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EXACT BOUNDARY ZERO CONTROLLABILITY OF THREE-DIMENSIONAL NAVIER–STOKES EQUATIONS

A. V. FURSIKOV

ABSTRACT. In a bounded three-dimensional domain Ω a solenoidal initial vector field $v_0(x) \in H^3(\Omega)$ is given. We construct a vector field $z(t,x)$ defined on the lateral surface $[0,T] \times \partial\Omega$ of the cylinder $[0,T] \times \Omega$ which possesses the following property: the solution $v(t,x)$ of the boundary value problem for the Navier–Stokes equation with the initial value $v_0(x)$ and the boundary Dirichlet condition $z(t,x)$ satisfies the relation $v(T,x) \equiv 0$ at the instant T. Moreover,

$$\|v(t,\cdot)\|_{H^3(\Omega)} \leq c\exp\bigl(-k/(T-t)^2\bigr) \quad \text{as} \quad t \to T,$$

where $c > 0$, $k > 0$ are certain constants.

We investigate an exact controllability problem for three-dimensional Navier– Stokes equations. Namely, for a given initial value $v_0(x)$ defined in the bounded domain $\Omega \subset R^3$, we seek a boundary Dirichlet condition $z(t,x), (t,x) \in [0,T] \times \partial\Omega$ such that the solution $v(t,x)$ of the boundary value problem for the Navier–Stokes equations mentioned above is equal to zero at the instant T. Moreover, the constructed control function $z(t,x)$ provides $v(t,x)$ with the high rate of decaying as $t \to T$:

$$\|v(t,\cdot)\|_{H^3(\Omega)} \leq c\exp\bigl(-k/(T-t)^2\bigr) \quad \text{as} \quad t \to T$$

with suitable constants $c > 0$, $k > 0$.

To solve the problem mentioned above, we use the method suggested in the papers of Fursikov and Imanuvilov [1], [2], where exact controllability problems were solved in the cases of the Burgers equation and the two-dimensional Navier–Stokes system. This method is based on the use of the optimality system of a certain extremal problem for the equation

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being investigated. As in the two-dimensional case ([2]), we reduce the exact controllability problem for three-dimensional Navier–Stokes equations to the same problem for the Helmholtz equation which describes the curl of the velocity vector field $v(t,x)$. But in contrast to the two-dimensional case, the three-dimensional Helmholtz equation is indeed a system defined on solenoidal vector fields. To overcome the difficulties arising in the investigation of this system we had to develop the method of [1], [2] in several directions and, in particular, to take a new minimized functional in the extremal problem mentioned above.

The case of simply connected bounded domains is the main one in this paper. Nevertheless, we can solve the controllability problem in the case of multiply connected domains as is shown in Remark 1.1 below.

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1. Statement of the problem and formulations of the results

1. In a bounded simply connected domain $\Omega \subset R^3$ with the C^∞-boundary $\partial\Omega$ we consider the Navier–Stokes equations

$$\partial_t v(t,x) + (v,\nabla)v - \Delta v + \nabla p(t,x) = 0, \ \operatorname{div} v = 0, \tag{1.1}$$

where $t \in (0,T)$, $x = (x_1, x_2, x_3) \in \Omega$, $v(t,x) = (v_1, v_2, v_3)$ is a velocity vector field, ∇p is a pressure gradient, $\partial_t = \partial/\partial t$, $(v,\nabla)v = \sum_j v_j \partial_j v$, $\partial_j v = \partial v/\partial x_j$, Δ is the Laplace operator, $\operatorname{div} v = \sum_j \partial_j v_j$.

Suppose that

$$v(t,x)|_{t=0} = v_0(x), \tag{1.2}$$

where $v_0(x) \in (L_2(\Omega))^3$ is a given solenoidal vector field.

The boundary zero controllability problem for the Navier–Stokes equations is to find the boundary value $z(t,x)$ of the velocity v:

$$v(t,x') = z(t,x'), \quad x' \in \partial\Omega, \quad t \in (0,T), \tag{1.3}$$

where $0 < T < \infty$, such that the solution $v(t,x)$ of the boundary value problem (1.1)–(1.3) satisfies the relation

$$v(T,x) \equiv 0 \tag{1.4}$$

at the instant T. To make this statement more precise and to formulate the results, we introduce certain functional spaces.

We denote by $W_2^k(\Omega)$, where k is a positive integer, the Sobolev space of functions defined on Ω which have the finite norm

$$\|v\|^2_{W_2^k(\Omega)} = \int_\Omega \sum_{|\alpha|\le k} |D^\alpha v(x)|^2 dx, \tag{1.5}$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ is a multi-index, $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$, $D^\alpha v = \partial^{|\alpha|} v / \partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \partial x_3^{\alpha_3}$. We shall also use the Sobolev spaces W_2^s with an arbitrary real s. These spaces are defined, for instance, in [3].

Let us define the space of solenoidal vector fields on Ω:

$$H^k(\Omega) = \{v(x) = (v_1, v_2, v_3) \in (W_2^k(\Omega))^3 : \operatorname{div} v = 0\}. \tag{1.6}$$

In addition, we shall also need the following spaces of functions defined on the cylinder $\Theta = (0,T) \times \Omega$:

$$W^{(m)}(\Theta) = \{v(t,x) \in L_2(0,T; W^{m+2}(\Omega)) : \partial_t v(t,x) \in L_2(0,T; W^m(\Omega))\}, \tag{1.7}$$

$$H^{(m)}(\Theta) = \{v \in L_2(0,T; H^{m+2}(\Omega)) : \partial_t v \in L_2(0,T; H^m(\Omega))\}. \tag{1.8}$$

The spaces $W^{(m)}(\Sigma)$ of vector fields defined on the lateral surface $\Sigma = (0,T) \times \partial\Omega$ are defined by analogy with (1.7) (in (1.7) Ω should be replaced by $\partial\Omega$).

Note that the solution of problem (1.1)–(1.4) can be reduced to the case where the initial value $v_0(x)$ from (1.2) satisfies the conditions

$$v_0(x) \in H^3(\Omega), \ \|v_0\|_{H^3(\Omega)} \le \varepsilon, \tag{1.9}$$

where $\varepsilon > 0$ is sufficiently small. In order to prove it, we set

$$z(t,x) \equiv 0$$

on time iterval $(0, T_1)$, where T_1 is sufficiently large. It is easy to get by energy estimate that the solution $v(t,x)$ of problem (1.1)–(1.3) with $z \equiv 0$ satisfies the conditions $v(T_1, x) \in H^3(\Omega)$, $\|v(T_1,\cdot)\|_{H^3(\Omega)} \ll 1$ at the instant $t = T_1$. Then we can solve the controllablity problem (1.1)–(1.4) with the initial value $v_0(x) \equiv v(T_1, x)$.

The principal result of this paper is the following.

Theorem 1.1. *Suppose that $\Omega \subset R^3$ is a bounded simply connected domain, $T > 0$ is given, and v_0 satisfies* (1.9) *with a sufficiently small ε. Then there exists a boundary control $z \in W^{(3/2)}(\Sigma)$ such that the solution $v(t,x)$ of problem* (1.1)–(1.3) *satisfies relation* (1.4). *Moreover,*

$$\|v(t,\cdot)\|_{H^3(\Omega)} \le c \, \exp(-k/(T-t)^2) \quad as \quad t \to T, \tag{1.10}$$

where $c > 0$, $k > 0$ are certain constants. In addition, the required control z can be found in the class of vector fields tangent to $\partial\Omega$:

$$(z(t,x), n(x)) = 0, \ x \in \partial\Omega, \ t \in (0,T), \tag{1.11}$$

where $n(x)$ is the vector field of the normals external to $\partial\Omega$.

On the whole, this paper is devoted to the proof of this result.

Remark 1.1. It is possible to get rid of simply connectedness of Ω by removing constraint (1.11) on control z in the interior parts of the boundary $\partial\Omega$. To do this, we replace Ω by simply connected domain Ω_1 with the boundary $\partial\Omega_1$ which coincides with the exterior part of $\partial\Omega$. Then we prolong the initial value from Ω up to Ω_1, apply Theorem 1.1 to this case, and thus solve problem (1.1)–(1.3) with the boundary condition z constructed in Theorem 1.1. Then the restriction imposed on the solution of (1.1)–(1.3) on $\partial\Omega$ will be the solution of the exact controllability problem in the case of multi-connected domain Ω.

2. First of all, we pass from the Navier–Stokes equations (1.1) to the Helmholtz equations for the curl of velocity $v(t,x)$. Since by definition $\operatorname{curl} v = (\partial_2 v_3 - \partial_3 v_2, \partial_3 v_1 - \partial_1 v_3, \partial_1 v_2 - \partial_2 v_1)$, the following well-known equality holds:

$$(y,\nabla)y = -y \times \operatorname{curl} y + \nabla(y^2/2), \tag{1.12}$$

where $y \times z = (y_2 z_3 - y_3 z_2, y_3 z_1 - y_1 z_3, y_1 z_2 - y_2 z_1)$ is the vector product of the vectors $y = (y_1, y_2, y_3)$, $z = (z_1, z_2, z_3)$. Therefore, applying the operator curl to the first equation in (1.1) and taking into account the relation $\operatorname{curl} \nabla f(x) \equiv 0$, we get the Helmholtz equations:

$$\partial_t \operatorname{curl} v(t,x) - \Delta \operatorname{curl} v - \operatorname{curl}(v \times \operatorname{curl} v) = 0. \tag{1.13}$$

When the operator curl is applied to the relations (1.2), (1.4), the latter take the form

$$\operatorname{curl} v(t,x)|_{t=0} = \operatorname{curl} v_0(x), \tag{1.14}$$

$$\operatorname{curl} v(t,x)|_{t=T} \equiv 0. \tag{1.15}$$

The following result will be proved later.

Theorem 1.2. *Let the condition of Theorem* 1.1 *be fulfilled. Then there exists a function $v(t,x) \in H^{(2)}(\Theta)$ satisfying Eqs.*(1.13)–(1.15) *and inequality* (1.10). *In addition, its boundary value $z = v|_\Sigma$ satisfies constraints* (1.11).

2. Unique solvability of the optimality system for one extremal problem

1. Let $G \subset R^3$ be a bounded domain with the C^∞-boundary ∂G and $Q = (0,T) \times G$. To prove Theorem 1.2, we consider the linearized analog of (1.13) defined in Q:

$$\partial_t \operatorname{curl} v(t,x) - \Delta \operatorname{curl} v - \operatorname{curl}(a \times \operatorname{curl} v) = f(t,x), \tag{2.1}$$

where $a(t,x) = (a_1, a_2, a_3) \in (W^{(2)}(Q))^3$ and f are given. The assumptions on f will be given below.

The exact zero controllability problem for (2.1) is to find $v(t,x) \in W^{(2)}(Q)$ satisfying (2.1), (1.14), (1.15). (Note that in the first phase of our costruction we do not require that the desired vector field $v(t,x)$ be solenoidal.) We shall solve this problem under the following further assumptions on $v_0(x)$:

$$v_0(x) \in (W_2^3(G))^3, \quad v_0(x)|_{\partial G} = \partial_n^j v_0(x)|_{\partial G} = 0, \quad j = 1,2, \tag{2.2}$$

where ∂_n^j is the derivative of order j with respect to the normal n external to ∂G.

First, we reduce problem (2.1), (1.14), (1.15) to the case where $v_0(x) \equiv 0$ in (1.14); for this we consider the boundary value problem

$$\partial_t \chi(t,x) - \Delta \chi - a(t,x) \times \operatorname{curl} \chi = 0, \tag{2.3}$$

$$\chi|_S = 0, \quad \text{where } S = (0,T) \times \partial G, \tag{2.4}$$

$$\chi|_{t=0} = v_0(x). \tag{2.5}$$

Since (2.3)–(2.5) is a linear parabolic boundary value problem, it follows that there exists a unique solution $\chi(t,x) \in W^{(2)}(Q)$ for v_0 satisfying (2.2) (see [4], [5]).

We make in (2.1) the change of the unknown function

$$v(t,x) = u(t,x) + \theta(t)\chi(t,x), \tag{2.6}$$

where $u(t,x)$ is the new unknown function, $\chi(t,x)$ is the solution of problem (2.3)–(2.5), $\theta(t) \in C^\infty(0,T)$, $\theta(t) \equiv 1$ for $t \in (0,T/3)$, $\theta(t) \equiv 0$ for $t \in (2T/3,T)$, $0 \le \theta(t) \le 1$. Substituting (2.6) into (2.1), (1.14), (1.15) and taking into account (2.3)–(2.5), we obtain the relations

$$Lu \equiv \partial_t \operatorname{curl} u(t,x) - \Delta \operatorname{curl} u - \operatorname{curl}(a \times \operatorname{curl} u) = g(t,x), \tag{2.7}$$

$$\operatorname{curl} u|_{t=0} = 0, \tag{2.8}$$

$$\operatorname{curl} u|_{t=T} = 0, \tag{2.9}$$

where

$$g = f + f_0 \quad \text{and} \quad f_0 = -(\partial_t \theta(t)) \ \operatorname{curl} \chi(t,x). \tag{2.10}$$

By virtue of the well-known estimates for the solution χ of problem (2.3)–(2.5), we get

$$\|f_0(t,\cdot)\|^2_{L_2(G)} \le |\partial_t\Theta(t)| \ \exp(\int_0^t \|\nabla a\|_{C(\Omega)}\,dt)\|v_o\|^2_{L_2(G)}, \tag{2.11}$$

$$\|\chi\|_{H^{(2)}(Q)} \le C\|v_o\|_{H^3(G)}, \tag{2.12}$$

where the constant C depends continuously on $\|a\|_{H^{(2)}(Q)}$.

2. To reduce (2.7)–(2.9) to a coercive problem we assume that the solution $u \in W^{(2)}(Q)$ of (2.7)–(2.9) exists and consider the extremal problem of minimizing the functional:

$$\frac{1}{2}\int_Q \varkappa^2(t,x)u^2(t,x)\,dx\,dt \quad \to \inf \tag{2.13}$$

on the set of functions $u \in W^{(2)}(Q)$ satisfying (2.7)–(2.9). Here $\varkappa(t,x) > 0$ is a certain weight function, which will be precisely defined below.

The optimality system of this problem is

$$L^*p \equiv \partial_t \operatorname{curl} p + \Delta \operatorname{curl} p - \operatorname{curl}(a \times \operatorname{curl} p) = \varkappa^2 u, \tag{2.14}$$

$$\operatorname{div} p = 0, \tag{2.15}$$

$$p|_S = 0\ ,\quad \partial_n p|_S = 0\ ,\quad \partial_n \operatorname{curl}\, p|_S = 0, \tag{2.16}$$

It will be clear later that it is not necessary to derive (2.14)–(2.16) in order to make a formal rigorous justification. Nevertheless, to make our presentation clearer, we give the draft of the optimality system derivation.

In order to apply the Lagrange principle ([6]), we must write the Lagrange function

$$\mathcal{L}(u,p) = \frac{1}{2}\int_Q \varkappa^2 u^2\,dx\,dt + \int_Q ((\partial_t \operatorname{curl} u - \\ - \Delta \operatorname{curl} u - \operatorname{curl}(a \times \operatorname{curl} u) - g), p)\,dx\,dt, \tag{2.17}$$

where $p(t,x)$ is a function defining a functional on the space coinciding with the image of the operator L from (2.7). Since the image of L consists of solenoidal vector fields, relation (2.15) holds.

The Lagrange principle asserts that there exists $p(t,x)$ such that the relation

$$\mathcal{L}^1_u(u,p) \equiv 0$$

holds for the solution u of the extremal problem (2.13), (2.7)–(2.9). This means that for an arbitrary $h \in W^{(2)}(Q)$ satisfying the conditions $h|_{t=0} = h|_{t=T} = 0$, we have

$$0 = \langle \mathcal{L}^1_u(u,p), h\rangle = \\ = \int\limits_Q [\varkappa^2(u,h) - ((\partial_t \operatorname{curl} h - \Delta \operatorname{curl} h - \operatorname{curl}(a \times \operatorname{curl} h)), p)]\, dx\, dt. \tag{2.18}$$

Taking $h \in C_0^\infty(Q)$, we obtain (2.14) understood in the sense of the theory of distributions. Since for any $\zeta \in C^\infty(\bar{Q})$, $\operatorname{div}\zeta = 0$ there exists $h \in (W^{(2)}(Q))^3$ such that $\operatorname{curl} h = \zeta$, we must substitute ζ into (2.18) instead of $\operatorname{curl} h$ and integrate by parts. Then we get $p|_S = 0$, $\partial_n p|_S = 0$. Now we return in (2.18) to the function h and, integrating by parts, first transform the operator curl from h into p and then obtain the third equality from (2.16).[1]

The optimality system (2.7)–(2.9), (2.14)–(2.16) will be the main object of our investigation. If we prove the unique solvability of this system, then we shall automatically get the solvability of controllability problem (2.7)–(2.9), taking the component u of the optimality system's solution (u,p) as its solution. To solve the optimality system (2.7)–(2.9), (2.14)–(2.16), we first exclude from it the unknown function u. Multiplying both parts of (2.14) by $\varkappa^{-2}$ and applying operator L from (2.7) to them, we get

$$L(\varkappa^{-2}L^* p) = g. \tag{2.19}$$

Taking into account (2.14), we can rewrite (2.8), (2.9) as follows:

$$\operatorname{curl}(\varkappa^{-2}L^*p)|_{t=0} = 0\ ,\ \operatorname{curl}(\varkappa^{-2}L^*p)|_{t=T} = 0. \tag{2.20}$$

Thus we reduce problem (2.7)–(2.9), (2.14)–(2.16) to problem (2.19), (2.15), (2.16), (2.20) with one unknown function p.

3. The main instrument that we use to solve problem (2.19), (2.20), (2.15), (2.16) is the Carleman estimates for the solution of the Cauchy problem (2.14)–(2.16). To derive them, we introduce the notations

$$\varphi(t,x) = \psi(t)\,(\sigma - \sum_{j=1}^{3}(x_j - x_j^0)^2),\quad \psi(t) = (T-t)^{-2} + t^{-2}, \tag{2.21}$$

where $x^0 = (x_1^0, x_2^0)$ lies outside of the closure $\bar{G}$ of the domain G and

$$\sigma > \max_{x \in G} |x - x^0|^2.$$

[1] This derivation of boundary condition (2.16) is not complete, of course. More detailed proof will be given in another paper.

Theorem 2.1. *Let p satisfy (2.14)–(2.16). Then there exists a positive continuous monotone nondecreasing function $s_0(\lambda)$, $\lambda > 0$, such that for an arbitrary $s > s_0(\|a\|_{C(\bar{Q})} + \|\nabla a\|_{C(\bar{Q})})$ the Carleman inequality*

$$\int\limits_Q ((s\psi(t))^{-1}(|\partial_t \operatorname{curl} p|^2 + |\Delta \operatorname{curl} p|^2) + s\psi(t)|\nabla \operatorname{curl} p|^2 +$$
$$+ (s\psi(t))^3|\operatorname{curl} p|^2)e^{-2s\varphi}\,dx\,dt + \int\limits_G ((s\psi(t))^{-1}|\nabla \operatorname{curl} p(t,x)|^2 +$$
$$+ s\psi(t)|\operatorname{curl}\ p(t,x)|^2)e^{-2s\varphi}\,dx \le C\int\limits_Q \varkappa^4 e^{-2s\varphi}u^2\,dx\,dt \tag{2.22}$$

holds, where ψ, φ are functions (2.21) and $C > 0$ does not depend on s, u, p.

Proof. We denote

$$q = e^{-s\varphi}p, \qquad w = e^{-s\varphi}\varkappa^2 u \tag{2.23}$$

and define the operator

$$Mq = e^{-s\varphi}((\partial_t + \Delta)(e^{s\varphi}q) - \operatorname{curl}(a \times (e^{s\varphi}q))). \tag{2.24}$$

This operator can be written as

$$\begin{aligned} Mq = (\partial_t + \Delta)q + s(\partial_t\varphi)q + 2s(\nabla\varphi, \nabla q) + \\ + s^2(\nabla\varphi)^2 q + s(\Delta\varphi)q - \operatorname{curl}(a \times q) - s(\nabla\varphi \times a \times q). \end{aligned} \tag{2.25}$$

We introduce the notations

$$M_1 q = \Delta q + s^2(\nabla\varphi)^2 q, \quad M_2 q = \partial_t q + 2s(\nabla\varphi, \nabla q) + s(\Delta\varphi)q, \tag{2.26}$$

$$f_s = \operatorname{curl}(a \times q) + s(\nabla\varphi \times a \times q - (\partial_t\varphi)q). \tag{2.27}$$

Obviously, Eq. (2.14) can be rewritten in the form

$$(M_1 + M_2)q = w + f_s, \tag{2.28}$$

where M_1, M_2, q, w, f_s are defined by (2.26), (2.23), (2.27). It follows from (2.28) that

$$\|w + f_s\|^2_{L_2(Q)} = \|M_1 q\|^2_{L_2(Q)} + \|M_2 q\|^2_{L_2(Q)} + 2(M_1 q, M_2 q)_{L_2(Q)}. \tag{2.29}$$

Taking into account (2.26), we have

$$2(M_1 q, M_2 q)_{L_2(Q)} = I_1 + I_2 + I_3 + I_4, \tag{2.30}$$

where

$$I_1 = 2\int_Q (\Delta q, \partial_t q)\,dx\,dt,$$

$$I_2 = 2s^2\int_Q (\nabla\varphi)^2(q, \partial_t q)\,dx\,dt,$$

$$I_3 = 2s\int_Q (\Delta q, (2(\nabla\varphi, \nabla q) + \Delta\varphi q))\,dx\,dt,$$

$$I_4 = 2s^3\int_Q (\nabla\varphi)^2(q, 2(\nabla\varphi, \nabla q) + \Delta\varphi q)\,dx\,dt. \tag{2.31}$$

It follows from (2.16), (2.21), (2.23) that

$$q|_S = 0, \quad \partial_n q|_S = 0, \quad \partial_n \operatorname{curl} v|_S = 0, \quad q|_{t=0} = q|_{t=T} = 0. \tag{2.32}$$

We transform $I_1, \ldots, I_4$ by integrating by parts and taking into account (2.32) and (2.21):

$$I_1 = 0,$$

$$I_2 = \frac{-2s^2}{2}\int_Q |q|^2\partial_t(\nabla\varphi)^2\,dt\,dx = -8s^2\int_Q |q|^2\psi(\partial_t\psi)|x - x^0|^2\,dt\,dx. \tag{2.33}$$

Using the agreement that the summation is taken over repeating indices, we find, with the aid of (2.21), that

$$I_3 = 2s\int_Q \partial_{ii}q_j(2\partial_k\varphi\partial_k q_j + \partial_{ll}\varphi q_j)\,dx\,dt =$$

$$= 2s\int_Q (-\partial_i q_j(2\partial_k\varphi\partial_k\partial_i q_j + 2\partial_i\partial_k\varphi\partial_k q_j + \partial_{ll}\varphi\partial_i q_j)\,dx\,dt =$$

$$= 2s\int_Q (-(\partial_k\varphi)\partial_k(\partial_i q_j)^2 - 2(\partial_{ii}\varphi)(\partial_i q_j)^2 + 6\psi(\partial_i q_j)^2)\,dx\,dt =$$

$$= 2s\int_Q (-6\psi(\nabla q)^2 + 12\psi(\nabla q)^2 + 6\psi(\nabla q)^2)\,dx\,dt = 24s\int_Q \psi(\nabla q)^2\,dx\,dt. \tag{2.34}$$

Finally we obtain the following expression for I_4:

$$I_4 = 2s^3 \int_Q (\nabla\varphi)^2(\partial_k\varphi\partial_k|q|^2 + \Delta\varphi|q|^2)\,dx\,dt =$$
$$= 2s^3 \int_Q (-\partial_k(\partial_j\varphi\partial_j\varphi))\partial_k\varphi|q|^2\,dx\,dt = 32s^3\int_Q |x-x_0|^2\psi^3|q|^2\,dx\,dt. \tag{2.35}$$

It follows from (2.27) that

$$\|w+f_s\|^2_{L_2(Q)} \le C(\|w\|^2_{L_2(Q)} + (\|\nabla a\|^2_{C(\bar Q)}\|q\|^2_{L_2(Q)} + \|a\|^2_{C(\bar Q)}\|\nabla q\|^2_{L_2(Q)} +$$
$$+ s^2(\|a\|^2_{C(\bar Q)}\int_Q |\nabla\varphi|^2|q|^2\,dx\,dt + \int_Q (\partial_t\varphi)^2|q|^2\,dx\,dt))). \tag{2.36}$$

Substituting (2.33)–(2.36) into (2.29) and making simple transformations, we get

$$\|M_1q\|^2_{L_2(Q)} + \|M_2q\|^2_{L_2(Q)} + 24s\int_Q \psi(\nabla q)^2\,dx\,dt +$$
$$+ 32s^3\int_Q |x-x_0|^2\psi^3|q|^2\,dx\,dt \le s^2c(\int_Q |q|^2\psi|\partial_t\psi||x-x^0|^2\,dx\,dt +$$
$$+ \|a\|^2_{C(\bar Q)}\int_Q \psi^2|x-x^0|^2|q|^2\,dx\,dt + \int_Q (\partial_t\psi)^2(\sigma-|x-x_0|^2)^2|q|^2\,dx\,dt) +$$
$$+ C(\|\nabla a\|^2_{C(\bar Q)}\|q\|^2_{L_2(Q)} + \|a\|^2_{C(\bar Q)}\|\nabla q\|^2_{L_2(Q)} + \|w\|^2_{L_2(Q)}). \tag{2.37}$$

If we choose a sufficiently large value of the parameter s (this value depends on $\|a\|^2_{C(\bar Q)} + \|\nabla a\|^2_{C(\bar Q)}$), then it will be possible to cancel all terms on the right-hand side of (2.37), except $\|w\|^2$, with the last two terms on the left-hand side. As a result, we obtain

$$\|M_1q\|^2_{L_2(Q)} + \|M_2q\|^2_{L_2(Q)} + s\int_Q \psi(\nabla q)^2\,dx\,dt +$$
$$+ s^3\int_Q |x-x_0|^2\psi^3|q|^2\,dx\,dt \le c\,\|w\|^2_{L_2(Q)}. \tag{2.38}$$

Taking into account (2.26), we can get from (2.38) (as was done by Fursikov and Imanuvilov in [1]) the following estimate:

$$\int_Q (s\psi)^{-1}(|\partial_t q|^2 + |\Delta q|^2)\,dx\,dt +$$
$$+ s\int_Q \psi(\nabla q)^2\,dx\,dt + s^3\int_Q \psi^3|q|^2\,dx\,dt \le c\,\|w\|^2_{L_2(Q)}. \tag{2.39}$$

Obviously, there exists a constant $c > 0$ which does not depend on t and q and is such that for any $t \in (0,T)$ we have

$$\int_G ((s\psi(t))^{-1}|\nabla q(t,x)|^2 + s\psi(t)|q(t,x)|^2)\,dx \le$$
$$\le \int_Q \frac{1}{s\psi}(|\partial_t q|^2 + |\Delta q|^2) + s^3\psi^3|q|^2\,dx\,dt. \tag{2.40}$$

When we return from q, w to curl $p, \varkappa^2 u$, inequalities (2.39), (2.40) imply (2.22). □

Instead of (2.21), we now take the function

$$\alpha(t,x) = (T-t)^{-2}(\sigma - |x - x^0|^2), \tag{2.41}$$

where σ is the same as in (2.21).

Corollary 2.1. *Let p satisfy (2.14) – (2.16) and $s_0(\lambda)$ be the function from Theorem 2.1. Then, for $s \ge s_0(\|a\|_{C(\bar{Q})} + \|\nabla a\|_{C(\bar{Q})})$, we have the inequality*

$$I(\operatorname{curl} p) \equiv \int_Q (\frac{(T-t)^2}{s}(|\partial_t \operatorname{curl} p|^2 + |\Delta \operatorname{curl} p|^2) +$$
$$+ s(T-t)^{-2}|\nabla \operatorname{curl} p|^2 + s^3(T-t)^{-6}|\operatorname{curl} p|^2)\,e^{-2s\alpha(t,x)}\,dx\,dt +$$
$$+ \int_G ((T-t)^2|\nabla \operatorname{curl} p(t,x)|^2 + (T-t)^{-2}|\operatorname{curl} p(t,x)|^2)\,e^{-2s\alpha(t,x)}\,dx \le$$
$$\le c\int_Q \varkappa^4 e^{-2s\alpha}u^2\,dx\,dt, \tag{2.42}$$

where the constant $c > 0$ depends continuously on s and $\|a\|_{C(\bar{Q})} + \|\nabla a\|_{C(Q)}$.

Proof. We denote

$$q(t,x) = \operatorname{curl} p(t,x). \tag{2.43}$$

It follows from (2.14)–(2.16) that q is a solution of the problem

$$\partial_t q + \Delta q - \operatorname{curl}(a \times q) = \varkappa^2 u, \tag{2.44}$$

$$q|_S = 0, \quad \partial_n q|_S = 0. \tag{2.45}$$

We provide this problem with the initial condition at the instant $t_0 \in (0,T)$. Then, as it is well known, the following inequality holds (see Ladyzhenskaya, Solonnikov, and Ural'tseva [4]):

$$\|q\|_{W^{(0)}(Q_0)} \le c(\|\varkappa^2 u\|_{L_2(Q_0)} + \|q(t_0,\cdot)\|_{W_2^1(G)}), \quad Q_0 = [0,t_0) \times G. \tag{2.46}$$

Relations (2.43), (2.46), (2.22) imply (2.42). □

4. Everywhere below we take, as the function $\varkappa^2(t,x)$, the function

$$\varkappa^2(t,x) = e^{2s\alpha(t,x)}, \tag{2.47}$$

where $\alpha(t,x)$ is the function (2.41) and $s > 0$ is the fixed number for which inequality (2.42) holds. Let $x_* \in \partial G$ be a point such that

$$|x_* - x^0| = \inf_{x \in G} |x - x^0|. \tag{2.48}$$

Obviously,

$$e^{-2s\alpha(t,x_*)} = \min_{x\in G} e^{-2s\alpha(t,x)}. \tag{2.49}$$

Proposition 2.1. *Let $p(t,x)$ satisfy (2.14) – (2.16). Then, for $s \ge s_0$, where s_0 is the same as in Corollary 2.1, the following inequality holds:*

$$J(p) \equiv \int_0^T \Big(\frac{(T-t)^2}{s}\|\partial_t p\|_{H^1(G)} + s(T-t)^{-2}\|p\|^2_{H^2(G)} + \\ + s^3(T-t)^6\|p\|_{H^1(G)}\Big) e^{-2s\alpha(t,x_*)}\, dt \le c \int_Q e^{2s\alpha(t,x)}|u|^2\, dx\, dt. \tag{2.50}$$

Proof. We extend $p(t,x)$ by zero beyond G, denoting the new function by p and set

$$\operatorname{curl} p = v(t,x). \tag{2.51}$$

By virtue of (2.16), (2.15), after acting on (2.51) by the operator curl, we obtain

$$-\Delta p = \operatorname{curl} v, \tag{2.52}$$

where $\operatorname{curl} v \in L_2(R^3)$, $v(t,x) \equiv 0$, $x \notin \bar{G}$. Applying the Fourier transformation to both parts of (2.52), we get

$$\hat{p}(t,\xi) = -i\,\frac{\xi \times \hat{v}(t,\xi)}{(\xi)^2}, \tag{2.53}$$

where $\hat{p}, \hat{v}$ are the Fourier transforms of the functions p, v respectively and $\times$ is the vector product in R^3. Since $\operatorname{supp} p(t,x) \subset G$, $\operatorname{supp} v(t,x) \subset \bar{G}$, it follows that $\hat{p}(t,\xi)$, $\hat{v}(t,\xi)$ are entire analytic functions with respect to ξ. Therefore, considering (2.53) for $\xi_1, \xi_2 \in R$ and $\xi_3 = \eta + i\zeta_0$, where $\eta \in R$ and $\zeta_0 \neq 0$ is a fixed real number, we easily obtain the estimate

$$(1 + |\xi|^2)|\hat{p}(t,\xi)|^2 \leq c|\hat{v}(t,\xi)|^2,$$

which, when we return to the x-space R^3, is transformed into the inequality

$$\int_G \sum_{|\alpha|\leq 1} |D_x^\alpha(p(t,x)\,e^{\zeta_0 x_3})|^2\,dx \leq c\int_G |e^{\zeta_0 x_3} v(t,x)|^2\,dx.$$

This inequality implies the estimate

$$\|p(t,\cdot)\|_{H^1(G)} \leq c\|v(t,\cdot)\|_{L_2(G)} \tag{2.54}$$

because the domain G is bounded. Note that the relations

$$\operatorname{curl} \partial_t p = \partial_t v, \quad \nabla \operatorname{curl} p = \nabla v, \quad \partial_t p|_S = \partial_t \partial_n p|_S = 0 \tag{2.55}$$

follow from (2.51), (2.16). By analogy with (2.54), we obtain from (2.55) the inequalities

$$\|\partial_t p\|_{H^1(G)} \leq c\|\partial_t v\|_{L_2(G)}, \quad \|p\|^2_{H^2(G)} \leq c\|v\|^2_{H^1(G)}. \tag{2.56}$$

Taking into account (2.49), we easily derive (2.50) from (2.54), (2.56), (2.42). □

5. We shall formulate the generalized solution of problem (2.19), (2.20), (2.15), (2.16) and prove the existence and uniqueness of this solution.

We define the space Φ of the solenoidal vector fields p determined on Q by the relation

$$\begin{aligned} \Phi = \{\, p : \|p\|^2_\Phi &\equiv \|e^{-s\alpha} L^* p\|^2_{L_2(Q)} + I(\operatorname{curl} p) + J(p) < \infty, \\ \operatorname{div} p = 0, \quad p|_S &= \partial_n p|_S = \partial_n \operatorname{curl} p|_S = 0\}, \end{aligned} \tag{2.57}$$

where α is the function (2.41), I, J are defined by (2.42), (2.50), and s is the same as in (2.50).

Definition 2.1. The function $p \in \Phi$ is called a generalized solution of problem (2.19), (2.20), (2.15), (2.16) if, for an arbitrary $q \in \Phi$, the following equality is fulfilled:

$$(e^{-2s\alpha}L^*p,\, L^*q)_{L_2(Q)} = (g,q)_{L_2(Q)}. \tag{2.58}$$

Theorem 2.2. *Let g satisfy the following condition: there exists a function $g_1(t,x)$ on Q such that $g = \operatorname{curl} g_1$ and*

$$\int_Q e^{2s\alpha}|g|\,dx\,dt + \int_Q e^{2s\alpha}|g_1|^2\,dx\,dt < \infty, \tag{2.59}$$

where α is the function (2.41) and s is mentioned in Corollary 2.1. Then there exists a unique generalized solution $p \in \Phi$ of problem (2.19), (2.20), (2.15), (2.16). The function p satisfies (2.19) in the distributions sense and (2.20) understood as equalities in $H^{-2}(G)$. If the function u is defined by p in (2.14), where $\varkappa^2$ is the function (2.47), then

$$\|e^{s\alpha}u\|^2_{L_2(Q)} \le c\|e^{s\alpha}g_1\|^2_{L_2(Q)}, \tag{2.60}$$

where c is the constant from (2.42).

Proof. Inequality (2.59) implies that

$$|(g,q)_{L_2(Q)}| \le \|e^{s\alpha}g_1\|^2_{L_2(Q)}\|e^{-s\alpha}\operatorname{curl} q\|_{L_2(Q)} \tag{2.61}$$

and, therefore, the functional $q \to (g,q)_{L_2(Q)}$ is continuous on Φ. By virtue of Corollary 2.1 and Proposition 2.1, the bilinear form $(e^{-2s\hat{\alpha}}L^*p,\, L^*q)_{L_2(Q)}$ is coercive on the space Φ. Hence, the Riesz theorem on the functional representation on a Hilbert space implies the existence and uniqueness of the generalized solution p. We substitute the expressions $g = \operatorname{curl} g_1$ and $u = e^{-2s\phi}L^*p$ from (2.14) into (2.58) and integrate by parts (in the distributions sense) with $q \subset \Phi \cap \left(C_0^\infty(Q)\right)^3$. As a result we get the equality

$$\int_Q \left(\partial_t u - \Delta u - (a \times \operatorname{curl} u) - g_1\right)\operatorname{curl} q_1\,dx\,dt = 0.$$

This relation yields the equation

$$\partial_t u - \Delta u - (a \times \operatorname{curl} u) - g_1 = \nabla r_1$$

where r_1 is a distribution. We apply the operator curl to the last equality and express u by means of p to show that p satisfies (2.19) in the distributions sense. Since $p \in \Phi$, it follows, by virtue of (2.57), that the function u appearing in (2.14) with $\varkappa$ from (2.47) satisfies the relation

$$e^{s\alpha}u \in L_2(Q).$$

Hence $e^{s\alpha}\operatorname{curl} u \in L_2(0,T;H^{-1}(G))$, $e^{s\alpha}(\Delta \operatorname{curl} u + \operatorname{curl}(a \times \operatorname{curl} u) \in$ $\in L_2(0,T;H^{-3}(\Omega)))$ and, therefore, by virtue of (2.7), (2.59), $\partial_t \operatorname{curl} u \in$ $L_2(0,T;H^{-3}(\Omega))$. These inclusions allow us to understand (2.20) as equalities in $H^{-2}(G)$. We set $q = p$ in (2.58). Taking into account (2.14) and the definition of g_1 as $g = \operatorname{curl}\, g_1$, we can rewrite (2.58) as

$$\begin{aligned} &||e^{s\alpha}u||^2_{L_2(Q)} = ||e^{-2s\alpha}L^*p||^2_{L_2(Q)} = (g_1, \operatorname{curl} p)_{L_2(Q)} \le \\ &\le ||g_1\ e^{s\alpha}||_{L_2(Q)}||e^{-s\alpha}\operatorname{curl} p||_{L_2(Q)} \le \\ &\le c\,||g_1\ e^{s\alpha}u||_{L_2(Q)}||e^{-s\alpha}L^*p||_{L_2(Q)}, \end{aligned} \tag{2.62}$$

where the last inequality in (2.62) follows from (2.42). Inequalities (2.62) imply (2.60). □

3. Solvability of the linear boundary zero controllability problem

1. Our first business is to investigate the smoothness of the generalized solution inside the cylinder Q. We shall restrict our consideration to the smoothness of the function u constructed by means of p with the aid of (2.14).

Lemma 3.1. *Let $g \in L_2(t_1,t_2;H^1(G_1))$ for an arbitrary $G_1 \subset G$ and $0 < t_1 < t_2 < T$, where g is the right-hand side of* (2.19). *Then, for the function u defined by* (2.14), *the inclusion* $\operatorname{curl} u \in H^{(1)}(Q_1)$, *where $Q_1 = (t_1,t_2) \times G_1$, holds.*

Proof. If u is defined by (2.14), then it satisfies (2.7) understood in the distributions theory sense. This assertion is derived simply from (2.19). We rewrite (2.7) in the form

$$\partial_t \operatorname{curl} u - \Delta \operatorname{curl} u = g + \operatorname{curl}(a \times \operatorname{curl} u). \tag{3.1}$$

Suppose that $b \in W^{(2)}(Q)$ and $\hat{b}$ is the linear operator of multiplication by b. Then the Sobolev embedding theorem implies that the operators

$$\hat{b} : L_2(0,T;H^k) \to L_2(0,T;H^k), \quad k = -1,0,1,2 \tag{3.2}$$

are continuous.

We know from Theorem 2.2 that $u \in L_2(Q_1)$. Hence, by virtue of (3.2), $g + \operatorname{curl}(a \times \operatorname{curl} u) \in L_2(0,T;H^{-2})$. Since the operator $\partial_t - \Delta$ is hypoelliptic, it follows, by virtue of (3.1), that $\operatorname{curl} u \in L_2(t_1,t_2;H^0(G_1))$ and, by virtue of (3.2), the right-hand side of (3.1) belongs to $L_2(t_1,t_2;H^{-1}(G_1))$. Repeating these arguments several times, we get the assertion of the theorem. □

We study now the smoothness of u near the boundary.

Let $\rho(x) \in C^\infty(G)$ be a function satisfying the conditions

$$\rho(x) > 0, \quad x \in G; \quad \rho(x') = 0, \quad x' \in \partial G, \quad \partial\rho/\partial n < 0,$$

where $n(x)$ is the vector field of the normals external to ∂G.

Lemma 3.2. *Let u be defined by* (2.14). *Then*

$$\sup_{t\in[0,T]} \int_G e^{2s\alpha}(T-t)^{12}\rho(x)^6 |\operatorname{curl} u(t,x)|^2\, dx +$$
$$+ \int_Q e^{2s\alpha}(T-t)^{12}\rho^6(x)|\nabla \operatorname{curl} u(t,x)|^2\, dx\, dt \le \gamma_1 \int_Q e^{2s\alpha}(g^2+u^2)\, dx\, dt, \tag{3.3}$$

where g is the right-hand side of (2.19) *and the constant γ_1 depends continuously and monotonically on $\|a\|_{C(\bar Q)} + \|\nabla a\|_{C(\bar Q)}$.*

Proof. We denote $v = \operatorname{curl} u$ and rewrite (2.7) in the form

$$\partial_t v - \Delta v - \operatorname{curl}(a \times v) = g. \tag{3.4}$$

Scaling (3.4) in L_2 by $(T-t)^{12}\rho(x)^6 e^{2s\alpha} v$ and performing some simple transformation, we get

$$\int_0^{t_1}\int_G (T-t)^{12}\Big(\rho^6 e^{2s\alpha}\frac{1}{2}\partial_t|v(t,x)|^2 + \rho^6 e^{2s\alpha}|\nabla v|^2 + \frac{1}{2}(\nabla v^2, \nabla(\rho^6 e^{2s\alpha})) -$$
$$- \big[(((v,\nabla)a - (a,\nabla)v), v) - \operatorname{div} a |v|^2\big]\rho^6 e^{2s\alpha}\Big)\, dx\, dt =$$
$$= \int_0^{t_1}\int_G e^{2s\alpha}(T-t)^{12}\rho^6(g,v)\, dx\, dt. \tag{3.5}$$

Note that to transform the term $\operatorname{curl}(a \times v)$ in (3.5), we use the well-known relation $\operatorname{curl}(a\times v) = (v,\nabla)\,a - (a,\nabla)\,v + a \operatorname{div} v - v \operatorname{div} a$ and the relation

$\operatorname{div} v = 0$. Integrating by parts once more in (3.5), we obtain

$$\frac{1}{2}\int_G (T-t_1)^{12}\rho^6 e^{2s\alpha}|v(t_1,x)|^2\,dx + \int_0^{t_1}\int_G (T-t)^{12}\big[\rho^6 e^{2s\alpha}\big(|\nabla v|^2 + \\ + \big(6(T-t)^{-1} - s\partial_t\alpha\big)|v|^2\big) - \frac{1}{2}|v|^2\Delta(\rho^6 e^{2s\alpha}) - \\ - \big((v,\nabla)\,a, v\big)\rho^6 e^{2s\alpha} + \frac{1}{2}\operatorname{div} a|v|^2\rho^6 e^{2s\alpha} - \frac{1}{2}|v|^2\big(a,\nabla\big(\rho^6 e^{2s\alpha}\big)\big)\big]\,dx\,dt \le \\ \le c\int_0^{t_1}\int_G e^{2s\alpha}|g|^2\,dx\,dt + \int_0^{t_1}\int_G e^{2s\alpha}(T-t)^{12}\rho^6|v(t,x)|^2\,dx\,dt. \tag{3.6}$$

Taking into account that $v = \operatorname{curl} u$ and integrating by parts, we get

$$\int_G |v|^2\varkappa(t,x)\,dx = \int_G [(\operatorname{curl}\, v, u)\varkappa + ((\nabla\varkappa\times v), u)]\,dx. \tag{3.7}$$

In our case

$$\varkappa(t,x) = (T-t)^{12}[(6(T-t)^{-1} - s\partial_t\alpha(t,x))e^{2s\alpha}\rho^6 - \\ - \frac{1}{2}\Delta(\rho^6 e^{2s\alpha}) - \frac{1}{2}(a,\nabla(\rho^6 e^{2s\alpha}))]$$

and, therefore, the following estimates are true:

$$|\varkappa| \le c\,e^{2s\alpha}(\rho^4(x)(T-t)^{12} + (T-t)^8\rho^6(x)), \\ |\nabla\varkappa| \le c\,e^{2s\alpha}(\rho^3(T-t)^{12} + (T-t)^6\rho^6(x)). \tag{3.8}$$

To obtain the necessary estimates, we transfer all terms on the left side of (3.6), exept the first two, to the right-hand side and apply (3.7), (3.8) and the Young inequality. We obtain

$$\frac{1}{2}\int_G (T-t_1)^{12}\rho^6 e^{2s\alpha}|v(t_1,x)|^2\,dx + \int_0^{t_1}\int_G (T-t)^{12}\rho^6 e^{2s\alpha}|\nabla v|^2\,dx\,dt \le \\ \le c\int_0^{t_1}\int_G e^{2s\alpha}(|g|^2 + (T-t)^{12}\rho^6|v|^2(1 + \|a\|_{C(Q)} + \|\nabla a\|_{C(Q)}))\,dx\,dt + \\ + \frac{1}{2}\int_0^{t_1}\int_G e^{2s\alpha}(\rho^6(T-t)^{12}|\nabla v|^2 + \rho^6(T-t)^{12}|v|^2 + c_\varepsilon|u|^2)\,dx\,dt. \tag{3.9}$$

Transferring the term with $|\nabla v|^2$ from the right-hand side of (3.9) to the left-hand side and applying the Gronwall inequality, we get (3.3). □

Lemma 3.3. *Let u be defined by* (2.14) *and, in addition to* (2.59), *the inclusion $e^{s\alpha}\nabla g \in L_2(Q)$ holds true.*

Then

$$\|e^{s\alpha}(T-t)^{10}\rho^5 \operatorname{curl} u\|^2_{H^{(0)}(Q)} \le \gamma_1(\|e^{s\alpha}u\|^2_{L_2(Q)} + \|e^{s\alpha}g\|^2_{L_2(Q)}), \quad (3.10)$$

where the constant γ_1 depends continuously on $\|a\|_{C(\bar{Q})} + \|\nabla a\|_{C(\bar{Q})}$. Moreover,

$$\begin{aligned} &\|c^{s\alpha}(T-t)^{12}\rho^6 \operatorname{curl} u\|^2_{H^{(1)}(Q)} \le \\ &\le \gamma_2(\|e^{s\alpha}u\|^2_{L_2(Q)} + \|e^{s\alpha}g\|^2_{L_2(Q)} + \|e^{s\alpha}\nabla g\|^2_{L_2(Q)}), \end{aligned} \quad (3.11)$$

where γ_2 depends continuously on $\|a\|_{H^{(2)}(Q)}$.

Proof. Suppose that $v = \operatorname{curl} u$ as before and, in addition, $w = (T-t)^\beta \rho^k e^{s\alpha} v$. We substitute $v = (T-t)^{-\beta}\rho^{-k}e^{-s\alpha}w$ into (3.4) and rewrite this equation in the form

$$\begin{aligned} &\partial_t w - \Delta w - \operatorname{curl}(a \times w) = \rho^k (T-t)^\beta e^{s\alpha} g + (-\beta(T-t)^{-1} + \\ &+ s\partial_t \alpha)w + 2(\theta, \nabla w) - \theta \times (a \times w) + (s^2|\nabla\alpha|^2 - s\Delta\alpha + \\ &+ 2s\,k\,\rho^{-1}\langle\nabla\alpha, \nabla\rho\rangle + k(k+1)\rho^{-2}|\nabla\rho|^2 - k\,\rho^{-1}\Delta\rho)w, \end{aligned} \quad (3.12)$$

where

$$\theta = -s\nabla\alpha - k\,\rho^{-1}\nabla\rho.$$

Substituting $w = (T-t)^\beta \rho^k e^{s\alpha} v$ into the right-hand side of (3.12) and taking (3.3) into account, we see that the right-hand side of (3.12) belongs to $L_2(Q)$ precisely for $\beta \ge 10$, $k \ge 5$. We apply to (3.12) the well-known upper bound of the solution of the parabolic boundary value problem via its right-hand side and then estimate the L_2-norm of this right-hand side by means of (3.3). As a result, we get (3.10). Using now (3.10), we can estimate the right-hand side of (3.12) in $L_2(0,T;H^1)$ if we set $k=6$, $\beta = 12$. Now the upper bound of the solution of the parabolic boundary value problems leads to (3.11). □

2. Now we prove the theorem on the boundary exact zero controllability for Eq.(2.1). It is convenient to assume that (2.1) is defined in the cylinder $\Theta = [0,T] \times \Omega$, where $\Omega \subset R^3$ is a bounded domain with the C^∞-boundary

$\partial\Omega$. Also suppose that (1.14), (1.15) are defined for $x \in \Omega$. We set

$$L_2^k(\Theta, s\alpha) = \{f \in L_2(0,T;(W_2^k(\Omega))) : \|f\|^2_{L_2^k(\Theta,s\alpha)} = \\ = \int_\Theta e^{2s\alpha}(\sum_{|\beta|\le k} |D_x^\beta f|^2)\, dx < \infty\}. \tag{3.13}$$

Theorem 3.1. *Let $a(t,x) \in H^{(2)}(\Theta)$, $v_0 \in H^3(\Omega)$, $f_1 \in L_2^2(\Theta, s\alpha)$, and $f = \operatorname{curl} f_1$, where s, α are the same as in Corollary 2.1. Then there exists a solution $v \in L_2^0(\Theta, s\alpha)$ of problem (2.1), (1.14), (1.15) which satisfies the inequality*

$$\|e^{s\alpha}(T-t)^{12} \operatorname{curl} v\|^2_{H^{(1)}(\Theta)} + \|v\|^2_{L_2^0(\Theta,s\alpha)} \le \\ \le \gamma(\|f\|^2_{L_2^1(\Theta,s\alpha)} + \|f_1\|^2_{L_2^0(\Theta,s\alpha)} + \|v_0\|_{H^3(\Omega)}), \tag{3.14}$$

where $\gamma > 0$ depends continuously on $\|a\|_{H^{(2)}(\Theta)}$.

Proof. We choose a domain $G \in R^3$ containing Ω and denote by $R : H^3(\Omega) \to (W_2^3(G))^3$ a linear continuous operator that extends the function $u(x)$, $x \in \Omega$ to the function $Ru(x)$, $x \in G$, where $Ru(x) = u(x)$, $x \in \Omega$ and $Ru(x) \equiv 0$ in a fixed neighborhood of the boundary $\partial\Omega$ of Ω. Let $R_1 : H^{(2)}(\Theta) \to W^{(2)}(Q)$ be a linear continuous operator of extention of the function from $\Theta = [0,T] \times \Omega$ to $Q = [0,T] \times G$ and $(R_1 u)(t,x) \equiv 0$ in a fixed neighborhood of the lateral surface $S = [0,T] \times \partial G$ of Q. In addition, by $R_2 : L_2^k(\Theta, s\alpha) \to L_2^k(Q, s\alpha)$ we denote a linear continuous extention operator. Thus

$$\|Ru_0\|_{(W_2^3(G))^3} \le c\|u_0\|_{H^3(\Omega)}, \\ \|R_2 f\|_{L_2^k(Q,s\alpha)} \le c\|f\|_{L_2^k(\Theta,s\alpha)}, \tag{3.15}$$

and we suppose that for R_1 the following inequalities hold true:

$$\|R_1 a\|_{(W^{(2)}(Q))^3} \le c\|a\|_{(W^{(2)}(\Theta))^3}, \\ \|R_1 a\|_{C(\bar Q)} + \|\nabla R_1 a\|_{C(\bar Q)} \le c(\|a\|_{C(\bar\Theta)} + \|\nabla a\|_{C(\bar\Theta)}). \tag{3.16}$$

We consider problem (2.1), (1.14), (1.15) defined on Q, where the coefficients $a = (a_1, a_2, a_3)$ are replaced by $R_1 a$, the initial value v_0 by Rv_0, and the right-hand side f by $R_2 f$. Applying to this problem the construction mentioned from the beginning of Sec. 2 to Lemma 3.3 inclusive and taking into account (3.15), (3.16), we get the assertion of Therem 3.1 after restricting the obtained solution v on the input cylinder $\Theta = (0,T) \times \Omega$. □

3. Our next step is to solve the boundary zero controllability problem for equation (2.1) in the class of solenoidal vector fields $v(t,x)$. For this purpose we apply the Weyl decomposition to the solution $v(t,x)$ obtained in Theorem 3.1:

$$v(t,x) = w(t,x) + \nabla q(t,x), \tag{3.17}$$

where for every $t \in [0,T]$ we have

$$w(t,x) \in L_2(\Omega), \quad \operatorname{div} w(t,x) \equiv 0, \quad (w(t,x), n(x)) = 0, \tag{3.18}$$

where $n(x)$ is an external normal to $\partial\Omega$, and the last two relations in (3.18) are understood in the well-known distribution theory sense (see [7], [8]). Since $\operatorname{curl} \nabla q(t,x) \equiv 0$, $w(t,x)$ is the solution of (2.1), (1.14), (1.15) together with $v(t,x)$.

Our main aim is to establish the smoothness of $w(t,x)$. Note that, for every $t \in (0,T)$, $w(t,x)$ satisfies the boundary value problem

$$\operatorname{curl} w(t,x) = \operatorname{curl} v(t,x), \tag{3.19}$$

$$\operatorname{div} w(t,x) = 0, \quad (w(t,x), n(x))|_{\partial\Omega} = 0, \tag{3.20}$$

where $v(t,x)$ is a given vector field. The following assertion holds (see Solonnikov [9]).

Lemma 3.4. *Let $w(t,x) \in L_2(\Omega)$ be a solution of (3.19), (3.20). Then for $k \geq 1$ we have*

$$\|w(t,\cdot)\|^2_{(W_2^k(\Omega))^3} \leq c\left(\|\operatorname{curl} v(t,\cdot)\|^2_{(W_2^{k-1}(\Omega))^3} + \|w(t,\cdot)\|^2_{(L^2(\Omega))^3}\right), \tag{3.21}$$

where the constant c does not depend on $\operatorname{curl} v$.

Theorem 3.1 and Lemma 3.4 imply the following statement

Theorem 3.2. *Let the hypothesis of Theorem 3.1 be satisfied. Then there exists a solution $w \in H^{(2)}(\Theta)$ (particularly, $\operatorname{div} w = 0$) of problem (2.1), (1.14), (1.15) satisfying the following conditions:*

(i) the normal component of w on $\partial\Omega$ is equal to zero:

$$(w(x,t),\ n(x))|_{x\in\partial\Omega} = 0, \tag{3.22}$$

(ii) the following estimate holds:

$$\begin{aligned}\|e^{s\alpha}(T-t)^{12}\operatorname{curl} w\|^2_{H^{(1)}(\Theta)} + \|e^{s\hat{\alpha}}(T-t)^{12} w\|^2_{H^{(2)}(\Theta)} \leq \\ \leq \gamma(\|f\|^2_{L_2^1(\Theta,s\alpha)} + \|f_1\|^2_{L_2^1(\Theta,s\alpha)} + \|v_0\|^2_{H^3(\Omega)}).\end{aligned} \tag{3.23}$$

Here $\gamma > 0$ depends continuously on $\|a\|_{H^{(2)}(\Theta)}$ and

$$\hat{\alpha}(t) = \alpha(t,\hat{x}), \quad \textit{where} \quad |\hat{x} - x_0|^2 = \max_{x\in G}|x - x_0|^2. \tag{3.24}$$

Remark 3.1. Taking into account (2.41), we see that $\hat{x} \in \bar{G}$ satisfies the condition

$$e^{s\hat{\alpha}(t)} = e^{s\alpha(t,\hat{x})} = \min_{x \in G} e^{s\alpha(t,x)}. \tag{3.25}$$

Proof of Theorem 3.2. Let $v(t,x)$ be the solution of problem (2.1), (1.14), (1.15) constructed in Theorem 3.1. We define $w(t,x)$ as a solenoidal component of $v(t,x)$ in the Weyl decomposition (3.17), (3.18). Since relation (3.19) holds, $w(t,x)$ satisfies (2.1), (1.14), (1.15) as well as $v(t,x)$. Relations (3.14), (3.19) imply the estimate of the first left-hand side term of inequality (3.23) by its right-hand side.

The right-hand side of (3.19) is differentiable with respect to t. Hence the left-hand side of (3.19) possesses the same property and, by virtue of Lemma 3.4,

$$\|\partial_t w(t,\cdot)\|^2_{(W_2^k(\Omega))^3} \le c\|\operatorname{curl} \partial_t v(t,x)\|^2_{(W_2^{k-1}(\Omega))^3} + \|\partial_t w(t,\cdot)\|^2_{(L_2(\Omega))^3}. \tag{3.26}$$

Multiplying both parts of (3.21), (3.26) by $(T-t)^{12}e^{2s\hat{\alpha}(t)}$, integrating with respect to t, and taking into account (3.14), (3.25), we get

$$\|e^{s\hat{\alpha}}(T-t)^{12}w\|^2_{H^{(2)}(\Theta)} \le \gamma(M + \|e^{s\hat{\alpha}}(T-t)^{12}\partial_t w\|^2_{L_{(2)}(\Theta)}), \tag{3.27}$$

where

$$M = \|f\|^2_{L_2^1(\Theta,s\alpha)} + \|f_1\|^2_{L_2^1(\Theta,s\alpha)} + \|v_0\|^2_{H^3(\Omega)}. \tag{3.28}$$

To estimate $\partial_t w$ in (3.27), we substitute (3.17) and the relation $f = \operatorname{curl} f_1$ into (2.1) and, because of the simple connectedness of the domain Ω, we get the equation

$$\partial_t w - \Delta w + \nabla p = f_1 - a \times \operatorname{curl} w \equiv f_2, \tag{3.29}$$

where the last equality in (3.29) is the definition of f_2 and p is a certain distribution. Applying the operator div to both parts of (3.29) and taking into account (3.20), we obtain the equalities

$$\Delta p = \operatorname{div} f_2, \qquad \partial_n p|_{\partial\Omega} = (f_2, n)|_{\partial\Omega} + (\Delta w, n)|_{\partial\Omega}. \tag{3.30}$$

By virtue of the hypothesis of Theorem 3.2 and (3.14),

$$\|e^{s\hat{\alpha}}(T-t)^{12}f_2\|^2_{L_2(0,T;W_2^1(\Omega))} \le cM, \tag{3.31}$$

where M is the magnitude of (3.28). Since $-\Delta w = \operatorname{curl}\operatorname{curl} w$ if $\operatorname{div} w = 0$, we have, by virtue of (3.14),

$$\begin{aligned} &\|e^{s\hat{\alpha}}(T-t)^{12}\Delta w\|^2_{L_2(0,T;W_2^1(\Omega))} \le cM, \\ &\|e^{s\hat{\alpha}}(T-t)^{12}\partial_n p\|^2_{L_2((0,T)\times\partial\Omega)} \le cM. \end{aligned} \tag{3.32}$$

Relations (3.30), (3.32) yield the estimate

$$\|e^{s\hat{\alpha}}(T-t)^{12}\nabla p\|^2_{L_2(\Theta)} \le c_1 M. \tag{3.33}$$

Equation (3.29) and bounds (3.31)–(3.33) imply the inequality

$$\|e^{s\hat{\alpha}}(T-t)^{12}\partial_t w\|^2_{L_2(\Theta)} \le c_2 M. \tag{3.34}$$

Hence, (3.27), (3.34) yield the estimate of the second term on the left-hand side of (3.23). □

4. Exact zero controllability of the Helmholtz equation and the Navier–Stokes system

1. We begin with problem (1.13)–(1.15), i.e., with the Helmholtz equation. First of all we prove the following assertion.

Lemma 4.1. *Let $\zeta(\lambda)$ be a positive continuous monotone nondecreasing function defined for $\lambda \ge 0$,*

$$\hat{\varepsilon} = \sup\{\varepsilon : \quad \text{equation } \ \lambda = \zeta(\lambda)\varepsilon \ \text{ has a solution } \ \hat{\lambda}\}. \tag{4.1}$$

Then, for the arbitrary $\varepsilon \in [0, \hat{\varepsilon})$, the numbers $a_0 = 0$, $a_n = \zeta(a_{n-1})\varepsilon$, $n = 1, 2, \dots$ satisfy the inequality

$$0 < a_n \le \lambda_0(\varepsilon), \tag{4.2}$$

where $\lambda_0(\varepsilon)$ is the minimal solution of the equation from (4.1). *In addition,*

$$\lambda_0(\varepsilon) \to 0 \quad as \ \varepsilon \to 0. \tag{4.3}$$

Proof. Since for an arbitrary $\lambda > 0$ there exists an $\varepsilon > 0$ such that $\lambda = \zeta(\lambda)\varepsilon$, it follows that $0 < \hat{\varepsilon} \le \infty$, where $\hat{\varepsilon}$ is the number (4.1). For any $\varepsilon > 0$ we have $\zeta(0)\varepsilon > 0$, which implies the inequality

$$a_0 = 0 < \lambda_0(\varepsilon).$$

Suppose that inequality (4.2) is established for a certain n. Then (4.2) and the monotonicity of $\zeta(\lambda)$ imply the inequality

$$a_{n+1} = \zeta(a_n)\varepsilon \le \zeta(\lambda_0(\varepsilon))\varepsilon = \lambda_0(\varepsilon) \tag{4.4}$$

which proves (4.2) for all n.

For an arbitrarily small $\lambda > 0$ there exists sufficiently small $\varepsilon > 0$ such that $\zeta(\lambda)\varepsilon = \lambda$. Since $\lambda_0(\varepsilon) \le \lambda$, (4.3) is proved. □

Proof of Theorem 1.2. In order to solve problem (1.13)–(1.15), we apply the iterations similar to that used in the two-dimensional case (see [2]). We set

$$w^0(t,x) = (w_1^0(t,x),\ w_2^0(t,x),\ w_3^0(t,x)) \equiv 0 \tag{4.5}$$

and define the iteration $w^n(t,x)$ as a solution of the controllability problem

$$\partial_t \operatorname{curl} w^n(t,x) - \Delta \operatorname{curl} w^n - \operatorname{curl}(w^{n-1} \times \operatorname{curl} w^n) = 0, \tag{4.6}$$

$$\operatorname{curl} w^n(t,x)|_{t=0} = \operatorname{curl} v_0(x), \quad \operatorname{curl} w^n|_{t=T} \equiv 0. \tag{4.7}$$

It is essential that the solution w^n of (4.6), (4.7) be constructed by the method used in Theorem 3.2. Therefore, by virtue of (3.23),

$$\begin{aligned} &\|e^{s\alpha}(T-t)^{12} \operatorname{curl} w^n\|^2_{H^{(1)}(\Theta)} + \|w^n\|^2_{H^{(2)}(\Theta)} \le \\ &\le \|e^{s\alpha}(T-t)^{12} \operatorname{curl} w^n\|^2_{H^{(1)}(\Theta)} + \|e^{s\hat\alpha}(T-t)^{12} w^n\|^2_{H^{(2)}(\Theta)} \le \\ &\le \gamma(\|w^{n-1}\|_{H^2(\Theta)})\|v_0\|^2_{H^3(\Omega)}, \end{aligned} \tag{4.8}$$

where $\gamma(\lambda)$ is a continuous function. Increasing $\gamma(\lambda)$ when necessary, we can assume that $\gamma(\lambda)$ is a monotone nondecreasing function. Estimate (4.8) and Lemma 4.1 imply that if

$$\|v_0\|^2_{H^3(\Omega)} \equiv \varepsilon \quad \text{is sufficiently small,} \tag{4.9}$$

then

$$\|w^n\|_{H^{(2)}(\Theta)} \le \varkappa < \infty, \quad \|e^{s\hat\alpha}(T-t)^{12} w^n\|_{H^{(2)}(\Theta)} \le \varkappa_1 < \infty, \tag{4.10}$$

$$\|e^{s\alpha}(T-t)^{12} \operatorname{curl} w^n\|_{H^{(1)}(\Theta)} \le \varkappa_2 < \infty, \tag{4.11}$$

where $\varkappa$, $\varkappa_1$, $\varkappa_2$ do not depend on n and tend to zero as $\varepsilon \to 0$.

We denote $y^n = w^{n+1} - w^n$. Subtructing (4.6), (4.7) from the analogous equation for w^{n+1}, we get

$$\partial_t \operatorname{curl} y^n - \Delta \operatorname{curl} y^n - \operatorname{curl}(w^n \times \operatorname{curl} y^n) = -\operatorname{curl}(y^{n-1} \times \operatorname{curl} w^n), \tag{4.12}$$

$$\operatorname{curl} y^n|_{t=0} = \operatorname{curl} y^n|_{t=T} = 0. \tag{4.13}$$

We want to show that y^n satisfies the analog of estimate (3.23).

Let (u^n, p^n) be a solution of problem (2.7)–(2.9), (2.14)–(2.16) with $a = R_1 w^{n-1}$, where R_1 is the extension operator from the proof of Theorem 3.1. We denote $z^n = u^{n+1} - u^n$, $q^n = p^{n+1} - p^n$. By analogy with (4.12) we get equations for z^n and q^n:

$$\partial_t \operatorname{curl} z^n - \Delta \operatorname{curl} z^n - \operatorname{curl}(R_1 w^n \times \operatorname{curl} z^n) = g, \tag{4.14}$$

$$\partial_t \operatorname{curl} q^n + \Delta \operatorname{curl} q^n - \operatorname{curl}(R_1 w^n \times \operatorname{curl} q^n) = e^{2s\alpha} z^n + h \tag{4.15}$$

with the boundary conditions as (4.13), (2.16), where

$$g = -\operatorname{curl}(R_1 y^{n-1} \times \operatorname{curl} u^n), \quad h = -\operatorname{curl}(R_1 y^{n-1} \times \operatorname{curl} p^n). \tag{4.16}$$

First we show that z^n satisfies the following analog of (2.60):

$$\|e^{2s\alpha} z^n\|^2_{L_2(Q)} \le \varkappa_3 \|e^{s\hat{\alpha}}(T-t)^{12} y^{n-1}\|_{H^2(\Theta)}, \tag{4.17}$$

where $\varkappa_3 \to 0$ as $\varepsilon \to 0$ and ε is (4.9). Indeed, scaling in $L_2(Q)$ Eq. (4.14) on q^n and (4.15) on z^n and summing up the obtained equalities, we get, after simple transformations, the relation

$$\int_Q e^{2s\alpha}|z^n|^2\,dx\,dt = -\int_Q ((g, q^n) + (h, z^n))\,dx\,dt. \tag{4.18}$$

Let us estimate the right-hand side of (4.18):

$$\begin{aligned}\|e^{s\alpha} z^n\|^2_{L_2(Q)} \le c_{\varepsilon_1}(\|e^{s\alpha}(R_1 y^{n-1} \times \text{curl } u^n)\|^2_{L_2(Q)} + \\ + \|e^{-s\alpha} h\|^2_{L_2(Q)}) + \varepsilon_1(\|e^{s\alpha} z^n\|^2_{L_2(Q)} + \|e^{-s\alpha} \text{curl } q^n\|^2_{L_2(Q)}),\end{aligned} \tag{4.19}$$

where $\varepsilon_1 > 0$ is sufficiently small. Applying the Carleman estimate (2.42) to the solution q^n of (4.15), we obtain

$$\varepsilon_1 \|e^{-s\alpha} \text{curl } q^n\|^2_{L_2(Q)} \le c\varepsilon_1(\|e^{s\alpha} z^n\|^2_{L_2(Q)} + \|e^{-s\alpha} h^n\|^2_{L_2(Q)}). \tag{4.20}$$

Inequalities (4.19), (4.20) yield the upper bound

$$\|e^{s\alpha} z^n\|^2_{L_2(Q)} \le c\,(\|e^{s\alpha}(R_1 y^{n-1} \times \text{curl } u^n)\|^2_{L_2(Q)} + \|e^{-s\alpha} h\|^2_{L_2(Q)}). \tag{4.21}$$

Taking into account (2.60), (2.10), (2.11), (3.16), we get

$$\begin{aligned}&\|e^{s\alpha}(R_1 y^{n-1} \times \text{curl } u^n)\|^2_{L_2(Q)} \le \\ &\le c\|e^{s\hat{\alpha}}(T-t)^{12} R_1 y^{n-1}\|^2_{(W^{(2)}(Q))^3} \|e^{s\alpha}(T-t)^2 \operatorname{curl} u^n\|^2_{L_2(0,T;H^{-1}(G))} \le \\ &\le c\|e^{s\hat{\alpha}}(T-t)^{12} y^{n-1}\|^2_{H^{(2)}(\Theta)} \|v_0\|^2_{L_2(\Omega)},\end{aligned} \tag{4.22}$$

where $\hat{\alpha}$ is defined by (3.25) and v_0 is the initial condition (2.5).

By analogy with (4.22), we apply (2.42) and estimate h:

$$\begin{aligned}\|e^{-s\alpha} h\|^2_{L_2(Q)} &\le c\|e^{s\hat{\alpha}}(T-t)^{12} y^{n-1}\|^2_{H^{(2)}(\Theta)} \cdot \\ &\cdot \|e^{-s\alpha}(\nabla(T-t)^2 \operatorname{curl} p^n)\|^2_{L_2(Q)}) \le \\ &\le c\|e^{s\hat{\alpha}}(T-t)^{12} y^{n-1}\|^2_{H^{(2)}(\Theta)} \|e^{s\alpha} u^n\|^2_{L_2(Q)}) \le \\ &\le c\|e^{s\hat{\alpha}}(T-t)^{12} y^{n-1}\|^2_{H^{(2)}(\Theta)} \|v_0\|_{L_2(\Omega)}.\end{aligned} \tag{4.23}$$

Now (4.17) follows from (4.21), (4.22), (4.23).

Since the equality $R_1 y^{n-1} = 0$ holds in the neighborhood of ∂G, we get, as in (4.22), with due account of (3.11), the estimate of g defined by (4.16):

$$\|g\, e^{s\alpha}\|^2_{L_2(0,T;\,W_2^1(G))} \leq$$
$$\leq c\|e^{s\hat{\alpha}}(T-t)^{12} R_1 y^{n-1}\|^2_{(W^{(2)}(Q))^3} \|e^{s\alpha}(T-t)^{12}\rho^6 \operatorname{curl} u^n\|^2_{H^{(1)}(G)} \leq$$
$$\leq c\|e^{s\hat{\alpha}}(T-t)^{12} y^{n-1}\|^2_{H(2)(\Theta)} \|v_0\|^2_{H^3(\Omega)}. \tag{4.24}$$

Taking into account (4.17), (4.24) and applying Lemmas 3.1–3.3 and Theorem 3.1 to the solution z^n of (4.14), we get the following analog of (3.21):

$$\|e^{s\alpha}(T-t)^{12} \operatorname{curl} z^n\|^2_{H^{(1)}(\Theta)} + \|z^n\|^2_{L_2(\Theta,s\alpha)} \leq$$
$$\leq \gamma(\|g\|^2_{L_2^1(\Theta,s\alpha)} + \|g_1\|^2_{L_2^0(\Theta,s\alpha)}) \leq$$
$$\leq c\|v_0\|^2_{H^3(\Omega)} \|e^{s\hat{\alpha}}(T-t)^{12} y^{n-1}\|^2_{H^{(2)}(\Theta)}. \tag{4.25}$$

Since z^n and y^n are connected by the Weyl decomposition $z^n = y^n + \nabla\psi$, estimate (4.25) yields the following analog of (3.23) due to Theorem 3.2:

$$\|e^{s\alpha}(T-t)^{12} y^n\|^2_{H^{(2)}(\Theta)} \leq \varkappa_4 \|e^{s\hat{\alpha}}(T-t)^{12} y^{n-1}\|^2_{H^{(2)}(\Theta)}, \tag{4.26}$$

where $\varkappa_4 \to 0$ as $\varepsilon \to 0$ and ε is defined by (4.9). Hence

$$\|e^{s\hat{\alpha}}(T-t)^{12} y^n\|^2_{H^{(2)}(\Theta)} \leq c\, 2^{-n}$$

and, therefore, a unique solution v of problem (1.13)–(1.15) exists. Moreover,

$$\|e^{s\hat{\alpha}}(T-t)^{12} v\|^2_{H^{(2)}(\Theta)} < \infty. \tag{4.27}$$

Estimate (4.27) implies the upper bound (1.10). □

As we can now show, Theorem 1.1 follows immediately from Theorem 1.2.

Proof of Theorem 1.1. Let $v(t,x)$ be the solution of the problem (1.13)–(1.15) constructed in the proof of Theorem 1.2. We can show that $v(t,x)$ is a solution of the Navier–Stokes equation (1.1) with a suitable ∇p. Indeed, (1.13) implies that

$$\operatorname{curl}(\partial_t v - \Delta v - v \times \operatorname{curl} v) = 0 \tag{4.28}$$

and (4.28) can be rewritten in the form

$$\partial_t v - \Delta v - v \times \operatorname{curl} v = \nabla p_1, \tag{4.29}$$

where $p_1 \in L_2(0,T;W^2(\Omega))$ is a certain function. Now (1.1) follows from (4.29), (1.12). Other assertions of Theorem 1.1 are obvious corollaries of Theorem 1.2. □

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Author's address:
Department of Mechanics and Mathematics
Moscow State University
119899 Moscow, Russia
E-mail: fursikov@dial01.msu.ru
fursikov@sci.fian.msk.su