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# EFFECTIVELY INFINITE CLASSES OF WEAK CONSTRUCTIVIZATIONS OF MODELS

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The study of computable classes of constructivizations is a central trend in the present-day constructive model theory [1-5]. The key pursuit in this direction is to construct numerations with specified properties, of which the most interesting is a uniform construction of an effective representation of a model, given its specification. The range of problems mentioned is of interest from the standpoint of both mathematical logic and foundations of the theory of information processing, which deals with the construction of semantics for specification and programming languages with abstract data types. One of the main methods used to prove the infinity for classes of constructive models is based on proving that they are effectively infinite.

The concept of an effectively infinite class of constructivizations was introduced in [6], where it was used to provide a characterization of nonautostable models, to study the computability of diverse classes of models, and to solve the problem of constructing a recursive model that satisfies a given specification or has a specified decidable problem. The concept is closely related to studies dealing with algorithmic dimensions of models and with the description of models of infinite algorithmic dimension and of noncomputable classes of constructivizations [5-12].

In the present paper, we use the method to give a complete solution of the problem raised by Nurtazin in [12], which asks whether or not the class of weak constructivizations of strongly constructivizable models admitting weak constructivizations is computable. In [6], it was proved that for 1-constructivizable models, the class of constructivizations that are not 1-constructivizations is not computable if such a model has at least one constructivization that is not a 1-constructivization. The notion of $n$-constructivity, $n \in \omega$, which is intermediate between constructivity and strong constructivity, was introduced in [13, 14].

We say that a model $(\mathfrak{M}, \nu)$ of signature $\Sigma$ is *constructive of finite type* $n$ if it is $n$-constructive, i.e., for the enrichment $\mathfrak{M}^*$ of $\mathfrak{M}$ by the constants $\{c_n \mid n \in \mathrm{N}\}$ such that the value of $c_n$ in $\mathfrak{M}^*$ is equal to $\nu(n)$, there exists an effective procedure verifying whether the formulas in $F_n$ (formulas with $n$ alternations of quantifiers in prenex normal form) are true.

The class of formulas in the signature $\Sigma\cup\{c_n \mid n \in \mathrm{N}\}$, true in $\mathfrak{M}^*$, is denoted by $\mathrm{Th}\,(\mathfrak{M}, \nu)$, and we write $\mathrm{Th}_n\,(\mathfrak{M}, \nu)$ for the class of formulas in $F_n$, true in $\mathfrak{M}^*$. A sequence of numerated models $(\mathfrak{M}_n, \lambda m\gamma(n, m))$ is called *k-computable* (*strongly computable*) if there exists an effective procedure enumerating all formulas from the corresponding set $\mathrm{Th}_k\,(\mathfrak{M}_n, \lambda m\gamma(n, m))$ $[\mathrm{Th}\,(\mathfrak{M}_n, \lambda m\gamma(n, m))]$ uniformly in $n$.

We note that the notions of a 0-constructivization and a 0-constructive model are equivalent to the standard notions of a constructivization and a constructive model, respectively. To study the algorithmic properties of the class of formulas in the signature $\Sigma \cup \{c_n \mid n \in \mathrm{N}\}$, we fix a Gödel numbering [15].

The question of whether all constructivizations of a model are strong is closely related to the completeness of the model; a similar connection exists between $n$-constructivity and $n$-completeness. Recall

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that if, for any formula $\varphi(x_0, \ldots, x_n)$ in prenex normal form with at most $n$ alternations of quantifiers and for a tuple of elements $a_0, \ldots, a_m$ of $\mathfrak{M}$, the relation $\mathfrak{M} \models \varphi(a_0, \ldots, a_m)$ implies that there exists a $\exists$-formula $\psi(x_0, \ldots, x_m)$ for which $\mathfrak{M} \models \psi(a_0, \ldots, a_m)$ and $\mathfrak{M} \models (\forall x_0, \ldots, x_m)(\psi \rightarrow \varphi)$, such a model is called *n-complete* [6].

In [6, 9], we asked if, for models having $(n+1)$-constructivizations, the class of constructivizations that are not $(n+1)$-constructivizations is effectively infinite. The question posed is connected to Nurtazin's problem on the noncomputability of the class of weak constructivizations for models having strong and weak constructivizations. In particular, in [9] the problem was positively solved for *limit-n-complete models*, i.e., models that are $n$-complete in some finite enrichment by constants but are not $(n+1)$-complete in any such finite enrichment.

We call a model $\mathfrak{M}$ *limit-ω-complete* if, for any $n$, there exists a finite $n$-complete enrichment of $\mathfrak{M}$ by constants, but there exists no finite complete enrichment by constants. A constructivization is said to be *weak* [12] if it is not strong.

The results concerning the noncomputability of a class of weak constructivizations were announced in [6]. In [9], the following theorem was proved.

THEOREM. If a model $\mathfrak{M}$ is limit-$n$-complete and has an $(n+1)$-constructivization, then, given a computable class $S$ of constructivizations of $\mathfrak{M}$, we can effectively build a constructivization of $\mathfrak{M}$ that is not autoequivalent to any constructivization in $S$ and is not an $(n+1)$-constructivization.

At the same time, it is easy to see [6] that every constructivization of an $n$-complete and $n$-constructivizable model is an $n$-constructivization, and every constructivization of a complete and strongly constructivizable model is strong. Note also that a computable class of strong constructivizations ($k$-constructivizations) is not always strongly computable ($k$-computable).

The present paper concludes our study of the class of weak constructivizations for strongly constructivizable models. We shall deal with constructivizations of limit-$\omega$-complete, strongly constructivizable models.

Let us introduce some definitions and notation, which will be used below. The definitions and the main results developed in algorithm theory are borrowed from [16, 17], and in the constructive model theory — from [15].

The set of natural numbers $\{0, \ldots, n, \ldots\}$ is denoted by N, and we often write $\bar{a}$ to denote a tuple of elements $(a_0, \ldots, a_n)$. If we have $(a_0, \ldots, a_n)$ or $\bar{a}$ and some unary function $g$, then $g(a_0, \ldots, a_n)$ or $g(\bar{a})$ will stand for the tuple $(g(a_0), \ldots, g(a_n))$.

We say that the function $f\colon \mathrm{N} \rightarrow \mathrm{N}$ *determines an automorphism* of the numerated model $(\mathfrak{M}, \nu)$ onto $(\mathfrak{M}, \mu)$ if the mapping $\varphi$ from $\mathfrak{M}$ to $\mathfrak{M}$, induced by this function in such a way that $\varphi(\nu(n)) \rightleftharpoons \mu f(n)$ for all $n$, determines an automorphism of $\mathfrak{M}$.

For a formula $\varphi(c_0, \ldots, c_n)$ in the signature $\Sigma \cup \{c_n |\ n \in \mathrm{N}\}$ and for a partial function $f\colon \mathrm{N} \rightarrow \mathrm{N}$, the formula obtained from $\varphi(c_0, \ldots, c_n)$ by replacing each occurrence of the constant $c_i$ by $c_{f(i)}$ is called an *f-image* in the model $(\mathfrak{M}, \mu)$ under the induced mapping from the numerated model $(\mathfrak{M}, \nu)$ to $(\mathfrak{M}, \mu)$. Similarly, every formula obtained from $\varphi(c_0, \ldots, c_n)$ by replacing each occurrence of $c_i$ by $c_j$ such that $f(j) = i$ is called an *f-preimage* in the model $(\mathfrak{M}, \mu)$ under the induced mapping from $(\mathfrak{M}, \nu)$ to $(\mathfrak{M}, \mu)$. If it is clear from the context which model is a "transmitter" and which one is a "receiver," the phrase "under the induced mapping from the numerated model $(\mathfrak{M}, \nu)$ to $(\mathfrak{M}, \mu)$" will be omitted.

If $\varphi(c_0, \ldots, c_n, \bar{x}, \bar{y})$ is a formula in the signature $\Sigma \cup \{c_n |\ n \in \mathrm{N}\}$ with tuples of variables and constants $\bar{x}$, $\bar{y}$, and $\bar{x} = (x_{i_0}, \ldots, x_{i_k})$, and $\bar{m}$ is a set of indices $(m_0, \ldots, m_k)$, by $\varphi(c_0, \ldots, c_n, c_{\bar{m}}, \bar{y})$ and

$[\varphi(c_0,\ldots,c_n,\bar{x},\bar{y})]^{\bar{x}}_{c_{\bar{m}}}$ we denote the formula obtained from $\varphi(c_0,\ldots,c_n,\bar{x},\bar{y})$ by replacing all free occurrences of $x_{i_s}$ by $c_{m_s}$.

The main result that concludes our study of computable classes of weak constructivizations is the following:

**THEOREM 1.** If a model $\mathfrak{M}$ is limit-$\omega$-complete and has a strong constructivization, then, given any computable class $S$ of constructivizations of $\mathfrak{M}$, we can effectively build a constructivization of $\mathfrak{M}$ that is not autoequivalent to any constructivization in $S$ and is not strong.

Proof. If $\nu$ is a strong constructivization of $\mathfrak{M}$, then $(\mathfrak{M},\nu)$ is a strongly constructive model such that for any $n$, the model $(\mathfrak{M},a_0,\ldots,a_n)$ is $n$-complete for some enrichment by the constants $a_0,\ldots,a_n$, but $\mathfrak{M}$ is not complete in any such finite enrichment. Without loss of generality, we may assume that the enriched model $(\mathfrak{M},a_0,\ldots,a_{n'})$ is 2-complete, and $\nu(i)=a_i$ for $i\leq n'$.

Let $(S,\gamma)$ be a computable class of constructivizations of $\mathfrak{M}$, where $\lambda n\gamma(m,n)\colon \mathrm{N}\to|\mathfrak{M}|$ is a constructivization for which, given $m$, we can effectively construct an algorithm verifying the truth of quantifier-free formulas at the elements of $\mathfrak{M}$, given a formula and a set of numbers of these elements.

In our construction, we also need a universal computable function $f_n(x)$ for the class of unary partial recursive functions [16]; moreover, the value $f_n^t(x)$ is defined and equals $f_n(x)$ if $f_n(x)$ is computed in no more than $t$ steps and $n,x\leq t$; otherwise, $f_n^t(x)$ is undefined.

We proceed step-by-step to define the following: a partial numeration $\mu^t$ of $\mathfrak{M}$, a function $\bar{\mu}^t$ which reduces this numeration to a strong constructivization $\nu$, and a partial diagram $D^t$ of $\mathfrak{M}$ consisting of finitely many quantifier-free formulas in the signature $\Sigma\cup\{c_0,\ldots,c_n,\ldots\}$. At any step $t$, for every formula $\varphi(c_0,\ldots,c_k)$ in $D^t$, a formula $\varphi(x_0,\ldots,x_k)$ is satisfied in $\mathfrak{M}$ at the tuple of elements $(\mu^t(0),\ldots,\mu^t(k))$, and $\mu^t(s)=\nu\bar{\mu}^t(s)$. Note that since $\nu$ is a strong constructivization, for any number $t$, tuple $\bar{m}$, and formula $\phi$, we can effectively determine whether or not $\phi$ is satisfied in $\mathfrak{M}$ at the elements $\mu^t(\bar{m})$.

In the construction, we will use markers of two kinds: $[m,k]$, $k$, $m\in\omega$ and $[m]$, $m\in\omega$, $[-1]$, which are attached to $\mu$-numbers at step $t$. A marker of the first kind — $[m,k]$ — is applied to violate the reducibility to $\lambda s\gamma(m,s)$ of the numeration $\mu$ constructed via the function $f_k$; a marker of the second kind — $[m]$ — is used to violate the decision procedure for $\mathrm{Th}(\mathfrak{M},\mu)$, defined by the function $f_m$; finally, $[-1]$ labels a set of numbers $\{0,\ldots,n'\}$ to the elements of which it is attached constantly.

The pairs $\langle\forall\bar{y}\Phi(\bar{c},\bar{x},\bar{y}),\bar{m}_0\rangle$ and the label $*$ will be attached to $[m,k]$; with each such $[m,k]$ having an attached pair $\langle\forall\bar{y}\Phi(\bar{c},\bar{x},\bar{y}),\bar{m}_0\rangle$ numbered $m_0$ we associate the sequence of triples

$$\langle\forall\bar{y}^1\Phi(\bar{c}^1,\bar{x}^1,\bar{y}^1),\bar{m}_1,t_1\rangle,\ldots,\langle\forall\bar{y}^s\Phi(\bar{c}^s,\bar{x}^s,\bar{y}^s),\bar{m}_s,t_s\rangle,$$

where the pairs

$$\langle\forall\bar{y}^1\Phi(\bar{c}^1,\bar{x}^1,\bar{y}^1),\bar{m}_1\rangle,\ldots,\langle\forall\bar{y}^s\Phi(\bar{c}^s,\bar{x}^s,\bar{y}^s),\bar{m}_s\rangle$$

have respective numbers $m_1,\ldots,m_s$ and $m_1,\ldots,m_s$ less than $m_0$.

The number of alternations of quantifiers in $\forall\bar{y}\Phi(\bar{c},\bar{x},\bar{y})$ is called the *complexity* of the formula $\forall\bar{y}\Phi(\bar{c},\bar{x},\bar{y})$ and the pair $\langle\forall\bar{y}\Phi(\bar{c},\bar{x},\bar{y}),\bar{m}_0\rangle$, attached to $[m,k]$.

The pair $\langle\forall\bar{y}\Phi(\bar{c},\bar{x},\bar{y}),\bar{m}_0\rangle$ on $[m,k]$ is chosen to be a formula that is potentially true at a tuple $\gamma(m,f_k(\bar{m},\bar{m}_0))$ in the model $(\mathfrak{M},\lambda n\gamma(m,n))$, where $\bar{m}$ are indices of constants from $\bar{c}$, and we will construct the numeration $\mu$ in such a way that $\forall\bar{y}\Phi(\bar{c},\bar{x},\bar{y})$ is false in the model at a tuple of elements numbered $\bar{m}_0$. For the formula $\forall\bar{y}\Phi(\bar{z},\bar{x},\bar{y})$, however, we cannot effectively verify whether it is true in $(\mathfrak{M},\ \lambda n\gamma(m,n))$ at the elements numbered $f_k(\bar{m},\bar{m}_0)$. What we can do is to determine at some step, under some additional conditions, that it is false. Let $\langle\forall\bar{y}^s\Phi(\bar{c}^s,\bar{x}^s,\bar{y}^s),\bar{m}_s\rangle$ be a pair with number less than $m_0$ such that we cannot identify its falsity, and at some step $t_s < t$, it will be impossible to force $\langle \forall \bar{y}^s \Phi(\bar{c}^s, \bar{x}^s, \bar{y}^s), \bar{m}_s \rangle$ to be false at $\bar{m}_0$ in the model $(\mathfrak{M}, \nu)$ under construction. Then, with $[m, k]$ to which the pair $\langle \forall \bar{y} \Phi(\bar{c}, \bar{x}, \bar{y}), \bar{m}_0 \rangle$ is attached, we associate the triple $\langle \forall \bar{y}^s \Phi(\bar{c}^s, \bar{x}^s, \bar{y}^s), \bar{m}_s, t_s \rangle$. Note that the existence of a tuple at which $\exists \bar{y}' D^{t_s}(\bar{c}^k, \bar{x}, \bar{y}')$ is true and $\forall \bar{y} \Phi(\bar{c}^k, \bar{x}, \bar{y})$ is false means that the function $f_k$ cannot determine an isomorphism of $(\mathfrak{M}, \mu)$ onto $(\mathfrak{M}, \lambda n\ \gamma(m, n))$. However, at step $t$, given a fixed set of numbers $\bar{m}$ and a natural number $n$, we cannot determine whether or not the model $(\mathfrak{M}, \gamma(m, \bar{m}))$ is $n$-complete. As opposed to the case of an $n$-complete model [9], here attaching a pair to $[m, k]$ does not guarantee the following. If a formula of the pair is not true, and at the elements of $\bar{m}$, $f_k$ determines a partial isomorphism extended to the automorphism, or if a wrong pair is chosen, which does not violate the reducibility, then the falsity of the formula can be identified. Therefore, we attach to $[m, k]$ several markers, but each of the succeeding pairs on $[m, k]$ contains a formula of lower complexity, i.e., markers will be attached in order of decreasing complexity. The markers attached to the pairs are ordered as follows: markers of different complexity are ordered by complexity; the ordering of markers of equal complexity respects the numeration of pairs, consisting of a Gödel number of a formula and a number of the set of numbers for the corresponding marker.

For our purposes, we need a procedure that correctly verifies whether the formulas are true in the models $(\mathfrak{M}, \lambda n \gamma(m, n))$ for all important cases.

Let us denote this procedure by $\mathrm{F}(\bar{m}, m, k, s, x, t)$. For $s \in N$, we define it as an "approximation" of the truth verification for formulas in $\mathrm{Th}_s(\mathfrak{M}, \lambda y \gamma(m, y))$ with respect to the function $f_k$; the latter determines a partial isomorphic embedding at a tuple of elements $\bar{m}$ for which $f_k^t$ is assumed to be defined.

For $k = 0$, the value of $\mathrm{F}(\bar{m}, m, k, 0, x, t)$ is defined if $f_k^t$ is defined for the elements of $\bar{m}$ and determines a partial isomorphic embedding of the model $(\mathfrak{M}, \mu^t)$ in $(\mathfrak{M}, \lambda n \gamma(m, n))$; $\mathrm{F}(\bar{m}, m, k, 0, x, t) = 1$ if $x$ is a Gödel number of a quantifier-free formula in the signature $\Sigma \cup \{c_0, c_1, \ldots, c_n, \ldots\}$, and the formula

$$[\varphi]^{c_0, \ldots, c_\alpha}_{\gamma(m,0), \ldots, \gamma(m,\alpha)}$$

is true in $\mathfrak{M}$; otherwise, $\mathrm{F}(\bar{m}, m, s, 0, x, t) = 0$.

Suppose that, for $p \leq k$, the values of $\mathrm{F}(\bar{m}, m, s, p, x, t)$ are defined. We define the values for $\mathrm{F}(\bar{m}, m, s, k+1, x, t)$ as follows. First, consider the set $\bar{m}$ of $\mu$-numbers. If $f_k^t$ is not defined at least for one element of $\bar{m}$ or does not determine a partial isomorphic embedding of $(\mathfrak{M}, \mu^t)$ in $(\mathfrak{M}, \lambda n \gamma(m, n))$, then, for all $x$, the value of $\mathrm{F}(\bar{m}, m, s, k+1, x, t)$ is undefined. Otherwise we consider the following four cases.

Case 1. For the tuple $(m, s, k+1, x, t, \bar{m})$, one of the following conditions holds:

(1) $x$ is a Gödel number of a formula of the form $\exists z_0, \ldots, z_q \varphi$, where $\varphi$ has at most $k$ alternations of quantifiers, and for some indices $i_0, \ldots, i_q$ and the Gödel number $z$ of the formula

$$[\varphi]^{z_0, \ldots, z_q}_{c_{i_0}, \ldots, c_{i_q}},$$

we have $\mathrm{F}(\bar{m}, m, k, s, z, t) = 1$.

(2) $x$ is a Gödel number of $\forall z_0, \ldots, z_q \varphi$, where $\varphi$ has at most $k$ alternations of quantifiers; there exists a $\exists$-formula

$$\exists y_0, \ldots, y_{m_0}\ \psi(x'_0, \ldots, x'_k, x_0, \ldots, x_{m_1}, y_0, \ldots, y_{m_0})$$

such that its Gödel number is less than $t$; there exists a tuple $s_0, \ldots, s_{m_0} \leq t$ for which

$$\mathfrak{M} \models \psi(\gamma(m, f_k(\bar{m})), \gamma(m, \bar{m}_1),\ \gamma(m, s_0), \ldots, \gamma(m, s_{m_0})),$$

and the formula

$$(\forall \bar{x}_{\bar{m}_1})\left((\exists y_0,\dots,y_{m_0})\psi(\mu^t(\bar{m}),\bar{x}_{\bar{m}_1},y_0,\dots,y_m)\to\left[[(\forall z_0,\dots,z_q)\varphi]^{\bar{c}_{\bar{m}_1}}_{\bar{x}_{\bar{m}_1}}\right]\right)$$

is true in $(\mathfrak{M},\nu\mu^t(\bar{m}))$, where $\bar{m}_1$ is the set of indices of constants in the formula $\forall z_0,\dots,z_q\varphi$.

Case 2. For the tuple $(m,s,k+1,x,t,\bar{m})$, one of the following conditions holds:

1) $y$ is a Gödel number of the formula in prenex normal form, obtained by the standard reducing procedure from the negation of the formula with Gödel number $x$, which has at most $k+1$ alternations of quantifiers in prenex normal form, and case 1 holds for $\mathrm{F}(\bar{m},m,s,k+1,y,t)$.

2) $y$ is a Gödel number of a formula with more than $k+1$ alternations of quantifiers.

In what follows, we denote by $\neg x$ a Gödel number of the formula in prenex normal form, obtained from the formula with Gödel number $x$ using the standard reducing procedure.

Case 3. For $(m,s,k+1,x,t,\bar{m})$, cases 1 and 2 fail.

Case 4. For $(m,s,k+1,x,t,\bar{m})$, cases 1 and 2 hold.

If, for $(m,s,k+1,x,t,\bar{m})$, case 1 holds but case 2 fails, we put $\mathrm{F}(\bar{m},m,s,k+1,x,t)=1$. If, to the contrary, case 2 holds but case 1 does not, we put $\mathrm{F}(\bar{m},m,s,k+1,x,t)=0$. If case 3 holds, put $\mathrm{F}(m,s,k+1,x,t,\bar{m})=1$ (an undefined value). If case 4 is satisfied, put $\mathrm{F}(\bar{m},m,s,k+1,x,t)=\top$ (an overdefined value). Consider the value of the function $\mathrm{F}(\bar{m},m,s,k+1,x,t)$. Suppose that the tuple $\bar{m}$ of elements is not specified. Then such a tuple should be assumed to consist of those elements which, at step $t$, either have markers less than $[m,k]$ or attached to them is $[m,k]$ itself, with markers of higher complexity than the candidate considered.

To define the ordering of type $\omega$ on the set of markers of the first and the second kinds, with $[m,k]$ we associate the Gödel number of the set $\langle m,k\rangle$, and with $[m]$ the number of the tuple $\langle m\rangle$; $[-1]$ is the least marker.

There is no loss of generality in assuming that the constructivizations $\lambda n\gamma(m,n)$ for any $m\in N$ and $\nu$ are one-to-one [18].

We proceed to define our construction by steps.

Step 0. Define $\mu^0(i)=\nu(i)$ for $i=k$, $\bar{\mu}^0(i)=i$ for $i\le k$, $D^0\ne\emptyset$, $r(m,k,s,0)=0$; attach $[-1]$ to $i\le k$.

Step $T+1=2t+1$. We verify if there exists a marker $M$ with number less than $t$ and such that one of the following cases holds:

A. $M$ is equal to $[m,k]$ and is not labeled by $*$. We have two possibilities.

Subcase (1): One of the following is satisfied:

(1.1) The mapping $\lambda n(f_k^{2t+1}(\bar{\mu}^t)^{-1}(n))$ does not determine an isomorphic embedding of a submodel of $\mathfrak{M}$ in $\mathfrak{M}$.

(1.2) For $x\le t$ a number of a formula in $F_s$, either $\mathrm{F}(\bar{m},m,k,s,x,t)=\top$ (i.e., an overdefined value) for some $s$, or the value of $\mathrm{F}(\bar{m},m,k,s,x,t)$ is not equal to the truth value of an $f_k$-preimage, where $f_k^T$ is defined at the elements of $\bar{m}$.

(1.3) $M$ has a marker $\langle\forall y_0,\dots,y_k\ \Phi(\bar{c},\bar{x}_0,\bar{y}),\bar{m}_0'\rangle$ with number $p$ of complexity $n+1$, and associated with them is a triple $\langle\forall y_0,\dots,y_k\ \Phi(\bar{c},\bar{x}_0,\bar{y}),\bar{m}_0',t_0\rangle$ with number $m_0$ of the pair $\langle\forall y_0,\dots,y_k\ \Phi(\bar{c},\bar{x}_0,\bar{y}),\bar{m}_0\rangle$ less than $p$. Moreover, the function $f_k^{2t}$ is defined at $\bar{m}_0'$ and there exist tuples $\bar{m},\bar{h}$ consisting of elements less than $t$, $n_0,\dots,n_s\le t$, such that for a Gödel number $G_m^n$ of the formula

$$\left[\neg\Phi\right]^{\bar{c},\ \bar{x},\ y_0,\dots,y_s,}_{\bar{c}_{f_k(\bar{m}_0)},\ \bar{c}_{\bar{m}},\ c_{n_0},\dots,c_{n_s},}$$

we have $\mathrm{F}(\bar{m}, m, k, n, G^n_m, t) = 1$, and the formula

$$[\& \, D^{t_0}(\bar{c}, \bar{c}_0, \bar{c}_1)]^{\bar{c}, \bar{c}_0, \bar{c}_1}_{\bar{x}, \bar{x}_0, \bar{x}_1}$$

is true in the constructive model $(\mathfrak{M}, \lambda n \gamma(m, n))$ at the elements with numbers $f_k(\bar{m}_0)$, $\bar{m}$, $\bar{h}$. Here $\bar{m}_0$ are indices of constants from $\bar{c}$; $\bar{x}_0$ is a tuple of variables with the same indices as in $\bar{c}$; the indices of constants from $\bar{c}$ consist of those elements which either have markers less than $M$, or they have $M$ to which a marker of complexity higher than $n$ is attached. The set $\bar{c}'$ of all constants of the formulas in $D^{2t}$ is divided into three subsets: $\bar{c}$ are defined immediately above; $\bar{c}_0$ are constants with indices from $\bar{m}_0'$; $\bar{c}_1$ are all the remaining constants. In the tuples $\bar{x}$ and $\bar{c}$, $\bar{x}_0$ and $\bar{c}_0$, $\bar{x}_1$ and $\bar{c}_1$, the corresponding variables and constants have equal indices.

(1.4) $M$ has a marker $\langle \forall y_0, \ldots, y_k \; \Phi(\bar{c}, \bar{x}_0, \bar{y}), \bar{m}_0' \rangle$ numbered $p$ of complexity $n+1$. Moreover, at step $T$, associated to $\langle \forall y_0, \ldots, y_k \; \Phi(\bar{c}, \bar{x}_0, \bar{y}), \bar{m}_0' \rangle$ is a subformula $t(M)(\bar{c}, \bar{x}, \bar{y})$ of $\forall y \Phi(\bar{c}, \bar{x}, \bar{y})$, obtained by the elimination of quantifiers $\forall y_0, \ldots, y_k$, and a tuple of elements $\bar{m}_0^T$ expanding $\bar{m}_0$ and satisfying the following conditions. The formula $\neg\Phi(\bar{c}, \bar{x}_0, \bar{y})$ is true in $(\mathfrak{M}, \mu^T)$ at the elements $\mu^T(\bar{m}_0^T)$; $f_k^T$ is defined for $\bar{m}_0^T$; for a Gödel number $G$ of the formula

$$\left[\Phi\right]^{\bar{c}, (\bar{x}, \bar{y}),}_{\bar{c}_{f_k(\bar{m}_0)}, \bar{c}_{f_k(\bar{m}_0^T)},}$$

$\mathrm{F}(\bar{m}', m, k, n, G, t) = 1$, with some tuple $\bar{m}'$ for the elements of which $f_k^T$ is defined. Here $\bar{m}_0$ are indices of constants from $\bar{c}$, consisting of those elements which have markers less than $M$ and $[m, k]$, with markers whose complexity is higher than the complexity of the formula $\forall y_0, \ldots, y_k \; \Phi(\bar{c}, \bar{x}_0, \bar{y})$.

We consider the second possibility.

Subcase (2): (1) fails; we have

$$\mathrm{Rang}\, f_k^t \cap \mathrm{Dom}\, f_k^t \supseteq \{0, \ldots, r(m, k, 0, 2t)\},$$

and there exist numbers $n$, $p \leq T$ such that one of the conditions (2.1) or (2.2) is satisfied.

(2.1) $M$ is marked by the pair $\langle \forall y_0, \ldots, y_k \; \Phi'(\bar{c}, \bar{x}_0', \bar{y}'), \bar{m}_0' \rangle$ with number $m_0'$ of complexity $n+1$, and there exists a tuple $n_0, \ldots, n_s \leq t$ such that for a Gödel number $G^n_m$ of the formula

$$\left[\neg\Phi'\right]^{\bar{c}, \bar{x}_0', y_0, \ldots, y_s,}_{\bar{c}_{f_k(\bar{m}, \bar{m}_0)}, c_{n_0}, \ldots, c_{n_s},},$$

we have $\mathrm{F}(\bar{m}', m, k, n, G^n_m, t) = 1$ for some tuple $\bar{m}'$, for the elements of which $f_k^T$ is defined. Here $\bar{m}$ are indices of constants in the tuple $\bar{c}$, with markers less than $M$.

(2.2) $M$ either is not marked or has a marker with number $i > p$ of complexity $n+1$. In addition, for the markers on $[m, k]$ with numbers less than $p$, the lowest complexity of a marker on $M$ is higher than $n+1$. For any $s \leq n$ and $z \leq r(m, k, s, t)$, where $z$ is a number of an $s$-formula, the value of $\mathrm{F}(\bar{m}, m, k, s, z, t)$ is defined and equals 0 or 1. Further, suppose that a number of $\langle \forall y_0, \ldots, y_k' \; \Phi(\bar{c}, \bar{x}, \bar{y}), \bar{m}_0 \rangle$ is equal to $p$, and its complexity — to $n+1$. Assume that there are no tuples $\bar{n}', n_0', \ldots, n_{k'}' \leq t$ such that $\mathrm{F}(\bar{m}, m, k, n, G^{n'}_p, t) = 0$, where $G^{n'}_p$ is a Gödel number of the formula

$$\left[\Phi\right]^{\bar{x}, \bar{x}_0, y_0, \ldots, y_{k'},}_{\bar{c}_{f_k(\bar{m}, \bar{m}_0')}, c_{n_0}', \ldots, c_{n_{k'}'}',}.$$

Finally, let $F(\bar{m}, m, k, n, G_p^{n'}, t) = 1$ for all $\bar{n}'$, $n'_0, \ldots, n'_{k'} \leq t$ such that $G_p^{n'} \leq r(m, k, 2t)$ and let the following formula be true in $(\mathfrak{M}, \mu^{2t})$:

$$\exists \bar{x}\, \exists \bar{x}_1\, \exists \bar{y}\, \left[\neg\Phi(\bar{c}, \bar{x}, \bar{y})\ \&\ \&\, D^{2t}\right]_{\bar{x}_1\ \bar{x}}^{\bar{c}_1\ \bar{c}_0},$$

where the set $\bar{c}'$ of all constants of the formulas in $D^{2t}$ is divided into three subsets: $\bar{c}_0$ are the constants whose indices lie in $\bar{m}'_0$; $\bar{c}$ are the constants whose indices either have markers less than $M$ or $M$ with a pair of complexity higher than $n+1$; $\bar{c}_1$ are all the remaining constants. The corresponding indices in $\bar{x}_1$ and $\bar{c}_1$ coincide; the tuples $\bar{c}_0$ and $\bar{c}_{\bar{m}'_0}$ are equal.

We proceed to condition B.

B. $M = [k]$, is not labeled by $*$, and satisfies the following condition:

There exists $G \leq t$ such that the function $f_k^{2t}$ is defined at the Gödel number $G$ of the formula $\forall z_0, \ldots, z_k\ \Phi_0(\bar{c}, \bar{z})$ in $\mathrm{Th}_n(\mathfrak{M}, \mu)$; its value is 1. Suppose that $(\exists \bar{x}\bar{y})(\&\, D^T(\bar{c}_0, \bar{x}, \bar{y})\ \&\ (\exists \bar{z})\ \neg\Phi_0(\bar{c}_0, \bar{x}, \bar{z})$ is satisfied in $(\mathfrak{M}, \mu^t)$ at the tuple $\bar{c}_0$, consisting of constants whose indices have markers less than $M$; $\bar{x}$ are variables with indices $i$ for which the function $\mu^{2t}$ is defined, but $c_i$ is not in $\bar{c}_0$.

If such an $M$ does not exist, we proceed to the final stage $Z$ of the step, described below.

If such markers $M$ exist, among them we choose one that is least.

If, for $M$, subcase (1) of A holds, and $M = [m, k]$, then we mark by $M$ all the numbers $i$ for which $\mu^{2t}(i)$ is defined, attach $*$ to $M$, and pass to the final stage $Z$ of the step.

If, for $M$, subcase (2) of A holds, we choose the least $p$ satisfying (2). Suppose that condition (2.1) holds for $M$ and the pair $\langle \forall y_0, \ldots, y'_k\ \Phi'(\bar{c}, \bar{x}'_0, \bar{y}'), \bar{m}'_0 \rangle$ with number $m'_0$ of complexity $n+1$. Then we must remove all markers of complexity at most $n+1$ from $M$ and remove $M$ with these markers from all numbers to which they are attached. Define $r(m, k, s, 2t+1) = r(m, k, s, 2t) + 1$ for $s \leq n$ and pass to the final stage $(Z)$.

If, for $M$, conditions (2.2) are satisfied, we consider the pair $\langle \forall y_0, \ldots, y_k\ \Phi(\bar{c}, \bar{x}, \bar{y}), \bar{m} \rangle$ with number $p$ for which these conditions are met. By (2.2), the formula

$$\exists \bar{x}\, \exists \bar{x}_1\, \exists \bar{y}\, \left[\neg\Phi(\bar{c}, \bar{x}, \bar{y})\ \&\ \&\, D^{2t}\right]_{\bar{x}_1\ \bar{x}}^{\bar{c}_1\ \bar{c}_0}$$

holds in $(\mathfrak{M}, \mu^{2t})$, provided the constants are appropriately divided into groups $\bar{c}_0$ and $\bar{c}_1$. We find the tuples $(\bar{n}_0, \bar{n}_1, \bar{n}_2)$ such that in $(\mathfrak{M}, \nu)$ the following formula is true:

$$\left[\neg\Phi(\bar{c}, \bar{x}, \bar{y})\ \&\ \left[\&\, D^{2t}\right]_{\bar{x}\bar{x}_1}^{\bar{c}_1\bar{c}_0}\right]_{\bar{x}_0}^{\bar{c}},$$

under the interpretation of the variables $(\bar{x}_0,\ \bar{x},\ \bar{x}_1,\ \bar{y})$ by the elements $(\nu\mu^{2t}(\bar{n}),\ \nu\bar{n}_0,\ \nu\bar{n}_1,\ \nu\bar{n}_2)$, where $\bar{n}$, $\bar{m}$, and $\bar{m}_1$ are indices of constants from $\bar{c}$, $\bar{c}_0$, and $\bar{c}_1$, respectively. We extend $\bar{\mu}^{2t}\restriction\bar{n}$ to $\bar{\mu}'$, so that $\bar{\mu}'(\bar{m}, \bar{m}_1) = (\bar{n}_0, \bar{n}_1)$, and then extend $\bar{\mu}'$ to $\bar{\mu}^{2t+1}$, so that

$$\mathrm{Rang}\,\bar{\mu}^{2t+1} \cap \mathrm{Dom}\,\bar{\mu}^{2t+1} \supseteq \{0, \ldots, 2t+1\}.$$

Now, define $\mu^{2t+1} = \nu\bar{\mu}^{2t+1}$.

Attach $M$ with $\langle \forall y_0, \ldots, y_k\ \Phi(\bar{c}, \bar{x}, \bar{y}), \bar{m} \rangle$ to all the numbers in $\mathrm{Dom}\,\mu^{2t+1}$; remove all markers greater than $M$ and remove from them all labels and associated triples. Attach the pair $\langle \forall y_0, \ldots, y_k\ \Phi(\bar{c}, \bar{x}, \bar{y}), \bar{m} \rangle$ to $M$. Further, with each pair $\langle \forall y'_0, \ldots, y'_k\ \Phi'(\bar{c}, \bar{x}, \bar{y}'), \bar{m}'_0 \rangle$ of complexity $n+1$ with a number less than $p$, we associate the triple

$$\langle \forall y'_0, \ldots, y'_s\ \Phi'(\bar{c}, \bar{x}, \bar{y}'), \bar{m}_0, t' \rangle,$$

where $t' < T$ is the least number such that the following formula is true in $\mathfrak{M}$:

$$\forall \bar{x}\ (\exists \bar{y}\ \&\ D^{t'}(\bar{x}_0, \bar{x}, \bar{y}) \to \forall \bar{z}\ \Phi(\bar{x}_0, \bar{x}, \bar{z})),$$

with the constants replaced by appropriate variables (if, of course, such $t'$ exists). As $t(M)$ we consider the formula $\Phi(\bar{c}, \bar{x}, \bar{y})$ and take $\bar{m}_0^{T+1}$ to be equal to the tuple $\bar{\mu}^{2t+1}(\bar{n}_0, \bar{n}_1)$. For all the other pairs attached to markers of the formula, we use the same tuples as at stage $T$ and pass to the final stage $(Z)$.

If $M$ is a marker of the second kind $M = [k]$ satisfying case B, we consider the formula $\forall z_0, \ldots, z_s$ $\Phi(\bar{c}, \bar{z})$ and the function $f_k^{2t}$, for which this case holds. Next, choose the tuples $\bar{n}_0$, $\bar{m}_1$, $\bar{m}_2$, $\bar{m}_3$ such that $\bar{n}_0$ is the set of indices from $\bar{c}_0$, and the formula

$$\&\ D^{2t}(\bar{x}_0, \bar{x}, \bar{y})\ \&\ \neg\Phi(\bar{x}_0, \bar{x}, \bar{z})$$

is true if the variables $\bar{x}_0$, $\bar{x}$, $\bar{y}$, and $\bar{z}$ are replaced by the elements $\nu\bar{\mu}^t(\bar{m}_0)$, $\nu(\bar{m}_1)$, $\nu(\bar{m}_2)$, and $\nu(\bar{m}_3)$, respectively. Here $\bar{x}_0$ have the same indices as $\bar{c}_0$, and $\bar{n}_1$ is the tuple of numbers, not lying in $\bar{n}_0$, for which $\bar{\mu}^{2t+1}$ is defined. Consider $\bar{\mu}^{2t+1} \supseteq \bar{\mu}^{2t}$ satisfying $\bar{\mu}^{2t+1}(n_i') = m_i'$ for $n_i$ and $m_i$ such that $\bar{n}_1 = (\bar{n}_0, \ldots, \bar{n}_k)$, $\bar{m}_1 = (m_0', \ldots, m_k')$ $\operatorname{Rang} \bar{\mu}^{2t+1} \supseteq \bar{n}_0 \cup \bar{m}_1 \cup \bar{m}_2 \cup \bar{m}_3$, and $\bar{n}_2$, $\bar{n}_3$ are the tuples such that

$$\mu^{2t+1}(\bar{n}_2) = \bar{m}_2 \text{ and } \mu^{2t+1}(\bar{m}_3) = \bar{m}_3.$$

Attach $M$ to all the elements in $\operatorname{Dom} \mu^{2t+1}$. Attach $\forall z_0, \ldots, z_s \Phi(\bar{c}, \bar{z})$ to $M$. Remove all the markers greater than $M$ and remove from them all labels and associated triples. Define $\mu^{2t+1} = \nu\bar{\mu}^{2t+1}$ and pass to the next step.

Remove all markers greater than $M$ and remove all labels from them.

Step $T + 1 = 2t + 2$. We find the quantifier-free sentence $\varphi$ with the least number such that $\varphi \notin D^{2t+1}$ and $\neg\varphi \notin D^{2t+1}$. If the function $\mu^{2t+1}$ is defined for indices of all the variables in $\varphi$, then we put $D^{2t+2} = D^{2t+1} \cup \{\varphi\}$ if $\varphi$ is true in $(\mathfrak{M}, \mu^{2t+1})$. If not, put $D^{2t+2} = D^{2t+1} \cup \{\neg\varphi\}$ and pass to the final stage $(Z)$.

Otherwise we pass immediately to the final stage $(Z)$ of the step.

## FINAL STAGE $Z$

If $\bar{\mu}^{T+1}$ is not yet defined, consider the extension $\bar{\mu}^{T+1}$ of the map $\bar{\mu}^T$ such that

$$\operatorname{Dom} \bar{\mu}^{T+1} \cap \operatorname{Rang} \bar{\mu}^{T+1} \supseteq \{0, \ldots, T+1\}.$$

Put $\mu^{T+1} = \nu\bar{\mu}^{T+1}$. For all pairs on the markers, the formulas and tuples remain the same as at stage $T$. Let $D^{T+1} = D^T$; if $r(m', k', s, T+1)$ is still undefined, we put $r(m', k', s, T+1) = r(m', k', s, T)$ and pass to the next step.

The description of the construction is completed. We are going to consider some of its properties.

Note that if, after step $t_0$, no markers less than $[m, k]$ are either attached or removed and the pair $[m, k]$ marked by $\langle \forall y_0, \ldots, y_k\ \Phi(\bar{c}, \bar{x}, \bar{y}), \bar{m}_0 \rangle$ constantly stays at the elements of $\bar{m}$, then, for any $s$, $t > t_0$ and $z$, if $\mathbf{F}(\bar{m}, m, k, s, z, t) = 1$, then, for $t' > t$, $\mathbf{F}(\bar{m}, m, k, s, z, t')$ is equal to 1 or $\top$; if $\mathbf{F}(\bar{m}, m, k, s, z, t) = \top$, then, for any $t' > t$, $\mathbf{F}(\bar{m}, m, k, s, z, t')$ equals $\top$.

LEMMA 1. The family of formulas $\{D^t \mid t \in \omega\}$ has the following properties:

1) $D^t$ is consistent;

2) $D^t \subseteq D^{t+1}$;

3) for every quantifier-free sentence $\varphi$ in the signature $\Sigma^*$, there exists a step $t$ such that $\varphi \in D^t$ or $\neg\varphi \in D^t$;

4) $\varphi \,\&\, \psi \in \cup D^t \Leftrightarrow \varphi \in \cup D^t$ and $\psi \in \cup D^t$;

5) $\varphi \in \cup D^t \Leftrightarrow \neg\varphi \notin \cup D^t$;

6) $\neg\varphi \in \cup D^t \Leftrightarrow \varphi \notin \cup D^t$;

7) $\varphi \vee \psi \in \cup D^t \Leftrightarrow \varphi \in \cup D^t$ or $\psi \in \cup D^t$;

8) $\varphi \to \psi \in \cup D^t \Leftrightarrow \psi \in \cup D^t$ or $\neg\varphi \in \cup D^t$;

9) $t \approx q \in \cup D^t$ and $q \approx p \in \cup D^t \Rightarrow t \simeq p \in \cup D^t$;

10) $t \approx t \in \cup D^t$;

11) $t \approx q \in \cup D^t \to q \approx t \in \cup D^t$;

12) if $t_1 \approx q_1, \ldots, t_n \simeq q_n \in \cup D^t$ and $P(t_1, \ldots, t_n) \in \cup D^t$, then $P(q_1, \ldots, q_n) \in \cup D^t$.

The Proof follows immediately from the definition of steps of type $2t + 2$.

We define the model $\mathfrak{M}(D)$, taking the quotient set $M(D) \rightleftharpoons \{c_0, \ldots, c_n, \ldots\}/\approx$, where $c_i \approx c_j$ if $c_i = c_j \in \cup D^t$, as the universe.

Evidently, the lemma implies that $\approx$ is indeed an equivalence relation on $C = \{c_0, \ldots, c_n, \ldots\}$.

We say that a formula $P(t_1/\approx, \ldots, t_n/\approx)$ is true in $\mathfrak{M}(D)$ if $P(t_1, \ldots, t_n) \in \cup D^t$. From the properties of the equality relation it follows that the definition does not depend on the choice of a representative. $c_n$ is interpreted by the equivalence class $c/\approx$. Thus, we obtain a model $\mathfrak{M}(D)$ of the signature $\Sigma^* = \Sigma \cup \{c_0, \ldots, c_n, \ldots\}$. Define $\mu(n) \rightleftharpoons c_n/\sim$. Since each step of the construction is effective, it is easy to see that the sequence $\{D^t | t \in N\}$ is computable, and we have an effective procedure to verify the property $\mathfrak{M}(D) \models P(\nu n_1, \ldots, \nu n_k)$.

Indeed, since $\mathfrak{M}(D) \models P(\nu n_1, \ldots, \nu n_k) \Leftrightarrow (\exists t) P(c_{n_1}, \ldots, c_{n_k}) \in D^t \Leftrightarrow (\forall t)(\neg P(c_n, \ldots, c_n) \notin D^t)$, the property mentioned is decidable.

Our present goal is to show that the model constructed is isomorphic to $\mathfrak{M}$, and that $\mu$ is the desired numeration.

The next two lemmas follow readily from the definition of step $2t + 1$.

LEMMA 2. If the marker $M = [m, k]$ is attached infinitely many times, then $\lim r(m, k, 0, t) = \infty$; hence $f_k$ is a total function and $\operatorname{Rang} f_k = N$.

LEMMA 3. If the marker $M = [m, k]$ is attached infinitely many times, then the function $f_k$ determines an isomorphism from $(\mathfrak{M}(D), \mu)$ to $(\mathfrak{M}, \lambda n \gamma(m, n))$.

LEMMA 4. Suppose that $[m, k]$ is the least pair such that $[m, k]$ is attached infinitely many times and the markers of complexity $n + 1$ and higher are attached to $[m, k]$ infinitely many times, whereas all markers of lower complexity are attached to $[m, k]$ finitely many times. Let $t_0$ be a step such that, after $t_0$, no markers less than $[m, k]$ are either attached or removed and no markers of complexity lower than $n + 1$ are either attached to or removed from $[m, k]$. Let $\bar{m}_0$ be the tuple of all elements at which $[m, k]$ with some marker or some smaller marker stay constantly after step $t_0$. Then, for all $t \geq t_0$, $\lambda x \lambda t$ $\mathrm{F}(\bar{m}_0, m, k, s, x, t)$ is a well-defined function for $s \leq n + 1$, i.e., for each $s$-formula $\varphi$ with number $x$, $\mathrm{F}(\bar{m}_0, m, k, s, x, t) = \top$ does not hold; if $\varphi$ is true in $(\mathfrak{M}, \lambda n \gamma(m, n))$, then $\mathrm{F}(\bar{m}_0, m, k, s, x, t)$ equals $\perp$ or 1, and if it is false, then $\mathrm{F}(\bar{m}_0, m, k, s, x, t)$ equals $\perp$ or 0. But if $\mathrm{F}(\bar{m}_0, m, k, s, x, t) \in \{0, 1\}$, then, for all $t' > t$, we have $\mathrm{F}(\bar{m}_0, m, k, s, x, t) = \mathrm{F}(\bar{m}_0, m, k, s, x, t')$. Moreover, for any $s$-formula $\varphi$ with number $x$ such that $s \leq n + 1$, there exists a step $t$ at which $\mathrm{F}(\bar{m}_0, m, k, s, x, t) \in \{0, 1\}$, and each marker can be attached to $[m, k]$ a finite number of times.

The Proof follows immediately from the definition of a counter, namely from the fact that the markers of complexity at least $n+1$ are attached to $[m,k]$ infinitely many times. In this case, $\bar{m}_0$ is taken to be a stable set of numbers, and we can remove a marker from $[m,k]$ if either some smaller marker is attached or the corresponding formula is identified as false, which ensures that it cannot be used twice.

LEMMA 5. If the conditions of Lemma 4 are met, then the model $(\mathfrak{M}, \gamma(m, f_k(\bar{m}_0)))$ is $(n+1)$-complete.

The Proof follows by contradiction. Let $s$ be the least number less than $n+1$ and such that $(\mathfrak{M}, \gamma(m, f_k(\bar{m}_0)))$ is not $(s+1)$-complete. We choose an $(s+1)$-formula $\forall\bar{y}\ \Phi(\bar{c}_0, \bar{x}, \bar{y})$ and a tuple $\bar{a}$, so that this formula is true at $\bar{a}$ when the constants $\bar{c}_0$ are interpreted by the elements of $\gamma(m, f_k(\bar{m}_0))$ and there does not exist a $\exists$-formula $\phi(\bar{c}_0, \bar{x})$ such that $\phi(\bar{c}_0, \bar{x})$ is true at $\bar{a}$, whereas $\forall x(\phi(\bar{c}_0, \bar{x}) \to \forall\bar{y}\Phi(\bar{c}_0, \bar{x}, \bar{y}))$ is true in $(\mathfrak{M}, \gamma(m, f_k(\bar{m}_0))$.

Suppose that $\bar{m}$ is a tuple for which $\gamma(m, f_k(\bar{m})) = \bar{a}$, and $\bar{c}$ is a tuple of constants with indices from $\bar{m}$. Now, for the $f_k$-image of the $(s+1)$-formula $(\forall\bar{y})\Phi(\bar{c}_0, \bar{c}, \bar{y})$ which has number $G$ and is true in $(\mathfrak{M}, \lambda x\gamma(m,x))$, there exists a step $t_1 > t_0$ such that $\mathbf{F}(\bar{m}_0, m, k, s+1, G, t_1) = 1$; the existence follows from the fact that $\lambda x \lambda t\ \mathbf{F}(\bar{m}_0, m, k, s+1, x, t)$ is well defined. By definition, this means that there exists a $\exists$-formula $\phi(\bar{c}_0, \bar{x})$, which is true at $\bar{a}$ when the constants $\bar{c}_0$ are interpreted by the elements of $\gamma(m, f_k(\bar{m}_0))$ and the formula $\forall x(\phi(\bar{c}_0, \bar{x}) \to \forall\bar{y}\Phi(\bar{c}_0, \bar{x}, \bar{y}))$ is true in $(\mathfrak{M}, \nu\bar{\mu}^t(m_0))$. The argument given above implies, however, that the $(s+1)$-formula $\forall x(\phi(\bar{c}_0, \bar{x}) \to \forall\bar{y}\Phi(\bar{c}_0, \bar{x}, \bar{y}))$ is false in $(\mathfrak{M}, \gamma(m, f_k(\bar{m}_0)))$; hence, for the number $G'$ of its $f_k$-preimage, there will exist a step $t > t_0$ such that $\mathbf{F}(\bar{m}_0, m, k, s+1, G', t) = 0$. But since $\forall x(\phi(\bar{c}_0, \bar{x}) \to \forall\bar{y}\Phi(\bar{c}_0, \bar{x}, \bar{y}))$ is true in $(\mathfrak{M}, \nu\bar{\mu}^t(m_0))$, it follows that $*$ should be attached to $[m,k]$, a contradiction.

LEMMA 6. If the conditions of Lemma 4 are met, then markers of arbitrarily high complexity cannot be attached to $[m,k]$.

**Proof.** By Lemma 5, if markers of arbitrarily high complexity are attached, then the model $(\mathfrak{M}, \gamma(m, f_k(\bar{m}_0)))$ is $(n+1)$-complete for any $n$, which is impossible by the assumption of the theorem.

LEMMA 7. If the conditions of Lemma 4 are met, we can find the greatest number $n$ such that the markers of complexity $n+1$ are attached to $[m,k]$ infinitely many times.

The Proof follows from Lemma 6.

LEMMA 8. Suppose that the conditions of Lemma 4 are met, and after step $t_1 > t_0$, no marker of complexity higher than $n+1$ is either attached to or removed from $[m,k]$. Let $\bar{m}_0$ be the tuple of all numbers which either have markers less than $[m,k]$, or $[m,k]$ with some marker is attached to them and never removed. Let $f_k$ be a total function, $\mathrm{Rang}\, f_k = N$, and the model $(\mathfrak{M}, \gamma(m, f_k(\bar{m}_0)))$ be $(r+1)$-complete for $r \geq n$. Then, for any number $z$ of an $s$-formula, $s \leq r+1$, there exists a step $t_2 \geq t_0$ such that $\forall t \geq t_2\ \mathbf{F}(m,k,n,z,t) = \mathbf{F}(m,k,n,z,t_2) \in \{0,1\}$; if $\mathbf{F}(m,k,n,z,t) = 1$, then the corresponding formula is true, and if $\mathbf{F}(m,k,n,z,t) = 0$, then it is false.

**Proof.** We will use Lemmas 5 and 7 to prove that the conclusion of this lemma is met for $n+1 < s \leq r+1$. Suppose that $s$ is the least number for which the desired conclusion fails with formulas of complexity $s+1 \leq r+1$. It suffices to show that for the number $G$ of any formula $\phi$ of complexity $s+1$, there exists a step $t$ such that $\mathbf{F}(m,k,n,z,t) \in \{0,1\}$ and this value coincides with the truth value of $\phi$. By the definition of $\mathbf{F}$, the value of $\mathbf{F}$ cannot vary after that step, because otherwise it would equal $\top$, from which it follows that $[m,k]$ will not be used again, a contradiction.

Let a formula $\phi$ be false in $(\mathfrak{M}, \lambda x\gamma(m,x))$ and let it be of the form $\forall\bar{y}\Psi(\bar{c}_0, \bar{c}, \bar{y})$. Since the $\phi$-formula is false, there exists a tuple $\bar{c}'$ such that the formula $\neg\Psi(\bar{c}_0, \bar{c}, \bar{c}')$ with number $G$ is true in $\mathfrak{M}$. But $\neg\Psi(\bar{c}_0, \bar{c}, \bar{c}')$

is now an $s$-formula, where F is well defined.

If the formula $\varphi$ with number $z$ is false, has form $\exists\bar{y}\Psi(\bar{c},\bar{y})$, and is true for the $f_k$-preimage, then there exists a tuple $\bar{c}'$ for which $\Psi(\bar{c},\bar{c}')$ is satisfied. But then that formula will be false for the $f_k$-preimage of this $\bar{c}'$, which, by the induction hypothesis, is identified by the procedure F. In this case, however, $*$ is attached to $[m,k]$, and after that step, it cannot be attached anywhere, a contradiction. In the opposite case, the negation of the formula in question will be true for the $f_k$-preimage and, by construction, the identification of falsity for $\varphi$ reduces to finding that its negation is true; this case will be treated below.

Let a formula $\phi$ be true in $(\mathfrak{M},\lambda x\gamma(m,x))$. Suppose that it has the form $\forall\bar{y}\Psi(\bar{c},\bar{y})$. Since the model $(\mathfrak{M},\bar{m}_0)$ is $(r+1)$-complete, there exists a $\exists$-formula $\delta$ which is true at the given tuple, whence the formula $\forall\bar{y}\Psi(\bar{c},\bar{y})$ follows. That $\delta$ is true will be identified after some step. It remains to prove that the $f_k$-preimage of the implication

$$\forall\bar{x}\ (\delta(\bar{c}_0,\bar{x})\rightarrow\forall\bar{y}\Psi(\bar{x},\bar{y}))$$

is true in $(\mathfrak{M},\mu)$. Suppose not. Consider the tuple $\bar{c}'$ for which the $f_k$-preimage is false. Then the procedure F determines that the $s$-formula $\Psi(\bar{c},\bar{c}')$ is true at the $f_k$-image of $\bar{c}'$ and, therefore, the negated formula will be true at the $f_k$-preimage. In this case, $*$ is attached to $[m,k]$, after which $[m,k]$ can no longer be attached elsewhere, a contradiction. Thus, the procedure F determines that a formula of the form $\forall\bar{y}\Psi(\bar{c},\bar{y})$ is true.

If the formula $\varphi$ with number $z$ has the form $\forall\bar{y}\Psi(\bar{c},\bar{y})$, then the fact that it is true implies that there exists a tuple $\bar{c}'$ for which $\Psi(\bar{c},\bar{c}')$ is satisfied; this allows us to define the value of F for this formula.

LEMMA 9. If all the markers less than $M=[m,k]$ are attached and removed finitely many times, and if labels are attached to each of these markers finitely many times, then both $M$ and the labels attached to $M$ can be used only finitely many times.

Proof. Suppose, to the contrary, that $M$ is attached infinitely many times. Consider step $t_0$, after which no markers less than $M$ are either attached or removed. Assume that no labels can be attached to or removed from these markers. Since only a finite number of labels can constantly stay on $M$, we consider a step $t_1>t_0$ after which no labels of this type can be attached to or removed from $M$. Let $\bar{m}_0$ be all numbers such that, at step $t_1$, they either have markers less than $M$, or $M$ with some marker constantly stays on them.

Note that, after step $t_1$, $*$ cannot be attached to $M$, since it can be removed only in the case where a smaller marker is attached to $M$. Therefore, whenever $M$ is attached after some step, it will have some pair attached. The conditions of Lemma 4 are thus met for $[m,k]$. Now, by Lemma 7 there exist a step $t_2>t_1$ and a greatest number $n$ such that, after step $t_2$, no markers of complexity $n+1$ (and higher) can be attached to $M$, whereas the markers of complexity $n$ are attached infinitely many times. By Lemma 6, there exists an $r\geq n$ such that the model $(\mathfrak{M},\gamma(m,f_k(\bar{m}_0)))$ is $r$-complete but not $(r+1)$-complete. By Lemma 8, the procedure F is defined for numbers of all $s$-sentences with $s\leq r$.

Consider the enrichment of $\mathfrak{M}$ by the elements $\bar{a}_0\rightleftharpoons\gamma(m,f_k(\bar{m}_0))$. We know that $(\mathfrak{M},\gamma(m,f_k(\bar{m}_0)))$ is not $(r+1)$-complete by assumption, from which it follows that there exist elements $\bar{a}$ and an $(r+1)$-formula $\forall\bar{y}\Psi(\bar{c},\bar{x},\bar{y})$ satisfying $\mathfrak{M}\models\forall\bar{y}\Psi(\gamma(m,f_k(\bar{m}_0),\bar{a},\bar{y}))$. At the same time, if, for every $\exists$-formula $\varphi$, we have $\mathfrak{M}\models\varphi(\gamma(m,f_k(\bar{m}_0),\bar{a}))$, then there exists a tuple $\bar{h}$ such that $\mathfrak{M}\models\varphi(\gamma(m,f_k(\bar{m}_0),\bar{h}))$ but the formula $\forall\bar{y}\Psi(\gamma(m,f_k(\bar{m}_0),\bar{h},\bar{y}))$ is false in $\mathfrak{M}$.

Now, we choose the tuple $\bar{m}$ such that $\gamma(m,f_k(\bar{m}))=\bar{a}$ and consider step $t_1\geq t_0$ at which the function $f_k^{t_1}$ is defined for all numbers from $(\bar{m}_0,\bar{m})$.

We observe that each pair can be attached to $M$ only a finite number of times. Indeed, the pair is removed only when a pair with a smaller number is attached, or if the corresponding formula $\neg\Psi(\gamma(m, f_k(\bar{m}_0)), \gamma(m, f_k(\bar{m})), \gamma(m, \bar{n}))$ with number $G_{\bar{n}}$ is true in $\mathfrak{M}$ and the procedure $\lambda z \lambda t\ \mathrm{F}(m, k, n, G_{\bar{n}}, t)$ yields 1 at some step. This ensures that the pair $\langle \forall \bar{y} \Phi(\bar{c}_0, \bar{x}, \bar{y}), \bar{m} \rangle$ can no longer be attached after that step.

Consider the pair $\langle \forall \bar{y} \Phi(\bar{c}_0, \bar{x}, \bar{y}), \bar{m} \rangle$ numbered $m_0$, where $\bar{c}_0$ is a tuple of constants with indices from $\bar{m}_0$. Choose a step $t_2 > t_1$ at which a pair with a number greater than the number of $\langle \forall \bar{y} \Phi(\bar{c}_0, \bar{x}, \bar{y}), \bar{m} \rangle$ is attached to $M$ and after which to $M$ we can attach only pairs with numbers greater than $m_0$.

By Lemma 8, the procedure $\lambda z \lambda t\ \mathrm{F}(m, k, s, z, t)$, $s \le n$, is well defined for all $t \ge t_0$. This implies that when $M$ is attached again at some odd step $t_3 > t_2$, for the pair $\langle \forall \bar{y} \Phi(\bar{c}_0, \bar{x}, \bar{y}), \bar{m} \rangle$ we have the following cases: all the conditions for $\langle \forall \bar{y} \Phi(\bar{c}_0, \bar{x}, \bar{y}), \bar{m} \rangle$ to be attached to $M$ are satisfied, which is impossible by assumption; a pair with a smaller number of complexity at most $r+1$ is attached to $\langle \forall \bar{y} \Phi(\bar{c}_0, \bar{x}, \bar{y}), \bar{m} \rangle$; or there exists a step $t < t_3$ such that the triple $\langle \forall \bar{y} \Phi(\bar{c}_0, \bar{x}, \bar{y}), \bar{m}, t \rangle$ is associated with $M$ and with the pair attached to $M$. If, in the first case, the corresponding formula is false, its falsity will be determined by Lemma 9. If the formula is true, then, by definition, we can find its $f_k$-preimage that is false. Whenever the $f_k$-image of this preimage is identified to be true, $*$ is attached to $[m, k]$, and the latter can no longer be attached elsewhere, a contradiction. In the second case, the formula

$$(\forall \bar{x})\ ((\exists \bar{z})\ (\&\, D^t(\bar{c}_0, \bar{x}, \bar{z})) \to (\forall \bar{y}) \Phi(\bar{c}_0, \bar{x}, \bar{y}))$$

must be true in $\mathfrak{M}$. By the choice of $\bar{a}$, however, there exist tuples $\bar{m}_1$ and $\bar{n}$ such that

$$\&\, D^t(\gamma(m, f_k(\bar{m}_0, \bar{m})), \gamma(m, \bar{m}_1))\ \&\ \neg\Phi(\gamma(m, f_k(\bar{m}_0, \bar{m})), \gamma(m, \bar{m}_1), \gamma(m, \bar{n}))$$

is true in $\mathfrak{M}$.

Finally, consider the number $G_{\bar{n}}$ of $\neg\Phi(\gamma(m, f_k(\bar{m}_0, \bar{m})), \gamma(m, \bar{m}_1),\ \gamma(m, \bar{n}))$. Note that in view of Lemma 8, there exists an odd step $t_n > t_3$ such that $\mathrm{F}(m, k, n, G_{\bar{n}}, t_n) = 1$. In this case, condition (1.3) is satisfied for $M$ at that step, and $*$ is attached to $M$ and never removed, a contradiction. This completes the proof of the lemma.

LEMMA 10. Each marker is attached a finite number of times.

**Proof.** Consider the least marker $M$ that is attached infinitely many times. Then all of the smaller markers can be attached and removed only a finite number of times. Consider a step $t_0$ after which neither of these markers is either attached or removed. In this case, $M$ cannot be a marker of the second kind, since such markers can be removed if a smaller marker is attached. Let $M = [m, k]$. By Lemma 9, $M$ can be attached a finite number of times, which contradicts the assumption.

LEMMA 11. There exist infinitely many markers that are attached and never removed.

**Proof.** Assume that there exist a finite number of steps at which some markers are used, never to be removed or attached again. Let $t_0$ be the step such that all such markers have been attached before, never to be removed. Consider the number $m$ such that the function numbered $m$ is total and identically equals 1, and $M = [m]$ is greater than all of the markers which stay constantly. We turn to step $t_1 > t_0$ after which the markers less than $M$ and $M$ itself are neither attached nor removed. Such a step exists by Lemma 10. If, at step $t_1$, $M$ is attached to some number, then it can no longer be removed, since markers of the second kind can be removed only when a smaller marker is attached. But we have assumed that now $M$ cannot stay constantly on this number. This means that, at step $t_1$, $M$ is not attached, nor will it be attached to some number after this step. We proceed by obtaining a contradiction to this statement.

Consider a quantifier-free formula $\varphi(c_0,\ldots,c_s)$ such that $\neg\varphi(c_0,\ldots,c_s)$ belongs to $D^{t_2}$, at some $t_2 > t_1$, which exists by Lemma 1. Let $G$ be a number of the formula $\varphi(c_0,\ldots,c_s)$. Choose $t_3 = 2t+1$ in such a way that $t_3 > \max\{t_2, G\}$ and $f_m^{t_3}(G)$ is defined and equals 1 by the choice of $m$. In this case, $M$ must be attached at step $t_3$; this contradicts the argument given above.

LEMMA 12. For every $n$, there exists a $t_0$ such that $\bar{\mu}^t(n) = \bar{\mu}^{t_0}(n)$ at all $t \geq t_0$.

The Proof follows immediately from Lemma 11 by virtue of the fact that $\bar{\mu}^t(n) = \bar{\mu}^{t_0}(n)$ holds for all $t \geq t_0$ for the numbers $n$ to which the marker is attached and is not removed after step $t_0$.

LEMMA 13. For every $m$, there exist $n$ and $t_0$ such that $\bar{\mu}^t(n) = m$ at all $t \geq t_0$.

The Proof follows immediately from Lemmas 11 and 12 in view of the fact that for a partial numeration $\bar{\mu}^t$, Rang $\bar{\mu}^t$ contains all $m < t$, and at step $t$, some marker is attached to all $n$ for which $\bar{\mu}^t(n)$ is defined.

LEMMA 14. The models $\mathfrak{M}$ and $\mathfrak{M}(D)$ are isomorphic.

The Proof follows readily from Lemmas 12 and 13 by virtue of the fact that the function $\bar{\mu} \rightleftharpoons \lim \bar{\mu}^t$ determines an isomorphism from $(\mathfrak{M}(D), \mu))$ to $(\mathfrak{M}, \nu)$.

LEMMA 15. Suppose that, after step $t_1$, no marker of complexity higher than $n+1$ is either attached to or removed from $[m,k]$, and that the markers less than $[m,k]$ are neither attached nor removed. Let $\bar{m}_0$ be all numbers such that, at step $t_1$, they either have markers less than $[m,k]$ or the marker $[m,k]$ (with some other label), which cannot be removed from these numbers after step $t_1$. Let $f_k$ be a total function, Rang $f_k = N$, and the model $(\mathfrak{M}, \gamma(m, f_k(\bar{m}_0)))$ be $(r+1)$-complete for $r \geq n$. Finally, suppose that there exists an isomorphism $\varphi$ from $\mathfrak{M}$ to $\mathfrak{M}$ such that $\varphi(\nu\bar{\mu}^{t_1}(\bar{m})) = \gamma(m,\bar{m})$. Then, for all $t \geq t_1$, $\lambda x \lambda t\ \mathrm{F}(\bar{m}, m, k, s, x, t)$ is a well-defined function, with any $s$, i.e., for each $s$-formula $\varphi$ with number $x$, $\mathrm{F}(\bar{m}, m, k, s, x, t) = \top$ does not hold; if $\varphi$ is true in $(\mathfrak{M}, \lambda n\gamma(m,n))$, then, starting from some step $t \geq t_1$, $\mathrm{F}(\bar{m}, m, k, s, x, t)$ is undefined or equals 1, and if $\varphi$ is false, then $\mathrm{F}(\bar{m}, m, k, s, x, t)$ is undefined or equals 0. Moreover, if $s \leq r+1$ and $\bar{m} \supset \bar{m}_0$, then $\mathrm{F}(\bar{m}, m, k, s, x, t) \in \{0,1\}$ for some $t \geq t_1$, provided the values of $\bar{\mu}^t(\bar{m})$ do not change and $f_k^{t_1}(\bar{m})$ is defined after step $t_1$.

We use induction on $s$ to prove that the procedure is well defined.

Let $s = 0$. Then all of the conclusions are met, since $\lambda z \lambda t \mathrm{F}(\bar{m}, m, k, 0, z, t)$ is defined as a decision procedure for quantifier-free formulas.

Let $s_0 > 0$ and suppose that the above statement holds for all $s < s_0$.

First, we consider true formulas. Let $z$ be a number of an $s_0$-sentence $\forall \bar{y}\varphi(\bar{c}, \bar{y})$, true in $(\mathfrak{M}, \lambda n\gamma(m,n))$, and suppose that, at some step $t > t_1$, $\mathrm{F}(\bar{m}, m, k, s_0, z, t)$ is equal to 0 or $\top$. By the definition of F, this means, however, that there exists a tuple $\bar{c}'$ such that for $z_0'$ a number of the formula $\neg\varphi(\bar{c}, \bar{c}')$, we have $\mathrm{F}(\bar{m}, m, k, s_0 - 1, z_0', t_1) = 1$ but the formula $\forall \bar{y}\varphi(\bar{c}, \bar{y})$ is true in $(\mathfrak{M}, \lambda n\gamma(m,n))$ and, consequently, $\neg\varphi(\bar{c}, \bar{c}')$ is false. This implies that F is not well defined already at stage $s_0 - 1$, a contradiction.

Next, let $z$ be a number of an $s_0$-sentence of the form $\exists \bar{y}\varphi(\bar{c}, \bar{y})$, true in $(\mathfrak{M}, \lambda n\gamma(m,n))$, and suppose that $\mathrm{F}(\bar{m}, m, k, s_0, z, t)$ is equal to 0 or $\top$, for some $t > t_1$. By definition, this means that for $\neg z$ a number of the negated formula $\neg\exists\bar{y}\varphi(\bar{c}, \bar{y})$, $\mathrm{F}(\bar{m}, m, k, s_0, \neg z, t)$ is equal to 1, from which it follows that there exists a $\exists$-formula $(\exists \bar{z})\Psi(\bar{c}_0, \bar{x}, \bar{z})$, where $\bar{c}_0$ is a tuple of constants with indices from $\bar{m}$, such that the formula

$$(\forall \bar{x})\ ((\exists \bar{z})\Psi(\bar{c}_0, \bar{x}, \bar{z}) \to \forall \bar{y} \neg\varphi(\bar{x}, \bar{y}))$$

is true in $(\mathfrak{M}, \nu\mu^{t_1}(\bar{m}))$, whereas $\Psi(\bar{c}_0, \bar{c}, \bar{c}')$ is true in $(\mathfrak{M}, \gamma(m, f_k(\bar{m})), \gamma(m, f_k(\bar{m}_1)))$ for some tuple $\bar{c}'$, where $\bar{m}_1$ are indices of constants from $\exists\bar{y}\varphi(\bar{c}, \bar{y})$. Now, it follows immediately that there does not exist an isomorphism $\varphi$ from $\mathfrak{M}$ to $\mathfrak{M}$ such that $\varphi(\nu\mu^{t_0}(\bar{m}_0)) = \gamma(m, f_k(\bar{m}_0))$.

Second, consider the formula $\forall\bar{y}\varphi(\bar{c}_0,\bar{c},\bar{y})$ numbered $z$, false in $\mathfrak{M}$. Suppose that $\mathbf{F}$ is not well defined for its number, i.e., that $\mathbf{F}(\bar{m}_0,m,k,s_0,z,t)$ is equal to 1 or $\top$ for some $t>t_1$. Then there exists a $\exists$-formula $\Psi(\bar{c}_0,\bar{x})$ which is true in $(\mathfrak{M},\lambda n\gamma(m,n))$, whereas an $f_k$-preimage of the formula

$$(\forall\bar{x})\ (\Psi(\bar{c}_0,\bar{x})\rightarrow\forall\bar{y}\varphi(\bar{c}_0,\bar{x},\bar{y}))$$

is true in $(\mathfrak{M},\nu\mu^{t_0}(\bar{m}_0))$. This means that there does not exist an isomorphism $\varphi$ from $\mathfrak{M}$ to $\mathfrak{M}$ such that $\varphi(\nu\mu^{t_0}(\bar{m}))=\gamma(m,f_k(\bar{m}))$, a contradiction.

Further, let $z$ be a number of a false formula $\exists\bar{y}\varphi(\bar{c},\bar{y})$. Suppose that $\mathbf{F}(\bar{m},m,k,s,z,t)$ is equal to 1 or $\top$ for some $t>t_1$. By definition, there then exists a tuple $\bar{c}'$ such that for $G$ a Gödel number of the formula $\varphi(\bar{c},\bar{c}')$, $\mathbf{F}(\bar{m},m,k,s-1,G,t)=1$, where $s_0$ is the complexity of $\exists\bar{y}\varphi(\bar{c},\bar{y})$, which means that $\mathbf{F}$ is not well defined already at stage $s_0-1$.

To complete the proof, by induction on $s$ we show that for $z$ a number of some formula of complexity $s$, if $s\leq r+1$, then $\mathbf{F}(\bar{m},m,k,s,x,t)\in\{0,1\}$ for some $t\geq t_1$.

If $s$ is equal to 0, the conclusion follows immediately from the definition of $\mathbf{F}$. Consider the least $s\leq r+1$ for which the conclusion is not true. For a formula of complexity $s$, which is true in $(\mathfrak{M},\lambda n\gamma(m,n))$ and is quantified $\exists$, we use the induction hypothesis and the fact that the procedure is well defined, proved above, to state that there exists a step $t\geq t_1$ at which the conditions of case (1) specified in the definition of $\mathbf{F}$ are satisfied for this formula, and that $\mathbf{F}(\bar{m},m,k,s,x,t)=1$ holds for its number $x$. Similar considerations show that the required condition will be met for the formulas of complexity $s$ which are false in $(\mathfrak{M},\lambda n\gamma(m,n))$ and are quantified by $\forall$. Now, suppose that the formula $\forall\bar{y}\varphi(\bar{c},\bar{y})$ of complexity $s$ is true in $(\mathfrak{M},\lambda n\gamma(m,n))$ and is quantified by $\forall$. Since $(\mathfrak{M},\gamma(m,f_k(\bar{m}_0)))$ is $(r+1)$-complete, it follows that there exists a $\exists$-formula $(\exists\bar{z})\Psi(\bar{c}_0,\bar{x},\bar{z})$, where $\bar{c}_0$ is a tuple of constants with indices from $\bar{m}$, and the formula

$$(\forall\bar{x})\,((\exists\bar{z})\Psi(\bar{c}_0,\bar{x},\bar{z})\rightarrow\forall\bar{y}\varphi(\bar{x},\bar{y}))$$

is true in $(\mathfrak{M},\gamma(m,f_k(\bar{m})))$, whereas $\Psi(\bar{c}_0,\bar{c},\bar{c}')$ is true in $(\mathfrak{M},\gamma(m,f_k(\bar{m})),\gamma(m,f_k(\bar{m}_1)))$ for some tuple $\bar{c}'$. Since there exists an isomorphism $\varphi$ from $\mathfrak{M}$ to $\mathfrak{M}$ such that $\varphi(\nu\mu^{t_1}(\bar{m}))=\gamma(m,\bar{m})$, it follows that

$$(\forall\bar{x})((\exists\bar{z})\Psi(\bar{c}_0,\bar{x},\bar{y})\rightarrow\forall\bar{y}\neg\varphi(\bar{x},\bar{y}))$$

is true in $(\mathfrak{M},\nu\mu^{t_1}(\bar{m}))$. Therefore, for the number $z$ of $\forall\bar{y}\varphi(\bar{c},\bar{y})$, the conditions of case (1) in the definition of $\mathbf{F}$ will be satisfied at some step, whence the required conclusion follows.

It remains to consider the case where a formula of complexity $s$, quantified by $\exists$, is false in $(\mathfrak{M},\lambda n\gamma(m,n))$. It follows by the argument given above that for a number $\neg x$ of its negation, $\mathbf{F}(\bar{m}_0,m,k,s,\neg x,t)=1$ at some step $t\geq t_1$, in which case for $x$, we have $\mathbf{F}(\bar{m}_0,m,k,s,x,t)=0$. The lemma is proved.

**LEMMA 16.** For any $m$ and $k$, if $f_k$ is a total function, then $f_k$ does not determine an isomorphism of $(\mathfrak{M}(D),\mu)$ onto $(\mathfrak{M},\lambda n\gamma(m,n))$.

**Proof.** Assume, to the contrary, that the constructive models $(\mathfrak{M},\lambda n\gamma(m,n))$ and $(\mathfrak{M}(D),\mu)$ are isomorphic. Then there exist a recursive function $f_k$ and an isomorphism $\varphi$ from $\mathfrak{M}(D)$ to $\mathfrak{M}$ such that $\varphi\mu(n)=\gamma(m,f_k(n))$ for all $n$. Consider the marker $[m,k]$. By Lemma 10, there exists a step $t$ after which no markers less than or equal to $[m,k]$ can be attached or removed. For $[m,k]$, we then have one of the following three cases: $*$ stays on $[m,k]$ constantly; at least one of the markers of the form $\langle\forall\bar{y}\Phi(\bar{c}_0,\bar{x},\bar{y}),\bar{m}\rangle$ of complexity $s+1$ constantly stays on $[m,k]$ and the model $(\mathfrak{M},\gamma(m,f_k(\bar{m}_0)))$ is $s$-complete, where $\bar{m}_0$ is a related tuple; neither of the two cases holds.

First, we prove that the last case is impossible. Suppose that, at step $t_1$, $\bar{m}_0$ consists of all numbers with markers less than $M$, or with $M$ itself to which a certain label was attached at some step, never to be removed.

Consider the tuple $\bar{a}_0 = (m, f_k(m_0))$. The model $\mathfrak{M}$ is not $(n+1)$-complete but is $n$-complete for some $n$ over $\bar{a}_0$. Therefore, there exist an $(n+1)$-formula $\forall\bar{y}\,\Phi(\bar{a}_0,\bar{x},\bar{y})$ and a tuple $\bar{b}$ such that $\mathfrak{M} \models \forall\bar{y}\Phi(\bar{a}_0,\bar{b},\bar{y})$. But if, for every $\exists$-formula $\varphi(\bar{a},\bar{x})$, we have $\mathfrak{M} \models \varphi(\bar{a},\bar{b})$, then there exists a tuple $\bar{c}$ such that $\mathfrak{M} \models \varphi(\bar{a},\bar{c})\ \&\ \forall\bar{y}\Phi(\bar{a},\bar{c},\bar{y})$. Since $f_k$ is a total function that determines an isomorphism of $(\mathfrak{M}(D),\mu)$ onto $(\mathfrak{M}, \lambda n\gamma(m,n))$, it follows that there exists an isomorphism $\varphi$ from $\mathfrak{M}$ to $\mathfrak{M}$ such that $\varphi(\nu\mu^{t_0}(\bar{m}_0)) = \gamma(m, f_k(\bar{m}_0))$. Consider the tuple $\bar{m}$, satisfying $\gamma(m, f_k(\bar{m})) = \bar{c}$, and the pair $\langle\forall\bar{y}\Phi(\bar{c}_0,\bar{x},\bar{y}),\bar{m}\rangle$. Since $f_k$ is a total function and $\operatorname{Rang} f_k = N$, there exists a step $t_1 > t_0$ such that

$$\operatorname{Rang} f_k^{t_1} \cap \operatorname{Dom} f_k^{t_1} \supseteq \{0,\ldots,r(m,k,t_0)\}.$$

By Lemma 15, the procedure $\lambda z\lambda t\ \mathrm{F}(\bar{m}_0,m,k,s,z,t)$ is well defined for $s \le n$. Therefore, for the marker $M$ and the pair $\langle\forall\bar{y}\Phi(\bar{c}_0,\bar{x},\bar{y}),\bar{m}\rangle$, the following conditions are met at step $2t+1 > t_1$:

(1) There does not exist a tuple $\bar{n}$ such that for $G^{\bar{n}}$ a number of the formula

$$\neg\Phi(\gamma(m, f_k(\bar{m}_0)), \gamma(m, f_k(\bar{m})), \gamma(m,\bar{n}))$$

and for any $t > t_0$, $\mathrm{F}(\bar{m},m,k,n,G^{\bar{n}},t)$ is not equal to 1 because

$$(\forall\bar{y})\Phi(\gamma(m, f_k(\bar{m}_0)), \gamma(m, f_k(\bar{m})), \bar{y})$$

is true in $\mathfrak{M}$ and $\lambda z\lambda t\ \mathrm{F}(\bar{m}_0,m,k,s,z,t)$ is well defined by Lemma 15.

(2) After step $t_0$, the values of $\mu^t$ and $\bar{\mu}^t$ at the elements of $\bar{m}_0$ do not change. Consequently, after step $t_1 > t_0$, the values of functions $\bar{\mu}^t$ and $\mu^t$ also remain unchanged at $\bar{m}_0$ because $f_k$ determines an isomorphism of $(\mathfrak{M}(D),\mu)$ onto $(\mathfrak{M},\lambda n\gamma(m,n))$ and $\bar{\mu} = \lim \bar{\mu}^t$ determines an isomorphism $\varphi$ of $(\mathfrak{M}(D),\mu)$ onto $(\mathfrak{M},\nu)$, defined by the rule $\varphi\mu^{t_1}(\bar{m}_0) = \gamma(m, f_k(\bar{m}_0))$. Now, for every $\exists$-formula $\phi(\bar{x},\bar{y})$ such that $\phi(\nu\mu^{t_1}\bar{m}_0, \gamma\mu^{t_1}\bar{m})$ is true in $\mathfrak{M}$, the formula

$$(\exists\bar{y})\ (\exists z)\neg\Phi(\nu\mu^{t_1}(\bar{m}_0),\bar{x},\bar{y})\ \&\ \phi(\nu\mu^{t_1}(\bar{m}_0),\bar{x})$$

is also true in $\mathfrak{M}$. Therefore, for the pair $\langle\forall\bar{y}\Phi(\bar{c}_0,\bar{x},\bar{y}),\bar{m}\rangle$ at an odd step $t_2 > t_1$ such that $f_k^{t_2}$ is defined at the elements from $\bar{m}_0$ and $\bar{m}$, if $M$ does not have a marker of complexity lower than $n+1$, then all of the conditions for $M$ to be attached and for the given pair to be attached to $M$ are satisfied. Since smaller markers cannot be used, it is the marker $M$ proper that is attached, a contradiction. If, starting from some step $T'$, a pair $\langle\forall\bar{y}\Phi'(\bar{c}_0,\bar{x},\bar{y}),\bar{m}'\rangle$ of complexity not higher than $n+1$ stays on $M$ constantly, we choose one with lowest complexity and consider the formula $t(\Phi,\bar{m}')$ and the tuple $\bar{m}'^{T'}$, constructed for this pair at step $T'$. By construction, for these $t(\Phi,\bar{m}')$ and $\bar{m}'^{T'}$, the formula $\neg t(\Phi,\bar{m}')$ is true in $\mathfrak{M}$ at the tuple $\nu\bar{\mu}^{t_1}(\bar{m}'^{T'})$, and after that step, $M$ with the pair $\langle\forall\bar{y}\Phi'(\bar{c}_0,\bar{x},\bar{y}),\bar{m}'\rangle$ constantly stays at the elements of $\bar{m}'^{T'}$. Since the function $f_k$ determines an isomorphism and $\bar{\mu}^t(\bar{m}'^{T'})$ stabilizes after step $T'$, it follows that the formula mentioned is also true for the $f_k$-image of these elements in the model $(\mathfrak{M},\lambda n\gamma(m,n))$. By virtue of the fact that $\lambda z\lambda t\ \mathrm{F}(\bar{m}_0,m,k,s,z,t)$ is a well-defined procedure for $s \le n$, it is easy to see that, starting from some step $t_2$, $\mathrm{F}(\bar{m}_0,m,k,n,G^n_m,t) = 1$ will be satisfied for the Gödel number $G^n_m$ of the formula

$$\left[\neg\Phi'\right]^{\bar{c},\bar{x}'_0,y_0,\ldots,y_s,}_{\bar{c}_{f_k(\bar{m},\bar{m}_0)},c_{n_0},\ldots,c_{n_s},}.$$

In this case, conditions (2.1) are satisfied at some step for such a marker, which, therefore, must be removed, a contradiction. This means that the case where $M$ has a marker of lower complexity is also impossible.

Now we prove that case (2) is unfeasible. Assume the contrary. Let at least one of the markers of the form $\langle \forall \bar{y} \Phi(\bar{c}_0, \bar{x}, \bar{y}), \bar{m} \rangle$ of complexity $s+1$ stay on $M$ constantly. Suppose that $(\mathfrak{M}, \gamma(m, f_k(\bar{m}_0)))$ is $s$-complete for a related tuple $\bar{m}_0$. Among such pairs, we choose one with lowest complexity and consider the formula $t(\Phi, \bar{m}')$ and the tuple $\bar{m}'^{T'}$, constructed for this pair at step $T'$. By construction, for these $t(\Phi, \bar{m}')$ and $\bar{m}'^{T'}$, the formula $\neg t(\Phi, \bar{m}')$ is true in $\mathfrak{M}$ at $\nu \bar{\mu}^{t_1}(\bar{m}'^{T'})$, and after that step, $M$ with the pair $\langle \forall \bar{y} \Phi'(\bar{c}_0, \bar{x}, \bar{y}), \bar{m}' \rangle$ constantly stays at the elements of $\bar{m}'^{T'}$. Now, since $\bar{\mu}^t(\bar{m}'^{T'})$ stabilizes after step $T'$, it follows that $\neg t(\Phi, \bar{m}')$ is true in $(\mathfrak{M}, \nu)$ at $\nu \bar{\mu}^t(\bar{m}'^{T'})$. The fact that $f_k$ determines an isomorphism implies that the $f_k$-image of this formula will be true at the tuple $\gamma(m, f_k(\bar{m}'^{T'}))$. Using Lemma 15 and the fact that $(\mathfrak{M}, \gamma(m, f_k(\bar{m}_0)))$ is $s$-complete, we assert that $\lambda z \lambda t\ \mathrm{F}(\bar{m}_0, m, k, s, z, t)$ is well defined for $s$, in which case after some step $t_2$, $\mathrm{F}(\bar{m}_0, m, k, n, G^n_m, t) = 1$ is satisfied for the Gödel number $G^n_m$ of the formula

$$\left[\neg \Phi'\right]^{\bar{c}, \bar{x}'_0, y_0, \ldots, y_s,}_{\bar{c}_{f_k(\bar{m}, \bar{m}_0)}, c_{n_0}, \ldots, c_{n_s},}.$$

But then conditions (2.1) will be satisfied at some step for such a marker, which, therefore, must be removed, a contradiction. This means that case (2) cannot hold.

We have shown that $*$ constantly stays on $M$ after some step. For $*$ to be attached to $M$, there are also several possibilities. Namely, they are listed in conditions (1.1)-(1.4) specified for the odd step $t_0$.

Condition (1.1) fails because $f_k$ determines the isomorphism of $(\mathfrak{M}(D), \mu)$ onto $(\mathfrak{M}, \lambda n \gamma(m, n))$.

(1.2) also cannot hold since it follows from Lemma 4 that the procedure verifying whether $\lambda z \lambda t \mathrm{F}(m, k, s, z, t)$ is true for $s \leq n$ is well defined at all $t \geq t_0$. Moreover, the isomorphism is determined by $f_k$ and, consequently, the truth values of sentences are preserved under taking their $f_k$-images.

We turn to condition (1.3). Suppose that, at step $t_0$, $M$ has an associated triple $\langle \forall \bar{y} \Phi(\bar{c}_0, \bar{x}, \bar{y}), \bar{m}, t \rangle$ and there exist tuples $\bar{m}_1$ and $\bar{n}$ such that the formula

$$\neg \Phi(\gamma(m, f_k(\bar{m}_0)), \bar{m}_1, \bar{n})) \,\&\, (\exists \bar{y}) D^t(\gamma m, f_k(\bar{m}_0)), \bar{m}_1, \bar{n})))$$

is true in $\mathfrak{M}$. This is so because $\lambda z \lambda t \mathrm{F}(m, k, s, z, t)$ is a well-defined procedure for $t \geq t_0$, from which it follows that $\mathrm{F}(m, k, n, G, t_0) = 1$ implies that the left conjunctive term is true. The truth of the right conjunctive term follows immediately from the definition of the step. Now, since the function $f_k$ determines an isomorphism of $(\mathfrak{M}(D), \mu)$, $(\mathfrak{M}, \nu)$, and $(\mathfrak{M}, \lambda n \gamma(m, n))$, the formula

$$\exists \bar{y} \exists \bar{x} (\neg \Phi(\nu \mu^{t_0} \bar{m}_0, \bar{x}, \bar{y}) \,\&\, \exists \bar{z} \,\&\, D^t(\nu \mu^{t_0} \bar{m}_0, \bar{x}, \bar{z}))$$

will be true in $\mathfrak{M}$. This contradicts the fact that with $M$ we associate only those triples for which

$$(\forall \bar{x})(\exists z \,\&\, D^t(\nu \mu^t \bar{m}_0, \bar{x}, \bar{z}) \to (\forall \bar{y}) \neg \Phi(\nu \mu^t \bar{m}_0, \bar{x}, \bar{y}))$$

is true in $\mathfrak{M}$. The fact that the pair with which the given triple is associated constantly stays at the elements of $\bar{m}_0$ until step $t_0$ is reached implies that

$$(\forall \bar{x})(\exists z \,\&\, D^t(\nu \mu^t \bar{m}_0, \bar{x}, \bar{z}) \to (\forall \bar{y}) \neg \Phi(\nu \mu^t \bar{m}_0, \bar{x}, \bar{y}))$$

is true in $\mathfrak{M}$. This is also a contradiction, which proves that condition (1.3) cannot be met.

It remains to consider condition (1.4). Suppose that a pair $\langle\forall\bar{y}\Phi'(\bar{c}_0,\bar{x},\bar{y}),\bar{m}'\rangle$ is attached to $M$ at step $T'$, and that $M$ with this pair stays constantly until step $t_0$ is reached. For the pair $\langle\forall\bar{y}\Phi'(\bar{c}_0,\bar{x},\bar{y}),\bar{m}'\rangle$, at step $T'$ we define a formula $t(\Phi,\bar{m}')$ and a tuple $\bar{m}'^{T'}$, for which, by construction, $\neg t(\Phi,\bar{m}')$ is satisfied in $\mathfrak{M}$ at the tuple $\nu\bar{\mu}^{t_1}(\bar{m}'^{T'})$, and after this step, $M$ with the pair $\langle\forall\bar{y}\Phi'(\bar{c}_0,\bar{x},\bar{y}),\bar{m}'\rangle$ constantly stays at the elements of $\bar{m}'^{T'}$. Since $\bar{\mu}^t(\bar{m}'^{T'})$ stabilizes after step $T'$, it is easy to see that $\neg t(\Phi,\bar{m}')$ is true in $(\mathfrak{M},\nu)$ at $\nu\bar{\mu}^t(\bar{m}'^{T'})$. The fact that the function $f_k$ determines an isomorphism implies that the $f_k$-image of the formula is true at $\gamma(m,f_k(\bar{m}'^{T'}))$. By Lemma 15, $\lambda z\lambda t\ \mathrm{F}(\bar{m}_0,m,k,s,z,t)$ is a well-defined procedure for all $s$. But if $*$ is attached to $M$ at step $t_0$, then, for the Gödel number $G^n_m$ of the formula

$$\left[\neg\Phi'\right]^{\bar{c},\bar{x}'_0,y_0,\ldots,y_s,}_{\bar{c}_{f_k(\bar{m},\bar{m}_0)},c_{n_0},\ldots,c_{n_s},}$$

$\mathrm{F}(\bar{m}_0,m,k,n,G^n_m,t_0)=0$ must hold, which contradicts the fact that F is well defined. This completes the proof of the lemma.

LEMMA 18. The constructivization $\mu$ of $\mathfrak{M}$ is not strong.

Proof. Assume the contrary. If the theory $\mathrm{Th}(\mathfrak{M},\mu)$ is decidable, then the characteristic function for the set of Gödel numbers of sentences in this theory is recursive and has a number, i.e., it is equal to $f_k$. First, we show that $M$ equal to $[k]$ is attached at some step, never to be removed. Assume it is not. Fix a step $t_0$ after which no markers less than $M$ are either attached or removed; this step exists by Lemma 10.

Let $\bar{m}_0$ be all numbers to which the markers less than $M$ are attached at step $t_0$. There exists an $n$ such that $(\mathfrak{M},\nu\bar{\mu}^{t_0}(\bar{m}_0))$ is $n$-complete but not $(n+1)$-complete; the models $(\mathfrak{M},\nu\bar{\mu}^{t_0}(\bar{m}_0))$ and $(\mathfrak{M},\mu(\bar{m}_0))$ are isomorphic by Lemma 14, in view of the fact that $\nu\bar{\mu}^{t_0}(\bar{m}_0)=\mu^t(\bar{m}_0)$ at all $t\geq t_0$. Therefore, there exist an $(n+1)$-formula $\forall\bar{y}\Phi(\bar{c}_0,\bar{x},\bar{y})$ and a tuple $\bar{m}'$ such that $\forall\bar{y}\Phi(\bar{c}_0,\bar{x},\bar{y})$ is true in $(\mathfrak{M},\nu\bar{\mu}^{t_0}(\bar{m}_0))$ at $\nu(\bar{m}')$, and for every $\exists$-formula $\phi(\bar{x}_0,\bar{x})$ which is true in $\mathfrak{M}$ at $\nu\bar{\mu}^{t_0}(\bar{m}_0)$, $\nu(\bar{m}')$, the formula

$$\exists\bar{y}\exists\bar{x}(\neg\Phi(\nu\mu^{t_0}\bar{m}_0,\bar{x},\bar{y})\ \&\ \phi(\nu\mu^{t_0}\bar{m}_0,\bar{x}))$$

is true in $\mathfrak{M}$. By Lemma 13, there exist a step $t_1\geq t_0$ and a tuple $\bar{m}$ such that $\bar{\mu}^t(\bar{m})=\bar{m}'$ at all $t\geq t_1$. By Lemma 14, the formula $\forall\bar{y}\Phi(\bar{c}_0,\bar{c},\bar{y})$, where the tuple of constants $\bar{c}$ is equal to $\bar{c}_{\bar{m}}$, is true in $(\mathfrak{M},\mu)$. Hence, for its Gödel number $G$, the value of $f_k$ equals 1.

We consider an odd step $t_2>t_1$ such that $f_k^{t_2}(G)=1$. By the choice of $\forall\bar{y}\Phi(\bar{c}_0,\bar{x},\bar{y})$ and $\nu(\bar{m}')$, the conditions of case B now hold for $M$ at step $t_2$. Since smaller markers cannot be used, it follows that the marker $M$ should be attached, an impossibility.

We have thus proved that $M$ will be attached at some step, never to be removed. In this case, $M$ is marked by the formula $\forall\bar{y}\Phi(\bar{c},\bar{y})$ with Gödel number $G$, and $f_k(G)=1$, i.e., the formula considered belongs to $\mathrm{Th}(\mathfrak{M},\mu)$. At the same time, we define $\bar{\mu}^{t_0}$ so that for some tuple $\bar{m}$, the formula $\neg\Phi(\bar{c}_0,\bar{y})$ is true in $(\mathfrak{M},\nu\bar{\mu}^{t_0}(\bar{m}_0))$ at $\nu\bar{\mu}^{t_0}(\bar{m})$. All the elements of $\bar{m}_0$ and $\bar{m}$ are marked by $M$, which cannot be removed after this step, and so $\neg\Phi(\bar{c}_0,\bar{y})$ is true in $(\mathfrak{M},\mu(\bar{m}_0))$ at $\mu(\bar{m})$ by Lemma 14. This conflicts with our previous considerations.

By the properties of the construction proved above, we obtain the required algorithm to build a constructivization that is not strong and does not lie in the given computable class of constructivizations. The theorem is proved.

THEOREM 2. If $\mathfrak{M}$ is strongly and weakly constructivizable, then, for a given computable class of its constructivizations, we can effectively build a weak constructivization of $\mathfrak{M}$ that is not autoequivalent to any constructivization belonging to this class.

**Proof.** If there exists a tuple $\bar{a}$ such that $(\mathfrak{M}, \bar{a})$ is complete, then every constructivization of a strongly constructive model $\mathfrak{M}$ is also strong (see [6]). But the model $\mathfrak{M}$ has a weak constructivization and, therefore, is not complete in any finite enrichment by constants. If, for every $n$, there exists a tuple $\bar{a}_n$ such that the model $(\mathfrak{M}, \bar{a}_n)$ is $n$-complete, then $\mathfrak{M}$ is limit-$\omega$-complete, and the result follows by Theorem 1. If the latter assumption does not hold, then there exists an $n$ such that $\mathfrak{M}$ is limit-$n$-complete, in which case the result follows from [9].

COROLLARY 1. The class of weak constructivizations of a strongly constructivizable model is either empty or effectively infinite, i.e., given a computable class of weak constructivizations, we can effectively build a weak constructivization that is not autoequivalent to any constructivization belonging to this class.

COROLLARY 2. The class of weak constructivizations of a strongly constructivizable model is either empty or infinite and not computable.

## REFERENCES


1. C. J. Ash and A. Nerode, "Intrinsically recursive relations. Aspects of effective algebra," in: *Proc. Conf. Monash Univ.*, Australia (1981), pp. 24-41.
2. C. J. Ash and J. F. Knight, "Pairs of recursive structure," *Logic Paper*, 64, 1-45 (1988).
3. C. J. Ash and S. S. Goncharov, "Strong $\Delta_2^0$-categoricity," *Algebra Logika*, 24, No. 6, 718-727 (1985).
4. S. S Goncharov, "The problem of the number of nonequivalent constructivizations," *Algebra Logika*, 19, No. 6, 621-639 (1980).
5. S. S. Goncharov, "The number of nonequivalent constructivizations," *Algebra Logika*, 16, No. 3, 257-289 (1977).
6. S. S. Goncharov, "Autostability and computable families of constructivizations," *Algebra Logika*, 14, No. 6, 647-680 (1975).
7. S. S. Goncharov, "Autostability of models and Abelian groups," *Algebra Logika*, 19, No. 1, 23-44 (1980).
8. S. S. Goncharov and V. D. Dzgoev, "Autostability of models," *Algebra Logika*, 19, No. 1, 45-58 (1980).
9. S. S. Goncharov, "Computable classes of constructivizations for models of finite type," *Sib. Mat. Zh.*, 34, No. 5, 23-37 (1993).
10. V. P. Dobritsa, "Computability of some classes of algebras," *Sib. Mat. Zh.*, 18, No. 3, 570-579 (1977).
11. V. P. Dobritsa, "Structural properties of computable classes of constructive models," *Algebra Logika*, 26, No. 1, 36-62 (1987).
12. A. T. Nurtazin, "Strong and weak constructivizations and computable families," *Algebra Logika*, 13, No. 3, 311-323 (1974).
13. S. S. Goncharov, "Bounded theories of constructive Boolean algebras," *Sib. Mat. Zh.*, 17, No. 4, 797-812 (1976).
14. S. S. Goncharov, "Some properties of constructivizations for Boolean algebras," *Sib. Mat. Zh.*, 16, No. 2, 264-278 (1975).
15. Yu. L. Ershov, *Decidability Problems and Constructive Models* [in Russian], Nauka, Moscow (1980).

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is true in $\mathfrak{M}$ at $\nu\bar{\mu}^{t_0}(\bar{m}_0)$, $\nu(\bar{m}')$, the formula

$$\neg\Phi(\nu\mu^{t_0}\bar{m}_0,\bar{x},\bar{y})\ \&\ \phi(\nu\mu^{t_0}\bar{m}_0,\bar{x}))$$

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bility, McGraw-Hill, New York

77).

3, 3-60 (1961).