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Computing

Approximation by Circles

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Abstract — Zusammenfassung

Approximation by Circles. The problem considered is to assign a measure of circularity to a given compact set in the plane. The measure adopted is the size of the smallest annulus containing the given set. Two different notions of the size of an annulus, that of area and that of difference of radii are studied.

Approximation durch Kreise. Das hier untersuchte Problem ist, einer kompakten Menge in der Ebene ein Maß der Kreisförmigkeit zuzuschreiben. Als Maß wird die Größe des kleinsten Kreisringes gewählt, der die gegebene Menge enthält. Zwei verschiedene Größenbegriffe für den Kreisring werden untersucht, nämlich dessen Oberfläche und die Differenz der Radien.

1. Introduction

How can one recognize a circle (or circular arc) in the plane? We present a precise formulation of this question of pattern recognition, and give some quantitative answers in the context of approximation theory. Thus the problem we consider is to assign a measure of circularity to a given compact set in the plane. Our approach is to determine the best annulus which contains the given set and judge the circularity of the given set according to the size of the annulus. Of course, we must make precise the sense of "best" and "size" in the preceding sentence. To this end let S be a compact set in the plane. We consider two different assignments of size to an annulus. The first is the area of the annulus, the second the difference of its radii. A notion of size having been fixed we define a best annular approximation to S to be any annulus of least size which contains S. It turns out that with the criterion of size corresponding to area the problem of best annular approximation of S is equivalent to a well-known linear uniform approximation problem.

This is discussed in the second section. The second notion of size is equivalent to a somewhat novel non-linear approximation problem as we see in Section 3.

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2. Least Area Approximation

Given a compact set S in the plane, for any complex number w put

$$r_2(w)=\max_{z\in S}|z-w|;\quad r_1(w)=\min_{z\in S}|z-w|. \tag{1}$$

Clearly, S is contained in the annulus

$$r_1(w)\le|z-w|\le r_2(w) \tag{2}$$

whose area is

$$A(w)=\pi\left(r_2^2(w)-r_1^2(w)\right).$$

If

$$\inf_w A(w)=A(w_0) \tag{3}$$

then the annulus described by (2) with $w=w_0$ is a best annulus in the sense of area which contains S. Its center is w_0 and its outer and inner radii are $r_2(w_0)$ and $r_1(w_0)$ respectively.

Now put $f(x,y)=x^2+y^2$ and let V be the linear space of linear functions $a\,x+b\,y+c$. $f(x,y)$ has a best uniform approximation out of V on S.

Theorem 1: *If $w_0:(x_0,y_0)$ is the center of a best annulus (in the sense of area) for S, then*

$$v_0(x,y)=2\,x_0\,x+2\,y_0\,y-\left(x_0^2+y_0^2-\frac{r_1^2(w_0)+r_2^2(w_0)}{2}\right) \tag{4}$$

is a best uniform approximation to $f(x,y)$ on S out of V, and

$$M=\|f-v_0\|=\frac{r_2^2(w_0)-r_1^2(w_0)}{2}.$$

Conversely, if $v_0(x,y)=2\,x_0\,x+2\,y_0\,y+c_0$ is a best uniform approximation to $f(x,y)$ on S out of V then $w_0:(x_0,y_0)$ is the center of a best annulus (in the sense of area) for S and if $M=\|f-v_0\|$,

$$r_2^2(w_0)=c_0+x_0^2+y_0^2+M;\quad r_1^2(w_0)=c_0+x_0^2+y_0^2-M,$$

and hence

$$r_2^2(w_0)-r_1^2(w_0)=2\,M.$$

Proof:

i) Suppose w_0 satisfies (3) and $r_1(w_0)$ and $r_2(w_0)$ are defined by (1). Consider v_0 given by (4), then if we put $e_0=f-v_0$ we have for $(x,y)\in S$

$$-\frac{r_2^2(w_0)-r_1^2(w_0)}{2}\le e_0(x,y)\le\frac{r_2^2(w_0)-r_1^2(w_0)}{2},$$

and, indeed, $M=\|e_0\|=\left(r_2^2(w_0)-r_1^2(w_0)\right)/2$.

If there exists $v\in V$ such that $e=f-v$ satisfies

$$\|e\|=K<\frac{r_2^2(w_0)-r_1^2(w_0)}{2}$$

then every point of S lies in the annulus

$$-K \leq e(x, y) \leq K$$

whose area is $2\pi K$, which is less than $\pi(r_2^2(w_0) - r_1^2(w_0))$.

This contradiction establishes that v_0 is a best approximation to f out of V on S.

ii) Suppose $v_0(x, y) = 2x_0 x + 2y_0 y + c_0$ is a best approximation to $f(x, y)$ on S out of V, and $e_0 = f - v_0$ satisfies

$$-M \leq e_0(x, y) \leq M \tag{5}$$

for $(x, y) \in S$. Note that there must exist points (x_1, y_1); $(x_2, y_2) \in S$ such that $e_0(x_1, y_1) = -M$ and $e_0(x_2, y_2) = M$. Every point of S lies in the annulus defined by (5) which is centered at (x_0, y_0) has radii

$$r_1^2(w_0) = c_0 + x_0^2 + y_0^2 - M; \; r_2^2(w_0) = c_0 + x_0^2 + y_0^2 + M,$$

and area $2\pi M = \pi(r_2^2(w_0) - r_1^2(w_0))$. We claim this must be a best annulus. If not, suppose that for all $(x, y) \in S$

$$r^2 \leq (x-h)^2 + (y-k)^2 \leq R^2$$

and $R^2 - r^2 < 2M$. Then

$$\left\| f(x, y) - \left(2hx + 2ky - \left(h^2 + k^2 - \frac{R^2 + r^2}{2}\right)\right)\right\| \leq \frac{R^2 - r^2}{2},$$

contradicting our assumption that v_0 is a best approximation, and proving the theorem.

Corollary: *Every compact set S has a best annular approximation in the sense of area.*

Remark 1: It is clear from the proof of Theorem 1 that there is a one-to-one correspondence between best uniform approximations to f on S out of V and best annular approximations to S in the sense of area. In particular, then, a best annular approximation is unique if, and only if, a best uniform approximation to f is unique.

Remark 2: It is shown in Rivlin and Shapiro [1] that if S is a compact convex set in the plane a best approximation to $x^2 + y^2$ on S out of V, is

$$2x_0 x + 2y_0 y + (r^2/2) - x_0^2 - y_0^2$$

where (x_0, y_0) is the center of the circle of minimal radius circumscribing S and r its radius. Thus for such S, there is a best annular approximation which center (x_0, y_0), outer radius r, and inner radius zero.

The characterization of best uniform approximations out of V can be nea ly described in terms of the notion of extremal signature. We recall (cf. Rivlin [2] for details) that, in the present context, a signature in S is a continuous function, Σ, whose domain (called the base of Σ) is a closed subset of S and whose range consists of the values ± 1. A signature, Σ, is *extremal* for V if there exist real numbers $\zeta_1, \ldots, \zeta_s$ and distinct points $y_1, \ldots, y_s$ of the base of Σ such that

$$\operatorname{sgn} \zeta_i = \Sigma(y_i), \quad i=1, \ldots, s.$$

and

$$\sum_{i=1}^{s} \zeta_i \, v(y_i) = 0, \text{ all } v \in V. \tag{6}$$

$\zeta_1, \ldots, \zeta_s$ are called weights for Σ. If $g \in C(S)$ let $E(g; S) = \{z \in S : |g(z)| = \|g\|\}$. Then we quote the characterization theorem.

Theorem A: *A best approximation on S out of V to $g \notin V$ is v^*, if and only if, there exists an extremal signature, Σ, for V whose base consists of at most four points of $E(g - v^*; S)$ such that $\Sigma(y) = \operatorname{sgn}(g(y) - v^*(y))$.*

In view of (6) it is easy to see that there can be no extremal signature for V based on less than 3 points. Thus for our purposes only extremal signatures based on 3 or 4 points are of interest and they must have the configurations schematized in Fig. 1. (The sign associated with each point of the base is the sign of the weight corresponding to the point, all signs for a given base may be multiplied by -1, of course.)

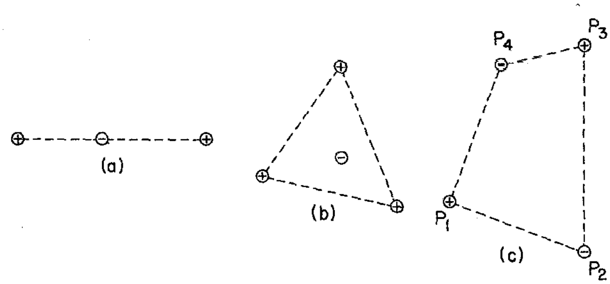

Fig. 1

Thus, for example, we can find a best annular approximation to S when S consists of 3 or 4 points.

i) If S consists of 3 collinear points, which may be taken with no loss of generality, to be $(\pm t, 0)$ and $(s, 0)$ where $0 \le s < t$, then it is easy to verify that for all u, $v_0(x, y) = u\,y + (s^2 + t^2)/2$ is a best approximation to $f(x, y)$ on S, and so every point on the y-axis is the center of a best annular approximation to S.

The case that S consists of 3 non-collinear points is trivial.

ii) If S consists of 4 collinear points, which may be taken, with no loss of generality, to be $(\pm t, 0)$, $(s, 0)$ and $(q, 0)$ where $0 \le |q| < s < t$, then it is easily seen that the situation is exactly the same as that of 3 collinear points.

iii) If S consists of 4 points, three (but not four) of which are collinear, and the collinear points are taken to be $(\pm t, 0)$, and $(s, 0)$ where $0 \le s < t$, and (a, b) is the fourth point, then every point $(0, u/2)$ where u satisfies

$$a^2 + b^2 - t^2 \le u\,b \le a^2 + b^2 - s^2$$

is the center of a best annular approximation to S.

iv) If S consists of 4 points, and one is inside the triangle having the other three as vertices, then (cf. Fig. 1 b) the best annular approximation has its center at the circumcenter of the triangle, the outer circle of the annulus is the circumcircle of the triangle and the inner circle passes through the point inside the triangle.

v) Finally, if S consists of the four vertices of a convex quadrilateral, then (cf. Fig. 1 c) the best annular approximation has a pair of diagonally opposite points on both its inner and outer circles. Which pair is on the inner circle and which on the outer depends on certain distances between the pairs of points and is easily determined. If the points are labeled as in Fig. 1 c, the origin is placed at P_4, say, and with obvious notation for the coordinates of the points, if

$$C=(x_1^2+x_1^2)\,\Delta_1-(x_2^2+y_2^2)\,\Delta_2+(x_3^2+y_3^2)\,\Delta_3$$

where Δ_i, $i=1,2,3,4$ is the area of the triangle obtained from the four points by omitting P_i, then if $C>0$, P_2 and P_4 are on the inner circle; if $C<0$ P_2 and P_4 are on the outer circle; and if $C=0$, the 4 points are concyclic. In a similar fashion, the area of the best annulus can be calculated directly in terms of the coordinates of the points by the formula, which reads in the present context,

$$A=2\pi\frac{C}{\Delta_4}.$$

Remark: These considerations immediately imply the following geometric fact. If S is an oval (boundary of a strictly convex body in the plane), then there is a pair of concentric circles of which the inner one lies inside the oval and touches it at at least two distinct points and the outer one lies outside the oval and touches it at at least two distinct points. The question of what is the most general class of closed curves having this property is open.

3. Least Difference of Radii

In this section, we retain the notation of Section 2, but here instead of $A(w)$ we consider

$$r(w)=r_2(w)-r_1(w), \tag{7}$$

and if

$$\inf_w r(w)=r(w_0), \tag{8}$$

we call the annulus described by (2) with $w=w_0$ a best approximation (in the uniform sense) to S, and $r(w_0)$ its *radius.*

A first observation is that a best uniform annular approximation need not exist, as is evidently the case if S consists of 3 or more collinear points. In this case $r(w)$ tends to zero as w tends to infinity. Indeed, if

$$\inf_w r(w)=\alpha \tag{9}$$

then either there is some sequence of centers $\{w_n\}$ such that

$$r(w_n) \downarrow \alpha \tag{10}$$

and $w_n \to \bar{w}$ as $n \to \infty$, in which case $\bar{w}$ is a center of an annulus of best uniform approximation to S, or $|w_n| \uparrow \infty$ as $n \to \infty$ for (a subsequence of) every sequence of centers satisfying (10). In the latter case we claim that α is the opening of the narrowest strip in the plane containing S. For, if S is contained in the strip

$$d_1(\theta) \leq \operatorname{Re}(z\, e^{-i\theta}) \leq d_2(\theta) \tag{11}$$

with $\alpha - (d_2 - d_1) = 2\varepsilon > 0$ and the (topological) diameter of S is D then there is an annulus centered at any $\rho\, e^{i\theta}$ with $\rho > d_2 + \varepsilon + (D^2/4\varepsilon)$ containing S and having a radius less than α, contradicting (9). Similarly, if for each θ, $0 \leq \theta < 2\pi$, the narrowest strip of the form (11) containing S satisfies $d_2(\theta) - d_1(\theta) > \alpha$, then given $\varepsilon > 0$ there exists $N(\varepsilon)$ such that for $n > N$ every annulus centered at w_n and containing S has radius greater than $\alpha + \varepsilon$, contradicting (10).

If (8) holds and we let $G_1(w_0)$ be the circle $|z - w_0| = r_1(w_0)$ and $G_2(w_0)$ be the circle $|z - w_0| = r_2(w_0)$ then we clearly have

Lemma 1: *If w_0 is the center of a best uniform annular approximation to S then each of G_1 and G_2 contains a point of S.*

Suppose now that S is a finite set of 4 or more points.

Lemma 2: *If w_0 is the center of a best uniform annular approximation to S then either $G_1 \cap S$ or $G_2 \cap S$ contains more than 1 point.*

Proof: Suppose, the Lemma false so that, in view of Lemma 1, each of $G_1 \cap S$ and $G_2 \cap S$ contains exactly one point, say $A = G_1 \cap S$ and $B = G_2 \cap S$. Let 0 denote the point w_0.

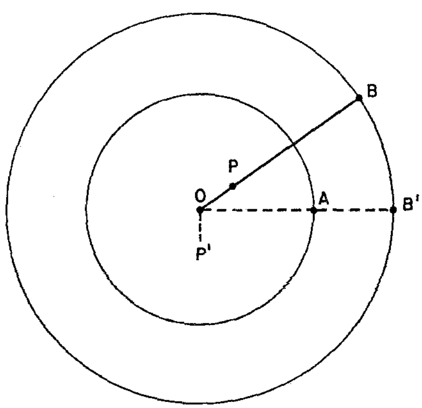

Fig. 2

i) A is not on the closed segment $0B$. Choose P on $0B$ near 0. Then $|0A| < |0P| + |PA|$ and $r = |0B| - |0A| = |0P| + |PB| - |0A| > |PB| - |PA|$. Thus by choosing P close enough to 0 there is an annulus centered at P containing S (recall that S is a finite set of points) and having radius $< r(w_0)$, a contradiction.

ii) A is on the closed segment $0B=0B'$. Let P' be near 0 on the line passing through 0 and perpendicular to $0B'$. Then

$$|P'B'|<|P'A|+|AB'|, r=|AB'|>|P'B'|-|P'A|,$$

and we conclude as in case i).

Theorem 2: *If S is a finite set of 4 or more points and w_0 is the center of a best annular approximation to S then each of G_1 and G_2 contains at least 2 points of S.*

Proof:

i) G_2 contains only one point of S, call it D. Then G_1 contains points $P_1, \ldots, P_k$ of S with $k \geq 2$.

a) None of $P_1, \ldots, P_k$ is on closed $0D$. Choose P on $0D$. Then

$$|0P|>|0P_i|-|PP_i|, \; i=1, \ldots, k.$$

But for $i=1, \ldots, k$

$$r=|0D|-|0P_i|=|0P|+|PD|-|0P_i|>|PD|-|PP_i|,$$

and we obtain a contradiction in a, by now, familiar fashion.

b) P_1, say, is on $0D$. Choose P on $0P_1$ then

$$|0P|+|PP_i|>|0P_i|=|0P|+|PP_1|, \; i=2, \ldots, k.$$

Now consider the annulus centered at P with inner circle containing P_1 and outer circle containing D. Its radius is $r(w_0)$ and it contains S when P is close enough to 0, thus it is a best annulus for S, but its boundary contains only the points P_1 and D of S, contradicting Lemma 2.

ii) The case that G_1 has only one point of S leads to a contradiction by a similar argument.

Corollary: *If S is a finite set of 4 or more points such that*

$$\inf_w r(w)=\alpha,$$

and S is contained within a strip of opening α, then each bounding line of the strip contains at least 2 points of S.

Proof:

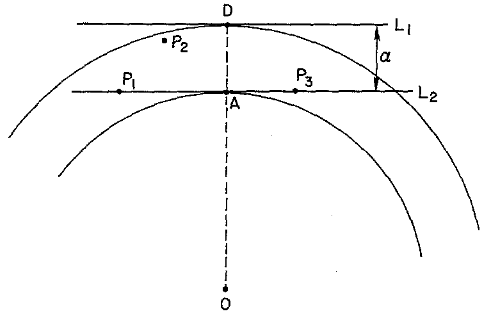

Fig. 3

Let L_1 and L_2 be the bounding lines of the strip. Suppose L_1 contains only point D of S. Let the perpendicular to L_1 at D meet L_2 in A. DA may be extended sufficiently far to 0 so that the annulus with center at 0, outer circle containing D, and inner circle containing A contains S, has radius α — thus making it a best annular approximation to S — yet its inner circle contains at most one point of S, contradicting Theorem 2.

Remark: If S is the set consisting of the unit circle $|z|=1$ and its center, $z=0$, then it is easy to see that a best annulus containing S has radius 1. Among these is the annulus centered at $w_0=1/3$ and having radii $r_2(w_0)=4/3$ and $r_1(w_0)=1/3$. For this annulus G_1 contains only one point of S, the origin, and G_2 contains only one point of S, the point $z=-1$. Thus even Lemma 2 fails if we drop the requirement that S be a finite set of points.

In order to say more about arbitrary compact S we relate the problem of best uniform annular approximation to a non-linear uniform approximation problem.

Consider

$$F(x, y; h, k, t) = ((x-h)^2 + (y-k)^2)^{1/2} - t.$$

The following equivalence is clear.

Lemma 3:

$$\mu = \min_{(h,k,t)} \max_{(x,y)\in S} |F(x, y; h, k, t)| = \max_{(x,y)\in S} |F(x, y; \bar{h}, \bar{k}, \bar{t})|$$

if, and only if, the annulus centered at $\bar{w}: (\bar{h}, \bar{k})$ *with* $r_1(\bar{w}) = \bar{t} - \mu$ *and* $r_2(\bar{w}) = \bar{t} + \mu$ *is a best uniform annular approximation to S.*

Suppose now that $(\bar{h}, \bar{k})$ is the center of a best annulus with $r_1 = \bar{t} - \mu$, $r_2 = \bar{t} + \mu$ and $0 < r_1 < r_2$. Let

$$D = \{(x, y) \in S : |F(x, y; \bar{h}, \bar{k}, \bar{t})| = \mu\}.$$

Suppose, further, that $(x, y) \in D$ and let $\phi(x, y) = (\phi_1(x, y),\ \phi_2(x, y),\ \phi_3(x, y))$ be the vector

$$\left(\operatorname{sgn} F(x, y; \bar{h}, \bar{k}, \bar{t}) \frac{\partial F}{\partial h}(x, y; \bar{h}, \bar{k}, \bar{t}),\ \operatorname{sgn} F(x, y; \bar{h}, \bar{k}, \bar{t}) \frac{\partial F}{\partial k}(x, y; \bar{h}, \bar{k}, \bar{t}),\right.$$

$$\left.\operatorname{sgn} F(x, y; \bar{h}, \bar{k}, \bar{t}) \frac{\partial F}{\partial t}(x, y; \bar{h}, \bar{k}, \bar{t})\right).$$

Then we have

Lemma 4: *Let p be any unit vector in* $\mathbb{R}^3$, *then*

$$\max_{(x,y)\in D} (p, \phi(x, y)) \geq 0. \tag{12}$$

Proof: Suppose (12) to be false. Then for the vector q

$$\max_{(x,y)\in D} (q, \phi(x, y)) \leq \delta < 0,$$

or,

$$(q, \phi(x, y; \bar{h}, \bar{k}, \bar{t})) \leq \delta < 0, \quad \text{all } (x, y) \in D.$$

Now $D=D_1 \cup D_2$ where $D_i \subset \{z: |z-\bar{w}|=r_i\}$, $i=1,2$.

Let U_1 and U_2 be open sets such that: $D_i \subset U_i \cap S = W_i$, $i=1,2$;

$$\sup_{z \in W_1} |z-\bar{w}| < \bar{t} < \inf_{z \in W_2} |z-\bar{w}|,$$

there exists $\varepsilon_0 > 0$ such that for $0 \le \varepsilon \le \varepsilon_0$

$$0 < \frac{\mu}{2} \le F(x, y; \bar{h}+\varepsilon q_1, \bar{k}+\varepsilon q_2, \bar{t}+\varepsilon q_3),\ (x, y) \in W_2,$$

$$0 > -\frac{\mu}{2} \ge F(x, y; \bar{h}+\varepsilon q_1, \bar{k}+\varepsilon q_2, \bar{t}+\varepsilon q_3),\ (x, y) \in W_1,$$

$$|F(x, y; \bar{h}+\varepsilon q_1, \bar{k}+\varepsilon q_2, \bar{t}+\varepsilon q_3)| \le \mu-\delta'$$

for $(x, y) \in S\ (W_1 \cup W_2)$ for some $\delta' > 0$, and

$$\big(q, \phi(x, y; \bar{h}+\varepsilon q_1, \bar{k}+\varepsilon q_2, \bar{t}+\varepsilon q_3)\big) \le \delta/2 < 0, \text{ all } (x, y) \in W_1 \cup W_2.$$

Then

$$F(x, y; \bar{h}+\varepsilon_0 q_1, \bar{k}+\varepsilon_0 q_2, \bar{t}+\varepsilon_0 q_3) = F(x, y; \bar{h}, \bar{k}, \bar{t}) + \varepsilon_0 \big(q, \phi(x, y; \bar{h}+\varepsilon q_1, \bar{k}+\varepsilon q_2, \bar{t}+\varepsilon q_3)\big)$$

for $(x, y) \in W_2$ and some ε in $[0, \varepsilon_0]$, and

$$F(x, y; \bar{h}+\varepsilon_0 q_1, \bar{k}+\varepsilon_0 q_2, \bar{t}+\varepsilon_0 q_3) = F(x, y; \bar{h}, \bar{k}, \bar{t}) - \varepsilon_0 \big(q, \phi(x, y; \bar{h}+\bar{\varepsilon} q_1, \bar{k}+\bar{\varepsilon} q_2, \bar{t}+\bar{\varepsilon} q_3)\big)$$

for $(x, y) \in W_1$ and some $\bar{\varepsilon}$ in $[0, \varepsilon_0]$. Thus

$$|F(x, y; \bar{h}+\varepsilon_0 q_1, \bar{k}+\varepsilon_0 q_2, \bar{t}+\varepsilon_0 q_3)| \le \mu + (\varepsilon_0\, \delta)/2$$

on $W_1 \cup W_2$, contradicting the definition of μ.

With notation and hypothesis the same as in Lemma 4 we have

Lemma 5: *If* (12) *holds then there exist points of* D, $(x_1, y_1), \ldots, (x_k, y_k)$, *with* $k \le 4$ *and*

$$0 = \sum_{j=1}^{k} \lambda_j \phi_i(x_j, y_j),\ i=1,2,3,$$

where $\lambda_1 + \ldots + \lambda_k = 1$ *and* $\lambda_j > 0, j=1, \ldots, k$.

Proof: Consider the compact set in $\mathbb{R}^3$

$$\Phi = \{(\phi_1(x, y), \phi_2(x, y), \phi_3(x, y)) : (x, y) \varepsilon D\}.$$

In order to prove the lemma it suffices to show that the origin (in $\mathbb{R}^3$) is in the convex hull of Φ in view of Caratheodory's theorem (cf. Rivlin [2]). But if the origin is not in the convex hull of Φ there exists a hyperplane

$$h(X, Y, Z) = c_0 + c_1 X + c_2 Y + c_3 Z = 0$$

such that $h(0, 0, 0) = c_0 > 0$ and

$$h(\phi_1(x, y), \phi_2(x, y), \phi_3(x, y)) \leq 0,$$

for all $(x, y) \in D$; that is for all $(x, y) \in D$

$$c_1 \phi_1(x, y) + c_2 \phi_2(x, y) + c_3 \phi_3(x, y) \leq -c_0 < 0,$$

contradicting (12).

Lemmas 3, 4 and 5 help establish the following necessary condition for a best uniform annular approximation.

Theorem 3: *If $\bar{w}:(h, k)$ is the center of an annulus which is a best uniform approximation to S, with $0 < r_1(\bar{w}) < r_2(\bar{w})$, then there exist a positive integer $k \leq 4$, positive numbers $\lambda_1, \ldots, \lambda_k$, distinct points $z_1, \ldots, z_k$ of S and disjoint subsets I_1 and I_2 of $\{1, \ldots, k\}$ such that $|I_2| + |I_1| = k$,*

$$\sum_{j=1}^{k} \lambda_j = 1, \tag{13}$$

$$\sum_{j \in I_1} \lambda_j = \sum_{j \in I_2} \lambda_j, \tag{14}$$

and

$$\sum_{j \in I_1} \lambda_j \frac{z_j - \bar{w}}{r_1(\bar{w})} = \sum_{j \in I_2} \lambda_j \frac{z_j - \bar{w}}{r_2(\bar{w})}, \tag{15}$$

where $|z_j - \bar{w}| = r_1$ for $j \in I_1$ and $|z_j - \bar{w}| = r_2$ for $j \in I_2$.

Proof: We need only observe that if

$$|z_j - \bar{w}| = r_2(\bar{w}),\ \phi_1(z_j) = (\bar{h} - x_j)/r_2(w),\ \phi_2(z_j) = (\bar{k} - y_j)/r_2(w),\ \phi_3(z_j) = -1;$$

and if

$$|z_j - \bar{w}| = r_1(\bar{w}),\ \phi_1(z_j) = -(\bar{h} - x_j)/r_1(w),\ \phi_2(z_j) = -(\bar{k} - y_j)/r_1(w),\ \phi_3(z_j) = 1.$$

Remark: Conditions (13), (14) and (15) have some interesting consequences. We must have $k > 1$. If $k = 2$ then $I_1 = \{1\}$, $I_2 = \{2\}$, say, and $\lambda_1 = \lambda_2 = \frac{1}{2}$. Thus $(z_1 - \bar{w})/r_1 = (z_2 - \bar{w})/r_2$ and the points z_1 and z_2 lie on the same ray issuing from $\bar{w}$. (This is precisely the situation in the case of the best annular approximation described in the Remark following Theorem 2.) $k = 3$ is impossible, for if $k = 3$ then we have, say, $\lambda_1 + \lambda_2 = \lambda_3 = \frac{1}{2}$ and

$$2\lambda_1 \frac{z_1 - \bar{w}}{r_1} + 2\lambda_2 \frac{z_2 - \bar{w}}{r_1} = \frac{z_3 - \bar{w}}{r_2},$$

a contradiction. Finally, if $k = 4$ we are led to $\lambda_1 + \lambda_2 = \lambda_3 + \lambda_4 = \frac{1}{2}$ and

$$2\lambda_1 \frac{z_1 - \bar{w}}{r_1} + 2\lambda_2 \frac{z_2 - \bar{w}}{r_1} = 2\lambda_3 \frac{z_3 - \bar{w}}{r_2} + 2\lambda_4 \frac{z_4 - \bar{w}}{r_2},$$

which has the geometric interpretation that if z_1, z_2, z_3, z_4 are viewed from $\bar{w}$ the two points of the outer circle and the two points of the inner circle are strictly interlaced angle-wise.

When S consists of 4 points, $k=2$ is impossible. For suppose the points are A, B, C, D and A and B lie on the same ray issuing from the center, 0, A and D are on the inner circle, B and C on the outer. (See Fig. 4.) Consider the annulus of which

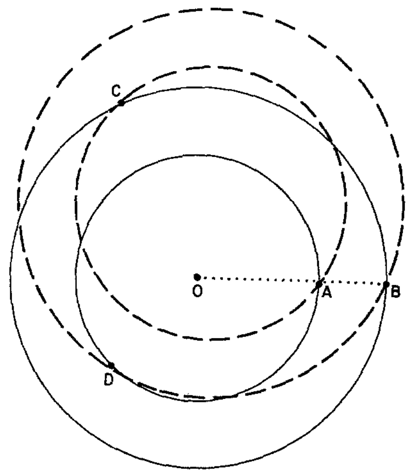

Fig. 4

the inner circle contains A and C and the outer circle contains B and D. (See Fig. 4.) (Suppose for the moment that AC is not parallel to BD.) The line segment AB joins the inner to the outer circle of this new annulus, and since the length of a line segment joining the inner to the outer circle of an annulus is minimal, if and only if, the segment is radial, the radius of the new annulus is less than $|AB|$ — which is impossible ($|AB|$ being the radius of a best annulus) — unless A, B, C, D are collinear, in which case the hypothesis of Theorem 3 is not satisfied, in contradiction to our assumption. If AC is parallel to BD (see Fig. 5) then S is contained in a strip whose opening is less than $|AB|$ leading to a contradiction of our assumption that the original annulus was a best approximation to S.

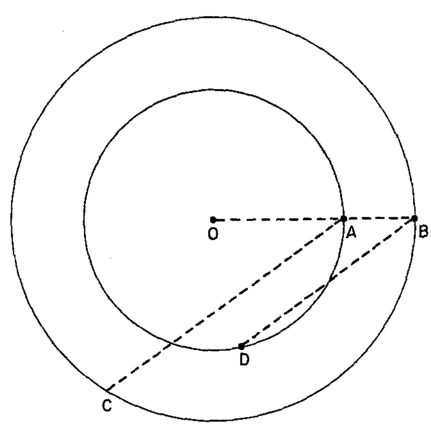

Fig. 5

Thus we have

Theorem 4: *If S consists of* 4 *points the two points of S on the outer circle of an annulus of best uniform approximation to S interlace angle-wise with the two points of S on the inner circle, as viewed from the center of the annulus.*

Proof: If the 4 points are concyclic the theorem is certainly true. Suppose the 4 points are not concyclic. An annulus of best uniform approximation has 2 points on its inner circle and two on its outer circle according to Theorem 2, hence the hypothesis of Theorem 3 is in effect, and as we have just seen, $k=4$, which proves the theorem.

Remark 1: A set of 4 points can fail to have an annulus of best approximation only if the four points are collinear or are the vertices of a (convex) quadrilateral with a pair of opposite sides parallel, in view of the Corollary to Theorem 2. For example, the points $(0, \pm 1)$, $(\pm 4, 0)$ have no annulus of best approximation.

Remark 2: If one of 4 points is in the convex hull of the other three a best annular approximation exists and it is only necessary to consider the 3 annuli determined by the 3 pairs of pairs of points. It is easy to see that the "inside" point is on an inner circle in each of the three cases.

Finally, we remark that the interlacing condition in Theorem 4 is not sufficient for best annular approximation to 4 points. This can be seen in the case of the points $(0,0)$, $(2,0)$, $(4,0)$, $(0,1)$. The best annular approximation is centered at $(2,5/2)$. The annulus centered at $(1, -7/2)$ is not best although the interlacing property obtains with respect to this center. However, the radius $r(w)$ has a *local* minimum at this point.

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