GEORGIAN MATHEMATICAL JOURNAL: Vol. 2, No. 3, 1995, 323-334

APPLICATIONS OF THE METHOD OF BARRIERS II. SOME SINGULARLY PERTURBED PROBLEMS

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ABSTRACT. The method of barriers is used to justify asymptotic representations of solutions of two-point boundary value problems for singularly perturbed quasilinear equations of the second and the third order. This paper is a continuation of [1].

1. In constructing and justifying solutions of singularly perturbed problems asymptotic with respect to the parameter, there frequently occur situations in which the most actively used methods of obtaining estimates, such as the method of successive approximations, the maximum principle, and the like, turn out to be unacceptable because the problem has singularities. In these situations it becomes necessary to apply the methods of proofs which are based only on using such properties and relations between the input data of the problem that are close to or even coincide with the conditions necessary and sufficient for the existence of its solution. In considering boundary value problems, the method of barrier functions and that of differential inequalities are exactly such tools.

The idea of using differential inequalities in investigating solutions of initial and boundary value problems originates from the method of a priori estimates developed in the works by S.N. Bernstein [2] and S.A. Chapligin [3]. This method was realized in the paper by M. Nagumo [4] who showed the connection between the fulfillment of certain inequalities and the existence of a solution of the corresponding boundary value problem, obtained in a number of papers also by other researchers [5]–[12]. Starting from the work by N.I. Brish [13], differential inequalities have been actively used in investigating solutions of singularly perturbed boundary value problems and in constructing their asymptotic representations. The most consistent and complete application of this method to singularly perturbed problems

1991 *Mathematics Subject Classification.* 34B15, 34E15.
Key words and phrases. Barrier function, singularly perturbed quasilinear differential equations of the second and third order, two-point boundary value problem, asymptotic representation.

1072-947X/95/0500-0323$07.50/0

can be found in the monograph [14]. Some aspects of the application of the method of differential inequalities to the so-called bisingular boundary value problems are treated in [15]–[16].

2. Let us consider the boundary value problem

$$\varepsilon x'' - \big(p(t)x' + h(t,x)\big) = 0, \quad t \in (0,1), \tag{1}$$
$$x(0) = A_0, \quad x(1) = B_0, \tag{2}$$

where $\varepsilon > 0$ is a fixed small parameter, the functions $p(t)$ and $h(t,x)$ as functions of the variable t may have, on the interval $(0,1)$, a finite number of first kind discontinuity points $t = t_r$, $r = \overline{1,m}$, while on each set $[t_r, t_{r+1}] \times R$, $r = \overline{0,m}$, $t_0 = 0$, $t_{m+1} = 1$, these functions, defined completely at the endpoints of the corresponding intervals with respect to continuity, are infinitely differentiable. The function $p(t)$ may have on the interval $(0,1)$ a finite number of zeros $t = t_k^0$, $k = \overline{1,n}$, not coinciding with the points $t = t_r$, and, what is more, may vanish identically on a finite number of some subintervals of $(0,1)$. Moreover, the inequalities $h'_x(t_k^0, x) \geq h_0 > 0$, $k = \overline{1,n}$, are fulfilled for all $|x| < \infty$ and the function $h(t,x)$ satisfies the conditions of Theorems 3 and 4 from [16]. Note that with the above-mentioned properties of the functions $p(t)$, $h(t,x)$, the methods of [14], the use of which enables one to make a conclusion on the existence and some properties of solutions of the boundary value problem (1), (2), are inapplicable in a straightforward manner.

In stating the results concerning problem (1), (2), in order to simplify the formulations we shall limit ourselves to the case in which, on the interval $(0,1)$, there are one zero t_1^0 of the function $p(t)$ and one point t_1 of the first kind of discontinuity with respect to the variable t of the functions $p(t)$, $h(t,x)$, $t_1 \neq t_1^0$. It appears that the singularities of solutions of the considered problem and, accordingly, the kind of asymptotic representations for the considered solutions depend essentially on the manner in which the signs of $p(t)$ alternate on subintervals obtained as a result of dividing the interval $(0,1)$ by the points t_1, t_1^0. Generally speaking, the problem of defining a solution of the degenerate equation, to which the solution of the original problem tends for $\varepsilon \to 0$, $t \in (0,1)\backslash\{t_1\}$, is formulated differently according to each sign alternation variant.

One can easily ascertain that under the above conditions there are 16 variants of the mutual arrangement of the points t_1, t_1^0 and of alternation of the signs of the function $p(t)$ on the respective subintervals of $(0,1)$. In that case a function, which is the limiting one when $\varepsilon \to 0$ for the solution of problem (1), (2), is, for $t \neq t_1$, a solution of the corresponding degenerate equation

$$p(t)y' + h(t,y) = 0, \quad t \in (0,1)\backslash\{t_1\}. \tag{3}$$

To single out the unique (discontinuous, in the general case, for $t = t_1$) curve from the set of integral curves $y = y(t)$ of equation (3) on the intervals $(0, t_1)$, $(t_1, 1)$, it is necessary to introduce additional conditions at the points $t = 0$, $t = t_1$, $t = 1$. It appears that for the above-mentioned 16 variants there are 10 different cases of the collection of additional conditions which are used to obtain the unique solution of equation (3) which for $\varepsilon \to 0$ is the limiting one for the solution of problem (1), (2).

Before we proceed to formulating the main results on asymptotic representations of solutions of problem (1), (2), we shall give the sufficient conditions for equation (3) to be solvable.

Theorem 1. *Let the function $p(t)$ vanish at a unique point t_1^0 of the interval (a, b). Let $h(t_1^0, 0) = 0$, $h'_x(t_1^0, x) \geq h_0 > 0$ for $(t, x) \in [a, b] \times R$. Let, finally, the functions $p(t)$, $h(t, x)$ be infinitely differentiable for $(t, x) \in [a, b] \times R$. Then the following statements are true:*

—if $(t - t_1^0)p(t) < 0$ for $t \neq t_1^0$, then on the segment $[a, b]$ there exists the unique solution of equation (3) satisfying the given Dirichlet boundary conditions at both ends of the segment;

—if $(t - t_1^0)p(t) > 0$ for $t \neq t_1^0$, then on the segment $[a, b]$ there exists a unique solution of equation (3);

—if $p(t) > 0$ (resp., $p(t) < 0$) for $t \neq t_1^0$, then on the segment $[a, b]$ there exists a unique solution of equation (3) satisfying the Dirichlet boundary condition at the left-hand (resp., right-hand) end of the segment.

All the above-mentioned solutions belong to the space $C^\nu[a, b]$, where ν is the largest natural number satisfying the inequality

$$\nu p'(t_1^0) + h'_x(t_1^0, 0) > 0. \tag{4}$$

Theorem 2. *Let the conditions of Theorem 1 be fulfilled on the segment $[a, b]$ and, in addition to this, the functions $p(t)$, $h(t, x)$ be infinitely differentiable for $(t, x) \in [c, d] \times R$, where $c < a$, $d > b$. Let $p(t) = 0$ for $t = t_1^0$, and let there exist barrier functions $\alpha(t)$, $\beta(t)$ such that the inequalities*

$$\begin{gathered}[p(t)\alpha'(t) + h(t, \alpha)] \operatorname{sign} p(t) \leq 0, \\ [p(t)\beta'(t) + h(t, \beta)] \operatorname{sign} p(t) \geq 0\end{gathered}$$

hold for $t \in [c, a] \cup [b, d]$. Then all the statements of Theorem 1 are valid on the segment $[c, d]$.

For the proofs of Theorems 1 and 2, see [16].

Remark 1. It will be assumed that the conditions of Theorems 1 and 2 are fulfilled for some $a \geq 0$, $b \leq 1$, $a < t_1^0 < b$ on each of the segments into which the segment $[0, 1]$ is divided by the point t_1.

In what follows, to characterize each of the above-mentioned variants of alternation of the signs of the function $p(t)$ and of the mutual arrangement of the points t_1, t_1^0 we shall use a collection of symbols of the form

$$(t_1^0, t_1, +, -, +),$$

which in our case means that $t_1^0 < t_1$ and the function $p(t)$ is positive on the intervals $(0, t_1^0)$ and $(t_1, 1)$ and negative on the interval (t_1^0, t_1). The notation $[z] \equiv z(t_1 + 0) - z(t - 0)$ will also be used.

We have

Theorem 3. *The function $x_0(t)$, which is the limiting one as $\varepsilon \to 0$ for the solution of problem* (1), (2), *is, for $t \neq t_1$, a unique solution of equation* (3) *satisfying the following conditions:*

Case	*Variant*	*Conditions;*
1	$(t_1^0, t_1, +, -, +)$	$x_0(0) = A_0$, $[x_0] = [x_0'] = 0$;
2	$(t_1, t_1^0, -, +, -)$	$x_0(1) = B_0$, $[x_0] = [x_0'] = 0$;
3	$(t_1^0, t_1, -, -, +)$, $(t_1, t_1^0, -, +, +)$	$[x_0] = [x_0'] = 0$;
4	$(t_1^0, t_1, -, +, +)$, $(t_1, t_1^0, -, -, +)$	$[x_0] = 0$;
5	$(t_1^0, t_1, +, +, +)$, $(t_1, t_1^0, +, +, +)$	$x_0(0) = A_0$, $[x_0] = 0$;
6	$(t_1^0, t_1, +, -, -)$, $(t_1, t_1^0, +, +, -)$	$x_0(0) = A_0, [x_0] = 0, x_0(1) = B_0$;
7	$(t_1^0, t_1, +, +, -)$, $(t_1, t_1^0, +, -, -)$	$x_0(0) = A_0$, $x_0(1) = B_0$;
8	$(t_1^0, t_1, -, -, -)$, $(t_1, t_1^0, -, -, -)$	$[x_0] = 0$, $x_0(1) = B_0$;
9	$(t_1^0, t_1, -, +, -)$	$x_0(1) = B_0$;
10	$(t_1, t_1^0, +, -, +)$	$x_0(0) = A_0$.

Moreover, in the first three cases the function $x_0(t)$ exists only if the additional condition below is fulfilled:

Condition A. *A system of equations*

$$\begin{aligned} p(t_1 - 0)b_1 + h(t_1 - 0, b_2) &= 0, \\ p(t_1 + 0)b_1 + h(t_1 + 0, b_2) &= 0 \end{aligned} \tag{5}$$

has a unique solution with respect to the unknowns b_1, b_2.

In the case where Condition A is not fulfilled, then, in general, there is no function which is limiting as $\varepsilon \to 0$ for the solution of problem (1), (2).

That the statement of Theorem 3 is valid follows from Theorems 1 and 2.

3. An asymptotic representation of the solution of problem (1), (2) in the general case is of the form

$$x_N(t, \varepsilon) = \sum_{n=0}^{N} \varepsilon^n \left[x_n(t) + y_n(t/\varepsilon) + z_n((1-t)/\varepsilon) + \right.$$

$$+ u_n((t_1 - t)/\varepsilon) + v_n((t - t_1)/\varepsilon)\big], \tag{6}$$

where the number N satisfies the condition $2N \leq \nu - 2$ (see inequality (4)), $x_n(t)$ are $\nu - 2n$-times continuously differentiable functions for $t \neq t_1$ which are uniformly bounded together with their derivatives, while the other functions in (6) are infinitely differentiable for non-negative values of the argument, are identically zero for negative values of the argument, and tend, together with all their derivatives, to zero as the argument tends to infinity. Functions $x_n(t)$, $n \geq 0$, make the asymptotic representation (6) approach the solution of problem (1), (2) everywhere except some neighborhoods of the points $t = 0$, $t = t_1$, $t = 1$. Note that the function $x_0(t)$ satisfies equation (3) and certain additional conditions in accordance with one of the cases mentioned in Theorem 3. The functions $y_n(t)$, $z_n(\rho)$, $u_n(\xi)$, $v_n(\eta)$ are defined as solutions of differential equations of second order which for $n \geq 1$ are linear and have constant coefficients. Depending on the specific considered case some of these functions are identically zero for any $n \geq 0$. For example, in case 6 representation (6) contains only the functions $x_n(t)$ and $v_n(\eta)$ when $t_1^0 < t_1$. In that case functions $v_n(\eta)$ are such that $v_0(\eta) = 0$, while for $n \geq 1$ they are solutions of the problems

$$v_n'' - p(t_1 + 0)v_n' = \phi_n(\eta), \quad \eta > 0, \tag{7}$$

$$v_n'(0) = -[x_{n-1}'], \tag{8}$$

where the functions $\phi_n(\eta)$ are defined by the procedure typical of the method of small perturbations with the aid of well-known recurrent formulas. For $n \geq 1$ the functions $x_n(t)$ are defined as solutions of the equation

$$p(t)x_n' + h_x'(t, x_0(t))x_n = f_n(t), \quad t \in (0, t_1) \cup (t_1, 1), \tag{9}$$

satisfying the additional conditions

$$x_n(0) = x_n(1) = 0, \quad [x_n] = -v_n(0); \tag{10}$$

as above, functions $f_n(t)$ are defined here by the standard procedure using well-known recurrent formulas. As follows from [15], problems (9), (10) are always solvable and their solutions are $\nu - 2n$ times continuously differentiable for $t \neq t_1$.

One can easily ascertain that the difference $w_N(t, \varepsilon) = x(t, \varepsilon) - x_N(t, \varepsilon)$ satisfies the equation

$$Lw_N \equiv w_N'' - p(t)w_N' - w_N \int_0^1 h_x'(t, x_N + \theta w_N)d\theta =$$

$$= \varepsilon^N \omega_N(t, \varepsilon), \quad t \neq t_1, \tag{11}$$

and the additional conditions

$$w_N(0,\varepsilon) = 0, \quad w_N(1,\varepsilon) = m_1\varepsilon^{N+1}, \quad [w_N] = 0,$$
$$[w'_N] = m_2\varepsilon^N, \quad |\omega_N(\tau,\varepsilon)| \le m_3,$$

where m_1, m_2, m_3 are constants.

In a similar manner one can also define the coefficients of asymptotic representations in the other cases mentioned in Theorem 3.

Theorem 4. *For representation* (6) *we have the estimate*

$$|w_N(t,\varepsilon)| + \varepsilon|w'_N(t,\varepsilon)| \le M\varepsilon^{N+1}, \tag{12}$$

where the constant M does not depend on ε.

Proof of Case 6. Estimate (11) is proved using the method of barrier functions. The continuous functions

$$\alpha_N(t,\varepsilon) = x_N(t,\varepsilon) - \gamma_N(t,\varepsilon)\varepsilon^{N+1},$$
$$\beta_N(t,\varepsilon) = x_N(t,\varepsilon) + \gamma_N(t,\varepsilon)\varepsilon^{N+1},$$

where the $\gamma_N(t,\varepsilon)$ function is to be defined, will be considered as corresponding to the lower and the upper barrier function, respectively. We shall construct the function $\gamma_N(t,\epsilon)$ as the sum of two terms

$$\gamma_N(t,\varepsilon) = \gamma_{N,1}(t,\varepsilon) + \varepsilon\gamma_{N,2}((t-t_1)/\varepsilon).$$

We shall define the function $\gamma_{N,1}(t,\varepsilon)$ as a solution of the problem

$$p(t)\gamma'_{N,1} + h'_x(t,x_0(t))\gamma_{N,1} = m, \quad t \in (0,t_1) \cup (t_1,1),$$
$$\gamma_{N,1}(0,\varepsilon) = M_1, \quad \gamma_{N,1}(1,\varepsilon) = M_2,$$
$$\gamma_{N,1}(t_1 - 0,\varepsilon) - \gamma_{N,1}(t_1 + 0,\varepsilon) = \varepsilon M_3,$$

where m, M_1, M_2, M_3 are some constants not depending on ε. As follows from [15], the function $\gamma_{N,1}(t,\varepsilon)$ is defined uniquely. We shall define the function $\gamma_{N,2}(\eta)$ as a solution of the problem

$$\gamma''_{N,2} - p(t+0)\gamma'_{N,2} = 0 \text{ for } \eta > 0, \quad \gamma_{N,2}(\eta) = 0 \text{ for } \eta < 0,$$
$$\gamma_{N,2}(0) = M_4.$$

It is obvious that the function $\gamma_N(t,\varepsilon)$ is continuous for $t \in (0,1)$. By choosing the constant M_4 sufficiently large, we succeed in fulfilling the inequality

$$\alpha'_N(t_1 + 0,\varepsilon) - \alpha'_N(t_1 - 0,\varepsilon) > 0.$$

Moreover, the function $\gamma_N(t,\varepsilon)$ will be positive for $t \in (0,1)$ provided that the constant M_4 is chosen sufficiently large. Thus for the appropriately chosen constants m, M_1, M_2, M_3, M_4 and sufficiently small values of $\varepsilon > 0$ the functions $\alpha_N(t,\varepsilon)$, $\beta_N(t,\varepsilon)$ will be the lower and the upper barrier function, respectively. Therefore there exists a solution $x = x(t,\varepsilon)$ of problem (1), (2) for which the inequalities

$$x_N(t,\varepsilon) - \varepsilon^{N+1}\gamma_N(t,\varepsilon) \le x(t,\varepsilon) \le x_N(t,\varepsilon) + \varepsilon^{N+1}\gamma_N(t,\varepsilon)$$

are fulfilled.

Now let us estimate the derivative of $w_N(t,\varepsilon)$. Consider the auxiliary function $v_N(t,\varepsilon) = \exp(-\lambda_0 t)w_N(t,\varepsilon)$, where λ_0 is some constant. This function satisfies the equation

$$L_1 v_N \equiv \varepsilon v_N'' + \left[2\varepsilon\lambda_0 - p(t)\right]v_N' + \Big[\varepsilon\lambda_0^2 - p(t)\lambda_0 -$$
$$- \int_0^1 h_x'(t, x_N + \theta w_N)d\theta\Big]v_N = \varepsilon^{N+1}\phi_N(t,\varepsilon)\exp(-\lambda_0 t), \quad t \ne t_1, \tag{13}$$

and the additional conditions

$$v_N(0,\varepsilon) = 0, \quad v_N(1,\varepsilon) = m_4\varepsilon^{N+1}\exp(-\lambda_0 t),$$
$$[v_N] = 0, \quad [v_N'] = m_5\varepsilon^N \exp(-\lambda_0 t).$$

For $t < t_1$ the function

$$\psi_N(t,\varepsilon) = \varepsilon^N M_5 \exp(-\lambda t) + v_N(t,\varepsilon),$$

where $\lambda = const$, satisfies the equation

$$L_1\psi_N = \Big\{\varepsilon\lambda^2 - \left[2\varepsilon\lambda_0 - p(t)\right]\lambda + \Big[\varepsilon\lambda_0^2 - p(t)\lambda_0 -$$
$$- \int_0^1 h_x'(t, x_N + \theta w_N)d\theta\Big]\Big\}\varepsilon^N M_5 \exp(\lambda t) + \varepsilon^N \phi_N(t,\varepsilon)\exp(-\lambda_0 t). \tag{14}$$

Choose λ_0 such that the coefficient of $v_N(t,\varepsilon)$ in expression (13) is negative for $t < t_1^0/2$. Next, choose λ such that the expression in the braces on the right-hand side of (14) is positive. Finally, choose M_5 by the requirement that the entire right-hand side of (14) is positive.

Let $t \le \lambda^{-1} < t_1^0/2$. Obviously, the function $\psi_N(t,\varepsilon)$ cannot have a maximum positive value for $o < t < \lambda$. Increasing, if necessary, the constant M_5, we obtain the inequality $\psi_N(0,\varepsilon) > \psi_N(\lambda^{-1},\varepsilon)$. Therefore, when $t = 0$, the function $\psi = \psi_N(t,\varepsilon)$ has maximum value for the segment $[0,\lambda^{-1}]$ and hence $\psi_N'(0,\varepsilon) \le 0$. We thus have the estimate

$$w_N'(0,\varepsilon) \le \lambda\varepsilon^N M_6 \le \varepsilon^{N-1} M_6.$$

In a similar manner one can derive an estimate for $w'_N(0,\varepsilon)$ from below. Integrating now the left- and right-hand sides of (11) from 0 to 1, we easily obtain the second part of estimate (12).

4. Let us consider the boundary value problems for the equations of third order

$$L_\varepsilon x \equiv \varepsilon^2 x''' + \varepsilon r(t)x'' - \left[p(t)x' + h(t,x)\right] = 0, \quad t \in (0,1), \tag{15}$$

$$x(0) = A_0, \quad x'(0) = A_1, \quad x'(1) = B_1, \tag{16}$$

where $p(t)$, $h(t,x)$ are the same as in problem (1), (2), while the function $r(t)$ may have a discontinuity of the first kind for $t = t_1$ and, being defined at the end points of $[0,t_1]$, $[t_1,1]$ with respect to continuity, is infinitely differentiable on these segments. To simplify our further discussion, it will be assumed that $r(t)$ may change sign only at the points t_1^0, t_1, taking into account that the one-sided limiting values of this function at the points $t = 0$, $t = t_1$, $t = 1$ differ from zero.

Under these conditions we have 128 variants of the mutual arrangement of the points t_1 t_1^0 and alternation of signs of the functions $r(t)$, $p(t)$ on the respective segments of $(0,1)$. The upper and lower barrier functions, uniformly bounded with respect to the parameter ε, can be constructed for 75 variants, of which 21 contain an infinite number of pairs of barrier functions uniformly bounded with respect to the parameter ε. Note that different pairs of barrier functions confine different solutions of problem (15), (16). Generally speaking, for $\varepsilon \to 0$ these solutions have different limiting functions which, for $t \neq t_1$, satisfy equation (3) and some additional conditions at the points $t = 0$, $t = t_1$, $t = 1$. The remaining 53 variants do not allow us, generally speaking, to construct barrier function bounded uniformly with respect to ε.

To illustrate the properties of solutions of the problem under consideration and the peculiar features of construction of respective asymptotic representations we shall consider one concrete variant. The other variants are considered similarly.

Let $t_1 < t_1^0$, the function $r(t)$ be negative on the segment $[0,1]$, and the function $p(t)$ be positive for $t \in (0,t_1^0)$ and negative on the interval $(t_1^0,1)$. An asymptotic representation of problem (15), (16) will be constructed in the form

$$x_N(t,\varepsilon) = \sum_{k=0}^{N} \varepsilon^k \left[x_k(t) + y_k(\lambda t/\varepsilon) + z_k\left(\mu(t_1 - t)/\varepsilon\right) + \right.$$
$$\left. + u_k\left(\nu(t - t_1)/\varepsilon\right) + v_k\left(\omega(1-t)/\varepsilon\right) + w_k\left(\sigma(1-t)/\varepsilon\right), \right. \tag{17}$$

where all functions with a quickly changing argument for $\varepsilon \to 0$ possess the same properties as analogous functions described in constructing an asymptotic representation of a solution of problem (1), (2). Note that λ, μ, ν are the positive roots of the equations $\lambda^2 - r(0)\lambda - p(0) = 0$, $\mu^2 - r(t_1-0)\mu - p(t_1-0) = 0$, $\nu^2 - r(t_1+0)\nu - p(t_1+0) = 0$, respectively, while ω and σ are the roots of the equation $\rho^2 - r(1)\rho - p(1) = 0$ (both these roots are positive for the variant under consideration). The coefficients of representation (17) are to be constructed so that for $t \neq t_1$ the function $x_N(t\epsilon)$ satisfies, to within values of order $O(\varepsilon^N)$, equation (15) and conditions (16), and so that for all $t \in (0,1)$ the function $x_N(t,\varepsilon)$ is continuous and possesses derivatives of first and second orders.

The function $x_0(t)$ is one of the solutions of equation (3). It should be chosen so that it is continuous and continuously differentiable for $t \in (0,1)$ and satisfies the conditions $x_0(0) = A_0$, $x_0(1) = B_0$, where B_0 is some constant.

Functions $x_m(t)$, $m \geq 1$, are defined as solutions of equation (9) for $t \neq t_1$, satisfying the conditions $x_m(0) = 0$, $x_m(1) = b_m$, where b_m are some constants. Other functions from representation (17) are defined as solutions of linear ordinary differential equations of third order with constant coefficients. All these functions are defined so that $x_m(t,\varepsilon)$ satisfies the above-formulated conditions. Due to the restrictions imposed on the functions $r(t)$, $p(t)$, $h(t,x)$, such a construction is possible for any $m \geq 1$.

Note that with the above approach the functions $y_k(\tau)$, $z_k(\xi)$, $u_k(\eta)$ are defined uniquely, but to find constants B_0, b_m, $v_m(0)$, $w_m(0)$, $m \geq 0$ we have only one condition, namely the right-hand boundary condition of (16), and therefore representation (17) describes a whole family of asymptotic representations. Assuming that the above-mentioned constants are defined, let us prove

Theorem 5. *Let the function $r(t)$ have a discontinuity of the first kind at one point $t = t_1$ at most, and let it be infinitely differentiable on each of the segments $[0,t_1]$, $[t_1,1]$ and not change sign on the intervals $(0,t_1)$, (t_1,t_1^0), $(t_1^0,1)$. Let the conditions of Theorems 1 and 2 and Remark 1 be fulfilled. Then in the neighborhood of each function $x_N(t,\varepsilon)$ from family (17) there is a solution $x(t,\varepsilon)$ of problem (15), (16) where the estimate*

$$\left|x(t,\varepsilon) - x_N(t,\varepsilon)\right| \leq M\varepsilon^{N+1} \tag{18}$$

holds, where the constant M does not depend on ε.

Proof. The theorem will be proved using the theorem on barrier functions [16]. It should be noted in advance that for all $t \in [0,1]$ the function $x_N(t,\varepsilon)$ is continuous, exactly satisfies the boundary conditons (16), for $t \neq t_1$ satisfies the equation

$$L_\varepsilon x_N = \varepsilon^N \psi_N(t,\varepsilon) \tag{19}$$

and for $t = t_1$ the first and second derivatives of $x_N(t,\varepsilon)$ have discontinuities and $x'_N(t_1+0,\varepsilon) - x'_N(t_1-0,\varepsilon) = \varepsilon^N a_N$, $x''_N(t_1+0,\varepsilon) - x''_N(t_1-0,\varepsilon) = \varepsilon^{N-1} c_N$, where a_N, c_N, $\psi_N(t,\varepsilon)$ remain bounded as $\varepsilon \to 0$.

Let $\chi_N(t,\varepsilon) = \exp[\vartheta(t-t_1)/\varepsilon]$ for $t \le 0$, $\chi_N(t,\varepsilon) = \exp(\varkappa(t_1-t)/\varepsilon]$ for $t \ge 0$, where the constants $\vartheta, \varkappa$ are chosen so that the function

$$y_N(t,\varepsilon) = x_N(t,\varepsilon) + \varepsilon^{N+1}\chi_N(t,\varepsilon)$$

is continuous together with its derivatives of first and second orders. Barrier functions will be constructed in the form

$$\alpha(t,\varepsilon) = y_N(t,\varepsilon) - \varepsilon^N\gamma(t) = y_N(t,\varepsilon) - \Gamma(t,\varepsilon),$$
$$\beta(t,\varepsilon) = y_N(t,\varepsilon) + \Gamma(t,\varepsilon),$$

where $\gamma(t)$ is to be defined. Let t_2 be a minimum and t_3 a maximum value of the variable t such that $h'_x(t_1, x_0(t_2)) = h'_x(t_3, x_0(t_3)) = h_0$ and $h'_x(t_2, x_0(t_2)) \ge h_0$ for all $t \in [t_2, t_3]$. However, if the inequality $h'_x(t, x_0(t) \ge h_0$ holds for $t \in [0, t_3]$ or $t \in [t_2, 1]$, then we will accordingly set $t_2 = 0$, $t_3 = 1$. Let

$$H = \sup_{[0,1]} \left|h'_x(t, x_0(t))\right|, \quad p_0 = \inf_{[0,t_2]} p(t).$$

On the segment $[0, t_1]$ the function $\gamma(t)$ will be constructed as a solution of the problem

$$p_0\gamma' - H\gamma = M_1(t), \quad \gamma(0) = \gamma_0,$$

where $M_1(t)$ is some positive function with a positive derivative, $\gamma_0 > 0$. Obviously, $\gamma(t) > 0$, $\gamma'(t) > 0$ for $t \in [0, t_1]$. For $t \in [t_1, t_2]$ the function $\gamma(t)$ will be defined as a solution of the problem

$$p_0\gamma' - H\gamma = M_2(t), \quad \gamma(t_1) = \gamma(t_1 - 0), \qquad .$$

where $M_2(t)$ is a positive function, $M_2(t_1) = M_1(t_1)$. It will be assumed that $M'_1(t_1) \ge M'_2(t_1)$. It is obvious that $\gamma(t) > 0$, $\gamma'(t) > 0$.

On the segment $[t_1, t_1^0]$ the function will be sought for as a solution of the problem

$$p(t)\gamma' + h'_x(t, x_0(t))\gamma = M_3(t), \quad \gamma(t_2) = \gamma(t_2 - 0),$$

where $M_3(t)$ is a positive function such that

$$M_3(t_2) = p(t_2+0)\left[M_2(t_2) + H\gamma(t_2)\right]/p_0 + h_0\gamma(t_2).$$

With such a choice of the value of $M_3(t_2)$ the function $\gamma(t)$ will be continuous together with the first derivative for $t = t_2$. If, in addition,

$$M_3(t) = h'_x(t, x_0(t))M_3(t_2)/h_0 + \left[h'_x(t, x_0(t))\right]^2 \int_{t_2}^{t} g(\tau)\left[h'_x(\tau, x_0(\tau))\right]^{-2} d\tau/h_0,$$

where $g(t)$ is an arbitrary positive function, then $\gamma'(t) > 0$ for $t \geq t_2$. It is easy to verify that if the value of $g(t_2)$ is sufficiently large, then the inequality

$$\gamma''(t_2 - 0) - \gamma''(t_2 + 0) > 0$$

will be fulfilled.

In a similar manner we define the function $\gamma(t)$ on the segment $[t_1^0, 1]$. It is not difficult to define this function so that it is continuous for $t = t_1^0$.

We can easily ascertain that with the function $\gamma(t)$ thus defined, $\alpha(t, \varepsilon)$, $\beta(t, \varepsilon)$ will respectively be the lower and the upper function of the problem under consideration. Therefore the solution of this problem exists and estimate (18) holds for it.

By a similar reasoning one can prove estimates of the nearness for derivatives of the solutions considered. Note that because functions $y_k(t)$ (for $t \in [t_1^0, 1]$) and values $v_k(0)$, $w_k(0)$ are given ambiguously, the solution of problem (15), (16) is defined nonuniquely. □

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(Received 27.01.1994)

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