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THE DYNAMIC INTERPOLATION PROBLEM: ON RIEMANNIAN MANIFOLDS, LIE GROUPS, AND SYMMETRIC SPACES

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ABSTRACT. We consider the dynamic interpolation problem for nonlinear control systems modeled by second-order differential equations whose configuration space is a Riemannian manifold M. In this problem we are given an ordered set of points in M and would like to generate a trajectory of the system through the application of suitable control functions, so that the resulting trajectory in configuration space interpolates the given set of points. We also impose smoothness constraints on the trajectory and typically ask that the trajectory be also optimal with respect to some physically interesting cost function. Here we are interested in the situation where the trajectory is twice continuously differentiable and the Lagrangian in the optimization problem is given by the norm squared acceleration along the trajectory. The special cases where M is a connected and compact Lie group or a homogeneous symmetric space are studied in more detail.

1. INTRODUCTION

The present work is motivated by the motion planning problem and the tracking problem for nonlinear systems. First of all, the trajectories are specified in terms of an ordered set of points through which selected dynamic variables must pass, together with smooth constraints and some performance measure. Then the problem is reduced to determining suitable controls which give rise to such trajectories. We call this procedure the "dynamic interpolation problem," a term that first appeared in a series of papers related to flight paths of aircraft by Crouch and Jackson [11], [12], [13]

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and Jackson [21]. In our previous paper [14], we started looking at the dynamic interpolation problem in a more abstract setting, in an attempt to understand the geometry of the problem. Here we continue to look at the mathematical and control theoretic aspects, using ideas from differential geometry and Lie theory. The organization of the paper is as follows. In Sec. 1 we derive the second variation formulas for a variational problem whose extremals turn out to be generalizations of cubic splines to Riemannian manifolds and which appeared before in Noakes, Heinzinger, and Paden [26] and Crouch and Silva Leite [14]. We also exhibit analogues of the Jacobi vector fields from the theory of geodesics. Section 2 contains the solution of a variational problem which differs from the one considered in the previous section, where it contains additional constraints. Using a result due to Nomizu [27], in Sec. 3, we simplify the formulas obtained in Secs. 1 and 2 for the special situation in which the Riemannian manifold is a connected and compact Lie group G. The analogues of cubic splines and of Jacobi fields are then given by equations involving the right-invariant vector fields on G. This procedure is partially repeated in Sec. 5 for Riemannian homogeneous symmetric spaces. For the sphere S^2, we present the equation of cubic splines. Finally, the last section is devoted to the $\mathcal{C}^2$-dynamic interpolation problem for systems evolving on connected and compact Lie groups. We show that it is possible to solve a number of optimal control problems in the context of the variational problems studied in the first part of the paper. These optimal control problems can be regarded as particular cases of minimum energy problems for systems evolving on general manifolds, by lifting the original system evolving on a manifold M to a system evolving on its tangent bundle TM. There are many papers dealing with minimum energy problems for systems without a drift term ([2], [5], [18], [21], [31], [33]). The situation in which the system has a drift term is not so well studied ([22] is one case). Our methods also apply to systems which have a drift term and in which the number of controls may be less than the dimension of M. When we consider homogeneous symmetric spaces rather than Lie groups, our methods still apply to systems with full control. We do not know yet how to handle the situation for restricted control systems evolving on homogeneous symmetric spaces. For systems evolving on the three-dimensional rotation group, we present nonlinear differential equations which the optimal controls must satisfy. This low-dimensional case has strong connections with applications to robotics and the path planning of aircraft [9], [12], [15], [16], [17], [21].

2. Second variation formulas for the cubic spline interpolation on Riemannian manifolds

Suppose that M is a Riemannian manifold, with Riemannian metric $\langle \cdot, \cdot \rangle$. Denote the symmetric connection on M, which is compatible with this metric, by ∇, and the covariant derivative along a curve $t \mapsto x(t)$ in M by DW_t/dt, where $W_t \in T_{x(t)}M$ is a vector field along x. Thus, by definition, $DW_t/dt = (\nabla_V W)(x(t))$, where W is a vector field defined in the neighborhood of the curve x satisfying $W(x(t)) = W_t$, and $V(x(t)) = dx(t)/dt \in T_{x(t)}M$. To simplify the notation, we sometimes write $\frac{Dx}{dt}$ for $\frac{dx}{dt}$ and $\frac{D^j x(t)}{dt^j}$ for $\frac{D}{dt}\left(\frac{D^{j-1}x(t)}{dt^{j-1}}\right)$, $j \geq 2$. We say that a function f defined on a closed interval $[a, b] \in \mathbb{R}$ is smooth on this interval if f is smooth on (a, b) and, in addition, has bounded limits $\lim_{t\to a^+} f^{(k)}(t)$ and $\lim_{t\to b^-} f^{(k)}(t)$, where $f^{(k)}(t)$ is the kth jet of f.

In [14] we considered the following **problem** $(\mathcal{P}_1)$:

Problem $(\mathcal{P}_1)$. "Find critical values of

$$J_1(x) = \frac{1}{2}\int_0^T \left\langle \frac{D^2x(t)}{dt^2}, \frac{D^2x(t)}{dt^2} \right\rangle dt, \tag{1}$$

over the class Ω of C^1-paths x on M, satisfying $x|_{[T_{i-1},T_i]}$ is smooth,

$$x(T_i) = x_i, \qquad 1 \leq i \leq N-1, \tag{2}$$

for a distinct set of points $x_i \in M$ and fixed times T_i, where $0 = T_0 < T_1 < \cdots < T_{N-1} < T_N = T$, and, in addition,

$$x(0) = x_0, \quad x(T) = x_N, \quad \frac{dx}{dt}(0) = v_0, \quad \frac{dx}{dt}(T) = v_N, \tag{3}$$

where $v_0 \in T_{x_0}M$ and $v_N \in T_{x_N}M$ are fixed tangent vectors."

A necessary condition for a curve x to be a critical point for problem $(\mathcal{P}_1)$ was given in [14, Theorem 1]. The same condition for problem $(\mathcal{P}_1)$ without the interpolating conditions (2) was first derived independently in Noakes et al. [26]. We include both results here for the sake of completeness.

Theorem 2.1 (Crouch and Silva Leite [14]). *A necessary condition for x to be an extremal for problem $(\mathcal{P}_1)$ is that x be of class C^2 and*

$$\frac{D^3V_t}{dt^3} + R\left(\frac{DV_t}{dt}, V_t\right)V_t \equiv 0, \qquad t \in [T_{i-1}, T_i], \qquad 1 \leq i \leq N, \tag{4}$$

where $V_t = dx(t)/dt$ and R is the curvature tensor of the connection ∇ on M.

The following **problem** $(\mathcal{P}_2)$ is a simplified version of problem $(\mathcal{P}_1)$.

Problem $(\mathcal{P}_2)$. "Find critical values of the cost functional $\mathcal{J}_1$, defined in (1), over the class $\overline{\Omega}$ of piecewise smooth and C^1-curves $\bar{x}$ on M satisfying $\bar{x}|_{[t_{i-1},t_i]}$ is smooth for some fixed times t_i satisfying $0 = t_0 < t_1 < \cdots < t_l = T$ and

$$\bar{x}(0) = x_0, \quad \frac{d\bar{x}(0)}{dt} = v_0, \quad \bar{x}(T) = x_N, \quad \frac{d\bar{x}(T)}{dt} = v_N." \tag{5}$$

Theorem 2.2 (Noakes et al. [26]). *A necessary condition for $\bar{x}$ to be an extremal solution for problem $(\mathcal{P}_2)$ is that $\bar{x}$ be smooth and*

$$\frac{D^3V_t}{dt^3} + R\left(\frac{DV_t}{dt}, V_t\right)V_t \equiv 0, \quad t \in [0,T], \tag{6}$$

where $V_t = \dfrac{d\bar{x}(t)}{dt}$.

Remark 2.3. The functional $\mathcal{J}_1$ can be viewed as a special case of the functional

$$\frac{1}{2}\int_0^T \left(\left\langle \frac{Dv}{dt}, \frac{Dv}{dt}\right\rangle + \left\langle \frac{dx}{dt}, \frac{dx}{dt}\right\rangle\right) dt, \qquad v = \frac{dx}{dt}.$$

The case where the constraint $v = \dfrac{dx}{dt}$ is ignored and the problem is treated as a variational problem on TM may be viewed as a traditional geodesic problem on TM endowed with the Sasaki metric [30]. However, the geodesic extremals of this new problem have a completely different structure from the extremals associated with the variational problem treated in this paper. See Camarinha et al. [8] for details of the comparison.

If $M = \mathbb{R}^n$ with the Euclidean inner product, the covariant derivative is just the usual derivative and the curvature tensor is zero. In this case

$$\mathcal{J}_1(x) = \frac{1}{2}\int_0^T ||\ddot{x}(t)||^2\, dt,$$

and the extremals for problems $(\mathcal{P}_1)$ or $(\mathcal{P}_2)$ satisfy $\overset{\dots}{x}(t) = 0$, giving the usual cubic splines in $\mathbb{R}^n$. Solutions of Eqs. (2), (3), and (4) may be viewed as "cubic splines" on a Riemannian manifold.

In the calculus of variations applied to geodesics, the theory of conjugate points is easily derived by evaluating the second variation of the energy functional at an extremal (see Milnor [24]). The existence of conjugate points along a geodesic is shown to be equivalent to the existence of nontrivial Jacobi fields along that geodesic. We would like to extend his theory to the present situation. As a first step we derive below the second variation formulas for our problem and obtain the analogue of the Jacobi fields in the theory of geodesics. The theory of conjugate points will appear in a forthcoming paper.

We define the tangent space $T_x\Omega$ to a C^1-path x satisfying (2) and (3) to be the vector space of C^1-vector fields $t \to W_t$ along x satisfying W_t is smooth on the domains $[T_{i-1}, T_i]$, $1 \le i \le N$, $W_{T_i} = 0$, $0 \le i \le N$, $\frac{DW_0}{dt} = 0$, $\frac{DW_T}{dt} = 0$. Note that, since x is a C^1-curve, it follows that the curve $t \to \frac{DW_t}{dt}$ in TM is continuous. Moreover, $W \in T_x\Omega$ is C^k if and only if $t \to \frac{D^kW_t}{dt^k}$ is continuous, for any k.

For the class of C^1-curves on M satisfying conditions (2) and (3), we introduce two parameter variations $\alpha : [0,T] \times (-\varepsilon, \varepsilon) \times (-\delta, \delta) \to M$ of $x(t) = \alpha(t, 0, 0)$ which are characterized infinitesimally by the vector space $T_x\Omega \times T_x\Omega$, by setting

$$\alpha(t, u_1, u_2) = \exp_{x(t)}(u_1 W_t^1 + u_2 W_t^2),$$

where $W^k = \partial\alpha/\partial u_k\big|_{u_1=u_2=0} \in T_x\Omega$, $k = 1, 2$. These variations satisfy the following properties:

$$\begin{aligned}
&\alpha \text{ is smooth on each domain } [T_{i-1}, T_i] \times (-\varepsilon, \varepsilon) \times (-\delta, \delta),\ 1 \le i \le N,\\
&\alpha(t, 0, 0) = x(t), \qquad 0 \le t \le T,\\
&\frac{\partial\alpha}{\partial u_k}(t, 0, 0) = W_t^k, \qquad 0 \le t \le T,\ k = 1, 2,\\
&\frac{\partial\alpha}{\partial u_k}(T_i, 0, 0) = 0, \qquad i = 0, 1, \dots, N,\ k = 1, 2,\\
&\frac{D}{dt}\frac{\partial\alpha}{\partial u_k}(t, 0, 0) = \frac{D}{dt}W_t^k \quad \text{is continuous on } [0,T], \quad k = 1, 2,\\
&\frac{D}{dt}\frac{\partial\alpha}{\partial u_k}(0, 0, 0) = \frac{D}{dt}\frac{\partial\alpha}{\partial u_k}(T, 0, 0) = 0, \qquad k = 1, 2.
\end{aligned} \tag{7}$$

To obtain a second variation formula corresponding to our problem $(\mathcal{P}_1)$, we have to calculate

$$\frac{\partial^2}{\partial u_2 \partial u_1} \mathcal{J}_1(\alpha_{u_1,u_2})\bigg|_{u_1=u_2=0},$$

where $\alpha_{u_1,u_2}(t) = \alpha(t, u_1, u_2)$, and $\alpha(t, 0, 0) = x(t)$ is an extremal of $\mathcal{J}_1$, so it satisfies (4) and is of class $\mathcal{C}^2$.

$$
\begin{aligned}
&\frac{\partial}{\partial u_1}\mathcal{J}_1(\alpha_{u_1,u_2}) = \\
&= \int_0^T \left\langle \frac{\partial}{\partial u_1}\alpha_{u_1,u_2}, \frac{D^4}{\partial t^4}\alpha_{u_1,u_2} + R\left(\frac{D^2}{\partial t^2}\alpha_{u_1,u_2}, \frac{\partial}{\partial t}\alpha_{u_1,u_2}\right)\frac{\partial}{\partial t}\alpha_{u_1,u_2}\right\rangle dt + \\
&+ \sum_{i=1}^{N}\left(\left\langle \frac{D}{\partial t}\frac{\partial}{\partial u_1}\alpha_{u_1,u_2}, \frac{D^2}{\partial t^2}\alpha_{u_1,u_2}\right\rangle - \left\langle \frac{\partial}{\partial u_1}\alpha_{u_1,u_2}, \frac{D^3}{\partial t^3}\alpha_{u_1,u_2}\right\rangle\right)\Bigg|_{T_{i-1}^+}^{T_i^-}.
\end{aligned} \tag{8}
$$

To obtain the second derivative with respect to u_2 evaluated at $u_1 = u_2 = 0$ involves the recursive use of the definition of curvature tensor given by the formula

$$
\begin{aligned}
\frac{D}{\partial u_2}\frac{D}{\partial t}\frac{D}{\partial u_1}\alpha_{u_1,u_2} &= \frac{D}{\partial t}\frac{D}{\partial u_2}\frac{D}{\partial u_1}\alpha_{u_1,u_2} + \\
&+ R\left(\frac{D}{\partial u_2}\alpha_{u_1,u_2}, \frac{D}{\partial t}\alpha_{u_1,u_2}\right)\frac{D}{\partial u_1}\alpha_{u_1,u_2},
\end{aligned} \tag{9}
$$

and also the following identities:

$$
\left.\frac{D}{\partial u_2}\frac{D^2}{\partial t^2}\alpha_{u_1,u_2}\right|_{u_1=u_2=0} = \frac{D^2}{\partial t^2}W_t^2 + R(W_t^2, V_t)V_t, \tag{10}
$$

$$
\begin{aligned}
\left.\frac{D}{\partial u_2}\frac{D^3}{\partial t^3}\alpha_{u_1,u_2}\right|_{u_1=u_2=0} &= \frac{D^3}{\partial t^3}W_t^2 + \frac{D}{\partial t}(R(W_t^2, V_t)V_t) + \\
&+ R(W_t^2, V_t)\frac{DV_t}{\partial t},
\end{aligned} \tag{11}
$$

together with the fact that W_t^1, W_t^2, and x are smooth on $[T_{i-1}, T_i]$, $(W_{T_i}^k = 0)$, $1 \le i \le N$, $k = 1, 2$, and $(\partial^2\alpha/\partial u_1\partial u_2)(T_i, 0, 0) = 0$, $i = 0, 1, \ldots, N$. We note that (9) is the definition of curvature tensor as in Nomizu [27], which differs from that in Milnor [24] by a minus sign. Then we get

$$\frac{\partial^2}{\partial u_2 \partial u_1}\mathcal{J}_1(\alpha_{u_1,u_2})\Big|_{u_1=u_2=0} = \int_0^T \Big\langle W_t^1, \frac{D}{\partial u_2}\Big[\frac{D^4}{\partial t^4}\alpha_{u_1,u_2} +$$

$$+R\left(\frac{D^2}{\partial t^2}\alpha_{u_1,u_2}, \frac{\partial}{\partial t}\alpha_{u_1,u_2}\right)\frac{\partial}{\partial t}\alpha_{u_1,u_2}\Big]\Big|_{u_1=u_2=0}\Big\rangle dt +$$

$$+\sum_{i=1}^{N}\Big\langle \frac{D}{\partial t}W_t^1, \frac{D}{\partial u_2}\frac{D^2}{\partial t^2}\alpha_{u_1,u_2}\Big|_{u_1=u_2=0}\Big\rangle\Big|_{T_{i-1}^+}^{T_i^-}.$$

To simplify the analysis further, we note that

$$\frac{D}{\partial u_2}\frac{D^4}{\partial t^4}\alpha_{u_1,u_2}\Big|_{u_1=u_2=0} = \frac{D^4}{\partial t^4}W_t^2 + \frac{D^2}{\partial t^2}\Big(R(W_t^2, V_t)V_t\Big) +$$

$$+\frac{D}{\partial t}(R(W_t^2, V_t)\frac{DV_t}{\partial t}) + R(W_t^2, V_t)\frac{D^2 V_t}{\partial t^2}, \tag{12}$$

and also make use of the following identities for the curvature tensor R and its covariant derivative $\nabla_W R$

$$\begin{aligned} R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0,\\ \nabla_W(R(X,Y)Z) = (\nabla_W R)(X,Y)Z + R(\nabla_W X, Y)Z +\\ + R(X, \nabla_W Y)Z + R(X,Y)\nabla_W Z. \end{aligned} \tag{13}$$

Taking into account that $W_{T_i}^k = 0$, $1 \le i \le N$, $k = 1, 2$, we obtain after many tedious manipulations

$$\frac{\partial^2}{\partial u_2 \partial u_1}\mathcal{J}_1(\alpha_{u_1,u_2})\Big|_{u_1=u_2=0} = -\sum_{i=1}^{N-1}\Big\langle \frac{D}{\partial t}W_{T_i}^1, \frac{D^2}{\partial t^2}W_{T_i^+}^2 - \frac{D^2}{\partial t^2}W_{T_i^-}^2\Big\rangle +$$

$$+\int_0^T \Big\langle W_t^1, K(W_t^2, V_t)\Big\rangle dt,$$

where

$$\begin{aligned}K(W,V) &= \nabla_V^4 W + (\nabla_V^2 R)(W,V)V + (\nabla_W R)(\nabla_V V, V)V + \\ &+ R(R(W,V)V,V)V + R(W, \nabla_V^2 V)V + \\ &+ 2[(\nabla_V R)(\nabla_V W, V)V + (\nabla_V R)(W, \nabla_V V)V + R(\nabla_V^2 W, V)V] + \\ &+ 3[(\nabla_V R)(W,V)\nabla_V V + R(W,V)\nabla_V^2 V + R(W,\nabla_V V)\nabla_V V] \\ &+ 4R(\nabla_V W, V)\nabla_V V.\end{aligned} \tag{14}$$

We summarize the result in the following

Theorem 2.4. *If $x \in \Omega$ is a critical path for problem $\mathcal{P}_1$ and α is a two-parameter variation of x, then*

$$\frac{\partial^2}{\partial u_2 \partial u_1} \mathcal{J}_1(\alpha_{u_1,u_2})\Big|_{u_1=u_2=0} = -\sum_{i=1}^{N} \left\langle \frac{D}{\partial t} W^1_{T_i}, \frac{D^2}{\partial t^2} W^2_{T_i^+} - \frac{D^2}{\partial t^2} W^2_{T_i^-} \right\rangle + \\ + \int_0^T \left\langle W_t^1, K(W_t^2, V_t) \right\rangle dt, \tag{15}$$

where $K(W^2, V)$ is given by (14).

It is clear that C^2-vector fields $W \in T_x\Omega$ such that $K(W,V) = 0$, $t \in [T_{i-1}, T_i]$, $1 \le i \le N$ play a similar role to the Jacobi vector fields. Note that $K(W,V) = 0$ defines a fourth-order differential equation for $W \in T_x\Omega$. Clearly, $(\partial^2/\partial u_2 \partial u_1)\mathcal{J}_1(\alpha_{u_1,u_2})|_{u_1=u_2=0}$ defines a symmetric bilinear form on $T_x\Omega$, which we denote by $B(W^1, W^2)$. At this point the symmetry is not evident, but we demonstrate this at the end of the section.

Theorem 2.5. *A vector field $W \in T_x\Omega$ belongs to the null space of the bilinear form $B(W^1, W^2)$ if and only if W is of class C^2 and $K(W,V) \equiv 0$, $t \in [T_{i-1}, T_i]$, $1 \le i \le N$.*

Proof. Clearly, if W^2 is any vector field of class C^2, then for all $W^1 \in T_x\Omega$ the first terms in (15) vanish. If, in addition to that, $K(W^2, V) \equiv 0$, $t \in [T_{i-1}, T_i]$, $1 \le i \le N$, then $B(W^1, W^2) = 0$. Conversely, if $B(W^1, W^2) = 0$, for all $W^1 \in T_x\Omega$, setting $W_t^1 = F(t)K(W^2, V)$, where $F(t) > 0$, $t \in (T_{i-1}, T_i)$, $1 \le i \le N$ and $F(T_i) = (dF/dt)(T_i) = 0$, $1 \le i \le N$, shows that $K(W^2, V) \equiv 0$, $t \in [T_{i-1}, T_i]$, $1 \le i \le N$. Now, setting $W_t^1 = F(t)Z_t$, where $Z_{T_i} = (D^2/\partial t^2)W^2_{T_i^+} - (D^2/\partial t^2)W^2_{T_i^-}$ and W_t^1 is a smooth vector field along $x(t)$ with $F(T_i) = 0$, $(dF/dt)(T_i) \neq 0$, $1 \le i \le N-1$, shows that $(D^2/\partial t^2)W^2_{T_i^+} = (D^2/\partial t^2)W^2_{T_i^-}$, $1 \le i \le N-1$. Thus W_t^2 is a C^2-vector field. □

For completeness, we now deduce the corresponding formula for the second variation of $\mathcal{J}_1$ along a smooth path which is a solution for the simplified problem $(\mathcal{P}_2)$. From (8) and taking into consideration that $W^k_{t_i}$, $1 \leq i \leq l-1$ does not necessarily vanish, we get

$$\begin{aligned}
\frac{\partial^2}{\partial u_1 \partial u_2}\mathcal{J}_1(\alpha_{u_1,u_2}\Big|_{u_1=u_2=0} &= \int_0^T \Big\langle W^1_t, \frac{D}{\partial u_2}\Big[\frac{D^4}{\partial t^4}\alpha_{u_1,u_2}+ \\
&+R\Big(\frac{D^2}{\partial t^2}\alpha_{u_1,u_2}, \frac{\partial}{\partial t}\alpha_{u_1,u_2}\Big)\frac{\partial}{\partial t}\alpha_{u_1,u_2}\Big]\Big|_{u_1=u_2=0}\Big\rangle\, dt + \\
&+\sum_{i=1}^{l}\Big(\Big\langle\frac{DW^1_t}{\partial t}, \frac{D}{\partial u_2}\frac{D^2}{\partial t^2}\alpha_{u_1,u_2}\Big|_{u_1=u_2=0}\Big\rangle - \\
&-\Big\langle W^1_t, \frac{D}{\partial u_2}\frac{D^3}{\partial t^3}\alpha_{u_1,u_2}\Big|_{u_1=u_2=0}\Big\rangle\Big)\Big|^{t_i^-}_{t^+_{i-1}} + \\
&+\sum_{i=1}^{l}\Big(\Big\langle\frac{D}{\partial u_2}\frac{D}{\partial u_1}\frac{\partial}{\partial t}\alpha_{u_1,u_2}\Big|_{u_2=u_2=0}, \frac{DV_t}{\partial t}\Big\rangle - \\
&-\Big\langle\frac{D}{\partial u_2}\frac{D}{\partial u_1}\alpha_{u_1,u_2}\Big|_{u_1=u_2=0}, \frac{D^2V_t}{\partial t^2}\Big\rangle\Big)\Big|^{t_i^-}_{t^+_{i-1}}.
\end{aligned} \tag{16}$$

The third term in this expression vanishes because the arguments turn out to be continuous since x is smooth and W^k_t, $k = 1, 2$ are C^1. If, in addition, we also use formulas (10), (11), (12), and (13), the following result can be deduced from (16).

Theorem 2.6. *If $\bar{x}$ is a critical path for problem $\mathcal{P}_2$ and α is a two-parameter variation of $\bar{x}$ with variation vector fields $W^k = \partial\alpha/\partial u_k|_{u_1=u_2=0}$, $k = 1, 2$, then*

$$\begin{aligned}
\frac{\partial^2}{\partial u_1 \partial u_2}\mathcal{J}_1(a_{u_1,u_2})\Big|_{u_1=u_2=0} &= -\sum_{i=1}^{l}\Big\langle\frac{D}{\partial t}W^1_{t_i}, \frac{D^2}{\partial t^2}W^2_{t_i^+} - \frac{D^2}{\partial t^2}W^2_{t_i^-}\Big\rangle + \\
&+\sum_{i=1}^{l}\Big\langle W^1_{t_i}, \frac{D^3}{\partial t^3}W^2_{t_i^+} - \frac{D^3}{\partial t^3}W^2_{t_i^-}\Big\rangle + \int_0^T \langle W^1_t, K(W^2_t, V_t)\rangle\, dt,
\end{aligned}$$

where $K(W, V)$ is given by Eq. (14).

Again $(\partial^2/\partial u_1\partial u_2)\mathcal{J}_1(\alpha_{u_1,u_2})|_{u_1=u_2=0}$ defines a symmetric bilinear form on $T_{\bar{x}}\overline{\Omega}$ which we denote by $\overline{B}(W^1, W^2)$. Arguing in a similar manner as in the proof of Theorem 2.5, we see that elements of the null space must be C^3 and hence smooth. Thus we obtain the following result.

Theorem 2.7. *A vector field $W \in T_{\bar{x}}\overline{\Omega}$ belongs to the null space of the bilinear form $B(W^1, W^2)$ if and only if W is smooth and $K(W, V) \equiv 0$, $t \in [t_{i-1}, t_i]$, $1 \le i \le l$.*

We point out that Theorems 2.1 and 2.5 are also true with larger classes of functions consisting of piecewise smooth C^1-interpolating functions on M satisfying (2) and (3). This can be demonstrated by incorporating both classes Ω and $\overline{\Omega}$ above and using the methods of Theorems 2.2 and 2.7 in the proofs of Theorems 2.1 and 2.5.

Finally in this section we demonstrate that the bilinear form $\overline{B}(W^1, W^2)$ is indeed symmetric in the arguments W^1 and W^2. First we note that

$$-\sum_{i=1}^{l}\left\langle \frac{D}{\partial t}W^1_{t_i}, \frac{D^2}{\partial t^2}W^2_{t_i^+} - \frac{D^2}{\partial t^2}W^2_{t_i^-}\right\rangle + \sum_{i=1}^{l}\left\langle W^1_{t_i}, \frac{D^3}{\partial t^3}W^2_{t_i^+} - \frac{D^3}{\partial t^3}W^2_{t_i^-}\right\rangle =$$

$$= \int_0^T \frac{d}{dt}\left(\left\langle \frac{D}{\partial t}W^1_t, \frac{D^2}{\partial t^2}W^2_t\right\rangle - \left\langle W^1_t, \frac{D^3}{\partial t^3}W^2_t\right\rangle\right) dt =$$

$$= \int_0^T \left(\left\langle \frac{D^2}{\partial t^2}W^1_t, \frac{D^2}{\partial t^2}W^2_t\right\rangle - \left\langle W^1_t, \frac{D^4}{\partial t^4}W^2_t\right\rangle\right) dt.$$

After adding this integrand to $\langle W^1_t, K(W^2_t, V_t)\rangle$ and considerable rearranging of terms, one can show that $\overline{B}(W^1, W^2)$ may be written as

$$\overline{B}(W^1, W^2) = \int_0^T (F_1(W^1_t, W^2_t, V_t) + F_2(W^1_t, W^2_t, V_t))dt,$$

where

$$F_1(W^1, W^2, V) = \langle \nabla_V^2 W^1 + R(W^1, V)V, \nabla_V^2 W^2 + R(W^2, V)V\rangle +$$

$$+ 2\langle \nabla_V V, (R(W^2, V)\nabla_V W^1 + R(W^1, V)\nabla_V W^2 + \frac{1}{2}R(W^2, \nabla_V V)W^1)\rangle,$$

$$F_2(W^1, W^2, V) = \langle \nabla_V V, (\nabla_V R)(W^2, V)W^1 + (\nabla_{W^2} R)(W^1, V)V\rangle.$$

It is clear that F_1 is symmetric in W^1 and W^2, whereas the symmetry of F_2 is not clear. To demonstrate the symmetry we must use the first Bianchi identity in the form

$$(\nabla_W R)(A, B)C + (\nabla_W R)(C, A)B + (\nabla_W R)(B, C)A = 0$$

and the second Bianchi identity (Hicks [20, p. 95]), for the 1-covariant, 3-contravariant tensor $K(\theta, A, B, C) = \theta(R(B, C)A)$, which states that

$$(\nabla_W K)(\theta, A, B, C) + (\nabla_C K)(\theta, A, W, B) + (\nabla_B K)(\theta, A, C, W) = 0,$$

which, in turn, yields an equivalent statment in the form

$$(\nabla_W R)(B, C)A + (\nabla_C R)(W, B)A + (\nabla_B R)(C, W)A = 0.$$

It follows that

$$(\nabla_V R)(W^2, V)W^1 = (\nabla_V R)(W^1, V)W^2 - (\nabla_V R)(W^1, W^2)V$$

and

$$(\nabla_{W^2} R)(W^1, V)V = (\nabla_{W^1} R)(W^2, V)V + (\nabla_V R)(W^1, W^2)V;$$

thus

$$(\nabla_V R)(W^2, V)W^1 + (\nabla_{W^2} R)(W^1, V)V = (\nabla_V R)(W^1, V)W^2 + \\ + (\nabla_{W^1} R)(W^2, V)V,$$

which establishes the symmetry of F_2. We can write F_2 in the symmetric form as

$$F_2(W^1, W^2, V) = \frac{1}{2}\langle \nabla_V V, (\nabla_{W^1} R)(W^2, V)V + (\nabla_{W^2} R)(W^1, V)V \rangle + \\ + \frac{1}{2}\langle \nabla_V V, (\nabla_V R)(W^2, V)W^1 + (\nabla_V R)(W^1, V)W^2 \rangle.$$

3. Interpolation on a Riemannian manifold with constraints

We now consider the following **problem** ($\mathcal{P}_3$) derived from the interpolation problems treated in Sec. 1 with some additional constraints.

Problem ($\mathcal{P}_3$). "Find critical values of

$$\mathcal{J}_1(x) = \frac{1}{2}\int_0^T \left\langle \frac{D^2 x(t)}{dt^2}, \frac{D^2 x(t)}{dt^2} \right\rangle dt, \tag{17}$$

over the class Ω of C^1-paths x on M, satisfying $x|_{[Ti-1, T_i]}$ is smooth,

$$x(T_i) = x_i, \quad 1 \le i \le N - 1 \tag{18}$$

for a distinct set of points $x_i \in M$ and fixed times T_i, where $0 = T_0 < T_1 < \cdots < T_N = T$ and, in addition,

$$x(0) = x_0, \quad x(T) = x_N, \quad \frac{dx}{dt}(0) = v_0, \quad \frac{dx}{dt}(T) = v_N, \tag{19}$$

and also

$$\left\langle \frac{dx}{dt}, X_i(x) \right\rangle = k_i, \quad i = 1, \dots, l \quad (l < n), \tag{20}$$

for X_i, $i = 1, \cdots, n$, linearly independent vector fields in some neighborhood of x and given constants k_i, $i = 1, \cdots, l$."

To deal with the constraints, we define the one forms $\omega_i(X) = \langle X_i, X \rangle$ and the two forms $d\omega_i$, $1 \leq i \leq l$, where d is the exterior derivative. We may also define tensors S_i; $S_{i_x} : T_x M \to T_x M$, by setting

$$d\omega_i(X, Y) = \langle S_i(X), Y \rangle = -\langle S_i(Y), X \rangle. \tag{21}$$

Similar problems to $\mathcal{P}_3$ have been dealt with extensively by many authors in relation to nonholonomic mechanics and control. See Bloch and Crouch [4] for further information. Constraints also give rise to abnormal extremals as solutions of variational problems. These abnormal extremals arise as solutions to the variational problem defined by the constraints alone, and are the subject of intense interest (see, for example, Agrachev and Sarychev [1], Bliss [3], Bryant and Hsu [7], Montgomery [25], and Sussmann [32]). These situations are not of principal concern in the current paper.

Theorem 3.1. *A necessary condition for $x \in \Omega$ to be a normal extremal for problem $(\mathcal{P}_3)$ is that x is C^2 and there exist smooth functions $\lambda_i(t)$ such that*

$$\begin{aligned} &\frac{D^3 V_t}{dt^3} + R\left(\frac{DV_t}{dt}, V_t\right) V_t - \sum_{i=1}^{l} \lambda_i S_i(V_t) - \sum_{i=1}^{l} \dot{\lambda}_i X_i \equiv 0, \\ &\langle V_t, X_i \rangle = k_i, \quad i = 1, \cdots, l, \end{aligned} \tag{22}$$

for $t \in [Ti - 1, T_i]$, $i = 1, \cdots, N$.

Any abnormal extremals for the problem $(\mathcal{P}_3)$ satisfy the system of equations

$$\begin{aligned} &\sum_{i=1}^{l} \dot{\lambda}_i X_i + \sum_{i=1}^{l} \lambda_i S_i(V_t) = 0, \\ &\langle V_t, X_i \rangle = k_i, \quad i = 1, \cdots, l, \end{aligned} \tag{23}$$

where $\lambda_1, \cdots, \lambda_l$ *are not all identically zero.*

Proof. To obtain the normal extremals, we simply augment the Lagrangian in the problem $(\mathcal{P}_1)$ by terms

$$\sum_{i=1}^{l} \lambda_i \left\langle \frac{dx}{dt}, X_i(x) \right\rangle.$$

Using arguments as in Bloch and Crouch [4], the variational procedure applied to these terms yields (with variational field W_t)

$$-\sum_{i=1}^{l} \dot{\lambda}_i \langle W_t, X_i \rangle - \sum_{i=1}^{l} \lambda_i \left(\left\langle \frac{DX_i}{\partial t}, W_t \right\rangle - \langle \nabla_{W_t} X_i, V_t \rangle \right). \tag{24}$$

But since $d\omega_i(V_t, W_t) = \langle \nabla_{V_t} X_i, W_t \rangle - \langle \nabla_{W_t} X_i, V_t \rangle$, using Eq. (21) we establish the required Eq. (22) for normal extremals. Equations (23) for the abnormal extremals are now simple consequence of their definition. □

Note that for normal extremals, differential equations for the Lagrange multipliers λ_i may be obtained by differentiating the constraints three times, assuming that the vector fields $X_1, \cdots, X_l$ are linearly independent. From Eq. (23) for abnormal extremals, the Lagrange multipliers are determined by the equations

$$\sum_{i=1}^{l} \dot{\lambda}_i \langle X_j, X_i \rangle + \sum_{i=1}^{l} \lambda_i \langle S_i(V_t), X_j \rangle = 0, \qquad 1 \le j \le l. \tag{25}$$

4. Interpolation on connected and compact Lie groups

In this section we study the variational problems of the previous sections for the case where the Riemannian manifold M is a connected and compact Lie group. The following theorem is used extensively in what follows.

Theorem 4.1 (Milnor [24], Nomizu [27]).

(i) *Every connected and compact Lie group admits a left-and right-invariant Riemannian metric* $\langle \cdot, \cdot \rangle$.

(ii) *If* ∇ *denotes the corresponding metric connection and* X, Y, *and* Z *are right-invariant vector fields on* G, *then*

$$\nabla_X Y = -\frac{1}{2}[Y, X], \tag{26}$$

$$R(X,Y)Z = -\frac{1}{4}[[X,Y],Z], \tag{27}$$

$$\nabla R = 0. \tag{28}$$

If $\{X_1, \ldots, X_n\}$ is a basis of the Lie algebra $\mathcal{L}$ of G and x is a curve on G, we may write any smooth vector field W_t along x as $W_t = \sum_1^n w_i(t) X_i(x(t))$, where w_i are smooth functions of time. In particular, the velocity vector field $V_t = (dx(t)/dt) = \sum_1^n v_i(t) X_i(x(t))$.

To simplify the notation, we may sometimes use W instead of W_t for a vector field along x and also write $\dot{W}$ for $\sum_i \dot{w}_i(t) X_i(x(t))$ and $W^{(k)}$ for $\nabla_V^k W$. If $W_t = \sum_i w_i(t) X_i(x(t))$ and $Z_t = \sum_i z_i(t) X_i(x(t))$, then $\sum_{i,j} w_i(t) z_j(t) [X_i, X_j](x(t))$, where $[X_i, X_j]$ is the Lie bracket in $\mathcal{L}$, denotes $[W_t, Z_t]$. Using (27) and (28) together with properties of ∇, the next result can be easily proven.

Lemma 4.2. *Let x be any curve on G, V the velocity vector field along x, and W any vector field along x. Then*

$$W^{(1)} = \dot{W} - \frac{1}{2}[W, V], \tag{29}$$

$$W^{(2)} = \ddot{W} - [\dot{W}, V] - \frac{1}{2}[W, \dot{V}] + \frac{1}{4}\Big[[W, V], V\Big], \tag{30}$$

$$\begin{aligned} W^{(3)} = \dddot{W} &- \frac{3}{2}[\ddot{W}, V] - \frac{3}{2}[\dot{W}, \dot{V}] - \frac{1}{2}[W, \ddot{V}] - \\ &- \frac{1}{8}\Big[\big[[W, V], V\big], V\Big] + \frac{3}{4}\Big[[\dot{W}, V], V\Big] + \\ &+ \frac{1}{2}\Big[[W, \dot{V}], V\Big] + \frac{1}{4}\Big[[W, V], \dot{V}\Big], \end{aligned} \tag{31}$$

$$\begin{aligned} W^{(4)} = \ddddot{W} &- 2[\dddot{W}, V] - 3[\ddot{W}, \dot{V}] - 2[\dot{W}, \ddot{V}] - \frac{1}{2}[W, \dddot{V}] + \\ &+ \frac{1}{16}\Big[\Big[\big[[W, V], V\big], V\Big], V\Big] - \frac{1}{2}\Big[\big[[\dot{W}, V], V\big], V\Big] - \\ &- \frac{1}{4}\Big[\big[[W, V], \dot{V}\big], V\Big] - \frac{1}{8}\Big[\big[[W, V], V\big], \dot{V}\Big] - \\ &- \frac{3}{8}\Big[\big[[W, \dot{V}], V\big], V\Big] + \frac{3}{2}\Big[[\ddot{W}, V], V\Big] + \\ &+ 2\Big[[\dot{W}, \dot{V}], V\Big] + \frac{3}{4}\Big[[W, \ddot{V}], V\Big] + \frac{3}{4}\Big[[W, \dot{V}], \dot{V}\Big] + \\ &+ \Big[[\dot{W}, V], \dot{V}\Big] + \frac{1}{4}\Big[[W, V], \ddot{V}\Big]. \end{aligned} \tag{32}$$

If $W = V$, formulas (29), (30), and (31) reduce to

$$\begin{aligned} V^{(1)} &= \dot{V}, \\ V^{(2)} &= \ddot{V} + \tfrac{1}{2}[V, \dot{V}], \\ V^{(3)} &= \dddot{V} + [V, \ddot{V}] + \tfrac{1}{4}[[\dot{V}, V]V], \end{aligned} \tag{33}$$

giving the result of Lemma 4.2 in [14]. Lemma 4 in [14] can also be reformulated in terms of this new notation using (28) and (33) as follows.

Lemma 4.3. *In the situation of Theorem* 4.1, *the following two statements are equivalent for a curve x on G with velocity vector field V_t:*

$$\text{(i)} \qquad \frac{D^4x}{dt^4} + R\left(\frac{D^2x}{dt^2}, \frac{Dx}{dt}\right)\frac{Dx}{dt} \equiv 0, \tag{34}$$

$$\text{(ii)} \qquad \dddot{V}_t + [V_t, \ddot{V}_t] \equiv 0. \tag{35}$$

The next theorem contains the analogue of the Jacobi differential equation when M is a connected and compact Lie group.

Theorem 4.4. *In the situation of Theorem* 4.1, *the differential equation $K(W,V) = 0$, where $K(W,V)$ is given by* (14) *and V satisfies* (34), *is equivalent to the following equation:*

$$\begin{aligned} &\ddddot{W} - 2[\dddot{W}, V] - 3[\ddot{W}, \dot{V}] - 2[\dot{W}, \ddot{V}] + \Big[[\ddot{W}, V], V\Big] + \\ &\quad + 2\Big[[\dot{W}, \dot{V}], V\Big] = 0. \end{aligned} \tag{36}$$

Proof. From (14), taking into consideration that in the present situation the covariant derivative of the curvature tensor is zero, and using identity (27), we get

$$\begin{aligned} K(W,V) = W^{(4)} &+ \frac{1}{16}\Bigg[\Big[\big[[W,V],V\big],V\Big],V\Bigg] - \frac{1}{4}\Big[[W, V^{(2)}], V\Big] - \\ &- \frac{1}{2}\Big[[W^{(2)}, V], V\Big] - \frac{3}{4}\Big\{\Big[[W,V], V^{(2)}\Big] + \\ &+ \Big[[W, V^{(1)}], V^{(1)}\Big]\Big\} - \Big[[W^{(1)}, V], V^{(1)}\Big]. \end{aligned}$$

This expression can be simplified further by using identities (29) to (32) and the Jacobi identity for the Lie bracket. After some tedious calculations, the terms with fourth- and fifth-order Lie brackets cancel and the result follows by applying (35). □

Example 4.5. When $G = SO(3)$ and $\{X_1, X_2, X_3\}$ is a cyclic basis for $so(3)$, i.e., $[X_1, X_2] = X_3$, $[X_3, X_1] = X_2$, $[X_2, X_3] = X_1$, any vector field W on $SO(3)$ can be written as $W = \sum_{i=1}^{3} w_i(t) X_i$. If "$\times$" denotes the cross product in $\mathbb{R}^3$ and $f : (\mathbb{R}^3, \times) \to so(3)$ is the Lie algebra isomorphism defined by $f(u) = S_u$, where $u = (u_1, u_2, u_3)^T$ and

$$S_u = \begin{pmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{pmatrix},$$

it is well known that $u \times v = S_u v$ and $f(u \times v) = [S_u, S_v]$. Using this isomorphism, Eqs. (35) and (36) reduce to

$$\frac{d^3 v}{dt^3} + v \times \frac{d^2 v}{dt^2} = 0 \tag{37}$$

and

$$\frac{d^4 w}{dt^4} - 2\frac{d^3 w}{dt^3} \times v - 3\frac{d^2 w}{dt^2} \times \frac{dv}{dt} - 2\frac{dw}{dt} \times \frac{d^2 v}{dt^2} + \\ + \left(\frac{d^2 w}{dt^2} \times v\right) \times v + 2\left(\frac{dw}{dt} \times \frac{dv}{dt}\right) \times v = 0, \tag{38}$$

respectively, where $w = (w_1, w_2, w_3)^T$ and $v = (v_1, v_2, v_3)^T$. Equation (37), the equation of a cubic spline in $SO(3)$, appeared in Noakes et al. [26] and Jackson [21] for the first time.

We finally prove the equivalent version of Theorem 3.1 when M is a connected and compact Lie group.

Theorem 4.6. *A necessary condition for a curve x on G to be a normal extremal for problem $(\mathcal{P}_3)$ is that x is C^2 and*

$$\dddot{V}_t + [V_t, \ddot{V}_t - Z_t] - \dot{Z}_t \equiv 0, \quad t \in [Ti-1, T_i], \qquad i = 1, \cdots, N, \\ \langle V_t, Z_t \rangle \equiv 0, \tag{39}$$

where $V_t = \dfrac{dx(t)}{dt}$ and $Z_t = \sum_1^l \lambda_i(t) X_i(x(t))$, for X_i, $i = 1, \cdots, l$ right-invariant vector fields on G. The abnormal extremals satisfy the equations

$$\dot{Z}_t + [V_t, Z_t] \equiv 0, \quad Z_t \neq 0, \\ \langle V_t, Z_t \rangle \equiv 0.$$

Proof. In this case, since G is a connected and compact Lie group with left-and right-invariant Riemannian metric, $\operatorname{ad} X$ is skew-symmetric with respect to $\langle \cdot, \cdot \rangle$ [29, p. 114]. That is, $\langle [X, Y], Z \rangle = -\langle [X, Z], Y \rangle$ for X, Y, and Z any left invariant vector fields on G. Now rewrite Eqs. (22) and (25) using the fact that in this case

$$d\omega_i(V_t, X_j) = \langle S_i(V_t), X_j \rangle = -\omega_i([V_t, X_j]). \quad \square$$

Note. The equations $\dot{Z}_t + [V_t, Z_t] \equiv 0$ are precisely the Eqs. (16) obtained by Montgomery [25].

5. Interpolation on Symmetric Spaces

We refer to Nomizu [27] and Helgason [19] for more details concerning Riemannian symmetric spaces. Let G be a connected Lie group, σ an involutive automorphism of G, and G/K a symmetric homogeneous space defined by σ. Here we assume that K is compact so that one can introduce a positive-definite Riemannian metric on G/K. In this case, the Lie algebra $\mathcal{L}$ of G admits a canonical decomposition $\mathcal{L} = \mathcal{S} \oplus \mathcal{M}$, where $\mathcal{M}$ is the eigenspace for the eigenvalue -1 of the involutive automorphism of the Lie algebra $\mathcal{L}$ induced by σ and $\mathcal{S}$ the Lie algebra of K. The following inclusions hold:

$$[\mathcal{S}, \mathcal{S}] \subset \mathcal{S}, \quad [\mathcal{M}, \mathcal{M}] \subset \mathcal{S}, \quad [\mathcal{M}, \mathcal{S}] \subset \mathcal{M}. \tag{40}$$

Let π denote the canonical projection of G onto G/K and $x_0 = \pi(e_G)$, where e_G is the identity element in G. Then $\mathcal{M}$ can be identified with the tangent space to G/K at x_0. Parallel vector fields play an important role in this section. If W_t is a vector field along a piecewise smooth curve x on G/K, W_t is said to be parallel along x if $\frac{DW_t}{dt} \equiv 0$. It is well known (see, for instance, Milnor [24]) that if $x_0 = x(0)$ is the initial point of a curve x and $W \in T_{x_0}(G/K)$ is an arbitrary tangent vector to G/K at x_0, then there exists a unique parallel vector field W_t along x having the value W at x_0. Hence, if $W_1, W_2, \ldots, W_n$ is an orthonormal frame at x_0, then there exists a unique parallel field of orthonormal frames along x which coincides with $W_1, W_2, \ldots, W_n$ at x_0. These vector fields are said to be obtained from $W_1, \ldots, W_n$ by parallel displacement along x and will be denoted hereafter by $\overline{W}_1, \ldots, \overline{W}_n$. If $Z_t = \sum_i z_i(t)\overline{W}_i(x(t))$ is an arbitrary vector field along x, its parallel pullback from x_t to x_0 is the vector $Z(t) = \sum_i z_i(t) W_i \in T_{x_0}M$.

Theorem 5.1 (Nomizu [27]). *Every symmetric homogeneous space G/K, with K compact, admits a Riemannian metric, invariant under the* *action of G, which induces the canonical affine connection ∇. This connection has trivial torsion tensor and if $\overline{X}, \overline{Y}, \overline{Z}$ are parallel displacements along a curve x on G/K satisfying $x(0) = x_0$, then*

$$(\nabla_{\overline{X}}\overline{Y})(x_0) = \frac{1}{2}[X, Y]_{\mathcal{M}} = 0, \qquad \forall X, Y \in \mathcal{M}, \tag{41}$$

$$(R(\overline{X}, \overline{Y})\overline{Z})(x_0) = -\big[[X, Y]_{\mathcal{S}}, Z\big]_{\mathcal{M}} = -[[X, Y], Z], \qquad \forall X, Y, Z \in \mathcal{M}, \tag{42}$$

$$\nabla_W(R) = 0, \qquad \textit{for any vector field } W \textit{ on } G/K. \tag{43}$$

We use the notation introduced in Sec. 3 for vector fields along a curve x. Note that if x is a curve on G/K which is a critical path for the functional $\mathcal{J}_1$ in (1) and $\overline{X}_1, \dots, \overline{X}_n$ an orthonormal frame of parallel vector fields along x, then, for any vector field along x, $W_t = \sum_{i=1}^{n} w_i(t)\overline{X}_i(x(t))$, we have

$$W_t^{(k)} = \sum_{i=1}^{n} \frac{d^k w_i(t)}{dt^k} \overline{X}_i(x(t)). \tag{44}$$

We also note that if $\overline{X}, \overline{Y}, \overline{Z}$ are parallel vector fields along a critical path x for $\mathcal{J}_1$, then $R(\overline{X}, \overline{Y})\overline{Z}$ is also a parallel vector field along x.

Lemma 5.2. *Let x be a curve on G/K which is a critical path for the functional $\mathcal{J}_1$ in (1). Assume that $x(0) = x_0$ and that*

$$V_t = \sum_{i=1}^{n} v_i(t)\overline{X}_i(x(t))$$

is the velocity vector field along x for $\{\overline{X}_1, \dots, \overline{X}_n\}$ the parallel displacement along x of a basis $\{X_1, \dots, X_n\}$ in $\mathcal{M}$. Then the following statements are equivalent:

$$\frac{D^3 V_t}{dt^3} + R\left(\frac{DV_t}{dt}, V_t\right) V_t \equiv 0, \tag{45}$$

$$\dddot{V}(t) + \big[V(t), [\dot{V}(t), V(t)]\big] \equiv 0, \tag{46}$$

where $V(t), \dot{V}(t)$, *and* $\dddot{V}(t)$ *are the parallel pullbacks from* x_t *to* x_0 *of* V_t, $\dot{V}_t$, *and* $\dddot{V}_t$, *respectively. That is,*

$$V(t) = \sum_i v_i(t) X_i, \quad \dot{V}(t) = \sum_i \dot{v}_i(t) X_i, \quad \dddot{V}(t) = \sum_i \dddot{v}_i(t) X_i.$$

Proof. Using (44) with $W_t = V_t$, we get

$$\frac{D^3 V_t}{dt^3} + R\left(\frac{DV_t}{dt}, V_t\right) V_t = \sum_i \dddot{v}_i \overline{X}_i + \sum_{j,k,l} \dot{v}_j v_k v_l R(\overline{X}_j, \overline{X}_k)\overline{X}_l.$$

According to the previous comment, $R(\overline{X}_j, \overline{X}_k)\overline{X_l}$ is parallel along x, thus determined by its value at x_0. From (40) and (42), $(R(\overline{X}_j, \overline{X}_k)\overline{X}_l)(x_0) = -[[X_j, X_k], X_l]$, and since $\overline{X}_1, \ldots, \overline{X}_n$ are linearly independent, we have that (45) is equivalent to

$$\dddot{v}_i X_i - \sum_{j,k,l} \dot{v}_j v_k v_l [[X_j, X_k], X_l] \equiv 0, \quad i = 1, 2, \cdots, n,$$

or

$$\dddot{V}(t) + [V(t), [\dot{V}(t), V(t)]] \equiv 0. \quad \square$$

Using arguments similar to those in the proof of Lemma 5.2, the analogue of the Jacobi equation can also be simplified to give the following result.

Theorem 5.3. *In the situation of Theorem* 5.1, *the analogue of the Jacobi differential equation,* $K(W, V) = 0$, *where* $K(W, V)$ *is given by* (44), $V = \sum_i v_i(t)\overline{X}_i$, $W = \sum_i w_i(t)\overline{X}_i$, *is equivalent to the following equation:*

$$\dddot{W}(t) + ad^4 V(t) \cdot W(t) - \Big[[W(t), \ddot{V}(t)], V(t)\Big] - 2\Big[[\ddot{W}(t), V(t)], V(t)\Big] -$$
$$- 3\{[W(t), \dot{V}(t)], \dot{V}(t)] + \Big[[W(t), V(t)], \ddot{V}(t)]\} -$$
$$- 4\Big[[\dot{W}(t), V(t)], \dot{V}(t)] = 0, \qquad (47)$$

where $V(t) = \sum_i v_i(t) X_i$, $\dot{V}(t) = \sum_i \dot{v}_i(t) X_i$, *etc., and similarly for* $W(t)$, $\dot{W}(t)$, *etc.*

The next example exhibits the equation of a cubic spline on the sphere S^2.

Example 5.4. (Cubic splines on the sphere.) Let $x(t)$ be a curve on the two-dimensional sphere S^2, satisfying the conditions of Lemma 5.2. The solution $V(t)$ of Eq. (46) can be seen as the velocity vector field of a curve $\gamma(t)$ in $T_{x_0}S^2$ satisfying the initial condition $\gamma(0) = x_0$. In order to obtain a cubic spline $x(t)$ on S^2 starting at x_0, we use the notion of parallel transport in the following way.

Suppose that the tangent plane $T_{x_0}S^2$ is rolling (without sliding) over the sphere S^2, touching S^2 at every instant of time t at the point $\gamma(t)$. This means that $T_{x_0}S^2$ rotates in $\mathbb{R}^3$ in such a way that its instantaneous axis of rotation is parallel to $T_{x_0}S^2$ and perpendicular to $V(t)$. In this case, the point of touching of S^2 and $T_{x_0}S^2$ draws on S^2 a curve $x(t)$ which is the development of $\gamma(t)$. If $R(t)$ represents such a rotation, then

$$x(t) = R(t)x_0, \quad \dot{x}(t) = R(t)V(t), \quad R(t)X_i = \overline{X}_i(x(t)). \tag{48}$$

Since the instantaneous axis of rotation of $R(t)$ is perpendicular to both $x(t)$ and $\dot{x}(t)$, we have

$$\dot{R}(t) = S_{x(t)\times\dot{x}(t)}R(t), \quad R(0) = I, \tag{49}$$

where "$\times$" is the cross product in $\mathbb{R}^3$ and $S_{x(t)\times\dot{x}(t)}$ is the skew-symmetric matrix associated with $x(t) \times \dot{x}(t)$, as defined in Example 4.5. Using (48), Eq. (49) can be written as

$$\dot{R}(t) = R(t)S_{x_0\times V(t)}, \quad R(0) = I. \tag{50}$$

Now, let x_1, x_2, and x_3 be defined by

$$x_1(t) = R(t)V(t), \quad x_2(t) = R(t)\dot{V}(t), \quad x_3(t) = R(t)\ddot{V}(t).$$

Then, from (46), (48), and (50) we obtain the following set of equations:

$$\begin{cases} \dot{x} = x_1 \\ \dot{x}_1 = -x(x_1^T x_1) + x_2 \\ \dot{x}_2 = -x(x_2^T x_1) + x_3 \\ \dot{x}_3 = -x(x_3^T x_1) + x_1(x_2^T x_1) - x_2(x_1^T x_1) \end{cases}$$

or

$$\frac{d}{dt}\begin{pmatrix} x \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -x_1^T x_1 & 0 & 1 & 0 \\ -x_2^T x_1 & 0 & 0 & 1 \\ -x_3^T x_1 & x_2^T x_1 & -x_1^T x_1 & 0 \end{pmatrix}\begin{pmatrix} x \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \tag{51}$$

with constraints

$$x^T x = 1, \quad x_1^T x = x_2^T x = x_3^T x = 0. \tag{52}$$

Equation (51) together with constraints (52) give the cubic spline on S^2.

Note. This example is worked out in detail in Crouch, Silva Leite, Yan, and Brunnett [10], and the case of constant velocity cubic splines in Brunnett and Crouch [6].

6. The C^2-dynamic interpolation problem

In this section we address the dynamic interpolation problem for nonlinear control systems. We first specify the trajectory in terms of a discrete ordered set of points through which the dynamical variables of state are forced to pass, and then determine suitable controls which give rise to such trajectories. Simultaneously, optimization of a cost function under some smooth constraints is imbedded in the problem to obtain certain desirable geometric properties of the resulting trajectory. In order for the problem to be well-posed, controllability is always assumed. We also require that the trajectory x be C^2 and satisfy $x|_{[T_{i-1},T_i]}$ is smooth, where $0 = T_0 < T_1 < \cdots < T_N = T$ is an ordered set of fixed times. We start with systems evolving on connected and compact Lie groups G and show the equivalence between some optimal control problems and the interpolation variational problems treated in the previous sections. In all that follows, $x_0, x_1, \ldots, x_N$ is a distinct set of points in G and $\{X_1, \ldots, X_n\}$ an orthonormal basis of right-invariant vector fields on G. We now consider the following **optimal control problem** $\mathcal{P}_4$:

Problem $(\mathcal{P}_4)$. "$\text{Min}_{u(\cdot)} \frac{1}{2}\int_0^T \sum_{i=1}^k u_i^2(t)\,dt$ subject to

$$\dot{x}(t) = \textstyle\sum_{i=1}^k v_i(t) X_i(x(t)), \quad \dot{v}_i(t) = u_i(t), \quad i = 1, \cdots, k,$$
$$x \in G, \ k \leq \dim G = n, \ \langle X_i, X_j \rangle = \delta_{ij},$$
$$x(T_i) = x_i, \quad i = 1, \cdots, N-1,$$
$$x(0) = x_0, \ x(T) = x_N, \ \dot{x}(0) = v_0, \ \dot{x}(T) = v_N."$$

The simplified problem, without the interpolation conditions $x(T_i) = x_i$, has already appeared in [14]. The extra interpolation conditions do not add any additional difficulties. In this case, using Lemma 4.2, we obtain

$$\frac{D^2 x}{dt^2} = \sum_{i=1}^k \dot{v}_i X_i = \sum_{i=1}^k u_i X_i$$

and

$$\mathcal{J}_1(x) = \frac{1}{2}\int_0^T \left\langle \frac{D^2x}{dt^2}, \frac{D^2x}{dt^2} \right\rangle dt = \frac{1}{2}\int_0^T \sum_{i=1}^k u_i^2(t)\, dt.$$

If $k = n$, the optimal control problem $(\mathcal{P}_4)$ is clearly equivalent to the variational problem $(\mathcal{P}_1)$ in the special situation of a connected and compact Lie group. For $k < n$, it is a particular case of the variational problem $(\mathcal{P}_3)$ with constraints

$$\left\langle \frac{Dx}{dt}, X_i(x) \right\rangle = 0, \qquad i = k+1, \dots, n.$$

Thus, the next result follows from Theorem 4.6.

Theorem 6.1. *The normal extremals of the optimal control problem $(\mathcal{P}_4)$ satisfy the equation*

$$\dddot{V}_t + \left[V_t, \ddot{V}_t - Z_t\right] - \dot{Z}_t \equiv 0, \;\; t \in [Ti-1, T_i], \;\; i = 1, \cdots, N, \quad (53)$$

where $V_t = \dfrac{dx(t)}{dt}$ *and* $Z_t = \sum_{i=k+1}^n \lambda_i(t) X_i(x(t))$ *for some smooth functions* $\lambda_i(t)$, $i = k+1, \dots, n$.

Remark 6.2. Note that if $k = n$, then $Z_t = 0$ and Eq. (53) reduces to (35), which is the equation of the extremals for problem $(\mathcal{P}_1)$ when M is a connected and compact Lie group.

A more realistic situation occurs when the control system contains a drift term. Our methods still apply in this case. Consider the following **optimal control problem** $\mathcal{P}_5$:

Problem $(\mathcal{P}_5)$**.** "$\text{Min}_{u(\cdot)} \frac{1}{2}\int_0^T \sum_{i=1}^k u_i^2(t)\, dt$ subject to

$$\begin{aligned}
&\dot{x}(t) = X_1(x(t)) + \textstyle\sum_{i=2}^k v_i(t) X_i(x(t)), \;\; \dot{v}_i(t) = u_i(t),\\
&i = 1, \cdots, k, \quad x \in G,\\
&k \le \dim G = n, \;\; \langle X_i, X_j \rangle = \delta_{ij},\\
&x(T_i) = x_i, \;\; i = 1, \cdots, N-1,\\
&x(0) = x_0, \;\; x(T) = x_N, \;\; \dot{x}(0) = v_0, \;\; \dot{x}(T) = v_N."
\end{aligned}$$

Here $\dfrac{D^2x}{dt^2} = \sum_{i=2}^k \dot{v}_i X_i = \sum_{i=2}^k u_i X_i$. Thus, the optimal control $(\mathcal{P}_5)$ is equivalent to the variational problem $(\mathcal{P}_3)$ with constraints

$$\left\langle \frac{Dx}{dt}, X_1(x) \right\rangle = 1 \;\text{ and }\; \left\langle \frac{Dx}{dt}, X_i(x) \right\rangle = 0, \; i = k+1, \cdots, n,$$

and the next result follows immediately from Theorem 4.6.

Theorem 6.3. *The normal extremals of the optimal control problem* ($\mathcal{P}_5$) *satisfy the equation*

$$\dddot{V}_t + \left[V_t, \ddot{V}_t - Z_t \right] - \dot{Z}_t \equiv 0, \quad t \in [Ti-1, T_i], \quad i = 1, \cdots, N, \qquad (54)$$

where $V_t = \dfrac{dx(t)}{dt}$ *and* $Z_t = \lambda_1(t)X_1(X(t)) + \sum_{i=k+1}^{n} \lambda_i(t)X_i(x(t))$, *for some functions* $\lambda_1(t)$ *and* $\lambda_i(t)$, $i = k+1, \cdots, n$.

The following example contains all interesting situations for control systems evolving on the rotation group $SO(3)$. As in Example 4.5, $\{X_1, X_2, X_3\}$ is the cyclic basis for $so(3)$. The interpolation conditions, as well as the boundary conditions, are always assumed to hold and will be omitted here.

Example 6.4.
Case I

$$\text{“Min}_{u(\cdot)} \frac{1}{2} \int_0^T (u_1^2 + u_2^2 + u_3^2)\, dt \quad \text{subject to}$$

$$\dot{x} = v_1 X_1(x) + v_2 X_2(x) + v_3 X_3(x), \quad \dot{v}_i = u_i, \quad i = 1, 2, 3.\text{”}$$

The extremals satisfy

$$\frac{d^3 v}{dt^3} + v \times \frac{d^2 v}{dt^2} = 0, \quad \text{where} \quad v = (v_1, v_2, v_3)^T$$

(Noakes et al. [26], and Jackson [21]).
Case II

$$\text{“Min}_{u(\cdot)} \frac{1}{2} \int_0^T (u_1^2 + u_2^2)\, dt \quad \text{subject to}$$

$$\dot{x} = v_1 X_1(x) + v_2 X_2(x), \quad \dot{v}_1 = u_1, \quad \dot{v}_2 = u_2.\text{”}$$

The extremals satisfy Eq. (53) in Theorem 6.1 with

$$Z_t = \lambda_3(t) X_3(x(t)).$$

That is,

$$\begin{cases} \dddot{v}_1 - \lambda_3 v_2 = 0, \\ \dddot{v}_2 + \lambda_3 v_1 = 0, \\ \dot{\lambda}_3 - v_1 \ddot{v}_2 + v_2 \ddot{v}_1 = 0. \end{cases}$$

Case III

$$\text{"Min}_{u(\cdot)}\tfrac{1}{2}\int_0^T u_2^2 dt \quad \text{subject to}$$
$$\dot{x} = X_1(x) + v_2X_2(x), \quad \dot{v}_2 = u_2."$$

The extremals satisfy Eq. (54) in Theorem 6.3 with

$$Z_t = \lambda_1(t)X_1(x(t)) + \lambda_3(t)X_3(x(t)).$$

That is,

$$\begin{cases} \dddot{v}_2 + \lambda_3 = 0, \\ \dot{\lambda}_1 + \lambda_3 v_2 = 0, \\ \dot{\lambda}_3 - \ddot{v}_2 - \lambda_1 v_2 = 0. \end{cases}$$

Finally, we consider an **optimal control problem** ($\mathcal{P}_6$) for a full control system evolving on a homogeneous symmetric space M as considered in Sec. 4.

Problem ($\mathcal{P}_6$). "$\text{Min}_{u(\cdot)}\frac{1}{2}\int_0^T \sum_{i=1}^n u_i^2(t)\, dt$ subject to

$\dot{x}(t) = \sum_{i=1}^n v_i(t)\overline{X}_i(x(t)), \ \dot{v}_i(t) = u_i(t), \ i = 1, \cdots, n, \ x \in M,$
$\dim M = n,$

$\{\overline{X}_1, \cdots, \overline{X}_n\}$ the parallel displacement along x of an orthonormal basis $\{X_1, \cdots, X_n\}$ in $T_{x_0}M$,

$x(T_i) = x_i, \quad i = 1, \cdots, N-1,$

$x(0) = x_0, \ x(T) = x_N, \ \dot{x}(0) = v_0, \ \dot{x}(T) = v_N$."

The following theorem follows immediately from Lemma 5.2.

Theorem 6.5. *The extremals of the problem* ($\mathcal{P}_6$) *satisfy the equation*

$$\dddot{V}(t) + [V(t), [\dot{V}(t), V(t)]] \equiv 0, \quad t \in [Ti-1, T_i], \quad i = 1, \cdots, N, \quad (55)$$

where $V(t)$ is the parallel pullback from x_t to x_0 of the velocity vector field V_t and similarly for $\dot{V}(t)$ and $\dddot{V}(t)$.

References

1. A. Agrachev and A. Sarychev, On abnormal extremals for Lagrange variational problems. *J. Math. Syst., Estimation and Control* (to appear).

2. J. Baillieul, Geometric methods for nonlinear optimal control problems. *J. Optimiz. Theory and Appl.* **25** (1978), No 4, 519-548.

3. A. Bliss, Lectures on calculus of variations. *Univ. Chicago Press*, 1946.

4. A. Bloch and P. Crouch, Nonholonomic and Vakonomic control systems on Riemannian manifolds. *Fields Inst. Commun.* **1** (1993).

5. R. Brockett, Control theory and singular Riemannian geometry. In: New Directions in Applied Mathematics. *Springer Verlag, New York*, 1981, 11-27.

6. G. Brunnett and P. Crouch, Elastic curves on the sphere. *Adv. in Comput. Math.* **2** (1994), 23-40.

7. R. Bryant and L. Hsu, Rigidity of integral curves of rank 2 distributions. *Duke Univ. Preprint.* (1993), No 5.

8. M. Camarinha, F. Silva Leite, and P. Crouch, On the relation between higher-order variational problems on Riemannian manifolds (to appear)

9. P. Chang, C. Lin, and J. Luh, Formulation and optimization of cubic polynomial joint trajectories for industrial robots. *IEEE Trans. Autom. Control* AC**28** (1983), 1066-1074.

10. P. Crouch, Y. Yan, F. Silva Leite, and G. Brunnett, On the construction of spline elements on spheres. *Proc. Second Europ. Control Conf., Groningen, June* 28–*July* 1 **4** (1993), 1930-1934.

11. P. Crouch and J. Jackson, A non-holonomic dynamic interpolation problem. *Proc. Conf. on Analysis of Control Dynam. Syst., Lyons, France,* 1990, *Birkhäuser*, ser. *Progress in Syst. and Control* (1991), 156-166.

12. ______, Dynamic interpolation and application to flight control. *J. Guidance, Control and Dynam.* **14** (1991), 814-822.

13. ______, Dynamic interpolation for linear systems. *Proc. 29th IEEE Conf. Decis. and Control, Hawaii,* 1990, 2312-2314.

14. P. Crouch and F. Silva Leite, Geometry and the dynamic interpolation problem. *Proc. Am. Control Conf., Boston,* 26 – 29 *July,* 1991, 1131-1136.

15. H. Erzberger, J. McLean, and J. Barmon, Fixed-range optimal trajectories for shot haul aircraft. *NASA TN. D*-8115. Dec., 1975.

16. H. Erzberger and H. Lee, Constrained optimum trajectories with specified range. *J. Guidance, Control and Dynam.* **3** (1980), 78-85.

17. H. Erzberger and J. McLean, Fuel-conservative guidance system for powered lift aircraft. *J. Guidance, Control and Dynam.* **4** (1981), 253-261.

18. L. Faibusovich, Explicitly solvable nonlinear optimal control problems. *Int. J. Control* **48**, (1988), No 6, 2507-2526.

19. S. Helgason, Differential geometry, Lie groups and symmetric spaces. *Academic Press*, 1978.

20. N. Hicks, Notes on differential geometry. *Van Nostrand Reinhold Math. Stud.* **3** 1971, 1978.

21. J. Jackson, Dynamic interpolation and application to flight control. PhD Thesis. *Arizona State Univ.*, 1990.

22. V. Judjevic, Non Euclidean elastica. (1993). *Am. J. of Math.* (to appear).

23. C. Liu and J. Luh, Approximate joint trajectories for control of industrial robots along cartesian paths. *IEEE Trans. Syst., Man and Cybernet.* **14** (1984), 444-450.

24. J. Milnor, Morse theory. *Ann. Math. Stud., Princeton Univ. Press* **51** 1969.

25. R. Montgomery, On the singularities in the space of curves tangent to a nonintegrable distribution and their relation to characteristics (to appear).

26. L. Noakes, G. Heinzinger, and B. Paden, Cubic splines on curved spaces. *IMA J. Math. Control and Inf.* **6** (1989), 465-473.

27. K. Nomizu, Invariant affine connections on homogeneous spaces. *Am. J. Math.* **76** (1954), 33-65.

28. T. Okubo, Differential geometry. In: Monographs and textbooks in pure and applied mathematics. **112** *Marcel Dekker, Inc., New York*, 1987.

29. W. Poor, Differential geometric structures. *McGraw Hill*, 1981.

30. S. Sasaki, On the differential geometry of the tangent bundle on Riemannian manifolds. I. *Tohôko Math. J.* **10** (1958), 338-354.

31. W. Shadwick, On the geometry of the Lagrange problem. *Preprint*, 1989.

32. H. Sussmann, A cornucopia of abnormal sub-Riemannian minimizers. (to appear).

33. R. Strichartz, Sub-Riemannian geometry. *J. Differ. Geom.* **24** (1986), 221-263.

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