatrix} \dfrac{\partial f}{\partial p} & \dfrac{\partial^2 f}{\partial p^2} \end{pmatrix} \equiv n \quad \text{in } S. \tag{4}$$

The Bruno condition (4) is weaker than the nondegeneracy of the Hessian (2) or the isoenergetic nondegeneracy (3).

However, if $\mathrm{rank}(\omega, \partial\omega/\partial p) = n$ at some point $p^0 \in S$, then at least one of the equalities $\mathrm{rank}(\partial\omega/\partial p) = n$ and $\mathrm{rank}\begin{pmatrix} \partial\omega/\partial p & \omega \\ \omega & 0 \end{pmatrix} = n+1$ exists at this point.

In the present paper, we prove the following result concerning the necessary and sufficient condition for the KAM-stability.

Theorem 1. *The unperturbed Hamilton function $f(p)$ is KAM-stable if and only if the image of the unperturbed frequency map $\omega = \partial f/\partial p : S \rightarrow \mathbb{R}^n$ does not lie in any linear hyperplane passing through the origin.*

The necessity of this condition is a rather simple fact whose proof we postpone to Sec. 3. The real problem is to prove that the condition of Theorem 1 is *sufficient* for the KAM-stability. In the sequel, we will call this hard half of Theorem 1 the *Rüssmann statement* because it was first announced by Rüssmann [9]–[11]. Similarly, we will say that the function $f(p)$ defined in $\mathcal{O}(S)$ is *nondegenerate in the sense of Rüssmann* if the image of the map $S \rightarrow \mathbb{R}^n$, $p \mapsto \partial f/\partial p$ does not lie in any linear hyperplane passing through the origin.

Remark. Rüssmann himself used the term "twist-Hamiltonians" instead of "KAM-stable Hamiltonians" [11]. In fact, paper [11] is devoted to a much more general context of *lower-dimensional* invariant tori (whose dimension is less than the number of degrees of freedom).

The Rüssmann statement was proved only very recently by Xiu *et al.* [12] and Cheng *et al.* [13]. These two proofs are independent but very similar. The main purpose of the present paper is to give an alternative proof of the Rüssmann statement.

The main difficulty of the problem is that the nondegeneracy in the sense of Rüssmann is a *very weak condition.* The image of the frequency map for an integrable Hamilton function nondegenerate in the sense of Rüssmann can be a variety in $\mathbb{R}^n$ of any positive dimension. It can even be a curve.

Example. Let k be an integer and $1 \le k \le n$. We denote by $u = u(p_1, \dots, p_k)$ the solution of the equation

$$\sum_{j=1}^{k} j u^{j-1} p_j = u,$$

which is defined and analytic in $p_1, \dots, p_k$ near the point $p_1 = \cdots = p_k = 0$ and vanishes at this point. The local existence and uniqueness of the function u are ensured by the implicit function theorem. If $k = 1$, then $u = p_1$, and if $k = 2$, then $u = p_1(1 - 2p_2)^{-1}$. It is easy to verify that

$$\frac{\partial u}{\partial p_j} = j u^{j-1} \frac{\partial u}{\partial p_1} = \frac{\partial (u^j)}{\partial p_1}, \quad 1 \le j \le k, \tag{5}$$

and that $u = 0$ for $p_1 = 0$. Now consider the Hamiltonian

$$f(p) = \int_0^{p_1} u(x, p_2, \dots, p_k)\, dx + \frac{1}{2} \sum_{j=k+1}^{n} p_j^2. \tag{6}$$

It follows from Eq. (5) that the frequencies corresponding to this Hamiltonian are

$$\omega_j(p) = \frac{\partial f(p)}{\partial p_j} = \left[\begin{array}{ll} (u(p_1, \dots, p_k))^j & \text{for } 1 \le j \le k, \\ p_j & \text{for } k+1 \le j \le n. \end{array} \right.$$

Thus, the integrable Hamiltonian (6) is nondegenerate in the sense of Rüssmann and the image of its frequency map is of dimension $n - k + 1$. A similar example for $k = n = 3$ can be found in [8].

Let the image of the frequency map for the integrable Hamiltonian $f(p)$ nondegenerate in the sense of Rüssmann be, say, a curve in $\mathbb{R}^n$. An arbitrarily small perturbation of this Hamiltonian of the form $f(p)+g(p)$ will distort this curve (note that we consider a perturbation leaving the Hamiltonian integrable). The new curve and the original one can be disjoint. Moreover, the situation is possible where there are no two points p^1 and p^2 such that the vectors $\partial f(p^1)/\partial p$ and $\partial[f(p^2)+g(p^2)]/\partial p$ are proportional. For an appropriate example in dimension $n = 3$, see [8]. The set of the *unperturbed* frequencies and that of the *perturbed* ones in Theorem 1 can therefore have nothing in common, irrespective of how small the perturbation is. One can guarantee that a perturbed system will admit many invariant tori and estimate the Lebesgue measure of the union of these tori, but no perturbed torus can in general be assigned to a particular unperturbed one. Unfortunately, Rüssmann himself in his landmark notes [9]–[11] spoke about the "survival" of unperturbed tori. This term is in fact quite unsuitable. In the context of his statement, the unperturbed tori do *not*, in general, survive a perturbation. They are broken up, and *new* tori appear nearby.

This circumstance prevents us from proving Theorem 1 by the classical KAM technique [2]–[5], where one fixes beforehand the frequency vector satisfying some Diophantine conditions and then looks for a perturbed torus with this frequency vector. However, the KAM iterative procedure can also be forced when it is impossible to choose the perturbed frequencies *a priori* [14], [15], and it is in this way that Xiu *et al.* [12] and Cheng *et al.* [13] proved the Rüssmann statement. Their proofs seem to be very cumbersome.

In the present paper, we introduce a different method of studying small perturbations of highly degenerate Hamiltonians. The essence of our approach is that we embed the perturbed Hamiltonian in a family of Hamiltonians depending on an external multidimensional parameter μ and thus achieve *full control of the frequencies*. The invariant tori of the original Hamiltonian system (corresponding to $\mu = 0$, say) can then be picked out using the fact that the invariant tori of the whole family constitute a Whitney-smooth foliation.

2. Proof of the Rüssmann Statement

Our proof is in fact based on the so-called parametric KAM theory developed by Broer, Huitema, and Takens [16], [17]. This theory deals with invariant tori of Hamiltonian systems depending on external parameters. The results of Broer, Huitema, and Takens pertain to n-tori as well as lower-dimensional tori (of dimensions $l < n$). The parametric KAM theorem for n-tori will be sufficient for our purpose.

In the sequel, the point $a \in \mathbb{R}^N$ is said to be of type (γ, ν) (γ and ν being some positive constants) if, for any integer vector $s \in \mathbb{Z}^N \backslash \{0\}$, the inequality

$$|\langle s, a\rangle| \geq \gamma(|s_1| + \ldots + |s_N|)^{-\nu}$$

holds. Here and henceforth, $\langle\cdot,\cdot\rangle$ denotes the standard inner product of vectors:

$$\langle s, a\rangle = s_1 a_1 + \ldots + s_N a_N.$$

Let the analytic Hamiltonian H in Eq. (1) depend analytically on the external multidimensional parameter $\mu \in \mathbb{R}^m$:

$$H = H(p, q, \mu) = F(p, \mu) + G(p, q, \mu),$$

the parameter μ varying in some neighborhood of the set $B \subset \mathbb{R}^m$ diffeomorphic to a closed m-dimensional ball. As before, the variables p and q are assumed to range in some real neighborhood of $S \subset \mathbb{R}^n$ and over $\mathbb{T}^n$, respectively. Moreover, we suppose that the functions F and G can be holomorphically extended to some fixed complex neighborhoods of their real definition domains.

Theorem 2 ([16], [17]). *Fix the function $F(p, \mu)$ and the number $\nu > n - 1$ and assume the frequency map*

$$\Omega : S \times B \to \mathbb{R}^n, \quad \Omega(p, \mu) = \partial F(p, \mu)/\partial p \tag{7}$$

to be submersive everywhere, i.e., its Jacobi $n \times (n + m)$ matrix to have rank n. Then, for any $\gamma > 0$ and any C^∞-neighborhood $\mathcal{U}$ of the identity mapping

$$I : S \times \mathbb{T}^n \times B \to \mathbb{R}^n \times \mathbb{T}^n \times \mathbb{R}^m, \quad I(p, q, \mu) \equiv (p, q, \mu), \tag{8}$$

there is a number $\varepsilon > 0$ such that whenever $|G(p, q, \mu)| < \varepsilon$ throughout the complex definition domain of the function G, there exists a C^∞-mapping

$$\Psi : S \times \mathbb{T}^n \times B \to \mathbb{R}^n \times \mathbb{T}^n \times \mathbb{R}^m$$

of the form

$$\Psi(p, q, \mu) = \big(P(p, q, \mu), Q(p, q, \mu), \Lambda(p, \mu)\big)$$

which lies in U, is analytic in q, and possesses the following property: for any pair $(p_0, \mu_0) \in S \times B$ for which the vector $\Omega(p_0, \mu_0) \in \mathbb{R}^n$ is of type (γ, ν), the set

$$\{(P(p_0, \varphi, \mu_0), Q(p_0, \varphi, \mu_0)) \mid \varphi \in \mathbb{T}^n\} \subset \mathbb{R}^n \times \mathbb{T}^n$$

is an invariant analytic n-torus of system (1) with the Hamiltonian

$$H = H(p, q, \Lambda(p_0, \mu_0)).$$

The motion on this torus is quasiperiodic with frequency vector $\Omega(p_0, \mu_0)$.

Theorem 2 is a direct generalization of an earlier result by Pöschel [5] who considered Hamiltonians $H = H(p, q) = f(p) + g(p, q)$ without an external parameter μ but with nondegenerate Hessian $\partial^2 f/\partial p^2$. When the parameter μ is absent, the submersivity condition on the frequency map $p \mapsto \omega(p) = \partial f(p)/\partial p$ is just the nondegeneracy in the sense of Kolmogorov (2).

To be brief, Theorem 2 asserts that if the frequency map (7) is submersive, then the unperturbed invariant tori with Diophantine frequencies do not disintegrate under small perturbations but just undergo a slight deformation (with a small shift along the parameter μ). Moreover, the mapping which projects unperturbed tori onto the corresponding perturbed ones can be extended to a C^∞-mapping defined everywhere rather than only on tori with Diophantine frequencies. The last property is referred to as the *smoothness* of the Cantor set of the perturbed tori *in the sense of Whitney* [5]. The first results concerning the Whitney smoothness of the Cantor families of invariant tori in the KAM theory (without external parameters) were due to Lazutkin [18]–[21] and Svanidze [22]. For a very recent exposition, see Lazutkin's book [23].

We also need some facts from the theory of Diophantine approximations on submanifolds of $\mathbb{R}^N$ (approximations of *dependent quantities* in Sprindžuk's terminology [24]). The literature devoted to Diophantine approximations on submanifolds is very extensive [24]–[32], and these approximations have already found important applications in the theory of dynamical systems. First, approximations of dependent quantities are necessary to construct invariant tori carrying quasiperiodic motions when the dimension of the tori is too large or when there are some degeneracies which pose obstacles to the control of the frequencies or their ratios [12], [13], [33]–[35]. The second class of applications pertains to the averaging theory [36]–[42]. In the latter, Diophantine approximations on submanifolds are used when $N_s < N_f - 1$ [37], where N_s is the number of slow variables and N_f is the number of fast angular variables; see the discussion in [38] (the case $N_s \geq N_f - 1$ has been examined in, e.g., [39]–[42]). We will use the following theorem.

Theorem 3 **([12], [32], [37]).** *Let the mapping $\omega : S \to \mathbb{R}^N$ of the class C^r possess the following property: for every point $p \in S$ the collection of $(n+r)!/n!r!$ vectors*

$$\frac{\partial^{|\alpha|}\omega(p)}{\partial p^{\alpha}}, \quad \alpha \in \mathbb{Z}_+^n, \ \ 0 \le |\alpha| = \alpha_1 + \ldots + \alpha_n \le r, \tag{9}$$

spans the linear space $\mathbb{R}^N$. Then for any fixed value of $\nu > Nr - 1$ the n-dimensional Lebesgue measure $M_{\gamma,\nu}(\omega^1)$ of the set of points $p \in S$, for which the vector $\omega^1(p)$ is of the type (γ, ν), tends to $\mathrm{meas}_n(S)$ as $\gamma \to 0$ uniformly with respect to all the C^r-functions $\omega^1 : S \to \mathbb{R}^N$ in some C^r-neighborhood of the mapping ω.

Finally, we will use the following very simple proposition.

Lemma 1 **([9], [10]).** *If the image of the analytic mapping $\omega : S \to \mathbb{R}^N$ does not lie in any linear hyperplane passing through the origin, then there exists a number $r \in \mathbb{N} = \mathbb{Z}_+ \backslash \{0\}$ such that for any point $p \in S$ the collection of vectors* (9) *spans $\mathbb{R}^N$.*

Proof of the Rüssmann statement. Consider system (1) with Hamiltonian $H = H(p,q) = f(p) + g(p,q)$, where the unperturbed Hamilton function $f(p)$ is nondegenerate in the sense of Rüssmann and $g(p,q)$ is a small perturbation. Let $\omega = \partial f/\partial p$ be the unperturbed frequency map. According to Lemma 1, there exists a number $r \in \mathbb{N}$ such that for any point $p \in S$ the collection of vectors (9) spans $\mathbb{R}^n$. Embed Hamiltonian H in the analytic family

$$F(p,\mu) + g(p,q), \quad \mu \in \mathbb{R}^m, \quad F(p,0) \equiv f(p) \tag{10}$$

such that the *extended* frequency map Ω defined by Eq. (7) is submersive. For instance, we can take $m = n$ and $F(p,\mu) = f(p) + \mu_1 p_1 + \ldots + \mu_n p_n$. Note that $\Omega(p,0) \equiv \omega(p)$. To apply Theorem 2 to family (10), we fix $\nu > nr - 1$ and consider a small number $\gamma > 0$ and a narrow C^∞-neighborhood $\mathcal{U}$ of mapping (8). In the sequel, the closeness of functions will be understood as the C^∞-closeness. Let $|g| < \varepsilon(\gamma, \mathcal{U})$. Since $\Lambda(p,\mu)$ is close to μ, we can solve the equation $\Lambda(p,\mu) = 0$ with respect to μ and obtain $\mu = \lambda(p)$, where the C^∞-function λ is close to 0. For any point $p_0 \in S$, for which the vector $\Omega(p_0, \lambda(p_0))$ is of the type (γ, ν), the set

$$\{ (P(p_0, \varphi, \lambda(p_0)), Q(p_0, \varphi, \lambda(p_0))) \mid \varphi \in \mathbb{T}^n \} \subset \mathbb{R}^n \times \mathbb{T}^n, \tag{11}$$

which is close to $\{p = p_0\}$, is an invariant analytic n-torus of system (1) with the original Hamiltonian $H = f(p) + g(p,q)$. The motion on this torus is quasiperiodic with the frequency vector $\Omega(p_0, \lambda(p_0))$. On the other hand, $\Omega(p, \lambda(p))$ is close to $\omega(p)$. Consequently, according to Theorem 3, the n-dimensional Lebesgue measure of the set of points $p_0 \in S$, for which the vector $\Omega(p_0, \lambda(p_0))$ is of the type (γ, ν), is arbitrarily close to $\mathrm{meas}_n(S)$ for sufficiently small γ. The $2n$-dimensional measure of the union of all the tori (11) is therefore arbitrarily close to $(2\pi)^n \mathrm{meas}_n(S)$ for sufficiently small γ and narrow $\mathcal{U}$. The proof is completed. □

A similar construction was exploited by Xia [35] to prove the existence of invariant n-tori in the volume preserving diffeomorphism A of $\mathbb{R} \times \mathbb{T}^n$. Xia considers an *augmented* diffeomorphism A_{aug} of $\mathbb{R}^n \times \mathbb{T}^n$, the original mapping A being realized as the restriction of A_{aug} to a certain $(n+1)$-dimensional invariant submanifold J. The existence of a Cantor family of invariant n-tori of A_{aug} can be proved with the use of ordinary KAM schemes. This family is C^∞-smooth in the sense of Whitney. Then the author verifies that many of these tori lie in J. For a much simplier version of this approach, see the proof [16], [43] of the KAM theorem for isoenergetically nondegenerate unperturbed Hamiltonians from the theorem for unperturbed Hamiltonians nondegenerate in the sense of Kolmogorov.

The difference $\mathrm{meas}_n(S) - M_{\gamma,\nu}(\omega^1)$ in Theorem 3 is of the order of $\gamma^{1/r}$ for fixed ν [12], [32], [37]. On the other hand, ε in Theorem 2 is of the order of γ^2 [16], [17]. Thus, in the context of Theorem 1, the Lebesgue measure of the resonant zone (the complement of the union of the invariant tori) is of the order of $\varepsilon^{1/2r}$, where ε is the perturbation magnitude and r is the number from Lemma 1. Recall that if the unperturbed Hamilton function satisfies the usual nondegeneracy conditions (2) or (3), then the measure of the resonant zone is of the order of $\varepsilon^{1/2}$ [5], [6].

Theorem 2 can be carried over to Hamiltonians of class C^∞ or even class C^R for sufficiently large $R \in \mathbb{N}$ [16], [17]. At the same time, the analyticity of the mapping ω in Lemma 1 is essential [12]. The Rüssmann statement can therefore be generalized to the smooth case as follows: the unperturbed Hamiltonian $f : S \to \mathbb{R}$ of class C^R ($R \in \mathbb{N} \bigcup \{\infty\}$ is sufficiently large) is KAM-stable if there exists a number $r < r_0(R) \le R$ such that for any point $p \in S$ the collection of vectors (9) with $\omega = \partial f / \partial p$ spans $\mathbb{R}^n$. Of course, the perturbations $g(p, q)$ are also assumed to be of class C^R. According to [12], one can take $r_0(R) = E(R/n) + 1$ for finite R, where E denotes the integral part (obviously, $r_0(\infty) = \infty$).

3. Completion of the proof of the main theorem

In this section we show that nondegeneracy in the sense of Rüssmann is not only a sufficient condition for the KAM-stability but also a necessary one. Although this fact is very simple, it seems to have never been stated in the literature.

Let the unperturbed Hamiltonian $f(p)$ be degenerate in the sense of

Rüssmann. This means that there exists a vector $c \in \mathbb{R}^n \backslash \{0\}$ such that

$$\langle c, \partial f(p)/\partial p\rangle \equiv 0 \quad \text{in } S.$$

Let $f^1(p) = f(K^{-1}p)$, where $K \in GL(n, \mathbb{R})$. Then

$$\langle c^1, \partial f^1(p)/\partial p\rangle \equiv 0 \quad \text{in } S,$$

where $c^1 = Kc$. One can choose K arbitrarily close to the identity matrix in such a way that c^1 will be proportional to an integer vector:

$$\langle s, \partial f^1(p)/\partial p\rangle \equiv 0, \quad s \in \mathbb{Z}^n \backslash \{0\}.$$

Now set

$$H(p, q) = f(K^{-1}p) + \varepsilon \cos\langle s, q\rangle, \quad \varepsilon \neq 0. \tag{12}$$

Then

$$\frac{d\langle s, q\rangle}{dt} = \langle s, \partial f^1(p)/\partial p\rangle = 0,$$

$$\frac{dp_j}{dt} = \varepsilon s_j \sin\langle s, q\rangle, \quad 1 \leq j \leq n.$$

Thus, the derivatives $dp_j/dt = C_j = \text{const}$ are time independent, and $C_j \neq 0$ provided that $s_j \neq 0$ and

$$\sin\langle s, q^{\text{initial}}\rangle \neq 0.$$

We can infer that the system with the Hamilton function H defined by Eq. (12) possesses *no* invariant tori. On the other hand, the perturbation $g(p, q) = H(p, q) - f(p)$ can be chosen arbitrarily small in the real analytic topology. The unperturbed integrable Hamiltonian $f(p)$ is therefore KAM-unstable, which completes the proof of Theorem 1.

4. Some generalizations

The concept of the KAM-stability and Theorem 1 can be carried over *mutatis mutandis* to exact symplectic diffeomorphisms, reversible vector fields, and reversible diffeomorphisms. Recall that the diffeomorphism A of $\mathcal{O}_{\text{real}}(S) \times \mathbb{T}^n$ is said to be *exact symplectic* with respect to the exact symplectic structure $dp \wedge dq = d(p\,dq)$ if the 1-form $A^*(p\,dq) - p\,dq$ is exact ($\mathcal{O}_{\text{real}}(S)$ being some real neighborhood of $S \subset \mathbb{R}^n$). Here, as before, $p \in \mathcal{O}_{\text{real}}(S)$ and $q \in \mathbb{T}^n$. Recall also that the vector field

$$V = X(p, q)\partial/\partial p + Y(p, q)\partial/\partial q$$

and the ordinary differential equation determined by this vector field are said to be *reversible* with respect to some involution Σ of the phase space

$\mathcal{O}_{\text{real}}(S) \times \mathbb{T}^n$ ($\Sigma^2 = identity$) if Σ transforms the field V into the opposite field $-V$. In the sequel, we will consider only the special involution

$$\Sigma : (p, q) \mapsto (p, -q), \tag{13}$$

in which case V is reversible with respect to Σ if and only if X is odd in q and Y is even in q. Analogously, the diffeomorphism A is said to be *reversible* with respect to the involution Σ if Σ conjugates A with A^{-1}, i.e., $A\Sigma A = \Sigma$. The reader is referred to [33], [44]–[53] for a survey of the main properties of reversible dynamical systems, examples, and physical applications.

To carry over Theorem 1 to exact symplectic mappings, reversible vector fields, and reversible mappings, one needs the corresponding analogs of Theorem 2. The analog for reversible vector fields can be found in [16], [53], and the theories for exact symplectic mappings and reversible mappings correspond to those for Hamiltonian vector fields and reversible fields. The Whitney smoothness of the Cantor families of invariant tori in reversible systems (without external parameters) was first proved by Pöschel [5].

We will confine our consideration to the formulation of the results and the verification that the corresponding conditions for the KAM-stability are indeed necessary. Their sufficiency can be proved just as in the case of Theorem 1. The necessity proofs are also similar to the proof given in Sec. 3. However, they differ in some details, which are worthwhile to point out. All the integrable systems and their perturbations are assumed to be analytic.

Theorem 4. *The integrable exact symplectic diffeomorphism*

$$(p, q) \mapsto (p, q + \partial f(p)/\partial p)$$

is KAM-stable if and only if the image of the frequency map $\omega = \partial f/\partial p : S \to \mathbb{R}^n$ *does not lie in any affine hyperplane in* $\mathbb{R}^n$.

Theorem 5. *The integrable reversible differential equation*

$$\frac{dp}{dt} = 0, \quad \frac{dq}{dt} = \omega(p)$$

is KAM-stable if and only if the image of the mapping $\omega : S \to \mathbb{R}^n$ *does not lie in any linear hyperplane passing through the origin.*

Theorem 6. *The integrable reversible diffeomorphism*

$$(p, q) \mapsto (p, q + \omega(p))$$

is KAM-stable if and only if the image of the mapping $\omega : S \to \mathbb{R}^n$ *does not lie in any affine hyperplane in* $\mathbb{R}^n$.

We will prove that the KAM-stability condition of Theorem 4 is necessary. Suppose that there exist a vector $c \in \mathbb{R}^n \backslash \{0\}$ and a number $c_0 \in \mathbb{R}$ such that

$$\langle c, \partial f(p)/\partial p\rangle \equiv c_0 \quad \text{in } S.$$

Let $f^1(p) = f(K^{-1}p)$, where $K \in GL(n, \mathbb{R})$. Then

$$\langle c^1, \partial f^1(p)/\partial p\rangle \equiv c_0 \quad \text{in } S,$$

where $c^1 = Kc$. One can choose K arbitrarily close to the identity matrix in such a way that the pair (c^1, c_0) will be proportional to $(s, 2\pi s_0)$, where s is an integer vector and s_0 is an integer:

$$\langle s, \partial f^1(p)/\partial p\rangle \equiv 2\pi s_0, \quad s \in \mathbb{Z}^n \backslash \{0\}, \; s_0 \in \mathbb{Z}. \tag{14}$$

Consider the mapping

$$(p, q) \mapsto (p', q') = (p + \varepsilon s \sin\langle s, q\rangle, q + \partial f^1(p)/\partial p), \quad \varepsilon \neq 0. \tag{15}$$

It is easy to verify, using Eq. (14), that this mapping is exact symplectic; the crucial fact is that

$$\sum_{i=1}^{n} s_i d\left(q_i + \frac{\partial f^1(p)}{\partial p_i}\right) = d(\langle s, q\rangle + 2\pi s_0) = d\langle s, q\rangle.$$

Then, again using Eq. (14), we observe that

$$\langle s, q'\rangle = \langle s, q + \partial f^1(p)/\partial p\rangle = \langle s, q\rangle + 2\pi s_0 = \langle s, q\rangle.$$

Thus we can infer by analogy with Sec. 3 that mapping (15) possesses no invariant tori.

Now we will prove that the KAM-stability condition of Theorem 5 is necessary. Suppose that there exists a vector $c \in \mathbb{R}^n \backslash \{0\}$ such that

$$\langle c, \omega(p)\rangle \equiv 0 \quad \text{in } S.$$

Let $\omega^1(p) = K^{-1}\omega(p)$ where $K \in GL(n, \mathbb{R})$. Then

$$\langle c^1, \omega^1(p)\rangle \equiv 0 \quad \text{in } S,$$

where $c^1 = K^T c$ (K^T being the transpose of K). One can choose K arbitrarily close to the identity matrix in such a way that c^1 will be proportional to an integer vector:

$$\langle s, \omega^1(p)\rangle \equiv 0, \quad s \in \mathbb{Z}^n \backslash \{0\}.$$

Consider the differential equation

$$\frac{dp}{dt} = \eta \sin\langle s, q\rangle, \quad \frac{dq}{dt} = K^{-1}\omega(p), \quad \eta \in \mathbb{R}^n \backslash \{0\} \tag{16}$$

reversible with respect to involution (13). Then

$$\frac{d\langle s,q\rangle}{dt} = \langle s, \omega^1(p)\rangle = 0,$$

and, therefore, Eq. (16) possesses no invariant tori.

Finally, we will prove that the KAM-stability condition of Theorem 6 is necessary. Suppose that there exist a vector $c \in \mathbb{R}^n \setminus \{0\}$ and a number $c_0 \in \mathbb{R}$ such that

$$\langle c, \omega(p)\rangle \equiv c_0 \quad \text{in } S.$$

Let $\omega^1(p) = K^{-1}\omega(p)$, where $K \in GL(n, \mathbb{R})$. Then

$$\langle c^1, \omega^1(p)\rangle \equiv c_0 \quad \text{in } S,$$

where $c^1 = K^T c$. One can choose K arbitrarily close to the identity matrix in such a way that the pair (c^1, c_0) will be proportional to $(s, 2\pi s_0)$, where s is an integer vector and s_0 is an integer:

$$\langle s, \omega^1(p)\rangle \equiv 2\pi s_0, \quad s \in \mathbb{Z}^n \setminus \{0\}, \ s_0 \in \mathbb{Z}. \tag{17}$$

Consider the mapping

$$(p,q) \mapsto (p', q') = (p + 2\beta(q), q + K^{-1}\omega[p + \beta(q)]), \tag{18}$$

where

$$\beta(q) = \eta \sin\langle s, q\rangle, \quad \eta \in \mathbb{R}^n \setminus \{0\}.$$

It is easy to verify, using Eq. (17), that this mapping is reversible with respect to involution (13); the crucial facts are that β is odd and $\beta(q + \omega^1(p)) \equiv \beta(q)$. Then, again using Eq. (17), we observe that

$$\langle s, q'\rangle = \langle s, q\rangle,$$

whence $\beta(q') = \beta(q)$ and, therefore, mapping (18) possesses no invariant tori.

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Note added in proof. After this paper went to press, two important articles came to my attention. The Whitney smoothness of the Cantor families of invariant tori in nearly integrable Hamiltonian systems was also obtained by L. Chierchia and G. Gallavotti [Smooth prime integrals for quasi-integrable Hamiltonian systems. *Nuovo Cimento B* **67** (1982), No. 2, 277–295]. Interesting topological conditions for the existence of invariant tori have been found by P.I. Plotnikov [The Morse theory for quasiperiodic solutions of Hamiltonian systems. (Russian) *Sibirsk. Mat. Zh.* **35** (1994), No. 3, 657–673]. To be more precise, Plotnikov constructs a one-to-one correspondence between invariant tori (with a fixed Diophantine frequency vector) of system (1) with Hamiltonian $H(p,q) = f(p) + g(p,q)$ and critical points of a certain analytic function defined in S (in our notations). His results imply, in particular, the existence of invariant tori in some cases where the unperturbed Hamilton function $f(p)$ does not satisfy the Bruno condition (4).

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