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# MODELING OF TRANSVERSE SHEARS OF PIECEWISE HOMOGENEOUS COMPOSITE BARS USING AN ITERATIVE PROCESS WITH ACCOUNT OF TANGENTIAL LOADS[1]. 1. CONSTRUCTION OF A MODEL

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***Keywords:*** *transverse shear, composite bar, piecewise homogeneity, nonclassical model, iterative process, tangential loads*

*A nonclassical model for the stress-strain state of a piecewise homogeneous composite bar is proposed. The model is based on an iterative process and takes into account the deplanation of cross sections of the bar caused by transverse shears. Based on the shear strains of some particular approximation, higher approximation models are constructed. The model accounts for both the normal and tangential loads.*

**Introduction.** The development of models for bent bars with regard to the effect of transverse (tangential) shears began with the solution of the vibration problem for bars suggested by S. P. Timoshenko, who was the first to introduce the concept of a shear-caused rotation angle of a cross section. The models taking into account the transverse shears were called refined, nonclassical, or Timoshenko-type models [1]. The construction of such models and the corresponding theories of bending of bars (beams), plates, and shells is widely described in the literature, including monographs and fundamental overviews [1-9]. Especially complicated is the construction of nonclassical models for nonuniform composite objects, particularly for composite bars with a discrete or so-called hybrid structure whose inhomogeneity is manifested both in their structure and cross-sectional shapes. From this it follows that the creation of nonclassical models for such bars is a high-priority task. The present study describes a nonclassical model for bent composite bars, allowing for the influence of transverse shears on the basis of an iterative process.

Since we are dealing with transverse (tangential) shears, it is important to take into account not only the normal but also tangential loads, which can significantly affect the stress-strain state of a bar.

## 1. Initial Prerequisites. Classical Model

**Structure of a Bar.** We consider a bar with a constant cross section along the length. The material of the bar consists of $n$ phases, generally composite ones, whose properties are represented by the generalized characteristics obtained experimentally or calculated by the methods of micromechanics of composites. It is assumed that, as found from experiments, the often used unidirectional fibrous composites, to sufficient accuracy, can be considered as linearly elastic bodies.

The bar is referred to a system of orthogonal coordinates $XYZ$ (Fig. 1), chosen so that $X$ — the longitudinal axis normal to the transverse cross section — passes through its stiffness center $C$. The $Y$ and $Z$ axes are the principal stiffness axes of the cross section, whereas the $XCY$ and $XCZ$ planes are the main planes of stiffness. The cross section has such a shape that the planes tangential

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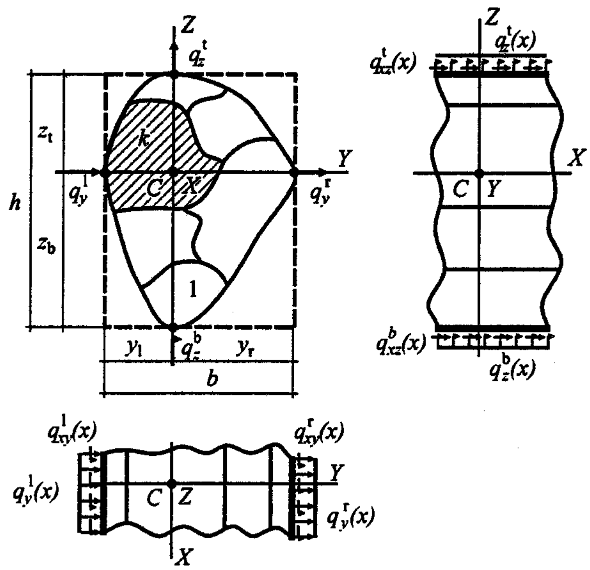


Fig. 1. Transverse section of a bar with the loading scheme.

to the contour surface of the bar and normal to the principal axes do not intersect the body of the bar, i.e., the contour of the bar cross section is inscribed in a rectangle whose sides are perpendicular to the principal axes, and the intersection points of the contour with the principal axes belong to this rectangle.

The bar is in equilibrium under the action of loads which can be applied to each of the main planes: in the $XCZ$ plane — normal $q_z^t(x)$, $q_z^b(x)$ and tangential $q_{xz}^t(x)$, $q_{xz}^b(x)$ on the upper $z = z_t$ and lower $z = z_b$ boundaries of the intersection of this plane with the surface of the bar; in the $XCY$ plane — normal $q_y^r(x)$, $q_y^l(x)$ and tangential $q_{xy}^r(x)$, $q_{xy}^l(x)$ on the right $y = y_r$ and left $y = y_l$ boundaries of the intersection of the plane with the surface of the bar.

For each phase of the material of the bar, $k = 1, 2, \ldots, n$, the elastic modulus $E_x^{(k)}$ and the transverse shear moduli $G_{xz}^{(k)}$ and $G_{xy}^{(k)}$ are given.

Account of the transverse shears under the action of a given load is modeled based on an iterative process, where the distribution law for transverse shear strains in a cross section of the bar found on a certain iteration $m$ is assumed as a hypothesis for the following step — iteration $m+1$. As the zero step of iteration ($m = 0$), we assume the model of the classical theory of bending based on the hypotheses of plane undeformed cross sections, which can be presented in the following known way:

— the shear strains in the cross section $YCZ$ and in the main planes $XCY$ and $XCZ$ are absent,

$$\gamma_{yz} = 0, \quad \gamma_{xy} = 0, \quad \gamma_{xz} = 0, \tag{1.1}$$

— the transverse strains along the principal axes are absent,

$$\varepsilon_y = 0, \quad \varepsilon_z = 0, \tag{1.2}$$

— and the pressure from the longitudinal fibers along the principal axes is neglected,

$$\sigma_y = 0, \quad \sigma_z = 0. \tag{1.3}$$

Let us consider, without loss of generality, the bend in the $XCZ$ plane. The strains in this plane are defined by the Cauchy relations

$$\varepsilon_x(x,z) = \frac{\partial u(x,z)}{\partial x}, \quad \varepsilon_z(x,z) = \frac{\partial w(x,z)}{\partial z}, \quad \gamma_{xz}(x,z) = \frac{\partial u(x,z)}{\partial z} + \frac{\partial w(x,z)}{\partial x}. \tag{1.4}$$

Integration of the second and third relations with respect to $z$ leads to displacements

$$w(x,z)=w(x)+\int_{\delta_z}^{z}\varepsilon_z\,dz,\quad u(x,z)=u(x)-\int_{\delta_z}^{z}\frac{\partial w(x,z)}{\partial x}dz+\int_{\delta_z}^{z}\gamma_{xz}\,dz. \tag{1.5}$$

Here, $w(x)$ are the unknown displacements (deflections) normal to the $X$ axis at an arbitrary level $z=\delta_z$ in the $XCZ$ plane and $u(x)$ are the unknown longitudinal (along the $X$ axis) displacements at the same level $z=\delta_z$.

Taking into account the third relation of hypotheses (1.1) and the second one from (1.2), we obtain from Eq. (1.5) the expressions for displacements

$$w(x,z)=w(x),\quad u(x,z)=u(x)-\frac{\partial w(x)}{\partial x}(z-\delta_z). \tag{1.6}$$

Obviously, the normal displacements are constant along the $Z$ axis, whereas the tangential ones vary linearly along this axis, which ensures the compatibility of longitudinal strains of fibers in all $n$ phases of the bar. These strains are defined by the first relation of Eqs. (1.4) with regard to Eq. (1.6):

$$\varepsilon_x(x,z)=\frac{du(x)}{dx}-\frac{d^2w(x)}{dx^2}(z-\delta_z). \tag{1.7}$$

Assuming that $E_x^{(k)}=E_k$ and using Hooke's law, we find the longitudinal normal stresses in each $k$th phase of the bar

$$\sigma_x^{(k)}(x,z)=E_k\,\varepsilon(x,z)=E_k\left[\frac{du(x)}{dx}-\frac{d^2w(x)}{dx^2}(z-\delta_z)\right]. \tag{1.8}$$

The expressions for the flexural stresses in the $XCY$ plane are the same.

Based on the principle of superposition, we write the total stresses

$$\sigma_x^{(k)}(x,y,z)=\sigma_x^{(k)}(x,z)+\sigma_x^{(k)}(x,y). \tag{1.9}$$

Consider next the transverse stresses. Since, according to the hypotheses introduced, the transverse shears are absent, these stresses cannot be obtained from Hooke's law. We find them from the equilibrium equations of a three-dimensional body related to an elementary volume of the $k$th phase:

$$\frac{\partial\sigma_x^{(k)}}{\partial x}+\frac{\partial\tau_{xy}^{(k)}}{\partial y}+\frac{\partial\tau_{xz}^{(k)}}{\partial z}=0,\quad \frac{\partial\tau_{yx}^{(k)}}{\partial x}+\frac{\partial\sigma_y^{(k)}}{\partial y}+\frac{\partial\tau_{yz}^{(k)}}{\partial z}=0, \tag{1.10}$$

$$\frac{\partial\tau_{zx}^{(k)}}{\partial x}+\frac{\partial\tau_{zy}^{(k)}}{\partial y}+\frac{\partial\sigma_z^{(k)}}{\partial z}=0,$$

where the variables $x$, $y$, and $z$ are referred to the chosen coordinates $XYZ$. In what follows, the superscript $k$ for stresses, strains, and displacements is omitted.

We assume that the first relation of hypotheses (1.1), according to which $\gamma_{yz}=0$ and respectively $\tau_{yz}=0$, is valid. Taking into account the independence of the stresses $\sigma_x(x,y)$ and $\sigma_x(x,z)$, we assign them to the transverse tangential stresses $\tau_{xy}(x,y)$ and $\tau_{xz}(x,z)$. Then, the following two equations are identical to the first equilibrium equation (1.10):

$$\frac{\partial\sigma_x(x,y)}{\partial x}+\frac{\partial\tau_{xy}(x,y)}{\partial y}=0,\quad \frac{\partial\sigma_x(x,z)}{\partial x}+\frac{\partial\tau_{xz}(x,z)}{\partial z}=0 \tag{1.11}$$

and, instead of the remaining equilibrium equations, we have

$$\frac{\partial\tau_{yx}(x,y)}{\partial x}+\frac{\partial\sigma_y(x,y)}{\partial y}=0,\quad \frac{\partial\tau_{zx}(x,z)}{\partial x}+\frac{\partial\sigma_z(x,z)}{\partial z}=0. \tag{1.12}$$

The unknown transverse stresses, which were originally neglected due to the hypotheses adopted, can be found from four equations (1.11) and (1.12).

For bending in the $XCZ$ plane, by integrating the second equations from (1.11) and (1.12) with respect to $z$, we obtain

$$\tau_{xz}(x,z) = -\int_{z_b}^{z} \frac{\partial \sigma_x(x,z)}{\partial x} dz + \Phi_{xz}(x), \tag{1.13}$$

$$\sigma_z(x,z) = -\int_{z_b}^{z} \frac{\partial \tau_{xz}(x,z)}{\partial x} dz + \Phi_z(x), \tag{1.14}$$

where $\Phi_{xz}$ and $\Phi_z$ are the integration functions.

Since the external load is aligned with the coordinate axes, the stresses (1.13) and (1.14), with regard for their sign conventions, must obey the conditions at the "lower" and "upper" (along the $Z$ axis) points of the cross section,

$$\begin{aligned} &\tau_{xz}^{b}(x,z_b) = -q_{xz}^{b}(x), \quad \tau_{xz}^{t}(x,z_t) = q_{xz}^{t}(x), \\ &\sigma_z^{b}(x,z_b) = -q_z^{b}(x), \quad \sigma_z^{t}(x,z_t) = q_z^{t}(x). \end{aligned} \tag{1.15}$$

Conditions (1.15) relate the tangential and normal loads to the corresponding stresses and determine the integration functions provided the following condition at the lower boundary ($z = z_b$) is satisfied:

$$\Phi_{xz}(x) = -q_{xz}^{b}(x), \quad \Phi_z(x) = -q_z^{b}(x). \tag{1.16}$$

Now, substituting expression (1.8) and the first value of (1.16) into Eq. (1.13), we have

$$\tau_{xz}(x,z) = -\frac{\partial^2 u(x)}{\partial x^2} \int_{z_b}^{z} E_k\, dz + \frac{d^3 w(x)}{dx^3} \int_{z_b}^{z} E_k (z - \delta_z) dz - q_{xz}^{b}(x)^{*}. \tag{1.17}$$

Account of the condition for the upper boundary point $z = z_t$ gives

$$\tau_{xz}(x,z_t) = -\frac{\partial^2 u(x)}{\partial x^2} \int_{z_b}^{z_t} E_k\, dz + \frac{d^3 w(x)}{dx^3} \int_{z_b}^{z_t} E_k (z - \delta_z) dz - q_{xz}^{b}(x) = q_{xz}^{t}(x). \tag{1.18}$$

Introducing the following designations for the constants:

$$B = \int_{z_b}^{z_t} E_k\, dz, \quad B_0 = \int_{z_b}^{z_t} E_k (z - \delta_z) dz, \tag{1.19}$$

we have from Eq. (1.18)

$$\frac{d^2 u(x)}{dx^2} = \frac{d^3 w(x)}{dx^3} \frac{B_0}{B} - \frac{1}{B}\left[q_{xz}^{b}(x) + q_{xz}^{t}(x)\right]. \tag{1.20}$$

This inhomogeneous equation is the equilibrium equation for the bar in projection on the $X$ axis. The displacement function $u(x)$, determined by integrating this equation, depends on the deflection function $w(x)$ and the tangential loads. The deflection dependence can be removed by assuming that $B_0/B = 0$. Introducing an auxiliary axis $Y_0$ parallel to the $Y$ axis and replacing the coordinates $z = z_0 - z_c$, we obtain from this condition

$$\int_{z_{0_b}}^{z_{0_t}} E_k z_0 dz_0 - z_c \int_{z_{0_b}}^{z_{0_t}} E_k dz_0 = \delta_z \int_{z_{0_b}}^{z_{0_t}} E_k dz_0. \tag{1.21}$$

From here, we find the coordinate of the stiffness center with respect to the $Y_0$ axis,

$$z_c = \left( \int\limits_{z_{0_b}}^{z_{0_t}} E_k z_0 dz_0 \Bigg/ \int\limits_{z_{0_b}}^{z_{0_t}} E_k dz_0 \right) - \delta_z . \tag{1.22}$$

In the particular case $\delta_z = 0$, we have the displacement function $u(x)$ at the level of the stiffness center.

Substituting expression (1.20) into Eq. (1.17) and collecting similar terms, we find the transverse tangential stresses

$$\tau_{xz}(x,z) = \frac{d^3 w}{dx^3} \left[ \int\limits_{z_b}^{z} E_k (z - \delta_z) dz - \frac{B_0}{B} \int\limits_{z_b}^{z} E_k dz \right]$$

$$+ q_{xz}^{b}(x) \left( \frac{1}{B} \int\limits_{z_b}^{z} E_k dz - 1 \right) + \frac{1}{B} q_{xz}^{t}(x) \int\limits_{z_b}^{z} E_k dz. \tag{1.23}$$

It follows from expression (1.23) that the tangential stresses at the level of the $z$ coordinate are the same in all phases. In this case, the influence of the phase properties is taken into account "integrally": the integral in formula (1.23) contains information on all the phases occurring in the range from $z_b$ to $z$, which corresponds to a rigid interfacial contact.

Let us introduce designations for the distribution functions of the tangential stresses along the $z$ coordinate:

$$f_0(z) = \int\limits_{z_b}^{z} E_k (z - \delta_z) dz - \frac{B_0}{B} \int\limits_{z_b}^{z} E_k dz, \tag{1.24}$$

$$f_q^{b}(z) = \frac{1}{B} \int\limits_{z_b}^{z} E_k dz - 1, \quad f_q^{t}(z) = \frac{1}{B} \int\limits_{z_b}^{z} E_k dz. \tag{1.25}$$

Then, the expression (1.23) takes the form

$$\tau_{xz}(x,z) = -\frac{d^3 w}{dx^3} f_0(z) + q_{xz}^{b}(x) f_q^{b}(z) + q_{xz}^{t}(x) f_q^{t}(z). \tag{1.26}$$

The transverse normal stresses can be determined from Eq. (1.14), where we take into account the transverse tangential stresses (1.26) and the integration function $\Phi_z(x) = -q_z^{b}(x)$ from Eq. (1.16). Then

$$\sigma_z(x,z) = -\frac{d^4 w}{dx^4} \int\limits_{z_b}^{z} f_0(z) dz - \frac{d q_{xz}^{b}(x)}{dx} \int\limits_{z_b}^{z} f_q^{b}(z) dz - \frac{d q_{xz}^{t}(x)}{dx} \int\limits_{z_b}^{z} f_q^{t}(z) dz - q_z^{b}(x). \tag{1.27}$$

Let us satisfy, for these stresses, the condition (1.15) at $z = z_t$:

$$\sigma_z(x,z_t) = -\frac{d^4 w}{dx^4} \int\limits_{z_b}^{z_t} f_0(z) dz - \frac{d q_{xz}^{b}(x)}{dx} \int\limits_{z_b}^{z_t} f_q^{b}(z) dz - \frac{d q_{xz}^{t}(x)}{dx} \int\limits_{z_b}^{z_t} f_q^{t}(z) dz - q_z^{b}(x) = q_z^{t}(x). \tag{1.28}$$

Now, we introduce the following designations:

$$D_{00} = \int\limits_{z_b}^{z_t} f_0(z) dz, \quad D_{0_b} = \int\limits_{z_b}^{z_t} f_q^{b}(z) dz, \quad D_{0_t} = \int\limits_{z_b}^{z_t} f_q^{t}(z) dz. \tag{1.29}$$

Then, from Eq. (1.28), we have

$$\frac{d^4 w(x)}{dx^4} = -\frac{1}{D_{00}} \left\{ \left[ q_z^{b}(x) + q_z^{t}(x) \right] + \frac{d}{dx} \left[ q_{xz}^{b}(x) D_{0_b} + q_{xz}^{t}(x) D_{0_t} \right] \right\}. \tag{1.30}$$

This is the equilibrium equation of the bar in the projection on the $Z$ axis, i.e., an equation in the required deflection functions $w(x)$, where $D_{00}$ is the bending stiffness of the bar. Equations (1.30) and (1.20) form a system of differential equilibrium equations for a piecewise homogeneous composite bar corresponding to the classical model, which is regarded as a zero iteration of the process of constructing a nonclassical model. We should note that, in these equations, both the normal and tangential loads are taken into account.

## 2. Nonclassical Model. The First Iteration

Based on expression (1.26) and Hooke's law at $G_{xz}^{(k)} = G_k'$, we obtain the transverse shear strains in the $k$th phase of the classical model (zero iteration),

$$\gamma_{0_{xz}}(x,z) = \frac{\tau_{xz}(x,z)}{G_k'} = \frac{d^3 w(x)}{dx^3}\frac{f_0(z)}{G_k'} + \frac{q_{xz}^{b}(x) f_q^{b}(z)}{G_k'} + \frac{q_{xz}^{t}(x) f_q^{t}(z)}{G_k'}$$

$$= \frac{d^3 w(x)}{dx^3}\varphi_{1k}(z) + q_{xz}^{b}(x)\varphi_{qk}^{b}(z) + q_{xz}^{t}(x)\varphi_{qk}^{t}(z). \tag{2.1}$$

It should be noted that, as a consequence, the shear strains in the $k$th phases at the level of the $z$ coordinate will be different since the shear moduli of separate phases are also different.

The functions $\varphi_{1k}(z)$, $\varphi_{qk}^{b}(z)$, and $\varphi_{qk}^{t}(z)$ define the distribution laws for shear strains over the height of a cross section, which are related to the sought-for deflection function and the given functions of tangential loading.

Let us assume the following relation:

$$\frac{d^3 w(x)}{dx^3} \Rightarrow \frac{d\chi(x)}{dx}, \tag{2.2}$$

which introduces a new sought-for function $\chi(x)$ — the so-called "shear function" [6] — in the irreversible correspondence. Then, instead of Eq. (2.1), we have

$$\gamma_{0_{xz}}(x,z) = \frac{d\chi(x)}{dx}\varphi_{1k}(z) + q_{xz}^{b}(x)\varphi_{qk}^{b}(z) + q_{xz}^{t}(x)\varphi_{qk}^{t}(z). \tag{2.3}$$

This expression is assumed as a hypothesis for constructing the first iteration (($m = 1$) of a nonclassical model. In addition, we retain the hypotheses $\gamma_{yz} = 0$, $\varepsilon_z = 0$, and $\sigma_z = 0$ of the classical model. With account of the hypotheses adopted, the displacements in a bar loaded in the $XCZ$ plane, according to Eq. (1.5), can be written as

$$w(x,z) = w(x),\; u(x,z) = u(x) - \frac{dw(x)}{dx}(z-\delta_z) - \frac{d\chi}{dx}\psi_{1k}(z) - q_{xz}^{b}(x)\psi_{qk}^{b}(z) - q_{xz}^{t}(x)\psi_{qk}^{t}(z) \tag{2.4}$$

where we assign the following functions:

$$\psi_{1k}(z) = -\int_{\delta_z}^{z}\varphi_{1k}(z)dz,\quad \psi_{qk}^{b}(z) = -\int_{\delta_z}^{z}\varphi_{qk}^{b}(z)dz,\quad \psi_{qk}^{t}(z) = -\int_{\delta_z}^{z}\varphi_{qk}^{t}(z)dz. \tag{2.5}$$

According to Eq. (1.4) and Hooke's law, the longitudinal strains and normal stresses take the form

$$\varepsilon_x(x,z) = \frac{du(x)}{dx} - \frac{d^2 w(x)}{dx^2}(z-\delta_z) - \frac{d^2\chi}{dx^2}\psi_{1k}(z) - \frac{dq_{xz}^{b}(x)}{dx}\psi_{qk}^{b}(z) - \frac{dq_{xz}^{t}(x)}{dx}\psi_{qk}^{t}(z), \tag{2.6}$$

$$\sigma_x(x,z) = E_k\varepsilon_x(x,z). \tag{2.7}$$

Let us now determine the tangential stresses corresponding to the first iteration of the model, i.e., to the normal stresses (2.7) with regard to Eq. (2.6). Substituting these stresses into Eq. (1.13), we write

$$\tau_{1xz}(x,z)=-\Bigg[\frac{d^2u(x)}{dx^2}\int\limits_{z_b}^{z}E_k dz-\frac{d^3w(x)}{dx^3}\int\limits_{z_b}^{z}E_k(z-\delta_z)dz-\frac{d^3\chi}{dx^3}\int\limits_{z_b}^{z}E_k\psi_{1k}(z)dz$$

$$-\frac{d^2q_{xz}^{b}(x)}{dx^2}\int\limits_{z_b}^{z}E_k\psi_{qk}^{b}(z)dz-\frac{d^2q_{xz}^{t}(x)}{dx^2}\int\limits_{z_b}^{z}E_k\psi_{qk}^{t}(z)dz\Bigg]-q_{xz}^{b}(x). \qquad (2.8)$$

In view of conditions (1.15), we have on the upper boundary surface

$$\tau_{1xz}(x,z_t)=-\Bigg[\frac{d^2u(x)}{dx^2}B-\frac{d^3w(x)}{dx^3}B_0-\frac{d^3\chi}{dx^3}B_1$$

$$-\frac{d^2q_{xz}^{b}(x)}{dx^2}B_q^{b}-\frac{d^2q_{xz}^{t}(x)}{dx^2}B_q^{t}\Bigg]-q_{xz}^{b}(x)=q_{xz}^{t}(x). \qquad (2.9)$$

Here, the following additional constants are introduced:

$$B_1=\int\limits_{z_b}^{z_t}E_k\psi_{1k}(z)dz,\quad B_q^{b}=\int\limits_{z_b}^{z_t}E_k\psi_{qk}^{b}(z)dz,\quad B_q^{t}=\int\limits_{z_b}^{z_t}E_k\psi_{qk}^{t}(z)dz. \qquad (2.10)$$

From Eq. (2.9), we obtain the equilibrium equation of the bar in projection on the $X$ axis,

$$\frac{d^2u(x)}{dx^2}=\frac{d^3w(x)}{dx^3}\frac{B_0}{B}+\frac{d^3\chi(x)}{dx^3}\frac{B_1}{B}$$

$$+\frac{d^2q_{xz}^{b}(x)}{dx^2}\frac{B_q^{b}}{B}+\frac{d^2q_{xz}^{t}(x)}{dx^2}\frac{B_q^{t}}{B}-\frac{1}{B}\Big[q_{xz}^{b}(x)+q_{xz}^{t}(x)\Big]. \qquad (2.11)$$

Substituting Eq. (2.11) into Eq. (2.8), we have

$$\tau_{1xz}(x,z)=\frac{d^3w(x)}{dx^3}\Bigg[\int\limits_{z_b}^{z}E_k(z-\delta_z)dz-\frac{B_0}{B}\int\limits_{z_b}^{z}E_k dz\Bigg]+\frac{d^3\chi(x)}{dx^3}\Bigg[\int\limits_{z_b}^{z}E_k\psi_{1k}(z)dz-\frac{B_1}{B}\int\limits_{z_b}^{z}E_k dz\Bigg]$$

$$+\frac{d^2q_{xz}^{b}(x)}{dx^2}\Bigg[\int\limits_{z_b}^{z}E_k\psi_{qk}^{b}(z)dz-\frac{B_q^{b}}{B}\int\limits_{z_b}^{z}E_k dz\Bigg]+\frac{d^2q_{xz}^{t}(x)}{dx^2}\Bigg[\int\limits_{z_b}^{z}E_k\psi_{qk}^{t}(z)dz-\frac{B_q^{t}}{B}\int\limits_{z_b}^{z}E_k dz\Bigg]$$

$$+q_{xz}^{b}(x)\Bigg[\frac{1}{B}\int\limits_{z_b}^{z}E_k\,dz-1\Bigg]+q_{xz}^{t}(x)\frac{1}{B}\int\limits_{z_b}^{z}E_k\,dz. \qquad (2.12)$$

Let us write expression (2.12) in a simplified form:

$$\tau_{1xz}(x,z)=\frac{d^3w(x)}{dx^3}f_0(z)+\frac{d^3\chi(x)}{dx^3}f_1(z)+\frac{d^2q_{xz}^{b}(x)}{dx^2}f_{1q}^{b}(z)+\frac{d^2q_{xz}^{t}(x)}{dx^2}f_{1q}^{t}(z)+q_{xz}^{b}(x)f_q^{b}(z)+q_{xz}^{t}(x)f_q^{t}(z), \qquad (2.13)$$

where, apart from $f_0(z)$, $f_q^{b}(z)$, and $f_q^{t}(z)$, we have introduced the functions $f_1(z)$, $f_{1q}^{b}(z)$, and $f_{1q}^{t}(z)$, whose expressions are given in square brackets in formula (2.12) at the respective factors.

According to Eq. (2.13) and Hooke's law, we obtain the transverse shear strain (as a result of the first iteration)

$$\gamma_{1xz}(x,z)=\frac{\tau_{1xz}(x,z)}{G_k'}=\frac{d^3w(x)}{dx^3}\frac{f_0(z)}{G_k'}+\frac{d^3\chi(x)}{dx^3}\frac{f_1(z)}{G_k'}+\frac{d^2q_{xz}^{b}(x)}{dx^2}\frac{f_{1q}^{b}(z)}{G_k'}$$

$$+\frac{d^2 q_{xz}^{t}(x)}{dx^2}\frac{f_{1q}^{t}(z)}{G_k'}+q_{xz}^{b}(x)\frac{f_q^{b}(z)}{G_k'}+q_{xz}^{t}(x)\frac{f_q^{t}(z)}{G_k'}$$

$$=\frac{d^3 w(x)}{dx^3}\varphi_{1k}(z)+\frac{d^3\chi(x)}{dx^3}\varphi_{2k}(z)+\frac{d^2 q_{xz}^{b}(x)}{dx^2}\varphi_{1qk}^{b}(z)$$

$$+\frac{d^2 q_{xz}^{t}(x)}{dx^2}\varphi_{1qk}^{t}(z)+q_{xz}^{b}(x)\varphi_{qk}^{b}(z)+q_{xz}^{t}(x)\varphi_{qk}^{t}(z). \tag{2.14}$$

The expression obtained is taken as a basis for continuation of the iterative process.

## 3. Nonclassical Iterative Model of Higher Approximations

The construction of a nonclassical model of higher approximations is regarded as an iterative process, where the shear strains obtained as a result of the $m$th iteration are taken as a basis for the following approximation — the $(m+1)$th iteration. For this purpose, to construct the second iteration, by analogy with the procedure for the first iteration, the following irreversible relations are introduced into Eq. (2.14):

$$\frac{d^3 w(x)}{dx^3}\Rightarrow\frac{d\chi(x)}{dx},\quad \frac{d^3\chi_1(x)}{dx^3}\Rightarrow\frac{d\chi_2(x)}{dx}, \tag{3.1}$$

where $\chi_1(x)$ and $\chi_2(x)$ are the shear functions for the second approximation. The function $\chi(x)$ of the first approximation (2.14) is not identical to the function $\chi_1(x)$ of the second one. The hypothesis for shear strains of the model of the second approximation, as a result of the first iteration, has the form

$$\gamma_{1xz}(x,z)=\frac{d\chi_1(x)}{dx}\varphi_{1k}(z)+\frac{d\chi_2(x)}{dx}\varphi_{2k}(z)+\frac{d^2 q_{xz}^{b}(x)}{dx^2}\varphi_{1qk}^{b}(z)$$

$$+\frac{d^2 q_{xz}^{t}(x)}{dx^2}\varphi_{1qk}^{t}(z)+q_{xz}^{b}(x)\varphi_{qk}^{b}(z)+q_{xz}^{t}(x)\varphi_{qk}^{t}(z). \tag{3.2}$$

We introduce a designation generalizing the summands from the tangential load,

$$\gamma_{1q}(x,z)=\left[\varphi_{qk}^{b}(z)+\varphi_{1qk}^{b}(z)\frac{d^2}{dx^2}\right]q_{xz}^{b}(x)+\left[\varphi_{qk}^{t}(z)+\varphi_{1qk}^{t}(z)\frac{d^2}{dx^2}\right]q_{xz}^{t}(x). \tag{3.3}$$

In addition, summation over the "dummy" (repeated) indices $p$ is assumed. Then, hypothesis (3.2) can be written in the form

$$\gamma_{(m-1)xz}(x,z)=\frac{d\chi_p}{dx}\varphi_{pk}(z)+\gamma_{(m-1)q}(x,z), \tag{3.4}$$

where, for the second iteration $(m=2)$, the "dummy" index $p$ takes the values $p=1,2$.

Proceeding with the iterative process, we obtain the hypotheses for the shear strains of an arbitrary $m$th iteration. In this case, the index $p$ takes the values $p=1,\dots,m$. The distribution functions for shear strains over the height of a cross section are defined as

$$\varphi_{pk}(z)=\frac{f_{(p-1)}(z)}{G_k'},\quad \varphi_{(p-1)qk}^{s}(z)=\frac{f_{(p-1)q}^{s}(z)}{G_k'},\quad p=1,\dots,m,\quad s=\mathrm{b,t}. \tag{3.5}$$

Hereinafter, $\varphi_{0qk}^{s}(z)=0$.

The summand of the load-induced shear strains has the form

$$\gamma_{(m-1)q}(x,z)=\left[\varphi^s_{qk}(z)+\varphi^s_{(p-1)qk}(z)\frac{d^{2(p-1)}}{dx^{2(r-1)}}\right]q^s_{xz}(x). \tag{3.6}$$

From here on, we assume that $p=1,\dots,m$, $r=p$, and $s=\mathrm{b},\mathrm{t}$. Summation is performed over the indices $p$ and $s$.

The transverse tangential stresses, corresponding to the hypothesis for shear strains (3.4), are defined by Hooke's law as

$$\tau_{(m-1)xz}(x,z)=\frac{d\chi_p(x)}{dx}f_{(p-1)}(z)+\left[f^s_q(z)+f^s_{(p-1)q}(z)\frac{d^{2(p-1)}}{dx^{2(r-1)}}\right]q^s_{xz}(x). \tag{3.7}$$

With account of the hypothesis assumed, the expression for longitudinal displacements takes the form

$$u_m(x,z)=u(x)-\frac{dw(x)}{dx}(z-\delta_z)-\frac{d\chi_p(x)}{dx}\psi_{pk}(z)$$

$$-\left[\psi^s_{qk}(z)+\psi^s_{(p-1)qk}(z)\frac{d^{2(p-1)}}{dx^{2(r-1)}}\right]q^s_{xz}(x), \tag{3.8}$$

where

$$\psi_{pk}(z)=-\int_{\delta_z}^{z}\varphi_{pk}(z)dz,\quad \psi^s_{qk}(z)=-\int_{\delta_z}^{z}\varphi^s_{qk}(z)dz,\quad \psi^s_{(p-1)qk}(z)=-\int_{\delta_z}^{z}\varphi^s_{(p-1)qk}(z)dz. \tag{3.9}$$

The longitudinal strains and normal stresses on an arbitrary $m$th iteration are given by

$$\varepsilon_{mx}(x,z)=\frac{du(x)}{dx}-\frac{d^2w(x)}{dx^2}(z-\delta_z)-\frac{d^2\chi_p(x)}{dx^2}\psi_{pk}(z)$$

$$-\left[\psi^s_{qk}(z)+\psi^s_{(p-1)qk}(z)\frac{d^{2(p-1)}}{dx^{2(r-1)}}\right]\frac{dq^s_{xz}(x)}{dx}, \tag{3.10}$$

$$\sigma_{mx}(x,z)=E_k\,\varepsilon_{mx}(x,z). \tag{3.11}$$

The transverse tangential stresses are obtained according to the procedure described previously,

$$\tau_{mxz}(x,z)=\frac{d^3w(x)}{dx^3}f_0(z)+\frac{d^3\chi_p(x)}{dx^3}f_p(z)+\left[f^s_q(z)+f^s_{pq}(z)\frac{d^{2p}}{dx^{2r}}\right]q^s_{xz}(x). \tag{3.12}$$

Thus, we have obtained the relations of a nonclassical iterative model for the stress-strain state, which take into account the transverse shear strains and the direct influence of an external tangential load. All the relations refer to a bar loaded in the main plane of stiffness, *XCZ*. For the *XCY* plane, these relations have the same form and are obtained by replacing the indices $z \to y$.

Account of tangential loads is of fundamental importance, which significantly contributes to all the model relations — the expressions of displacements, strains, as well as normal and transverse tangential stresses. Along with some complication of the model compared with that presented in [10], this modification expands the class of problems which can be solved on the basis of the constructed iterative model for the stress-strain states of piecewise homogeneous composite bars. A realization of the model developed and the solution of the problem will be presented in Part 2.

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