BOOLEAN FAMILIES OF VALUATION RINGS

Yu. L. Ershov

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The reader is expected to be familiar with elements of field valuation theory (see, e.g., [1,3]). By a Boolean space we mean a compact totally disconnected space (every Boolean space is homeomorphic to the space of maximal ideals of a certain Boolean algebra). The interest in Boolean families of valuation rings and their "lift" to algebraic extensions arose as a result of studying the property of being regularly closed with respect to a family of valuation rings for fields (see [4], specifically, Theorem 2.2).

Let F be a field, W a family of valuation rings of F; we call W weakly Boolean if the collection of subsets of W of the type $V_A \rightleftharpoons \{R_v | R_v \in W,\ A \subseteq R_v\}$, where A is a finite subset of F, forms a closed-open base of a Boolean topology on W. If $A = \{a\}$, we write V_a instead of $V_{\{a\}}$. Note that $V_a = \bigcap_{i \leq k} V_{a_i}$ if $A = \{a_0, \ldots, a_k\}$, i.e., the family of subsets of the type V_a forms a subbase of the topology with base V_A, $A \subseteq F$, and is finite. For notational convenience we write V_A^F in place of V_A and V_a^F in place of V_a, when considering different fields.

THEOREM 1. Let W be a weakly Boolean family of valuation rings of a field F, and $F_0 \geq F$ an algebraic extension of F; let $W_0 \rightleftharpoons \{R_{v_0} | R_{v_0}$ is a valuation ring of F_0 such that $R_{v_0} \cap F \in W\}$. Then W_0 is a weakly Boolean family of valuation rings of F_0.

First we prove the theorem, assuming that F_0 is a Galois extension of F.

By letting $\alpha_0 \in F_0^*$, we show that the set $V_0 \rightleftharpoons W_0 \setminus V_{\alpha_0}^{F_0}$ is representable as a union of a finite family of basic open sets. Let $f_{\alpha_0} = x^k + a_1 x^{k-1} + \ldots + a_k \in F[x]$ be a minimal polynomial of α_0 over F. Suppose $\alpha_0, \alpha_1, \ldots, \alpha_{k-1}$ are all elements from F_0 conjugate to α_0 in F (equivalently, $\{\alpha_0, \alpha_1, \ldots, \alpha_{k-1}\}$ are all roots of f_{α_0} in F_0). For any $i \in I \rightleftharpoons \{i | 1 \leq i \leq k,\ a_i \neq 0\}$, define

$$V_i \rightleftharpoons \Big(\bigcap_{j=1}^{k} V^{F_0}_{a_j a_i^{-1}}\Big) \cap \Big(\bigcap_{\substack{j=0 \\ a_j \neq 0}}^{i-1} (W_0 \setminus V^{F_0}_{a_i a_j^{-1}})\Big), \text{ where } a_0 \rightleftharpoons 1;$$

$$\mathcal{E}_i \rightleftharpoons \{E | E \subseteq \{1, \ldots, k-1\},\ |E| = k - i\};$$

$$V_i^* \rightleftharpoons \bigcup_{E \in \mathcal{E}_i} \Big(V_i \cap \Big(\bigcap_{j \in E} V_{\alpha_j}^{F_0}\Big)\Big);\ i < k;$$

$$V_k^* \rightleftharpoons V_k.$$

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Prior to proving that $V_0 = \bigcup_{i\in I} V_i^*$, we introduce some definitions. Let $f = x^k + a_1x^{k-1} + \ldots + a_k \in F[x]$ be a unitary polynomial, $k > 1$, and $f = \prod_{i<k}(x-\alpha_i)$ a factorization of f in some extension $F_0 \geq F$ of F. Let v be a certain valuation of F and v_0 its arbitrary extension to F_0. Define

$$\gamma_0 \rightleftharpoons \min\{v(a_i)|i=1,\ldots,k\},\quad M \rightleftharpoons \{i|v(a_i)=\gamma_0,\quad i=1,\ldots,k\},\quad i_0 \rightleftharpoons \min M,\quad i_1 \rightleftharpoons \max M;$$

then the following is valid.

LEMMA 1 (on root values). 1) If $\gamma_0 < 0$, there are exactly i_0 indices $i < k$ such that $v_0(\alpha_i) < 0$ and exactly $i_1 - i_0$ indices $i < k$ such that $v_0(\alpha_i) = 0$;

2) if $\gamma_0 = 0$, then $v_0(\alpha_i) \geq 0$ for every $i < k$, and there are exactly i_1 indices $i < k$ such that $(v_0\alpha_i) = 0$;

3) if $\gamma_0 > 0$, then $v_0(\alpha_i) > 0$ for every $i < k$.

Proof. Let $S_i \rightleftharpoons \{s|s \subseteq \{0,1,\ldots,k-1\},\ |s| = i\}$, $i = 1,\ldots,k$; $s \rightleftharpoons \bigcup_{i=1}^{k} s_i$; $\alpha_s \rightleftharpoons \prod_{i\in s}\alpha_i$ for $s \in S$. Then $a_i = (-1)^i \sum_{s\in S_i}\alpha_s$, $i = 1,\ldots,k$. Set $s_0 \rightleftharpoons \{i|i<k,\ v_0(\alpha_i) < 0\}$ and assume that $s_0 \neq \emptyset$. Then for any $s \in S$, $v_0(\alpha_{s_0}) \leq v_0(\alpha_s)$ is true; moreover, $v_0(\alpha_{s_0}) < v_0(\alpha_s)$ is true for $s_0 \neq s \in S$ such that $|s| \leq i_0' \rightleftharpoons |s_0|$. This follows easily from the validity of $v_0(\alpha_s) = \sum_{i\in s} v_0(\alpha_i)$ for $s \in S$ and from the definition of s_0. This in turn implies that $v(a_i) = v_0(a_i) \geq \min\{v_o(\alpha_s)|s \in S_i\} \geq \min\{v_0(\alpha_s)|s \in S_{i_0'}\} = v_0(\alpha_{s_0}) = v(a_{i_0'})$ and $v(a_i) > v(a_{i_0'})$ for $i < i_0'$; hence $i_0' = i_0 = \min M$, where $M = \{i|v(a_i) = \gamma_0\ (= \min\{v(a_i)|i=1,\ldots k\})\}$. Moreover, in this case, i.e., when $s_0 \neq \emptyset$, we have $\gamma_0 < 0$.

Let $s_1 \rightleftharpoons \{i|i<k\},\ v_0(\alpha_i) \leq 0\}$ and suppose that $s_1 \neq \emptyset$. Then $v_0(\alpha_{s_1}) \leq v_0(\alpha_s)$ for every $s \in S$ and $v_0(\alpha_{s_1}) < v_0(\alpha_s)$ for every $s \in S$ such that $|s| \geq i_1' \rightleftharpoons |s_1|$, $s \neq s_1$; hence $v(a_{i_1'}) = v_0(a_{i_1'}) = \min_{s\in S_{i_1'}} v_0(\alpha_s) = v_0(\alpha_{s_1}) \leq \min_{s\in S_i} v_0(\alpha_s) \leq v_0(a_i) = v(a_i)$ for every $i = 1,\ldots,k$ and $v(a_{i_1'}) < v(a_i)$ for $i_1' < i \leq k$. This implies $i_1' = i_1 = \max M$ ($M = \{i|v(a_i) = \gamma_0\ (= \min\{v(a_i)|i=1,\ldots,k\})\}$). In the present case where $s_1 \neq \emptyset$, we have $\gamma_0 \leq 0$.

Conversely, if $s_1 = \emptyset$, i.e., $v_0(\alpha_1) > 0$ for all $i < k$, then $v(a_i) = v_0(a_i) \geq \min\{v_0(\alpha_s)|s \in S_i\} > 0$ and $\gamma_0 \rightleftharpoons \min\{v(a_i)|i = 1,\ldots,,k\} > 0$.

The conclusions of the lemma follow from the above.

We apply the lemma to the case under consideration. Let $R_{v_0} \in V_i^*$ for some $i \in I$. Then

$$R_{v_0} \in V_i = (\bigcap_{j=1}^{k} V^{F_0}_{a_ja_i^{-1}}) \bigcap (\bigcap_{\substack{j=0\\ a_j\neq 0}}^{i=1} (W_0 \setminus V^{F_0}_{a_ia_j^{-1}}));$$

$R_{v_0} \in V^F_{a_ja_i^{-1}}$ entails $v_0(a_ja_i^{-1}) = v_0(a_j) - v_0(a_i) \geq 0$, $v_0(a_i) \leq v_0(a_j)$ for all $j = 1,\ldots,k$; so $v_0(a_i) = \min\{v_0(a_j)|j = 1,\ldots,k\}$. $R_{v_0} \in W_0 \setminus V^{F_0}_{a_ia_j^{-1}}$ implies $v_0(a_ia_j^{-1}) = v_0(a_i) - v_0(a_j) < 0$, $v_0(a_i) < v_0(a_j)$ for all $j < i$. In particular, $v_0(a_i) < v_0(1) = 0$, and so i has the properties of i_0 from Lemma 1 for the case $f = f_{\alpha_0}$. Further, for the case $i < k$, there exists $E \in \mathcal{E}_i$ such that $R_{v_0} \in \bigcap_{j\in E} V^{F_0}_{\alpha_j}$, i.e., $v_0(\alpha_j) \geq 0$ for $j \in E$, and since $E \in \mathcal{E}_i$, $|E| = k - i_0$. By virtue of Lemma 1, there are exactly i_0 roots α of the polynomial f_{α_0} such that $v_0(\alpha) < 0$, from which we conclude that E is precisely the set of all roots α such that $v_0(\alpha) \geq 0$, and $\{j|j<k, v_0(\alpha_j) < 0\} = \{0,\ldots,k-1\} \setminus E$. Since $E \subseteq \{1,\ldots,k-1\}$, $0 \notin E$

and $v_0(\alpha_0) < 0$, i.e., $R_{v_0} \in W_0 \setminus V_{\alpha_0}^{F_0} = V_0$. For the case $i = k$, $v_0(\alpha) < 0$ for all roots of f_{α_0}, hence $v_0(\alpha_0) < 0$ and $R_{v_0} \in W_0 \setminus V_{\alpha_0}^{F_0} = V_0$.

We have thus proved that $\bigcup_{i\in I} V_i^* \subseteq V_0$. Assume, on the contrary, that $R_{v_0} \in V_0$; then $v_0(\alpha_0) < 0$. Let i_0 be defined as in Lemma 1 for v_0 and f_{α_0}, and $E \rightleftharpoons \{j | j \leq k,\ v_0(\alpha_i) \geq 0\}$. Suppose $0 \notin E$, and by Lemma 1 $|E| = k - i_0$; then $E \in \mathcal{E}_{i_0}$ if $i_0 < k$. It is also easy to check that $i_0 \in I$, $R_{v_0} \in V_{i_0}$, and $R_v \in V_{i_0} \cap (\bigcap_{j\in E} V_{\alpha_j}^{F_0}) \subseteq V_{i_0}^*$. Thus $V_0 = \bigcup_{i\in I} V_i^*$ holds.

Up to this point we note that for $a \in F$ the set $W_0 \setminus V_a^{F_0}$ can be represented as a union of a finite number of basic open sets. Since W is weakly Boolean, $W \setminus V_a^F = \bigcup_{s\leq l} V_{A_s}^F$ for appropriate finite $A_s \subseteq F$, $s \leq l$. It is straightforward to verify that $W_0 \setminus V_a^{F_0} = \bigcup_{s\leq l} V_{A_S}^{F_0}$ too.

It follows from this remark and the equality $V_0 = \bigcup_{i\in I} V_i^*$ that $W_0 \setminus V_{\alpha_0}^{F_0} = V_0$ is representable as a union of a finite number of open basic sets. Note that this fact readily implies the following more general assertion:

The family B^*, consisting of all finite unions of basic open sets, is closed under union, intersection, and complementation.

It is obvious that B^* is closed under union and intersection. To check that B^* is closed under complementation, we should keep in mind the following:

a) $W_0 \setminus V_A^{F_0} = W_0 \setminus \bigcap_{\alpha_0\in A} V_{\alpha_0}^{F_0} = \bigcup_{\alpha_0\in A} (W_0 \setminus V_{\alpha_0}^{F_0})$;

b) $W_0 \setminus \bigcup_{s\leq l} V_{A_s}^{F_0} = \bigcap_{s\leq l}(W_0 \setminus V_{A_s}^{F_0})$,

which in view of the relation $W_0 \setminus V_{\alpha_0}^{F_0} \in B^*$, proved above, validate the present case.

We proceed to establish that W_0 with a given topology is a Hausdorff space. Let $R_{v_0} \neq R_{v_0'} \in W_0$. If $R_{v_0} \cap F \neq R_{v_0'} \cap F$, then the facts that W is Hausdorff and that $R_{v_0} \cap F$ and $R_{v_0'} \cap F \in W$ imply the existence of $a \in F$ such that $R_{v_0} \cap F \in V_a^F$ and $R_{v_0'} \cap F \notin V_a^F$; whence $R_{v_0} \in V_a^{F_0}$ and $R_{v_0'} \notin V_a^{F_0}$. If $R_{v_0} \cap F = R_{v_0'} \cap F$, then R_{v_0} and $R_{v_0'}$ are two distinct valuation rings dominating $R_{v_0} \cap F$; whence, as is well known (see [1, p.187]), $R_{v_0} \not\leq R_{v_0'}$ and $R_{v_0'} \leq R_{v_0}$. If $a_0 \in R_{v_0} \setminus R_{v_0'}$, then $R_{v_0} \in V_{a_0}^{F_0}$, $R_{v_0'} \notin V_{a_0}^{F_0}$.

It remains to establish the compactness of W_0.

First we prove:

Proposition 1. The restriction map $\varepsilon : R_{v_0} \mapsto R_{v_0} \cap F$ from W_0 to W is continuous and closed-open.

The continuity of ε is evident since $\varepsilon^{-1}(V_A^F) = V_A^{F_0}$ for every finite $A \subseteq F$.

Now we check that $\varepsilon(V_{A_0}^{F_0})$ is closed-open in W (A_0 is a finite subset of F_0). Examine the case $A_0 = \{\alpha\}$, $\alpha \neq 0$. By letting $f = x^k + a_1x^{k-1} + \ldots + a_k \in F[x]$ be a minimal polynomial of α^{-1} over F and letting $V \rightleftharpoons \cap\{V_{a_i}^F \setminus V_{a_i^{-1}}^F | \ i = 1, \ldots, k,\ a_i \neq 0\}$, we prove that $\varepsilon(V_\alpha^{F_0}) = W \setminus V$. To do this, we must use Lemma 1.

Let $R_{v_0} \in V_\alpha^{F_0}$, i.e., $\alpha \in R_{v_0}$, $v_0(\alpha) \geq 0$, and $v_0(\alpha^{-1}) \leq 0$; then by Lemma 1, there exists $1 \leq i \leq k$ such that $v(a_i) = v_0(a_i) \leq 0$, and so $R_{v_0} \notin V_{a_i}^{F_0} \setminus V_{a_i^{-1}}^{F_0}$, $\varepsilon(R_{v_0}) \notin V_{a_i}^F \setminus V_{a_i^{-1}}^F \supseteq V$, and $\varepsilon(R_{v_0}) \in W \setminus V$. Thus $\varepsilon(V_\alpha^{F_0}) \subseteq W \setminus V$. Assume, on the contrary, that $R_v \in W \setminus V$. Since $R_v \notin V$, for some $1 \leq i \leq k$ we have $R_v \notin V_{a_i}^F \setminus V_{a_i^{-1}}^F$ and $v(a_i) \leq 0$; by Lemma 1 $v_0(\beta) \leq 0$ holds for a suitable extension v_0 of the valuation v to F and for a certain root β of f. There exists an F-automorphism φ of F_0 such that $(\alpha^{-1})^\varphi = \beta$, and so for the valuation v_0^φ $(v_0^\varphi(a) \rightleftharpoons v_0(a^\varphi))$ of F_0 extending v, we have $v_0^\varphi(\alpha^{-1}) \leq 0$, $v_0^\varphi(\alpha) \geq 0$, and $R_{v_0^\varphi} \in V_\alpha^{F_0}$, $\varepsilon(R_{v_0^\varphi}) = R_{v_0^\varphi} \cap F = R_v$; consequently, $W \setminus V \subseteq \varepsilon(V_\alpha^{F_0})$ and $\varepsilon(V_\alpha^{F_0}) = W \setminus V$.

We turn to the case of an arbitrary basic closed-open set $V_A^{F_0}$, $A = \{\alpha_0, \ldots, \alpha_n\}$. Let $F_1 \leq F_0$ be the least Galois extension of F containing A. Let $k \rightleftharpoons [F_1 : F]$ and $G = G(F_1/F) = \{g_0 = e,\ g_1, \ldots, g_{k-1}\}$. Define $S_n \rightleftharpoons \{1, \ldots, k+1\}^n$ and

$$\delta_{\bar{s}} \rightleftharpoons \alpha_0 + \alpha_1^{s_1} + \alpha_2^{s_1 s_2} + \ldots + \alpha_n^{\pi\bar{s}(\rightleftharpoons s_1 s_2 \ldots s_n)}$$

for $\bar{s} = (s_1, \ldots, s_n) \in S_n$.

We show that $\varepsilon(V_A^{F_0}) = \bigcap_{\bar{s} \in S_n} \varepsilon(V_{\delta_{\bar{s}}}^{F_0})$. Check the inclusion $\subseteq$. If $R_v = \varepsilon(R_{v_0})$ and $R_{v_0} \in V_A^{F_0}$, then $R_{v_0} \in V_{\delta_{\bar{s}}}^{F_0}$ and $R_v \in \varepsilon(V_{\delta_{\bar{s}}}^{F_0})$ for all $\bar{s} \in S_n$; thus $R_v \in \bigcap_{s \in S_n'} \varepsilon(V_{\delta_{\bar{s}}}^{F_0})$, and so

$$\varepsilon(V_A^{F_0}) \subseteq \bigcap_{\bar{s} \in S_n} \varepsilon(V_{\delta_{\bar{s}}}^{F_0}).$$

To prove the inverse we need the following lemma.

LEMMA 2. Let B be an arbitrary subset of F_1, w a valuation of F_1 such that for every $\bar{s} \in S_n$, there exists $j < k$ such that $w(\beta^{g_j}) \geq 0$ for all $\beta \in B$ and $w(\delta_{\bar{s}}^{g_j}) \geq 0$. Then there exists $j < k$ such that $w(\beta^{g_j}) \geq 0$ for every $\beta \in B$, and $w(\alpha_i)^{g_j} \geq 0$ for all $i \leq n$.

Proof. The proof follows by induction on n. The case $n = 0$ is trivial. Let $n > 0$. Assume that for $(n-1)$ the lemma is true and its assumptions are satisfied. Then for any $\bar{s} \in S_{n-1}$ and $s \in \{1, \ldots, k+1\}$, there exists $j = j_{\bar{s},s} < k$ such that $w(\beta^{g_j}) \geq 0$ for any $\beta \in B$ and $w(\delta_{\bar{s},s}^{g_j}) \geq 0$; hence for every $\bar{s} \in S_{n-1}$, there exist s_0, s_1 such that $1 \leq s_0 < s_1 \leq k+1$, and $j_{\bar{s},s_0} = j_{\bar{s},s_1}$. Set $j_{\bar{s}} \rightleftharpoons j_{\bar{s},s_0} = j_{\bar{s},s_1}$. Then $w(\delta_{\bar{s},s_0}^{g_{j\bar{s}}}) \geq 0$, $w(\delta_{\bar{s},s_1}^{g_{j\bar{s}}}) \geq 0$, $w(\delta_{\bar{s},s_0}^{g_{j\bar{s}}} - \delta_{\bar{s},s_1}^{g_{j\bar{s}}}) \geq 0$; $\delta_{\bar{s},s_0}^{g_{j\bar{s}}} - \delta_{\bar{s},s_1}^{g_{i\bar{s}}} = (\alpha_n^{\pi\bar{s}\cdot s_0} - \alpha_n^{\pi\bar{s}\cdot s_1})^{g_{j\bar{s}}} = (\alpha_n^{\pi\bar{s}\cdot s_0} \cdot (1 - \alpha_n^{\pi\bar{s}(s_1 - s_0)}))^{g_{j\bar{s}}}$. This implies that $w(\alpha_n^{g_{j\bar{s}}}) \geq 0$ and $w(\delta_{\bar{s}}^{g_{j\bar{s}}}) \geq 0$, since $w(\delta_{\bar{s},s_0}^{g_{j\bar{s}}}) = w(\delta_{\bar{s}}^{g_{j\bar{s}}} + (\alpha_n^{\pi\bar{s}\cdot s_0})^{g_{j\bar{s}}}) \geq 0$. Consequently, for every $\bar{s} \in S_{n-1}$, there exists j $(= j_{\bar{s}}) < k$ such that $w(\beta^{g_j}) \geq 0$ for all $\beta \in B$, $w(\alpha_n^{g_j}) \geq 0$, and $w(\delta_{\bar{s}}^{g_i}) \geq 0$. Applying the induction hypothesis to the case $n-1$ and to $B' \rightleftharpoons B \cup \{\alpha_n\}$, we find a $j < k$ such that $w(\beta^{g_i}) \geq 0$ for all $\beta \in B$, $w(\alpha_n^{g_i}) \geq 0$ and $w(\alpha_i^{g_j}) \geq 0$ for $i < n$, as desired.

We prove the inverse inclusion $\bigcap_{\bar{s} \in S_n} \varepsilon(V_{\delta_{\bar{s}}}^{F_0}) \subseteq \varepsilon(V_A^{F_0})$. Let $R_v \in \bigcap_{\bar{s} \in S_n} \varepsilon(V_{\delta_{\bar{s}}}^{F_0})$, and let w be an arbitrary extension of v to F_1 (note that every extension w' of v to F_1 is of the form $w' = w^g$ for an appropriate $g \in G = G(F_1/F)$ $(w'(\alpha) = w(\alpha^g)$ for all $\alpha \in F_1)$. Then $R_v \in \bigcap_{\bar{s} \in S_n} \varepsilon(V_{\delta_{\bar{s}}}^{F_0})$ implies that, for any $\bar{s} \in S_n$, there exists a $j < k$ such that $w(\delta_{\bar{s}}^{g_j}) \geq 0$. Then by Lemma 2, there exists $j < k$ such that $w(\alpha_i^{g_j}) \geq 0$ for all $i \leq n$, and hence for v_0 extending w^{g_i} to F_0 we have $R_{v_0} \in V_A^{F_0}$ and $R_v = \varepsilon(R_{v_n})$. Thus the equality $\varepsilon(V_A^{F_0}) = \bigcap_{\bar{s} \in S_n} \varepsilon(V_{\delta_{\bar{s}}}^{F_0})$ is valid. It follows from this and from the above that $\varepsilon(V_A^{F_0})$ is closed-open. Hence ε is open.

Now we check the closeness of ε. Let $\Phi \subseteq W_0$ be closed; then $W_0 \setminus \Phi$ is open and $W_0 \setminus \Phi = \bigcup_{i \in I} V_{A_i}$ for a suitable family of finite subsets $A_i \subseteq F_0$, $i \in I$. For a finite $I_0 \subseteq I$, we let $\Phi_{I_0} \rightleftharpoons W \setminus \bigcup_{i \in I_0} V_{A_i}$, and note that $I_0 \subseteq I_1 \subseteq I$ entails $\Phi_{I_0} \supseteq \Phi_{I_1} \supseteq \Phi$. Since $\Phi_{I_0} \in B^*$ for a finite I_0, $\varepsilon(\Phi_{I_0})$ is closed-open. Further, $\Phi = \cap\{\Phi_{I_0} | I_0 \subseteq I,\ I_0 \text{ is finite }\}$ and $\varepsilon(\Phi) = \cap\{\varepsilon(\Phi_{I_0}) | I_0 \subseteq I,\ I_0 \text{ is finite }\}$ since the family $\{\Phi_{I_0} | I_0 \subseteq I,\ I_0 \text{ is finite }\}$ is directed under the inclusion $\Phi_{I_0} \cap \Phi_{I_1} \supseteq \Phi_{I_0 \cup I_1}$.

So $\varepsilon(\Phi)$, which is an intersection of closed sets, is closed.

The proposition is proved.

We proceed to show that the set $\varepsilon^{-1}(R_v) \subseteq W_0$ is compact for any $R_v \in W$. Let

$$\varepsilon^{-1}(R_v) \subseteq \bigcup_{i\in I} V_{A_i}^{F_0},$$

and suppose that for a finite $I_0 \subseteq I$, $\varepsilon^{-1}(R_v) \setminus \bigcup_{i\in I_0} V_{A_i}^{F_0} \neq \emptyset$. If every finite subset $\emptyset \neq I_0 \subseteq I$, assume $\mathcal{B}_{I_0} \rightleftharpoons \{B | B \subseteq \bigcup_{i\in I_0} A_i,\ B \cap A_i \neq \emptyset$ for all $i \in I_0$ and there exists $R_{v_0} \in \varepsilon^{-1}(R_v)$ such that $v_0(\beta) < 0$ for all $\beta \in B\}$; $\mathcal{B}_{I_0}$ is finite and nonempty by assumption. Let $\emptyset \neq I_0 \subseteq I_1 \subseteq I$ and I_1 be finite; then there exists a map $\pi_{I_1 I_0} : \mathcal{B}_{I_1} \to \mathcal{B}_{I_0}$, defined as $\pi_{I_1 I_0}(B) \rightleftharpoons B \cap (\bigcup_{i\in I_0} A_i)$, where $B \in \mathcal{B}_{I_1}$. The family $\{\mathcal{B}_{I_0},\ \pi_{I_1 I_0} | \emptyset \neq I_0 \subseteq I_1 \subseteq I,\ I_1$ is finite $\}$ is an inverse spectrum of finite nonempty sets, so $\mathcal{B} \rightleftharpoons \varprojlim \mathcal{B}_{I_0}$ is not empty, and every element $B \in \mathcal{B}$ can be identified with a subset of F_0 such that $B \cap A_i \neq \emptyset$ for all $i \in I$.

Choose $B \in \mathcal{B}$ and show that there exists $R_{v_0} \in \varepsilon^{-1}(R_v)$ such that $v_0(\beta) < 0$ for all $\beta \in B$. Consider the ring $R^* \rightleftharpoons R_v[\beta^{-1};\ \beta \in B]$ and its ideal $J \rightleftharpoons (m(R_v),\ \beta^{-1};\ \beta \in B)$. We show that $J \neq R^*$, i.e., J is a proper ideal. If $J = R^*$, then $1 \in J$ and there exists a representation $1 = m_0 r_0^* + \sum_{i=1}^{k} r_i^* b_i^{-1}$, where $m_0 \in m(R_v)$; $r_i^* \in R^*$, $i \le k$, $b_i \in B$, $1 \le i \le k$. Since $B_{I_0} \rightleftharpoons B \cap (\bigcup_{i\in I_0} A_i) \in \mathcal{B}_{I_0}$, for any finite $I_0 \neq \emptyset$ and $B = \bigcup_{I_0 \subseteq I} B_{I_0}$, there exists a nonempty finite subset $I_0 \subseteq I$ such that $r_0^*, \ldots, r_k^* \in R_v[\beta^{-1};\ \beta \in B_{I_0}]$ and $b_i \in B_{I_0}$ for all $i = 1, \ldots, k$. Since $B_{I_0} \in \mathcal{B}_{I_0}$, there exists $R_{v_0} \in \varepsilon^{-1}(R_v)$ such that $v_0(b) < 0$ for all $b \in B_{I_0}$; then $v_0(b^{-1}) > 0$ for $b \in B_{I_0}$, $R_v[\beta^{-1},\ \beta \in B_{I_0}] \le R_{v_0}$, so $\{m_0\} \cup \{\beta^{-1} | \beta \in B_{I_0}\} \subseteq m(R_{v_0})$, and hence $1 = m_0 r_0^* + \sum_{i=1}^{k} r_i^* b_i^{-1} \in m(R_{v_0})$, an impossibility. Thus J is a proper ideal and there exists a valuation ring R_{v_0} of F_0 such that $R_{v_0} \ge R^*$ and $m(R_{v_0}) \cap R^* \ge J \ge m(R_{v_0})$. This implies that v_0 extends v, $R_{v_0} \in \varepsilon^{-1}(R_v)$; but $R_{v_0} \notin \bigcup_{i\in I} V_{A_i}^{F_0}$ since there exists $\beta_i \in B \cap A_i$ for any $i \in I$; $v_0(\beta_i) < 0$, $\beta_i \notin R_{v_0}$, $R_{v_0} \notin V_{A_i}^{F_0}$. This is a contradiction to the fact that $\varepsilon^{-1}(R_v) \subseteq \bigcup_{i\in I} V_{A_i}^{F_0}$, which proves the compactness of $\varepsilon^{-1}(R_v)$.

We argue to establish that W_0 is compact. Let $W_0 = \bigcup_{i\in I} V_{A_i}^{F_0}$, and suppose that for any finite $I_0 \subseteq I$, $W_0 \neq \bigcup_{i\in I_0} V_{A_i}^{F_0}$, i.e., $\Phi_{I_0} \rightleftharpoons W_0 \setminus \bigcup_{i\in I_0} V_{A_i}^{F_0} \neq \emptyset$; further, $\varepsilon(\Phi_{I_0})$ is closed and nonempty; $\varepsilon(\Phi_{I_0}) \cap \varepsilon(\Phi_{I_1}) \supseteq \varepsilon(\Phi_{I_0} \cap \Phi_{I_1}) = \varepsilon(\Phi_{I_0 \cup I_1}) \neq \emptyset$; hence, there exists $R_v \in \cap\{\varepsilon(\Phi_{I_0}) | I_0 \subseteq I, I_0$ is finite $\}$. By the above $\varepsilon^{-1}(R_v)$ $(\subseteq W_0 = \bigcup_{i\in I} V_{A_i}^{F_0})$ is compact; hence, there exists a finite $I_1 \subseteq I$ such that $\varepsilon^{-1}(R_v) \subseteq \bigcup_{i\in I_1} V_{A_i}^{F_0} = W_0 \setminus \Phi_{I_1}$; but then $\varepsilon^{-1}(R_v) \cap \Phi_{I_1} = \emptyset$ and $R_v \notin \varepsilon(\Phi_{I_1}) \supseteq \cap\{\varepsilon(\Phi_{I_0}) | I_0 \subseteq I,\ I_0$ is finite $\}$. The contradiction obtained proves the compactness of W_0 and the theorem for the case where F_0 is a Galois extension of F.

Now we turn to the case where F_0 is a separable extension of F. We show that every basic open set $V_A^{F_0}$ ($A \subseteq F_0$, A is finite) is closed-open. It suffices to prove that the set $V_0 \rightleftharpoons W_0 \setminus V_{\alpha_0}^{F_0}$ is open for an arbitrary $\alpha_0 \in F_0$. Let $R_{v_0} \in V_0$; then $\alpha_0 \notin R_{v_0}$, $v_0(\alpha_0) < 0$, $v_0(\alpha_0^{-1}) > 0$. There are $k \ge 1$ and $a \in F$ such that $v_0(\alpha_0^{-k}) = v_0(a)$; $v_0(a) > 0$; $R_{v_0} \cap F \in V_a^F \setminus V_{a^{-1}}^F$. Since $V_a^F \setminus V_{a^{-1}}^F$ is closed-open in

$W,\ V_a^F \setminus V_{a^{-1}}^F = \bigcup_{i \le l} V_{B_i}^F$ for suitable finite sets $B_i \subseteq F,\ i \le l$. We have

$$R_{v_0} \in V \rightleftharpoons (\bigcap_{i \le l} V_{B_i}^{F_0}) \cap V_{(\alpha_0^k a)^{-1}}^{F_0}$$

We show that $V \subseteq V_0$. Let $R_{v_0'} \in V$, then $R_{v_0'} \in V_{(\alpha_0^k a)^{-1}}^{F_0}$ entails $0 \le v_0'(\alpha_0^{-k} a^{-1}) = k v_0'(\alpha_0^{-1}) - v_0'(a)$; $k v_0'(\alpha_0^{-1}) \ge v_0'(a)$; $R_{v_0'} \in \bigcap_{i \le l} V_{B_i}^{F_0}$ entails $R_{v'} \rightleftharpoons R_{v_0'} \cap F \in \bigcap_{i \le l} V_{B_i}^F = V_a^F \setminus V_{a^{-1}}^F$, i.e., $v_0'(a) = v'(a) > 0$; whence $k v_0'(\alpha_0^{-1}) \ge v_0'(a) > 0,\ v_0'(\alpha_0^{-1}) > 0,\ v_0'(\alpha_0) < 0,\ \alpha_0 \notin R_{v_0'},\ R_{v_0'} \in V_0$.

Thus V_0 is open, and so $V_{\alpha_0}^{F_0}$ is closed-open; hence every basic open set $V_A^{F_0}$ is closed-open.

Let F_1 be a Galois extension of F, containing F_0; then $W_1 \rightleftharpoons \{R_{v_1} | R_{v_1}$ is a valuation ring of F_1 such that $R_{v_1} \cap F \in W\}$ is a Boolean space. The map $\varepsilon' : W_1 \to W_0$, defined as $\varepsilon'(R_{v_1}) \rightleftharpoons R_{v_1} \cap F_0$, is continuous and onto. This, in view of the compactness of W_1, implies that $W_0 = \varepsilon'(W_1)$ is compact. The compactness of W_0 and the existence of a base of topology consisting of closed-open sets entail that W_0 is a Boolean space.

To end the proof of the theorem, we consider the case of a purely inseparable extension F_0 of F. Note that W_0 and W are homeomorphic in this situation: for every $R_v \in W$, there exists a unique $R_{v_0} \in W_0$ such that $R_{v_0} \cap F = R_v$, namely $R_{v_0} \rightleftharpoons \{\alpha_0 | \alpha_0 \in F_0$, and there is $k > 0$ such that $\alpha_0^{p^k} \in R_v\}$, where $p \rightleftharpoons \chi(F)$ is a characteristic of F, and $V_\alpha^{F_0} = V_{\alpha^n}^{F_0}$ for any $\alpha \in F_0,\ n > 0$.

If F_0 is an arbitrary algebraic extension of F, F_1 is a separable closure of F in F_0. Then F_0 is purely inseparable over F_1, and by the remark above W_0 is homeomorphic to $W_1 = \{R_{v_1} | R_{v_1}$ is a valuation ring of F_1 such that $R_{v_1} \cap F \in W\}$, and W_1 is Boolean by the above arguments.

The theorem is proved.

A weakly Boolean family of valuations W is called Boolean if the following hold:

1) for any $a, b \in F$, there exists $c = c(a,b) \in F$ such that $V_a^F \cap V_b^F = V_c^F$;

2) for any $a \in F$, there exists $a^* \in F$ such that $W \setminus V_a^F = V_{a^*}^F$.

COROLLARY. If W is Boolean, any closed-open subset of W is of the form V_a^F for a suitable $a \in F$.

We show that given the ring $R = R_W \rightleftharpoons \cap\{R_v | R_v \in W\}$, every Boolean family W is amenable to reconstruction.

Recall ([5], p.583) that the integral domain R with 1 is a Prüfer ring if and only if, for every maximal ideal $m < R$, the ring of fractions R_m is a valuation ring of F.

Proposition 2. If R is a Prüfer ring with a field of fractions F, F_0 is an algebraic extension of F, and R_0 is an integral closure of R in F_0, then R_0 is a Prüfer ring with a field of fractions F_0. This is excersise 16, p.584 in [5]. For completeness, we give here its proof.

Let m_0 be a maximal ideal of R_0; then $m \rightleftharpoons m_0 \cap R$ is a maximal ideal of R. Let R_m^0 be an integral closure of R_m in F_0. Since R_{0m_0} is integrally closed and $R_m \le R_{0m_0}$, then $R_m^0 \le R_{0m_0}$; $m^0 \rightleftharpoons m_0 R_{0m_0} \cap R_m^0$ is a prime ideal of R_m^0 containing mR_m; then m^0 is a maximal ideal of R_m^0 (since R_m^0 is integral over R_m) and $(R_m^0)_{m^0}$ is a valuation ring of F_0, containing R_m. But $(R_m^0)_{m^0} \le R_{0m_0}$, and so R_{0m_0} is a valuation ring of F_0 and R_0 is a Prüfer ring.

The Prüfer ring R is said to be regularly Prüfer if the factor ring of R with respect to a Jacobson radical $J(R) = \cap\{m | m$ is a maximal ideal of $R\}$ is a regular ring, i.e., for any $\bar{a} \in R/J(R)$, there exists $\bar{b} \in R/J(R)$ such that $\bar{a}^2 \bar{b} = \bar{a}$.

Proposition 3. If W is a Boolean family of valuation rings of a field F, then $R = R_W = \cap\{R_v \in W\}$ is a regularly Prüfer ring with a field of fractions F, and $\{p_v \rightleftharpoons m(R_v) \cap R | R_v \in W\}$ coincides with the set of all maximal ideals of R.

Note that $R \setminus \cup\{p_v | R_v \in W\}$ is exactly the set R^* of all invertible elements in R. Indeed, if $a \in R \setminus \cup\{p_v | R_v \in W\}$, for every $R_v \in W$, $a^{-1} \in R_v$ because $a \notin m(R_v)$ (if $a \in m(R_v)$, then $a \in m(R_v) \cap R = p_v$); consequently, $a^{-1} \in \cap\{R_v | R_v \in W\} = R$, i.e., a is invertible in R. Conversely, if a is invertible in R, then a does not lie in any proper ideal, whereas every ideal p_v is proper ($1 \notin p_v = m(R_v) \cap R$).

Now we prove that if J is a proper ideal of R, there exists $R_v \in W$ such that $J \leq p_v$. Assume the contrary; then for every $R_v \in W$, there exists $a_v \in J \setminus p_v$. We can, and do, show that $W = \bigcup_{R_v \in W} V^F_{a_v^{-1}}$. Choose an arbitrary $R_v \in W$. It follows then that $a_v \in J \setminus p_v$ implies $a_v \notin m(R_v)$, $a_v^{-1} \in R_v$, and $R_v \in V^F_{a_v^{-1}}$. In view of the compactness of W, there exist $R_{v_0}, \ldots, R_{v_k} \in W$ such that $W = \bigcup_{i \leq k} V^F_{a_i^{-1}}$. Since W is Boolean, there are $\alpha_0, \ldots, \alpha_k$ such that $W = \bigcup_{i \leq k} V^F_{\alpha_i}$; $V^F_{\alpha_i} \cap V^F_{\alpha_j} = \emptyset$ for $i < j \leq k$, and $V^F_{\alpha_i} \subseteq V^F_{a_{v_i}^{-1}}$, $i \leq k$.

To continue we need the following auxiliary lemma.

LEMMA 3. Let $a, a^* \in F$ and $W \setminus V^F_a = V^F_{a^*}$. Then $a_* \rightleftharpoons a^*(a + a^*)^{-1} \in R$, $aa_* \in R$, and $V^F_a = V^F_{a^{-1}}$.

Let $R_v \in V^F_a$; then $a \in R_v$, $v(a) \geq 0$, $a^* \notin R_v$, $v(a^*) < 0$, $v(a + a^*) = v(a^*)$, and $v(a_*) = v(a^*(a + a^*)^{-1}) = v(a^*) - v(a + a^*) = v(a^*) - v(a^*) = 0$. This implies $V^F_a \subseteq V^F_{a_*^{-1}}$; $v(aa_*) = v(a) + v(a_*) = v(a) \geq 0$.

Let $R_v \in V^F_{a^*}$; then $v(a^*) \geq 0$, $v(a) < 0$; $v(a + a^*) = v(a)$ and $v(a_*) = v(a^*(a + a^*)^{-1}) = v(a^*) - v(a) > 0$, $v(aa_*) = v(a) + v(a^*) - v(a) = v(a^*) \geq 0$.

It follows from $v(a_*) > 0$ that $a_*^{-1} \notin R_v$ and $V^F_{a^*} \cap V^F_{a_*^{-1}} = \emptyset$, i.e., $V^F_{a_*^{-1}} \subseteq W \setminus V^F_{a^*} = V^F_a$. Hence $V^F_a = V^F_{a_*^{-1}}$. Moreover, for every $R_v \in W$ we have $v(a_*) \geq 0$, $v(aa_*) \geq 0$; consequently, a_*, $aa_* \in R$.

Consider an element $a \rightleftharpoons a_{v_0}(\alpha_0)_* + \ldots + a_{v_k}(\alpha_k)_*$, where $(\alpha_i)_*$ is constructed from α_i, as in Lemma 3. Since $a_{v_i} \in J$ and $(\alpha_i)_* \in R$ for $i \leq k$, $a \in J$. We show, however, that a is invertible in R. Assume the contrary; then $a \in p_v = m(R_v) \cap R$ for some $R_v \in W$. Since $V^F_{\alpha_0}, \ldots, V^F_{\alpha_k}$ is a partition of W, there is a unique $i \leq k$ such that $R_v \in V^F_{\alpha_i}$. $V^F_{\alpha_i} \subseteq V^F_{a_{v_i}^{-1}}$ and $V^F_{\alpha_i} = V^F_{(\alpha_i)_*^{-1}}$, consequently, $v(a_{v_i}) = 0$ and $v((\alpha_i)_*) = 0$. For $j \neq i$, $j \leq k$, we have $R_v \notin V^F_{\alpha_j} = V^F_{(\alpha_j)_*^{-1}}$, $(\alpha_j)_*^{-1} \notin R_v$, $(\alpha_j)_* \in m(R_v)$, $v((\alpha_j)_*) > 0$, $v(a_{v_j}) \geq 0$, $v(a_{v_j}(\alpha_j)_*) > 0$. Then $v(a) = v(a_i(\alpha_i)_* + \sum_{i \neq j} a_j(\alpha_j)_*) = v(a_i(\alpha_i)_*) = 0$, and so $a \notin m(R_v)$, which contradicts the assumption that $a \in p_v = m(R_v) \cap R$. Thus a is invertible, but $a \in J$ for a proper ideal J. This is a contradiction, which proves that $J \leq p_v$ for a suitable $R_v \in W$.

It is straightforward from the above that every maximal ideal is of the form p_v for an appropriate $R_v \in W$.

Our immediate aim is to show that every ideal p_v is maximal. Assume p_v is not maximal and m is a maximal ideal containing p_v. By the above $m = p_{v'}$ for a suitable $R_{v'} \in W$. Thus we have $p_v < p_{v'}$, where $R_v, R_{v'} \in W$. Since W is Hausdorff, there exists $a \in F$ such that $R_{v'} \in V^F_a$ and $R_a \notin V^F_a$. By Lemma 3, there exists $a_* \in R$ such that $V^F_a = V^F_{a_*^{-1}}$, and the relation $R_v \notin V^F_a = V^F_{a_*^{-1}}$ implies $a_*^{-1} \notin R_v$, $a_* \in m(R_v) \cap R = p_v < p_{v'}$, $a_*^{-1} \notin R_{v'}$, $R_{v'} \notin V^F_{a_*^{-1}} = V^F_a$, a contradiction with the choice of a. Thus every ideal of the form p_v is maximal, and so the set of all maximal ideals of R coincides with $\{p_v | R_v \in W\}$. Hence $J(R) = \cap\{p_v | R_v \in W\}$.

We argue to establish that $R/J(R)$ is a regular ring. Let $a \in R \setminus \{0\}$, and for a^{-1}, let an element $(a^{-1})_*$ be defined as in Lemma 3. By virtue of this lemma, $(a^{-1})_*$, $a^{-1}(a^{-1})_* \in R$. We show that

$a - a^2(a^{-1}(a^{-1})_*) \in J(R)$; $a - a^2(a^{-1}(a^{-1})_*) = a(1-(a^{-1})_*) = a(1 - \frac{(a^{-1})^*}{a^{-1}+(a^{-1})^*}) = \frac{1}{a^{-1}+(a^{-1})^*}$.

Let $R_v \in W$. If $a^{-1} \in R_v$, then $(a^{-1})^* \notin R_v$ and $v(a^{-1}) \geq 0$, $v((a^{-1})^*) < 0$, $v(a^{-1} + (a^{-1})^*) = v((a^{-1})^*) < 0$, $v((a^{-1} + (a^{-1})^*)^{-1}) > 0$; if $a^{-1} \notin R_v$, then $(a^{-1})^* \in R_v$ and $v(a^{-1}) < 0$, $v((a^{-1})^*) \geq 0$, $v(a^{-1} + (a^{-1})^*) = v(a^{-1}) < 0$, $v((a^{-1}) + (a^{-1})^*)^{-1} > 0$. Thus for every $R_v \in W$ we have $v((a^{-1} + (a^{-1})^*)^{-1}) > 0$; then $(a^{-1} + (a^{-1})^*)^{-1} \in R$ and $(a^{-1} + (a^{-1})^*)^{-1} \in \cap\{p_v | R_v \in W\} = J(R)$. This implies the regularity of $R/J(R)$.

It remains to prove that R is a polyvaluation ring. We show that the equality $R_v = R_{p_v}$ holds for every $R_v \in W$. The inclusion $R_{p_v} \leq R_v$ is evident. Assume $a \in R_v$; then $v(a) \geq 0$, $v(a^*) < 0$, $v(a + a^*) = v(a^*)$, $v(\frac{a^*}{a+a^*}) = v(a^*) - v(a + a^*) = 0$. Since $a_* = \frac{a^*}{a+a^*} \in R$, $a_* \in R \setminus m(R_v) = R \setminus P_v$, but $aa_* \in R$ by Lemma 3, so $a = aa_*a_*^{-1} \in R_{p_v}$; hence $R_v \leq R_{p_v}$ and $R_v = R_{p_v}$.

This completes the proof.

The next proposition demonstrates that Boolean families W contrast with weakly Boolean ones by virtue of being closely connected to a ring $R_W = \cap\{R_v | R_v \in W\}$.

Proposition 4. If W is a weakly Boolean family of valuation rings of a field F, $R = R_W = \cap\{R_v | R_v \in W\}$ is a Prüfer ring with a field of fractions F such that the set of all maximal ideals coincides with $\{p_v\ (= m(R_v) \cap R) | R_v \in W\}$, then W is Boolean.

First we show that the family consisting of all sets of the type $H(b) \rightleftharpoons V^F_{b^{-1}}$, $b \in R \setminus \{0\}$ forms a base of the canonical topology on W. It is routine to check that $H(a) \cap H(b) = H(ab)$ for $a, b \in R \setminus \{0\}$. By letting $a \in F^*$ and $B = \{b | b \in R \setminus \{0\},\ H(b) \subseteq V^F_a\}$, we show that $V^F_a = \bigcup\limits_{b \in B} H(b)$. Assume the contrary and choose an arbitrary $R_v \in V_a \setminus \bigcup\limits_{b \in B} H(b)$. Then $H(c) \cap (W \setminus V^F_a) \neq \emptyset$ for any $c \in R \setminus p_v$. Since we have $c_0c_1 \in R \setminus p_v$, $H(c_0) \cap H(c_1) = H(c_0c_1)$ for $c_0, c_1 \in R \setminus p_v$, and $W \setminus V^F_a$ is closed, $(\bigcap\limits_{c \in R \setminus p_v} H(c)) \cap (W \setminus V^F_a) \neq \emptyset$. Let $R_{v'} \in (\bigcap\limits_{c \in R \setminus p_v} H(c)) \cap (W \setminus V^F_a)$. If $c \in R$, $R_{v'} \in H(c) = V^F_{c^{-1}}$ implies $c \in R \setminus p_v$; so $(R \setminus p_v) \subseteq (R \setminus p_{v'})$ and $p_{v'} \subseteq p_v$; but $p_{v'}$ is maximal; hence $p_v = p_{v'}$. On the other hand, $R_v \in V^F_a$, $R_{v'} \in W \setminus V^F_a$, and $R_v \neq R_{v'}$. Since R is a polyvaluation ring, R_{p_v} is a valuation ring of F; moreover, $R_{p_v} \leq R_v$, $R_{p_v} \leq R_{v'}$. All super-rings of a valuation ring are linearly ordered (see [3]), so either $R_v \leq R_{v'}$ or $R_{v'} \leq R_v$. Since $R_v \neq R_{v'}$, this is in conflict with the fact that W is Hausdorff. Thus $V^F_a = \bigcup\limits_{b \in B} H(b)$, and so $\{H(b) | b \in R \setminus \{0\}\}$ is a base of the canonical topology on W.

Next we prove that for every $a \in R \setminus \{a\}$, there exists $a' \in R \setminus \{0\}$ such that $H(a') = W \setminus H(a)$. Let $B \rightleftharpoons \{b | b \in R \setminus \{0\},\ H(a) \cap H(b) = \emptyset\}$, then $W = H(a) \cup (\bigcup\limits_{b \in B} H(b))$. Indeed, $W \setminus H(a)$ is open and $W \setminus H(a) = \cup\{H(b) | b \in R \setminus \{0\},\ H(b) \subseteq W \setminus H(a)\}$, by the above. Note that for any $A \subseteq R \setminus \{0\}$, $W = \bigcup\limits_{a \in A} H(a)$ implies that the ideal (A) of R generated in R by the set A is not proper.

In fact, if $(A) < R$, then there exists a maximal ideal m such that $(A) \leq m < R$, and m is of the form p_v and $R_v \notin \bigcup\limits_{a \in A} H(a)$. (The converse also holds: if $(A) = R$, then $W = \bigcup\limits_{a \in A} H(a)$.) Since $W = H(a) \cup (\bigcup\limits_{b \in B} H(b))$, $(\{a\} \cup B) = R$, and unity has representation $1 = r_0a + \sum\limits_{i=1}^{n} r_ib_i$, where $r_i \in R$, $i \leq n$; $b_i \in B$, $i = 1, \ldots, n$. Set $a' \rightleftharpoons \sum\limits_{i=1}^{n} r_ib_i$ and show that $H(a') = W \setminus H(a)$. Let $R_v \in H(a')$; then $a' \in p_v$. It follows from $H(a) \cap H(b_i) = \emptyset$ that $ab_i \in \cap\{p_{v'} | R_{v'} \in W\}$, $i = 1, \ldots, n$, so $aa' =$

$\sum_{i=1}^{n} r_i(ab_i) \in \cap\{p_{v'}|R_{v'} \in W\}$, $aa' \in p_v$, $a \in p_v$, and $R_v \notin H(a)$, i.e., $H(a') \subseteq W \setminus H(a)$. Let $R_v \notin H(a')$; then $a' \in p_v$, $1-a' \notin p_v$; but $1-a' = r_0 a$, consequently, $r_0 a \notin p_v$, $a \notin p_v$, $R_v \in H(a)$, i.e., $W \setminus H(a') \subseteq H(a)$, $W \setminus H(a) \subseteq H(a')$. Thus $H(a') = W \setminus H(a)$.

The arguments above entail that for every $a \in F^*$, there exists $b \in R \setminus \{0\}$ such that $V_a^F = H(b)$. Indeed, $V_a^F = \cup\{H(b)|b \in R \setminus \{0\},\ H(b) \subseteq V_a^F\}$. Since V_a^F is compact, there exist $b_0, \ldots, b_n \in R \setminus \{0\}$ such that $V_a^F = \bigcup_{i\leq n} H(b_i)$. If $b \rightleftharpoons (\prod_{i\leq n} b_i')'$, where the primed symbols signify complementation, it is immediate that $V_a^F = H(b)$. Since the family consisting of all sets of the form $H(b)$, where $b \in R \setminus \{0\}$, is closed under intersection and complementation, and forms a base of the canonical topology on the compact (Boolean) space W, every closed-open set in W is of the form $H(a) = V_{a^{-1}}^F$, $a \in r \setminus \{0\}$, and W is a Boolean family of valuation rings of F. The proposition is proved.

Below we show that Proposition 3 admits inversion.

Proposition 5. If R is a regularly Prüfer ring with a field of fractions F, then $W_R \rightleftharpoons \{R_m|m$ is a maximal ideal of $R\}$ is a Boolean family of valuation rings of F.

The notation adopted here is the same as in Proposition 4. We show that the family $\{H(a)|a \in R\setminus\{0\}\}$ of subsets of W_R forms a base of the canonical topology on W_R. Let $a \in F$ and $R_m \in V_a^F$; then $a \in R_m$ and there exist $b, c \in R$, $c \in R \setminus m$ such that $a = bc^{-1}$; hence $R_m \in H(c) = V_{c^{-1}}^F$, and obviously, $H(c) \subseteq V_a^F$.

Verify that the canonical topology is Hausdorff: if $m_0 \neq m_1$ are maximal ideals of R, then for $a \in m_1 \setminus m_0$, we have $R_{m_0} \in H(a)$, $R_{m_1} \notin H(a)$.

Check that W_R is compact. Suppose $W_R = \bigcup_{a\in A} H(a)$, and let (A) be the ideal of R generated by A. If $(A) \neq R$ and m is a maximal ideal of R such that $(A) \leq m < R$, then $R_m \notin \bigcup_{a\in A} H(a)$; so $(A) = R$ and there exists a representation of the form $1 = \sum_{i\leq n} r_i a_i$, where $r_i \in R$, $a_i \in A$; hence $W_R = \bigcup_{i\leq n} H(a_i)$. In fact, if $R_m \in W_R \setminus \bigcup_{i\leq n} H(a_i)$, then $(a_0, \ldots, a_n) \leq m$ and $1 \in m$, a contradiction.

It remains to prove that for every $a \in R \setminus \{0\}$, there exists $a' \in R \setminus \{0\}$ such that $H(a') = W_R \setminus H(a)$. Since $R/J(R)$ is regular, there exists $b \in R$ such that $a - a^2 b \in J(R)$. Set $a' \rightleftharpoons 1 - ab$.

If $R_m \in H(a')$, then $a' \notin m$; $aa' \in J(R) \leq m$; so $a \in m$, and $R_m \notin H(a)$.

If $R_m \notin H(a')$, then $a' \in m$, $1 - a' = ab \notin m$, $a \notin m$ and $R_m \in H(a)$. Thus $H(a') = W_R \setminus H(a)$. As in the proof of Proposition 4, we infer from this that W_R is a Boolean family of valuation rings of F.

Our goal now is to establish the main theorem on a "lift" of Boolean families to algebraic extensions.

THEOREM 2. Let W be a Boolean family of valuation rings of a field F, $F_0 \geq F$ an algebraic extension of F, $W_0 \rightleftharpoons \{R_{v_0}|R_{v_0}$ is a valuation ring of F_0 such that $R_{v_0} \cap F \in W\}$. Then W_0 is a Boolean family of valuation rings of F_0.

By Theorem 1 W_0 is weakly Boolean. We show that $R_{W_0} \rightleftharpoons \cap\{R_{v_0}|R_{v_0} \in W_0\}$ is an integral closure R_W^0 of the ring $R_W \rightleftharpoons \cap\{R_v|R_v \in W\}$ in F_0. Since $R_W \leq R_{W_0}$ and R_{W_0} is integrally closed, $R_W^0 \leq R_{W_0}$. If $\alpha \in R_{W_0} \setminus R_W^0$ and $f = x^n + a_1 x^{n-1} + \ldots + a_n$ is a minimal polynomial of α over F, then $\{a_1, \ldots, a_n\} \not\subseteq R_W$, and hence there exists $R_v \in W$ such that $\{a_1, \ldots, a_n\} \not\subseteq R_v$. For α not integral over R_v, there exists $R_{v_0} \in W_0$ such that $R_{v_0} \cap F = R_v$ and $\alpha \notin R_{v_0}$, but then $\alpha \notin R_{W_0}$, a contradiction.

Since R_W is the Prüfer ring with a field of fractions F by Proposition 3, $R_{W_0} = R_W^0$ is a polyvaluation ring of F_0 by Proposition 2. We check whether the assumption of Proposition 4 is valid for the family W_0. It suffices to establish that $\{m(R_{v_0}) \cap R_{W_0}|R_{v_0} \in W_0\}$ coincides with the set of all maximal ideals of R_{W_0}. Let m_0 be a maximal ideal of $R_{W_0} = R^0_W$; then $m \rightleftharpoons m_0 \cap R_W$ is a maximal ideal of R_W and $(R_W)_m \in W$; hence $(R_{W_0})_{m_0}$ is a polyvaluation ring dominating $(R_W)_m$, consequently, $(R_{W_0})_{m_0} \in W_0$ and $m_0 = m((R_{W_0})_{m_0}) \cap R_{W_0}$. Conversely, if $R_{v0} \in W_0$, then $m(R_{v0}) \cap R_W$ is a maximal ideal, and so $m(R_{v0}) \cap R_{W_0} = m(R_{v0}) \cap R^0_W$ is maximal as a prime ideal lying over the maximal ideal in the integral extension. By Proposition 4, W_0 is a Boolean family of valuation rings of F_0. This completes the proof.

Now we give a few instances of Boolean families of valuation rings. Unfortunately, our attempts to find an example of a weakly Boolean family that is not Boolean have as yet been unsuccessful. In going through the details of the next proposition, the reader will get some idea about the difficulties impeding the construction of such an example.

Proposition 6. Let W be weakly Boolean, and suppose that for every $a \in F^*$, there exists a unitary polynomial with integer coefficients $f_a(x) \in Z[x] \setminus Z$ such that $f_a(o) = \pm 1$ and $f_a(a) \notin m(R_v)$ for any $R_v \in V^F_a$. If, for every $R_v \in W$, the field $F_v \rightleftharpoons R_v/m(R_v)$ is an algebraic extension of a simple field of characteristic $p_v \neq 0$, then W is Boolean.

In view of Proposition 4, it suffices to show that the set $\{p_v\ (= m(R_v) \cap R) | R_v \in W\}$ coincides with the set of all maximal ideals of the ring $R\ (= \cap\{R_v | R_v \in W\})$, and that R is a polyvaluation ring.

Set $a_* \rightleftharpoons f_a(a)^{-1}$ for any $a \in F^*$; then $a_*,\ aa_* \in R$, and $V^F_a = V^F_{a_*^{-1}}$. Indeed, let $R_v \in W$ and $v(a) \geq 0$, i.e., $R_v \in V^F_a$. Then $a_*^{-1} = f_a(a) \in R_v \setminus m(R_v)$ and $v(a_*^{-1}) = 0$, by the above condition. If $a \notin R_v$, $v(a) < 0$, then $v(a_*^{-1}) = v(f_a(a)) = deg f_a \cdot v(a) < 0$; $v(a_*) = -deg f_a \cdot v(a) > 0$; $v(aa_*) = v(a) - deg f_a \cdot v(a) = -(deg f_a - 1)v(a) \geq 0$ since $deg f_a \geq 1$. Thus $v(a_*) \geq 0$ and $v(aa_*) \geq 0$ for all $R_v \in W$; hence, $a_*, aa_* \in R$. Moreover, $v(a) \geq 0$ implies $v(a_*^{-1}) = 0$, and $v(a) < 0$ implies $v(a_*^{-1}) < 0$; hence, $V^F_a = V^F_{a_*^{-1}}$.

Since, for any $a, b \in F^*$, we have $V^F_a = V^F_{a_*^{-1}}$, $V^F_b = V^F_{b_*^{-1}}$ and $V^F_a \cap V^F_b = V^F_{a_*^{-1}} \cap V^F_{b_*^{-1}} = V^F_{(a_*b_*)^{-1}}$, every basic open set is of the form $V^F_{a^{-1}}$ for a suitable $a \in R \setminus \{0\}$.

First we show that $R_v = R_{p_v}$ for every $R_v \in W$. The inclusion $R_{p_v} \leq R_v$ is evident. Let $a \in R_v \setminus \{0\}$; then $v(a) \geq 0$, $V(a_*^{-1}) = 0$, $v(a_*) = 0$; $a_*, aa_* \in R$, $a_* \notin p_v$, and $a = (aa_*)a_*^{-1} \in R_{p_v}$; thus $R_v = R_{p_v}$.

Next we prove that every maximal ideal m of R is of the form p_v for an appropriate $R_v \in W$. For every $a \in R \setminus m$, $V^F_{a^{-1}} \neq \emptyset$ holds. Indeed, if $V^F_{a^{-1}} = \emptyset$, then $a \in p_v$ for all $R_v \in W$, and so $a \in \cap\{p_v | R_v \in W\}$. At the same time, the maximality of m implies the existence of $b \in R$ such that $1 - ab \in m$; but $1 - ab \in R_v \setminus m(R_v)$, $(1 - ab)^{-1} \in R_v$ for all $R_v \in W$ (since $a, b \in R \leq R_v$, $a, ab \in p_v \leq m(R_v)$), then $(1 - ab)^{-1} \in R = \cap\{R_v | R_v \in W\}$ and $1 = (1 - ab) \cdot (1 - ab)^{-1} \in m$. Contradiction. Thus $V^F_{a^{-1}} \neq \emptyset$ for every $a \in R \setminus m$, and so $\bigcap_{a \in R\setminus m} V^F_{a^{-1}} \neq \emptyset$. Let $R_v \in \bigcap_{a \in R\setminus m} V^F_{a^{-1}}$, then $R \setminus m \subseteq R \setminus p_v$ and $p_v \leq m$.

Now we show that the ring of fractions R_m is a valuation ring. Let $R_{v0} \geq R$ be an arbitrary valuation ring of F such that $m(R_{v0}) \cap R = m$. Then $R_m \leq R_{v0}$. Let $a \in R_{v0}$, $v_0(a) \geq 0$, and $v_0(f_a(a)) \geq 0$, $v_0(a_*) = v_0(f_a(a)^{-1}) \leq 0$, but $a_* \in R \leq R_{v0}$ entails $v_0(a_*) \geq 0$; thus $v_0(a_*) = 0$ and $a = (aa_*)a_*^{-1} \in R_m$ since $aa_* \in R$, $a_* \in R \setminus m$, and so $R_m = R_{v0}$ is a valuation ring. The inclusion $p_v \leq m$ implies an inclusion $R_{v0} = R_m \leq R_{p_v} = R_v$. If $R_{v0} < R_v$, the valuation v_0 is representable as the composition $v \circ \bar{v}$, where $\bar{v}$ is a nontrivial valuation of F_v. By assumption, however, F_v is an absolutely algebraic field of nonzero characteristic, and such fields have no proper valuations. This implies that $p_v = m$, i.e., $m \in \{p_v | R_v \in W\}$.

The above arguments also show that p_v is maximal for every $R_v \in W$. Indeed, if m is a maximal ideal of R such that $p_v \leq m$, then $p_v = m$, as has been proved above. In view of Proposition 4, W is Boolean.

COROLLARY. If W is weakly Boolean and there exists $k > 0$ such that for any $R_v \in W$ the field

$F_v = R_v/m(R_v)$ is finite and $|F_v| \leq k$, then W is Boolean.

Let p be a prime number greater than k and $f_a(x) \rightleftharpoons x^{p-1} + x^{p-2} + \ldots + 1$ for every $a \in F^*$; then all assumptions of Proposition 6 are satisfied, and so W is Boolean.

Proposition 7. Let W be a finite family of valuations that are mutually incomparable with respect to inclusion. Then W is Boolean.

This, in essence, was established in [2, Sec. 3, Proposition 1].

The latter example will be detailed in subsequent papers. Let $\pi \in F^*$ be an arbitrary element distinct from 1. We call the valuation ring R_v of F a π-valuation ring, and the corresponding valuation v a π-valuation if $v(\pi)$ is the least positive element in the valuation group Γ_v. A field F is said to be formally π-adic if there exists at least one π-valuation of F.

Let F be formally π-adic and $W_\pi \rightleftharpoons \{R_v | R_v$ is a π-valuation ring of $F\}$.

Proposition 8. The family W_π of all π-valuation rings of F is Boolean.

Let $R_\pi \rightleftharpoons \cap\{R_v | R_v \in W_\pi\}$.

Note the following important property.

0. For every $a \in F^*$, $1 + \pi a^2 \neq 0$ and $\gamma(a) \rightleftharpoons \frac{a}{1+\pi a^2} \in R_\pi$.

Let v be an arbitrary π-valuation of F. If $1 + \pi a^2 = 0$, then $v(\pi a^2) = 0$, $v(\pi) + 2v(a) = 0$, $2v(a^{-1}) = v(\pi)$, and $0 < v(a^{-1}) < v(\pi)$, which is impossible if v is a π-valuation.

Further, if $v(a) \geq 0$, then $v(\pi a^2) > 0$, $v(1 + \pi a^2) = 0$, $v(\gamma(a)) = v(a) - v(1 + \pi a^2) = v(a) \geq 0$; if $v(a) < 0$, then $v(\pi a^2) = v(\pi) + 2v(a) = (v(\pi) + v(a)) + v(a) \leq v(a) < 0$; $v(1 + \pi a)^2 = v(\pi a^2) \leq v(a)$, $v(\gamma(a)) = v(a) - v(1 + \pi a^2) = v(a) - v(\pi a^2) \geq v(a) - v(a) = 0$. Thus for every $R_v \in W_\pi$ we have $v(\gamma(a)) \geq 0$, $\gamma(a) \in R_v$, and $\gamma(a) \in R_\pi$.

We establish a number of properties of basic sets of the canonical topology.

1. For $a \in F^*$, let $a^* \rightleftharpoons (\pi a)^{-1}$. Then $V^F_{a^*} = W_\pi \setminus V^F_a$.

Indeed, $R_v \in V^F_{a^*}$ implies $a^* \in R_v$, $v(a^*) = -v(\pi a) = -v(a) - v(\pi) \geq 0$; $v(a) \leq -v(\pi) < 0$, $a \notin R_v$, $R_v \notin V^F_a$; $R_v \notin V^F_{a^*}$ implies $a^* \notin R_v$, $v(a^*) < 0$; $v((a^*)^{-1}) = v(\pi a) > 0$; $v(a) = v(\pi a) - v(\pi) \geq 0$ (since $v(\pi a) > 0$ and $v(\pi)$ is the least positive element of Γ_v), so $R_v \in V^F_a$.

2. For any $a \in F^*$ we have $a_* \rightleftharpoons a^*(a + a^*)^{-1} \in R_\pi$ and $V^F_a = V^F_{a_*^{-1}}$.

Let $R_v \in V^F_a$; then $v(a) \geq 0$, $v(a^*) < 0$, $v(a + a^*) = v(a^*)$; $v(a_*) = v(a^*) - v(a + a^*) = v(a^*) - v(a^*) = 0$; $v(a_*^{-1}) = 0$, and so $R_v \in V^F_{a_*^{-1}}$.

Conversely, let $R_v \notin V^F_a$; then $R_v \in V^F_{a^*}$, $v(a^*) \geq 0$, $v(a) < 0$, $v(a + a^*) = v(a)$, $v(a_*) = v(a^*) - v(a + a^*) = v(a^*) - v(a) > 0$; consequently, $R_v \notin V^F_{a_*^{-1}}$. This implies that $V^F_a = V^F_{a_*^{-1}}$; moreover, $a_* \in R_\pi$ since $v(a_*) \geq 0$ for all $R_v \in W_\pi$.

3. For any $a, b \in F^*$, let $\delta(a, b) \rightleftharpoons (a_* b_*)^{-1}$. Then $V^F_a \cap V^F_b = V^F_{\delta(a,b)}$.

In fact, $V^F_a = V^F_{a_*^{-1}} = H(a_*)$, $V^F_b = V^F_{b_*^{-1}} = H(b_*)$, $V^F_a \cap V^F_b = H(a_*) \cap H(b_*) = H(a_* b_*) = V^F_{(a_* b_*)^{-1}} = V^F_{\delta(a,b)}$ (see the notation $H(a)$ and the properties of H in the proof of Proposition 4).

Thus, the family V^F_a, $a \in F^*$, is closed under finite intersections and complementations, hence under finite unions.

We prove that W_π endowed with canonical topology is Hausdorff. Letting $R_v \neq R_{v'} \in W_\pi$, we see that the inclusion $R_v < R_{v'}$ entails the existence of $a \in R_{v'} \setminus R_v$; $v(a) < 0$, $v'(a) \geq 0$; $v(\pi a) \leq 0$, $v'(\pi a) > 0$, $a^* = (\pi a)^{-1} \in R_v < R_{v'}$; $(\pi a)^{-1} \in R_{v'}$, $v'(a^*) \geq 0$, but $v'(a^*) = -v'(\pi a) < 0$, a contradiction. So $R_v \not\leq R_{v'}$, and if $a \in R_v \setminus R_{v'}$, $R_v \in V^F_a$ and $R_{v'} \notin V^F_a$.

It remains to establish the compactness of W_π.

To do this we prove:

LEMMA 4. If m is a maximal ideal of R_π, then $\pi \in m$. If $R_v \geq R_\pi$ is a valuation ring of F such that $m(R_v) \cap R_\pi = m$, then $R_v \in W_\pi$.

Note that $(1+\pi a)^{-1} \in R_\pi$ for every $a \in R_\pi$. This follows from the fact that if, for any π-valuation v, $a \in R_v$, then $v(a) \geq 0$, $v(\pi a) > 0$, $v(1+\pi a) = 0$, and $v((1+\pi a)^{-1}) = 0$, so $(1+\pi a)^{-1} \in R_v$.

If $\pi \notin m$, then $1 - \pi a \in m$ would be valid for some $a \in R_\pi$; but $(1-\pi a)^{-1} \in R_\pi$, hence $1 = (1-\pi a)(1-\pi a)^{-1} \in m$, a contradiction.

Let $R_v \geq R_\pi$ and $m(R_v) \cap R = m$. Since $\pi \in m$, $v(\pi) > 0$. If v is not a π-valuation, then there exists $a \in R_v$ such that $0 < v(a) < v(\pi)$. Consider an element $\gamma(a^{-1})$. By property 0, $\gamma(a^{-1}) \in R_\pi \leq R_v$, so $v(\gamma(a^{-1})) \geq 0$. On the other hand, $v(\pi a^{-2}) = v(\pi) - 2v(a) = (v(\pi) - v(a)) - v(a) > -v(a)$. If $v(1+\pi a^{-2}) < 0$, then $v(1+\pi a^{-2}) = v(\pi a^{-2}) > -v(a)$ and $v(\gamma(a^{-1})) = v(a^{-1}) - v(\pi a^{-2}) = -v(a) - v(\pi a^{-2}) < 0$. But if $v(1+\pi a^{-2}) \geq 0$, then $v(\gamma(a^{-1})) = -v(a) - v(1+\pi a^{-2}) \leq -v(a) < 0$. Thus $v(\gamma(a^{-1})) < 0$, $\gamma(a^{-1}) \notin R_v \geq R_\pi$, an impossibility. The lemma is proved.

Let $W_\pi = \bigcup_{a \in A} V_a^F = \bigcup_{a \in A} V_{a_*^{-1}}^F = \bigcup_{a \in A} H(a_*)$. Consider the ideal (A_*) generated by the set $A_* \rightleftharpoons \{a_* | a \in A\}$. If (A_*) is a proper ideal, we let m be a maximal ideal of R_π containing (A_*), and let $R_v \geq R_\pi$ be a valuation ring of F such that $m(R_v) \cap R_\pi = m$. Then, in view of the lemma, $R_v \in W_\pi$, $A_* \subseteq m \subseteq m(R_v)$, and $R_v \in W_\pi \setminus \bigcup_{a_* \in A_*} H(a_*)$, a contradiction.

Thus $1 \in (A_*)$ and there exist $a_0, \ldots, a_n \in A$ and $r_0, \ldots, r_n \in R_\pi$ such that $1 = \sum_{i \leq n} r_i (a_i)_*$. We show that $W_\pi = \bigcup_{i \leq n} H((a_i)_*) = \bigcup_{i \leq n} V_{a_i}^F$. If $R_\pi \leq R_v \notin \bigcup_{i \leq n} H((a_i)_*)$, $(a_i)_* \in m(R_v)$, $i \leq n$, but then $1 = \sum_{i \leq 1} r_i (a_i)_* \in m(R_v)$, a contradiction. We have thus proved the compactness of W_π.

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