LIE SUPERALGEBRAS

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Results pertaining to the theory of representations of "classical" Lie superalgebras are collected in the survey.

PREFACE

A new area of mathematics — the theory of supermanifolds — arose in the 1970s. Its rapid growth was stimulated by fantastic prospects in physics: the possibilities of combining bosons and fermions into a single multiplet, of combining groups of inner and dynamical symmetries, and, finally, of combining all fundamental forces into a single field theory (see [13, 45, 47, 49, 50, 59, 62, 64, 90]). Moreover, in 1982 it was found that it was possible to formulate a model of field theory not containing singularities in the language of supersymmetries. An introduction to the theory of supermanifolds is presented in [5, 6, 32]. For an improved and corrected exposition and a list of some problems pertaining to this survey see [39, 45, 46]. The part of the theory of supermanifolds which now finds the greatest number of applications is the theory of Lie supergroups and superalgebras. Here we shall give a brief survey of results concerning the theory of representations of "classical" Lie superalgebras. For facts from linear algebra on superspaces see [32, 39]. We assume that the elements of the theory of representations of Lie algebras are known (see [12, 15, 21]).

The basic features of the theory of representations of simple Lie superalgebras make them kindred to Lie algebras in characteristic p, while if p = 2 there is almost no difference between algebras and superalgebras (see [113]). In particular, there is no complete reducibility and the Laplace—Casimir operators, which are of great help in describing representations of Lie algebras, play a modest role in the case of superalgebras [4, 82-86]. Methods from the theory of representations of infinite-dimensional Lie algebras of vector fields — the special vectors of Rudakov [53, 54] and analogues of Poincare's lemma — occupy center stage. By means of these methods it was possible, at least in principle, to determine how to solve the problem of O. Veblen on describing invariant differential operators acting on tensor fields on a manifold [22, 23], to refine it, to greatly generalize it, and in some cases to obtain a complete answer (see [9-11, 14, 26-29, 33-38, 65-72]).

0. Recollections

Regarding Algebras. As usual, we write **C**, **R**, **Z**, $\mathbf{Z}^+$, **N**, **H** and **O** for the complex numbers, real numbers, integers, nonnegative integers, positive integers, quaternions, and Cayley numbers, respectively. We denote by $|S|$ the power of a set S and by $\langle S\rangle$ the linear space generated by the set S. The base field is **C**.

Any finite-dimensional Lie algebra over C is the semidirect sum of a semisimple algebra and a maximal solvable ideal, while the semisimple algebra is the direct sum of simple algebras. The simple Lie algebras form the 3 classical series $\mathfrak{sl}$, $\mathfrak{o}$ and $\mathfrak{sp}$ and 5 exceptional Lie algebras.

All simple Lie algebras have the same structure. The Cartan subalgebra $\mathfrak{h}$ in a simple Lie algebra $\mathfrak{g}$ (i.e., the maximal nilpotent subalgebra coinciding with its normalizer) is commutative, and all Cartan subalgebras are conjugate relative to the action of the adjoint group. The Cartan subalgebra $\mathfrak{h}$ prescribes an $\mathfrak{h}^*$-gradation in $\mathfrak{g}=\bigoplus_{\alpha\in R}\mathfrak{g}_\alpha$ and in finite-dimensional $\mathfrak{g}$-modules $M=\bigoplus_{\alpha\in P}M_\alpha$, whereby $\dim\mathfrak{g}_\alpha=\dim\mathfrak{g}_{-\alpha}$, where $\alpha\in\mathfrak{h}^*$. The elements of the sets $R, P\subset\mathfrak{h}^*$ are called the roots and weights, respectively.

The $\mathfrak{h}$-gradation in $\mathfrak{g}$ can be extended to a natural Z-gradation (in $\mathfrak{sl}(n)$ the degree of an

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element g is the number of the diagonal on which g lies over the main, zeroth diagonal), and it is possible to choose elements $X_i^{\pm}$, where $1 \leqslant i \leqslant \operatorname{rk} \mathfrak{g}$ of degree ±1, which generate $\mathfrak{g}$, whereby if $H_i = [X_i^+, X_i^-]$, then

$$[X_i^+, X_j^-] = \delta_{ij} H_i, \ [H_i, H_j] = 0,$$
$$(\operatorname{ad} X_i^{\pm})^{1-a_{ij}} (X_j^{\pm}) = 0. \qquad \text{(DR)}$$

The matrix (a_{ij}) is called the Cartan matrix and is conveniently assigned a Dynkin graph. The equations (DR) are the defining relations in $\mathfrak{g}$.

The weights are lexicographically ordered (relative to a fixed basis in $\mathfrak{h}$). In each finite-dimensional, irreducible module over a simple Lie algebra $\mathfrak{g}$ there is a 1-dimensional space (of leading vectors) corresponding to the highest (leading) weight.

The leading weight of a finite-dimensional module satisfies conditions that it be integral. For example, for $\mathfrak{sl}(n)$ the Cartan subalgebra consists of diagonal matrices; the index of the leading weight relative to the basis $\{e_{ii} - e_{i+1,i+1}\}$ must belong to Z^+. The leading weight uniquely determines the irreducible modules; in particular, on the basis of χ it is possible to compute the character of the $\mathfrak{g}$-modules $L(\chi)$ with leading weight χ, i.e., the function $\operatorname{ch} M(\chi) = \sum_{\lambda \in P} \dim L(\chi)_\lambda e^\lambda$, where $L(\chi)_\lambda$ is the eigensubspace of the weight λ and $e^\lambda(h) = e^{\lambda(h)}$ for $h \in \mathfrak{h}$. Let $W(\mathfrak{g})$ be the Weyl group of the Lie algebra $\mathfrak{g}$, i.e., the group generated by reflections in hyperplanes of the space $\mathfrak{h}$ given by the roots. Then the following formula of H. Weyl holds:

$$\operatorname{ch} M(\chi) = \sum_{w \in W} \operatorname{sgn} w e^{w(\chi+\rho)} \Big/ \sum_{w \in W} \operatorname{sgn} w e^{w\rho} = \prod_{\alpha \in R^+} (1 + e^{-\alpha})^{\dim \mathfrak{g}_\alpha} \sum_{w \in W}' \operatorname{sgn} w e^{w\chi}, \text{ where } \rho(H_i) = 1.$$

It was found that finite-dimensional representations of simple Lie algebras are most simply described within the framework of the category $\mathcal{O}$ consisting of infinite-dimensional modules satisfying some natural conditions [15, 93]. An analogous category of modules can also be defined over Kac–Moody algebras. It is composed of nontrivial central extensions of Kac algebras consisting of the following two series of infinite-dimensional simple Lie algebras:

1) algebras of currents or loops $\mathfrak{g}^{(1)} = \mathfrak{g} \otimes \mathbf{C}[t, t^{-1}]$ (the first name came from physicists, while the second is explained by the fact that $\mathfrak{g}^{(1)}$ = {mappings of the circle $S^1 \to \mathfrak{g}$ which can be expanded in a Fourier series where $\mathfrak{g}$ is a simple, finite-dimensional Lie algebra });

2) the Lie algebras $\mathfrak{g}_\varphi^{(m)} = \bigoplus_{k,j} \mathfrak{g}_j t^{mj+k}$ where $k = 0, \ldots, m-1$; $j \in Z$, φ is an outer automorphism of order m of the simple, finite-dimensional Lie algebra $\mathfrak{g}$, and $\mathfrak{g}_j = \{g \in \mathfrak{g} \mid \varphi(g) = e^{2\pi i j/m} g\}$.

For a survey of Kac–Moody algebras see [110]. In particular, simple Kac algebras are given by the formula (DR) with an extended Cartan matrix (Dynkin graph), and for irreducible modules over them an analogue of H. Weyl's formula holds with the necessary alteration in the definition of W and restrictions on the leading weight.

In 1966 Kac distinguished an important class of Lie algebras related to the most different areas of mathematics and physics: simple Z-graded algebras of finite growth. (We recall that the growth or Gel'fand–Kirillov dimension of a Z-graded algebra A is $\lim_{n\to\infty}$((log $\dim \bigoplus_{|i|<n} I = \bigoplus_i I \cap A_i$, while simplicity of the algebra A means that there are no graded ideals).

Conjecture [107, 109, 110]. The simple Z-graded Lie algebras of finite growth over C are:

1) simple, finite-dimensional algebras;

2) Kac algebras;

3) Lie algebras of formal vector fields of types W, S, H, K;

4) the Witt algebra $W = \operatorname{Der} \mathbf{C}[t, t^{-1}]$.

The structure of the Lie algebras of these 4 classes is very different, which is naturally reflected in the theory of representations. The simplest representations — irreducible representations — have been best studied, while for infinite-dimensional algebras the choice of the class of representations from which we extract irreducible representations is itself nontrivial [15, 80, 89, 90].

It turns out that finite-dimensional representations of finite-dimensional simple Lie algebras are completely reducible, i.e., it suffices to study irreducible modules. This and a large part of the other results regarding Lie algebras and modules over them were obtained by means of (co)homology theory (see [64]). For irreducible objects L_χ of the category $\mathcal{O}$ it is possible to construct a resolution of Verma modules, i.e., of modules induced from the character of the maximal solvable (or parabolic) Lie subalgebra $\mathfrak{n}_+$, and by means of this resolution we compute the cohomology of the Lie algebra $\mathfrak{n}_+$ with coefficients in L_χ. Here a major role is played by the elements of the center $Z(U(\mathfrak{g}))$ of the universal enveloping Lie algebra $\mathfrak{g}$ (the Casimir operators), and frequently a single quadratic operator is sufficient which can be canonically constructed both for a finite-dimensional Lie algebra and for a Kac algebra (see [93, 110]).

It recently became clear that the objects of the category $\mathcal{O}$ are conveniently studied within the framework of a broader approach — as part of the theory of sheaves of modules over rings of differential operators [80, 90]. This approach has so far been developed only for finite-dimensional Lie algebras, but it has yielded very strong results, among which is a proof of the Kazhdan–Lusztig conjecture regarding the structure of Verma modules.

Simple Lie algebras $\mathcal{L}$ of formal vector fields have a structure quite different from Lie algebras of types 1) and 2). In particular, for them $Z(U(\mathcal{L}))=C$. In these Lie algebras $\mathcal{L}$ there is a natural (Weisfeiler) filtration of the form

$$\mathcal{L}=\mathcal{L}_{-d}\supset\ldots\supset\mathcal{L}_0\supset\mathcal{L}_1\supset\ldots,$$

where $\mathcal{L}_0$ is the unique maximal subalgebra of finite codimension d = 1 or 2, and in the adjoint graded simple Lie algebra $L=\oplus L_i$, where $L_i=\mathcal{L}_i/\mathcal{L}_{i+1}$, there are no generators of degree ±1. Nevertheless, setting $\mathfrak{n}_\pm=\underset{i\geqslant 0}{\oplus}L_i$ it is possible to compute the defining relations in $\mathfrak{n}_\pm$; these are $H_2(\mathfrak{n}_\pm)$ (they are naturally considered as an L_0-module). For $\mathfrak{n}_-$ this computation is trivial, while for $\mathfrak{n}_+$ it is more involved. It was found that if the dimension of the space on which $\mathcal{L}$ is realized as a space of vector fields is greater than a certain amount, then the degree of all elements of $H_2(\mathfrak{n}_+)$ is equal to 2, i.e., all relations are "trivial." For part of these results see [63]; the cases of low dimensions have also all been treated.

Rudakov [54, 55] developed the theory of representations of Lie algebras of formal vector fields. He and Kostrikin [26, 27] described all irreducible modules over these Lie algebras in two natural classes of modules — spaces with discrete or linear-compact topology. It was hereby found that all such irreducible representations are produced (or induced) from finite-dimensional representations of Lie algebras of linear vector fields with the exception of representations in the space of differential forms, which is related to the existence of an exterior differential. It was also found that the problem of describing invariant differential operators is part of the problem of describing irreducible modules over Lie algebras of vector fields and the problem of resolving a tensor product of modules over these algebras into irreducible components [11]. For finite-dimensional modules over finite-dimensional simple algebras resolutions of a tensor product of irreducible modules into irreducible components are known [12]. If the algebra or module is infinite-dimensional, then the problem abruptly becomes more complicated [14].

Modules of tensor fields over Lie algebras $L=\oplus L_i$ of types 3) and 4) have finite functional dimension. Over these algebras there are further the modules $M(V)=\mathrm{Hom}_{U(\mathcal{L}_-)}(U(\mathcal{L}),V)$, where $\mathcal{L}_-=\underset{i<0}{\oplus}L_i$, and V is a finite-dimensional L_0-module having infinite functional dimension. Irreducible representations in the modules M(V) have been described only for the algebra W (see [61]).

Moreover, Kac and Witt algebras have other natural representations: in tensor fields on the circle (to these belong such an important representation as the adjoint representation), and their central extensions have spinor representations with which Feigin and Fuks associated semiinfinite forms [44, 61].

It is important to note that in applications not only and not so much are simple Lie algebras of types 1)-4) of interest as their "derivatives" — central extensions, deformations, Lie algebras of differentiations, nilpotent and solvable subalgebras, and also real forms of these Lie algebras and modules over them.

Finally, irreducible finite-dimensional modules over finite-dimensional solvable Lie algebras are described by Lie's theorem.

On Superspaces. The elements of the field $\mathbf{Z}_2$ — the residue field modulo 2 — are denoted by $\bar{0}$ and $\bar{1}$ in order to distinguish them from the elements of the ring $\mathbf{Z}$. We set $(-1)^{\bar{0}} = 1$ and $(-1)^{\bar{1}} = -1$. In working with $\mathbf{Z}_2$-graded objects it is useful to remember the following rule of signs: "when something of parity p is moved past something of parity q the sign $(-1)^{pq}$ jumps out," while it suffices to define formulas only on homogeneous (relative to the $\mathbf{Z}_2$-gradation) elements and extend by linearity to other elements. This rule makes it possible to immediately "super" the definition of a Lie algebra, commutativity, the Leibniz rule, etc.

A linear space V is called a superspace if it is equipped with a $\mathbf{Z}_2$-gradation, i.e., an expansion $V = V_{\bar{0}} \oplus V_{\bar{1}}$. The nonzero elements of the spaces $V_{\bar{0}}$ and $V_{\bar{1}}$ are called homogeneous (even and odd respectively) elements of the superspace V. If $v \in V_i$, where $i \in \mathbf{Z}_2$ and $v \neq 0$, then we write $p(v) = i$ and call $p(v)$ the parity of the element v. A subsuperspace is a $\mathbf{Z}_2$-graded subspace $W \subset V$ such that $W_i = W \cap V_i$. Let V and W be superspaces. The structure of a superspace on $V \oplus W$, $V \otimes W$, and $\mathrm{Hom}(V, W)$ is introduced in the natural manner. We denote by $\pi(V)$ the superspace defined by the formulas $\pi(V)_i = V_{i+\bar{1}}$; its elements are noted by $\pi(v)$ where $v \in V$. A superalgebra is a superspace A equipped with an even homomorphism mult: $A \otimes A \to A$. A morphism of superalgebras $\varphi : A \to B$ is an even algebra homomorphism of A into B.

Examples of Commutative Superalgebras. 1) The superalgebra $\mathbf{C}[x]$ consists of polynomials in the variables $x = (u, \xi)$, where $u = (u_1, \ldots, u_n)$ are even and $\xi = (\xi_1, \ldots, \xi_m)$ are odd, with the relations

$$x_i x_j = (-1)^{p(x_i)p(x_j)} x_j x_i, \quad 1 \leqslant i j \leqslant n+m.$$

For n = 0 the superalgebra $\mathbf{C}[x]$ is finite-dimensional. It is called the Grassmann superalgebra, which we also denote by $\Lambda(\xi)$ or $\Lambda(m)$.

2) The center Z(A) of an associative superalgebra A consists of the set of its elements which commute with any element of A.

3) If V is a superspace, then T(V) denotes the tensor algebra of the superspace V, i.e., the superspace $\bigoplus_{i \geqslant 0} T^i(V)$ where $T^i(V) = V \otimes \ldots \otimes V$ (i times) for $i > 0$, while $T^0(V) = \mathbf{C}$, with the natural $\mathbf{Z}$-gradation, which is called the degree, and with multiplication given by the formula $v \cdot w = v \otimes w$, where $v, w \in T(V)$. The symmetric algebra S(V) of a superspace V is defined as the factor of the superalgebra T(V) by the ideal generated by the elements $v \otimes w - (-1)^{p(v)p(w)} w \otimes v$, where $v, w \in V$. The exterior algebra E(V) of the superspace V is the superalgebra $S(\Pi(V))$. If C is a commutative superalgebra and V is a superspace, then we set $C[V] = C \otimes_{\mathbf{C}} S(V)$. It is obvious that if the superalgebras A and B are commutative, then $A \otimes B$ is a commutative superalgebra.

A left module over an associative superalgebra A is a superspace M with an even homomorphism a_l: $A \otimes M \to M$ such that $a(bm) = (ab)m$, where $a, b \in A$, $m \in M$.

If M is an A-module, then the structure of an A-module in the superspace $\Pi(M)$ is introduced by the formula $a(\pi(m)) = (-1)^{p(a)}\pi(am)$ where $a \in A$, $m \in M$.

A Lie superalgebra is a superalgebra $\mathfrak{g}$ with multiplication denoted usually by [,] which satisfies (rule of signs!) the conditions

$$[x, y] = -(-1)^{p(x)p(y)} [y, x],$$
$$[x, [y, z]] = [[x, y], z] + (-1)^{p(x)p(y)} [y, [x, z]].$$

Below we shall encounter only Lie superalgebras or associative superalgebras with a 1. We shall call the latter simply superalgebras.

We note the important circumstance that modules over commutative superalgebras are two-sided. Namely, suppose M is a left (respectively, right) module over a commutative superalgebra C. We convert it into a right (respectively, left) module by setting $mc = (-1)^{p(m)p(c)} cm$ where $c \in C$, $m \in M$ (see also [39]).

The universal enveloping algebra $U(\mathfrak{g})$ of a Lie superalgebra $\mathfrak{g}$ is defined by the rule of signs and satisfies the obvious analogue of the Poincare–Birkhoff–Witt theorem [15].

A (left) module over a Lie superalgebra $\mathfrak{g}$ is a superspace M equipped with a $\mathfrak{g}$-action, i.e., a homomorphism $\mathfrak{g} \otimes M \to M$ denoted simply $(g, m) \mapsto gm$, such that $[g_1, g_2]m = g_1(g_2m) - (-1)^{p(g_1)p(g_2)} g_2(g_1m)$, where $g_1, g_2 \in \mathfrak{g}$, $m \in M$.

A homomorphism of A-modules (over a superalgebra or Lie superalgebra A) or an A-invariant mapping is a homomorphism $F: M \to N$ such that $F(am) = (-1)^{p(a)p(F)} aF(m)$ where $a \in A$, $m \in M$. We denote the superspace of such homomorphisms by $\mathrm{Hom}_A(M, N)$. If A is a commutative superalgebra, then, setting $(Fa)m = F(am)$, we convert $\mathrm{Hom}_A(M, N)$ into an A-module.

We note that if M is a module over a Lie superalgebra $\mathfrak{g}$, then M is equipped with a natural structure of a $U(\mathfrak{g})$-module.

Let C be a commutative superalgebra, and let M be a module over C. Let $I = I_0 \cup I_1$ be a set consisting of the union of nonintersecting subsets I_0 and I_1. A basis of the C-module M is a collection of homogeneous elements $m_i \in M$ such that $p(m_i) = \bar{0}$ if $i \in I_0$ and $p(m_i) = \bar{1}$ if $i \in I_{\bar{1}}$, whereby each element $m \in M$ can be written uniquely in the form $m = \Sigma c_i m_i$, where $c_i \in C$, and all c_i except a finite number are equal to zero. A module M over C is called free if it has a basis. The dimension of a free module takes its value in the ring $Z[\varepsilon]/(\varepsilon^2 - 1)$. It is defined by the formula $\dim M = p + \varepsilon q$ where $p = |I_0|$, $q = |I_1|$. In particular, for modules over a field there is the formula $\dim M = \dim M_{\bar{0}} + \varepsilon \dim M_{\bar{1}}$. The motivation for this definition comes from K-theory. Usually we write for brevity $\dim M = (p, q)$ or $p|q$. The dimension of a free module does not depend on the choice of basis.

Let $\{m_i\}$ and $\{n_j\}$ be bases of the C-modules M and N, respectively. To each operator $F: M \to N$ we assign the matrix ${}^mF = ({}^mF_{ij})$, where $Fm_i = \sum_j n_j({}^mF_{ij})$.

A supermatrix structure (or simply a matrix) is a matrix with a prescribed parity for each row and column. Usually a supermatrix structure will be chosen so that even rows and columns go first and odd rows and columns after. (This structure is called a standard structure.) We denote the parity of the i-th row by $p_{row}(i)$ and the parity of the j-th column by $p_{col}(j)$.

For a matrix ${}^mF$ we set $p_{row}(j) = p(n_j)$, $p_{col}(i) = p(m_i)$.

If a matrix contains r even and s odd rows and p even and q odd columns, then we say that the dimension of the matrix is equal to $(r, s) \times (p, q)$. The order of a matrix of dimension $(p, q) \times (p, q)$ is the pair (p, q). The set of matrices of order (p, q) with elements in a superalgebra A we denote by $\mathrm{Mat}(p|q; A)$.

In $\mathrm{Mat}(p|q; A)$ we introduce the structure of a superspace by setting for a matrix $X = \begin{pmatrix} R & S \\ T & U \end{pmatrix}$

$$p(X) = \bar{0}, \quad \text{if} \quad p(R_{ij}) = p(U_{rs}) = \bar{0}, \quad p(S_{is}) = p(T_{ij}) = \bar{1};$$
$$p(X) = \bar{1}, \quad \text{if} \quad p(R_{ij}) = p(U_{rs}) = \bar{1}, \quad p(S_{is}) = p(T_{ij}) = \bar{0},$$

Let $X: M \to M$ and $Y: N \to N$ be two (even) automorphisms of the modules M and N with matrices ${}^mX$ and ${}^mY$, respectively (in bases $\{m_i\}$ and $\{n_j\}$, respectively). Then the matrix $({}^mF)'$ of the operator F defined relative to the bases $\{Xm_i\}$ and $\{Yn_j\}$ can be expressed in terms of the matrix ${}^mF$ by the formula

$$({}^mF)' = ({}^mY)^{-1} \cdot {}^mF \cdot {}^mX.$$

The module $M^* = \mathrm{Hom}_C(M, C)$ over a commutative superalgebra C is called adjoint or dual to the module M. The pairing of the modules M^* and M we denote by $(,)$, i.e., (m^*, m) is the image in C of the element $m \in M$ under the action of the functional $m^* \in M^*$.

To each operator $F \in \mathrm{Hom}_C(M, N)$ there corresponds an adjoint operator $F^* \in \mathrm{Hom}_C(N^*, M^*)$ defined by the formula

$$(F^*n^*, m) = (-1)^{p(n^*)p(F)} (n^*, Fm).$$

Let $\{m_i^*\}$ and $\{n_j^*\}$ be bases in M^* and N^* dual to the bases $\{m_i\}$ and $\{n_j\}$ of the modules M and N, i.e., $(m_i^*, m_l) = \delta_{il}$ and $(n_j^*, n_s) = \delta_{js}$. It follows from the definitions that the matrix of the operator F^* in the bases $\{m_i^*\}$ and $\{n_j^*\}$ has the form $({}^mF)^{st}$, where the supertranspose st is defined by the formulas (for $X = \begin{pmatrix} R & S \\ T & U \end{pmatrix}$)

$$X^{st}=\begin{pmatrix} R^t & T^t \\ -S^t & U^t\end{pmatrix}, \quad \text{if} \quad p(X)=\bar{0},$$

$$X^{st}=\begin{pmatrix} R^t & -T^t \\ S^t & U^t\end{pmatrix}, \quad \text{if} \quad p(X)=\bar{1}.$$

We note that supertransposition has order 4.

Let M and N be free modules over a commutative superalgebra C. A bilinear form is a mapping $B:M \times N \to C$ linear in each argument such that $B(mc, n) = B(m, cn)$, $B(m, nc) = B(m, n)c$, where $m\in M$, $n\in N$, $c\in C$.

We denote the superspace of bilinear forms by $Bil_C(M, N)$ or simply $Bil_C(M)$, if $M = N$. If $\{m_i\}$ and $\{n_j\}$ are bases of the modules M and N respectively, then the matrix of the form B is the matrix ${}^mB$, where $({}^mB)_{ij} = (-1)^{p(m)p(B)}B(m_i, n_j)$.

Let $X:M \to M$ and $Y:N \to N$ be two (even) automorphisms of the modules M and N with matrices ${}^mX$ and ${}^mY$, respectively. Then the matrix $({}^mB)'$ of the form B relative to the bases $\{Xm_i\}$ and $\{Yn_j\}$ has the form

$$({}^mB)'=({}^mX)^{st}({}^mB)({}^mY).$$

Let $u:Bil_C(M, N) \to Bil_C(N, M)$ be the inversion of bilinear forms given by the formula

$$B^u(n, m)=(-1)^{p(n)p(m)}B(m, n).$$

In terms of matrices, u is given by the following formula. Let $({}^mB) = \begin{pmatrix} R & S \\ T & U\end{pmatrix}$. Then

$$({}^mB^u)=\begin{pmatrix} R^t & T^t \\ S^t & -U^t\end{pmatrix}, \quad \text{if} \quad p(B)=\bar{0}$$

and

$$({}^mB^u)=\begin{pmatrix} R^t & -T^t \\ -S^t & -U^t\end{pmatrix}, \quad \text{if} \quad p(B)=\bar{1}.$$

A bilinear form $B \in Bil_C(M)$ is called (skew) symmetric if $B^u = (-)B$. We assign to a bilinear form $B \in Bil_C(M)$ it itself but considered as an element of the superspace $Bil_C(\pi(M))$. Under this correspondence symmetric forms become skew-symmetric, and conversely.

1. Classical Lie Superalgebras over C

1. Matrix Lie Superalgebras. The Lie superalgebras $\mathfrak{gl}(m|n)=\mathrm{Mat}(m|n, \mathbf{C})$ and $\mathfrak{sl}(m|n)=\{X\in\mathfrak{gl}(m|n)\,|\,\mathrm{str}\,X=0\}$, where $\mathrm{str}\begin{pmatrix} A & B \\ C & D\end{pmatrix} = \mathrm{tr}\,A - \mathrm{tr}\,D$, are called the general and special linear Lie superalgebras. The Lie superalgebra $\mathfrak{q}(n)=\{X\in\mathfrak{gl}(n|n)\,|\,[X, J_{2n}]=0\}$, where $J_{2n} = \begin{pmatrix} 0 & 1_n \\ -1_n & 0\end{pmatrix}$, preserves the complex structure given by the odd operator J_{2n}. This Lie superalgebra is called the general strange algebra, and its Lie subalgebra $\mathfrak{sq}(n)=\{X\in\mathfrak{q}(n)\,|\,\mathrm{otr}\,X=0\}$, where $\mathrm{otr}\begin{pmatrix} A & B \\ B & A\end{pmatrix} = \mathrm{tr}\,B$, is called the special strange subalgebra.

If d is a Lie algebra of scalar matrices, and $\mathfrak{g}\subset\mathfrak{gl}(n|n)$ is a Lie subsuperalgebra containing d, then the projective Lie superalgebra of type $\mathfrak{g}$ is $\mathfrak{pg}=\mathfrak{g}/d$. Projectivization sometimes leads to new Lie superalgebras: $\mathfrak{pgl}(n|n)$, $\mathfrak{psl}(n|n)$, $\mathfrak{pq}(n)$, $\mathfrak{psq}(n)$.

Let $B_{m,2n} = \mathrm{diag}(\sigma_m, J_n)$, where $\sigma_m = \mathrm{antidiag}(1, \ldots, 1)$ (m times). The Lie superalgebra

$$\mathfrak{osp}(m|2n)=\{X\in\mathfrak{gl}(m|2n)\,|\,X^{st}B_{m,2n}+B_{m,2n}X=0\},$$

preserving the even nondegenerate bilinear form with matrix $B_{m,2n}$, is called the orthogonal-symplective algebra, while the Lie superalgebra

$$\Pi(n)=\{X\in\mathfrak{gl}(n|n)\,|\,X^{st}J_{2n}+(-1)^{p(X)}J_{2n}X=0\},$$

preserving the odd nondegenerate bilinear form with matrix J_{2n}, is called the Palamodov algebra. Let $S\Pi(n)=\{X\in\Pi(n)\,|\,\mathrm{str}\,X=0\}$.

2. Exceptional Lie Superalgebras (See [103, 112, 127, 130]). $D(\alpha)$ is a deformation of the Lie superalgebra $\mathfrak{osp}(4|2)$. We identify $D(\alpha)_{\bar{0}}$ with $\mathfrak{sl}(2)\oplus\mathfrak{sl}(2)\oplus\mathfrak{sl}(2)=\{(a_1, a_2, a_3)\}$, and $D(\alpha)_{\bar{1}}$ with $\mathrm{id}_1\otimes\mathrm{id}_2\otimes\mathrm{id}_3$ where id_j is the identity representation of the j-th copy of $\mathfrak{sl}(2)$. Let ψ be an invariant form on id with matrix J_1. We define the mapping $p:\mathrm{id}\otimes\mathrm{id}\to\mathfrak{sl}(2)$ by the formula

$p(u, v)w = \lambda(\psi(v, w)u - \psi(w, u)v)$, where $\lambda \in C$. We define $[,]: S^2 D(\alpha)_{\bar 1} \to D(\alpha)_{\bar 0}$ by the formula

$$[u_1 \otimes u_2 \otimes_3, v_1 \otimes v_2 \otimes v_3] \mapsto \sum_{(i,j,k)=\sigma(1,2,3), \sigma \in S_3} \psi_i(u_i, v_i)\psi_j(u_j, v_j) p_k(u_k, v_k)$$

Exercise. 1) The Jacobi identity holds if and only if $\lambda_1 + \lambda_2 + \lambda_3 = 0$.

2) The numbers $(\lambda_1, \lambda_2, \lambda_3)$ are defined up to permutation and a common factor. We set $\alpha = \lambda_1/\lambda_2$. For $\alpha = 0, -1, \infty$ the Lie superalgebra $D(\alpha)$ is not simple and is simple for $\alpha \neq 0, -1, \infty$.

3) $D(\alpha) = D(\alpha')$ if $\alpha' = \alpha^{-1}$ or $\alpha' = -\alpha/(\alpha + 1)$. In the Lie superalgebras $D(-1)$ and $D(0)$ there are the ideals $\mathfrak{psl}(2|2)$ and $\mathfrak{sl}(2)$ respectively, while $D(-1)/\mathfrak{psl}(2|2) \cong \mathfrak{sl}(2)$, and $D(0)/\mathfrak{sl}(2) \cong \mathfrak{psl}(2|2)$.

AG_2: Let O be the algebra of Cayley numbers (octonions), and let $G_2 = \mathrm{Der}\, O$. The form $(x, y) = xy + yx$ where $x, y \in O$ is nondegenerate and G_2-invariant on O and $O^\circ = O/\mathbb{R}\cdot 1$.

Let $L_x(y) = xy$, $R_x(y) = yx$. Then the formula $D_{x,y} = [L_x, L_y] + [R_x, R_y] + [L_x, R_y]$, where $x, y \in O$, gives a G_2-invariant relation $D: O \otimes O \to G_2$. Let $D^\circ = D/O^\circ$. We set $(AG_2)_{\bar 0} = \mathfrak{sl}(2) \oplus G_2$, $(AG_2)_{\bar 1} = \mathrm{id} \otimes O^0$ and define $[,]: S^2(AG_2)_{\bar 1} \to (AG_2)_{\bar 0}$ by the formula

$$[x \otimes u, y \otimes v] = (x, y)\, p(u, v) - \psi(u, v) D^0_{x,y},$$

where p and ψ were defined in the description of $D(\alpha)$.

AB_3: We set $(AB_3)_{\bar 0} = \mathfrak{sl}(2) \oplus \mathfrak{o}(7)$, $(AB_3)_{\bar 1} = \mathrm{id} \otimes \mathfrak{spin}_7$, see [12]. The mapping $[,]: S^2(AB_3)_{\bar 1} \to (AB_3)_{\bar 0}$ we define by the formula

$$[\Gamma_1 \otimes u_1, \Gamma_2 \otimes u_2] = (\Gamma_1, \Gamma_2) \otimes p(u_1, u_2) + \sum_{j,k} \psi(u_1, u_2)(\Gamma_1, \gamma_j \gamma_k \Gamma_2)(E_{jk} - E_{kj}),$$

where p and ψ were defined in the description of $D(\alpha)$, and $(,)$ is an invariant form on $\mathfrak{spin}_7$, $\gamma_i\gamma_j + \gamma_j\gamma_i = \delta_{ij}$ for $1 \leqslant i, j \leqslant 7$.

We shall describe the rough structure of some Lie superalgebras. Let $\mathrm{id}(\langle 1 \rangle)$ be the standard (trivial) representation, and let $\mathfrak{cg}$ be the trivial central extension of the Lie algebra $\mathfrak{g}$. We call the following Z-gradations standard:

$\mathfrak{g}$	$\mathfrak{g}_{-2}$	$\mathfrak{g}_{-1}$	$\mathfrak{g}_0$	$\mathfrak{g}_1$	$\mathfrak{g}_2$
$\mathfrak{sl}(n\|m)$		$\mathrm{id} \otimes \mathrm{id}^*$	$\mathfrak{sl}(n) \oplus \mathfrak{gl}(m)$	$\mathrm{id}^* \otimes \mathrm{id}$	
$\mathfrak{osp}(n\|2m)$	$\langle 1 \rangle \otimes S^2 \mathrm{id}^*$	$\mathrm{id} \otimes \mathrm{id}^*$	$\mathfrak{o}(n) \oplus \mathfrak{sp}(2m)$	$\mathrm{id}^* \otimes \mathrm{id}$	$\langle 1 \rangle \otimes S^2 \mathrm{id}$
$\mathfrak{osp}(2\|2m)$		id^*	$\mathfrak{csp}(2m)$	id	
$(S)\pi(n)$		$\Lambda^2 \mathrm{id}^*$	$(\mathfrak{sl})\mathfrak{gl}(n)$	$S^2 \mathrm{id}$	
AG_2	$\langle 1 \rangle$	id^*	CG_2	id	$\langle 1 \rangle$
AB_3	$\langle 1 \rangle$	id^*	$\mathfrak{co}(4)$	id	$\langle 1 \rangle$

3. Lie Superalgebras of (Formal) Vector Fields. The presence in simple Lie superalgebras of vector fields of some maximal subalgebras of finite codimension is unexpected and surprising.

Standard Realizations. The Lie superalgebra $W(n|m) = \mathrm{Der}\, C[[x]]$, where $x = (u_1, \ldots, u_n, \xi_1, \ldots, \xi_m)$, is called the general Lie superalgebra of vector fields. The divergence of the field $D = \Sigma f_i \frac{\partial}{\partial u_i} + \Sigma g_j \frac{\partial}{\partial \xi_j}$ is the series $\mathrm{div}\, D = \Sigma \frac{\partial f_i}{\partial u_i} + \Sigma (-1)^{p(g_j)} \frac{\partial g_j}{\partial \xi_j}$. The Lie superalgebra

$$S(n|m) = \{D \in W(n|m) \mid L_D v_x = 0\} = \{D \in W(n|m) \mid \mathrm{div}\, D = 0\},$$

where v_x is the volume form with constant coefficients in the coordinates x [8, 39], is called a special or divergence-free Lie superalgebra.

Let

$$\alpha_1 = dt + \sum_{1<i<n} (p_i dq_i - q_i dp_i) + \sum_{1<j<m} \xi_j d\xi_j$$

(sometimes in place of α_1 it is more convenient to take

$$\tilde{\alpha}_1 = dt + \sum_{1<i<n}(p_i dq_i - q_i dp_i) + \sum_{1<i<r}(\xi_i d\eta_i + \eta_i d\xi_i)\ (+\theta d\theta)),$$

$$\alpha_0 = d\tau + \sum_{1<i<n}(q_i d\xi_i + \xi_i dq_i), \quad \omega_0 = d\alpha_1$$

$$(\text{ resp. } \quad \tilde{\omega}_0 = d\tilde{\alpha}_1), \ \omega_1 = d\alpha_0.$$

The Lie superalgebra

$$K(2n+1|m) = \{D \in W(2n+1|m) \mid L_D\alpha_1 = f_D\alpha_1\}$$

is called a contact superalgebra, while the Lie superalgebra

$$M(n) = \{D \in W(n|n+1) \mid L_D\alpha_0 = f_D\alpha_0\}$$

is called an odd contact superalgebra. The Lie superalgebra

$$\mathrm{Po}(2n|m) = \{D \in K(2n+1|m) \mid L_D\alpha_1 = 0\}$$

is called the Poisson superalgebra, while

$$B(n) = \{D \in M(n) \mid L_D\alpha_0 = 0\}$$

is the Butane superalgebra. The Lie superalgebras

$$SM(n) = \{D \in M(n) \mid \operatorname{div} D = 0\}, \quad SB(n) = \{D \in B(n) \mid \operatorname{div} D = 0\}$$

are called divergence-free odd contact and Butane Lie superalgebras.

The Lie superalgebras of the series K, M, SM, PO, B, SB are more conveniently given in terms of generating functions. For K(2n + 1/m) we set

$$K_{p_i} = p_i \frac{\partial}{\partial t} - \frac{\partial}{\partial q_i}, \quad K_{q_i} = q_i \frac{\partial}{\partial t} + \frac{\partial}{\partial p_i}, \quad K_{\xi_i} = \xi_i \frac{\partial}{\partial t} - \frac{\partial}{\partial \xi_i},$$

while for a series $f \in C[t, p, q, \xi]]$ we set

$$K_f = 2f \frac{\partial}{\partial t} + \sum_{i<n}(K_{p_i}(f)K_{q_i} - K_{q_i}(f)K_{p_i}) + (-1)^{p(f)} \sum_{j<m} K_{\xi_j}(f)K_{\xi_j}.$$

For any series $f \in C[[\tau, q, \xi]]$ we set

$$M_f = 2f \frac{\partial}{\partial \tau} + \sum_{i<n}(M_{\xi_i}(f) M_{q_i} + (-1)^{p(f)} M_{q_i}(f) M_{\xi_i}),$$

where $M_{q_i} = \frac{\partial}{\partial \xi_i} - q_i \frac{\partial}{\partial \tau}$, $M_{\xi_i} = \frac{\partial}{\partial q_i} + \xi_i \frac{\partial}{\partial \tau}$.

To the commutator of vector fields — elements of Lie superalgebras of the series K and M — there correspond contact brackets in the generating functions

$$\{f, g\}_{K.B.} = \Delta(f) \frac{\partial q}{\partial t} - \frac{\partial f}{\partial t} \Delta(g) - \{f, g\}_{P.B.}$$

and

$$\{f, g\}_{M.B.} = \Delta(f) \frac{\partial g}{\partial \tau} - (-1)^{p(f)} \frac{\partial f}{\partial \tau} \Delta(g) - \{f, g\}_{B.B.},$$

where $\Delta(f) = 2f - \Sigma y_i \frac{\partial f}{\partial y_i}$, y are all coordinates except t (respectively, τ), and the Poisson and Butane brackets are given, respectively, by the formulas

$$\{f, g\}_{P.B.} = \sum_{i<n}\left(\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i}\right) - (-1)^{p(f)} \sum_{j<m} \frac{\partial f}{\partial \xi_j}\frac{\partial g}{\partial \xi_j}$$

(in the realization with form ω_0) and

$$\{f, g\}_{B.B.} = \sum_{i<n}\left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial \xi_i} + (-1)^{p(f)} \frac{\partial f}{\partial \xi_i}\frac{\partial g}{\partial q_i}\right).$$

The Lie superalgebras of the Hamiltonian fields

$$H(2n|m) = \{D \in W(2n|m) \mid L_D\omega_0 = 0\}$$

and their analogues

$$\mathrm{Le}(n)=\{D\in W(n|n)\mid L_D\omega_1=0\} \text{ and } \mathrm{SLe}(n)=\{D\in \mathrm{Le}(n)\mid \operatorname{div} D=0\}$$

are conveniently given in terms of generating functions by setting

$$H_f=\sum_{i<n}\left(\frac{\partial f}{\partial p_i}\frac{\partial}{\partial q_i}-\frac{\partial f}{\partial q_i}\frac{\partial}{\partial p_i}\right)+(-1)^{p(f)}\sum_{j<m}\frac{\partial f}{\partial \xi_j}\frac{\partial}{\partial \xi_j},$$

$$\mathrm{Le}_f=\sum_{i<n}\left(\frac{\partial f}{\partial q_i}\frac{\partial}{\partial \xi_i}+(-1)^{p(f)}\frac{\partial f}{\partial \xi_i}\frac{\partial}{\partial q_i}\right).$$

Then $K(2n+1|m)=\langle K_f\rangle$, $M(n)=\langle M_f\rangle$, $H(2n|m)=\langle H_f\rangle$, $\mathrm{Le}(n)=\langle \mathrm{Le}_f\rangle$, $\mathrm{SLe}(n)=\langle \mathrm{Le}_f\mid \Sigma\frac{\partial^2 f}{\partial q_i\partial \xi_i}=0\rangle$, $\mathrm{SM}(n)=\langle M_f\left|\frac{\partial(\Delta(f)-f)}{\partial\tau}-\sum\frac{\partial^2 f}{\partial q_i\partial\xi_i}=0\right.\rangle$ where f runs through C[[x]].

We denote by $\mathrm{SLe}^0(n)=\mathrm{SLe}(n)/\langle \mathrm{Le}_{\xi_1\ldots\xi_n}\rangle$ and $S^0(n)=S(1|n)/\langle \xi_1\ldots\xi_n\frac{\partial}{\partial t}\rangle$ the ideals in SLe(n) and S(1|n), respectively.

Remarks. 1) It is obvious that the Lie superalgebras of the series W, S, H and Po for n = 0 are finite-dimensional.

2) A Lie superalgebra of the series H (respectively, Le and SLe) is a factor of the Lie superalgebra Po (respectively, B and SB) by the one-dimensional center Z which in the realization by generating functions consists of constants. We set

$$\mathrm{SPo}(m)=\left\{K_f\in \mathrm{Po}(0|m)\left|\int f\Delta_\xi=0\right.\right\},\quad \mathrm{SH}(m)=\mathrm{SPo}(m)/Z.$$

Nonstandard Realizations. Let $\mathscr{L}_0$ be a maximal subalgebra of finite codimension in a Lie superalgebra of formal vector fields $\mathscr{L}$. Let $\mathscr{L}_{-1}\subset\mathscr{L}$ be the minimal subsuperspace containing $\mathscr{L}_0$ which is $\mathscr{L}_0$-invariant. For i > 0 we set

$$\mathscr{L}_{i+1}=\{l\in\mathscr{L}_i\mid[l,\mathscr{L}_{-1}]\subset\mathscr{L}_i\},\quad \mathscr{L}_{-(i+1)}=[\mathscr{L}_{-1},\mathscr{L}_{-i}]\cup\mathscr{L}_{-i}.$$

We set $L_i=\mathscr{L}_i/\mathscr{L}_{i+1}$. The filtration by superspaces $\mathscr{L}_i$ is called with Weisfeiler filtration. We shall enumerate all Z-gradations in $\mathscr{L}$ associated with the Weisfeiler filtrations constructed on the basis of a maximal subsuperalgebra of finite codimension. The standard realization is labeled (*); we note that to it there corresponds the case where the codimension of the algebra $\mathscr{L}_0$ is minimal.

We set deg x = $^\circ x$ and note that the gradation in the series W (respectively, M or K) induces a gradation in the series S, S° (respectively, SM, Le, SLe, B, SB or Po, H). Suppose that the contact structure is given by the form $\tilde{\alpha}_1$. The Z-gradations are:

Lie superalgebra	Z-gradation
$W(n\mid m)$	$^\circ u_i={}^\circ\xi_i=1$ (*) $^\circ\xi_j=0$ for $1\leqslant j\leqslant r\leqslant m$, $^\circ u_i={}^\circ\xi_{r+j}=1$ for $j\geqslant 1$
$K(1\mid 2n)$	$^\circ t={}_0\xi_i=1$, $^\circ\xi_{n+i}=0$ for $1\leqslant i\leqslant n$
$M(n)$	$^\circ\tau=2$, $^\circ q_i={}^\circ\xi_i=1$ (*) $^\circ\tau={}^\circ u_i=1$, $^\circ\xi_i=0$ $^\circ\tau={}^\circ u_i=2$, $^\circ\xi_i=0$ for $1\leqslant i\leqslant r\leqslant n$. $^\circ u_{2+j}={}^\circ\xi_{r+j}=1$
$K(2n+1/m)$	$^\circ t=2$, $^\circ p_i={}^\circ q_i={}^\circ\xi_j=1$ (*) $^\circ t={}^\circ\xi_i=2$, $^\circ\xi_{r+i}=0$ for $1\leqslant i\leqslant r\leqslant[m/2]$ $^\circ p_i={}^\circ q_i={}^\circ\xi_{2r+j}=1$ for $j\geqslant 1$

The Lie superalgebras corresponding to nonstandard realizations we denote, respectively, by W(n|m; r); K(1|2n; 0); M(n; 0), M(n; r); K(2n + 1|m; r), where $r\geqslant 1$.

Digression: the Cartan and Shchepochkina Extensions. We recall that the Cartan extension of the pair $(\mathfrak{g}_{-1},\mathfrak{g}_0)$, where $\mathfrak{g}_{-1}$ is a module over the Lie superalgebra $\mathfrak{g}_0$, is the Lie superalgebra $(\mathfrak{g}_{-1},\mathfrak{g}_0)_*$ or briefly $\mathfrak{g}_*=\underset{i\geqslant -1}{\oplus}\mathfrak{g}_i$ where for i > 0

$$\mathfrak{g}_{i+1}=\{F\in \mathrm{Hom}(\mathfrak{g}_{-1},\mathfrak{g}_i)\mid F(v)(w)=(-1)^{p(v)p(w)}F(w)(v)\}$$

with the obvious commutation given by the imbedding $\mathfrak{g}_*\subset W(\dim\mathfrak{g}_{-1})$.

We shall now describe a generalization of the construction of the Cartan extension — the Shchepochkina extension. Let $\mathfrak{g}=\mathfrak{g}_{-1}\oplus\mathfrak{g}_0$ be a Lie superalgebra. Then a conformally $\mathfrak{g}_0$-invariant skew-symmetric form ω on $\mathfrak{g}_{-1}$ gives an extension $E\mathfrak{g}$ of depth 2 with center of dimension $\varepsilon^{p(\omega)}$ where

$$E\mathfrak{g}_0=\mathfrak{g}_0,\quad E\mathfrak{g}_{-1}=\mathfrak{g}_{-1},\quad E\mathfrak{g}_{-2}=\mathbf{C}y$$

and

$$[v, w]=w(v, w)\,y, \quad \text{if} \quad v, w\in E\mathfrak{g}_{-1}, \quad [u, y]=\alpha(u)\,y,$$

where $u\in E\mathfrak{g}_0$ and $\alpha(u)\omega = u\omega$.

Let ω be a nondegenerate form. Then $E\mathfrak{g}$ is contained in $K(\dim\mathfrak{g}_{-1})$ if $p(\omega) = \bar{0}$ or in $M(n)$, where $\dim\mathfrak{g}_{-1} = (n, n)$, if $p(\omega) = \bar{1}$. We call the maximal graded subalgebra $\mathfrak{h}$ of depth 2 in $K(\dim\mathfrak{g}_{-1})$ or $M(n)$ such that $\mathfrak{h}_i=E\mathfrak{g}_i$ the Shchepochkina extension of the pair $(\mathfrak{g}_{-1}, \mathfrak{g}_0)$ and write $(\mathfrak{g}_{-1}, \mathfrak{g}_0)^{MK}$ or simply $(\mathfrak{g}_0)^{MK}$.

4. Exceptional Lie Superalgebras (see [1, 73]) and Deformations. We set $W_*(n;\lambda) = (\Pi(T(\lambda)),\ W(0|n))_*$, $W_*(n)=W_*(n;0)$; $CW_*(n)=(\Pi(T(0))/\langle 1\rangle,\ CW(0|n))_*$. The Lie superalgebras $W_*(n;\lambda)$ are deformations of the Lie superalgebra $B(n)$ (to see this it is necessary to go over to the nonstandard realization in which $^{\circ}\xi_i = 0$) and are simple for $n > 1$ and $\lambda\neq 0$. It is obvious that $W_*(2;\lambda) = H(2|2;\lambda)$ (see [68]).

LEMMA. $CW_*(2)\cong W(2|1)$, while $CW_*(3)$ is an exceptional, simple Lie superalgebra. For $n > 3$ we have $CW_*(n) = W_*(n)\oplus \mathbf{C}z$ where $z|W_*(n)_i = i\cdot\mathrm{id}$.

We set 1) $\mathfrak{g}_0=CW(0|n)$, $\mathfrak{g}_{-1}=\Pi(T(-1/2))$, and suppose that z is an element of the center of the Lie superalgebra $\mathfrak{g}_0$, such that $z|\mathfrak{g}_{-1}=-\mathrm{id}$; 2) $x = \Sigma\xi_i(\partial/\partial\xi_i)$ (this is a grading operator in $S(0|n)$), $T^{\circ}(0) = \{f\in T(0)\,|\,\int f=0\}$ and $\mathfrak{g}_{-1}=\Pi(T^0(0)/\langle 1\rangle)$. Let $S_{\alpha,\beta}(n) = S(0|n)\oplus \mathbf{C}(\alpha x + \beta z)$, where z is the same kind of element as in 1).

In cases 1) and 2) we define a form ω on $\mathfrak{g}_{-1}$ by setting $\omega(f, g) = \int fg$ and $\int fgv_\xi$ respectively.

Among the Lie superalgebras $S_{\alpha,\beta}(n)^{MK}$ and $CW(0|n)^{MK}$ only three are simple: $S_{6,-5}(4)^{MK}$, $CW(0|2)^{MK}\cong K(3|2)$ and $CW(0|3)^{MK}$.

We set

$$S_t(0|2n)=\{D\in W(0|2n)\,|\,D[(1+t\xi_1\ldots\xi_{2n})v_\xi]=0\},$$

$$S_t(0|2n+1)=\{D\in\Lambda[\theta]W(0|2n+1)\,|\,D[(1+t\theta\xi_1\ldots\xi_{2n+1})v_\xi]=0\}.$$

Exercise. $S_t\cong S_{t'}$ for $t, t'\neq 0$. We set $S'(n) = S_1(0|n)$.

A deformation Q_h with parameter h of the Poisson superalgebra $Po(2n|2m)$ into the Lie superalgebra $\mathrm{Diff}(n|m)_L$ of differential operators on a space of half the number of dimensions we call a quantization. In particular,

$$Q_h(\mathrm{Po}(o|2m))\cong(\mathrm{Cliff}_{\pi_n}(2n))_L\cong\mathfrak{gl}(2^{n-1}|2^{n-1})\quad\text{for}\quad h\neq 0,$$

where $\mathrm{Cliff}_B(n)$ is the Clifford superalgebra of $(0, n)$-dimensional superspace with a skew-symmetric even form having matrix B.

What to do if there are an odd number of odd variables? Let $\pi\in\mathrm{Diff}(n|m)_{\bar{1}}$ with $\pi^2 = 1$ (for example, $\pi = \xi_1 + \partial/\partial\xi_1$). Let $\mathrm{QDiff}(n|m)=\{D\in\mathrm{Diff}(n|m)\,|\,[D,\pi]=0\}$. Then $Q_h(Po(2n|2m-1)) = \mathrm{QDiff}(n|m)$ for $h\neq 0$. In particular, $\mathrm{QDiff}(0|m)\cong\mathfrak{q}(2^{m-1})\cong\mathrm{Cliff}_{\pi_n^+}(2n+1))$, where $\pi^+_n = \mathrm{diag}(1, \pi_n)$. This a new interpretation of the Clifford algebra considered as a superalgebra [40].

There is so far no complete description of deformations (see only [1, 73]).

5. Lie Superalgebras of String Theories [44, 74, 75, 126]. In recent years in physics the idea has arisen that elementary particles can sometimes be considered very elastic springs — strings (see [47]). The Witt algebra W — the Lie algebra of vector fields on the circle S with coefficients expandable in a finite Fourier series — is used in describing these models. Let $t = \exp(ia)$, where a is the parameter on the circle. The algebra W can be realized as the Lie algebra of differentiations of Laurent polynomials $\mathbf{C}[t^{-1}, t]$. This algebra has several series of superanalogues.

On the superspace $R(n) = C[t^{-1}, t, \xi_1, \ldots, \xi_n]$ we define a contact bracket, and the Lie superalgebra obtained we denote by $\mathcal{K}(n)$. Let

$$\mathcal{W}(n) = \operatorname{Der} R(n), \quad \mathcal{S}(n) = \{D \in \mathcal{W}(n) \mid \operatorname{div} D = 0\}, \qquad \mathcal{S}^0(n) = \mathcal{S}(n) / \langle \xi_1 \ldots \xi_n \frac{\partial}{\partial t} \rangle .$$

These Lie superalgebras preserve the structure on the supermanifold $\mathcal{S}^{1,n}$, associated with the trivial bundle over S. The Möbius bundle gives one more example: the contact structure is reduced from $\mathcal{S}^{1,n}$ to the supermanifold associated with the Whitney sum of the trivial bundle of rank n − 1 and the Möbius bundle. The corresponding bracket has the form

$$\{f, g\}^{+}_{K.B.} = \Delta(f) \frac{\partial g}{\partial t} - \frac{\partial f}{\partial t} \Delta(g) - (-1)^{p(f)} \Big(\sum_{i \leqslant n-1} \frac{\partial f}{\partial \xi_i} \frac{\partial q}{\partial \xi_i} + t \frac{\partial f}{\partial \xi_n} \frac{\partial g}{\partial \xi_n} \Big),$$

and the Lie superalgebra obtained is denoted by $\mathcal{K}^{+}(n)$.

The most interesting thing to physicists in these (super) algebras is that they sometimes have nontrivial central extensions. They have all been enumerated [44, 126]. The central extensions for $\mathcal{K}(n)$ and $\mathcal{K}^{+}(n)$ are given by the same cocyle Res c(f, g), where Res f is the coefficient of $\prod \xi_i / t$, and all the c are

n	0	1	2	3
c	$fK_1{}^3(g)$	$fK_{\xi}K_1{}^2(g)$	$fK_{\xi_1}K_{\xi_2}K_1(g)$	$fK_{\xi_1}K_{\xi_2}K_{\xi_3}(g)$.

The cocyles for $\mathcal{W}(2)$, $\mathcal{S}(2)$ and $\mathcal{S}^0$ (2) can be found in [44], and there are no other extensions.

Gel'fand and Fuchs and Virasoro found a central extension for $\mathcal{K}(0)$, for $\mathcal{K}(1)$ one was found by Nevieux and Schwartz, and for $\mathcal{K}^{+}(1)$ one was found by Ramon; the corresponding Lie superalgebras are denoted by V, NS(1), and R(1) in their honor. We denote by NS(n) and R(n) for n = 2, 3 the extensions of the Lie superalgebras $\mathcal{K}(n)$ and $\mathcal{K}^{+}(n)$, respectively. Ademollo and others [74, 75] found extensions of the Lie superalgebra $\mathcal{Y}^0(2)$. We denote the central extensions of the Lie superalgebras $\mathcal{W}(2)$, $\mathcal{S}(2)$ and $\mathcal{S}^0$ (2) by $A\mathcal{W}$, $A\mathcal{S}$ and $A\mathcal{S}^0$.

6. Kac Superalgebras $\mathfrak{g}_{\varphi}^{(m)}$ and Kac–Moody Algebras [43]. Let $\mathfrak{g}$ be a simple, finite-dimensional Lie superalgebra. Let $G_{\bar{0}}$ be the associated group of the Lie algebra $\mathfrak{g}_{\bar{0}}$. The elements of the group Out $\mathfrak{g}$ = Aut $\mathfrak{g} / G_{\bar{0}}$, where the group $G_{\bar{0}}$ is imbedded in the group Aut $\mathfrak{g}$ of automorphisms of the Lie superalgebra $\mathfrak{g}$ is a natural way, are called outer automorphisms.

We call the superalgebra $\mathfrak{g}_{\varphi}^{(m)}$ the Kac superalgebra connected with the automorphism φ, and its nontrivial central extension with cocyle c — the Kac–Moody superalgebra — we denote by ${}^{c}\mathfrak{g}_{\varphi}^{(m)}$.

We define $d_3 \in \operatorname{Aut} D((-1 + i\sqrt{3})/2)$ by setting $d_3(a, u) = ((a_3, a_1, a_2), (-1 + i\sqrt{3}) u_3 \otimes u_1 \otimes u_2 / 2)$. If $\alpha + \bar{\alpha} = -1$, then we define $d_{23} \in \operatorname{Aut} D(\alpha)$ by setting $d_{23}(a, u) = ((\bar{a}_1, \bar{a}_3, \bar{a}_2), \bar{u}_1 \otimes \bar{u}_3 \otimes \bar{u}_2))$.

We define $A, B \in \operatorname{Aut} Sh(0|2n)$ by setting $A(\xi_i) = (-1)^{\delta_{1i}} \xi_i$, $B(\xi_i) = \xi_i + (\partial \xi_1 \ldots \xi_{2n}) / \partial \xi_i$ for i = 1, ..., 2n.

We define $\delta_\lambda \in \operatorname{Aut} W(0|n)$ by setting $\delta_\lambda(\xi_i) = \lambda \xi_i$, where $\lambda \in C \setminus 0$.

Let $J_{k,n} = \operatorname{diag}(A, 1_{2n})$, where A is an orthogonal transformation of 2n-dimensional space such that det A = −1.

For $X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ we set $st(X) = \begin{pmatrix} -A^t & C^t \\ -B^t & -D^t \end{pmatrix}$, $\Pi(X) = \begin{pmatrix} D & C \\ B & A \end{pmatrix}$, $\delta_\lambda(X) = \begin{pmatrix} A & \lambda B \\ \lambda^{-1} C & D \end{pmatrix}$, where $\lambda \in C \setminus 0$.

Let $q = -st \circ \delta_{\sqrt{-1}}$, and let (g) be the cyclic group generated by the element g. We carry over an automorphism from an algebra to a subfactor algebra without special mention and denote the automorphism obtained by the same symbol.

THEOREM. If $\mathfrak{g}$ is in the series W, S', $\mathfrak{osp}(2n+1|2m)$, $D(\alpha)$ for $\alpha \neq 1, (-2)^{\pm 1}$, $(-1 \pm i\sqrt{3})/2$ and AG_2 and AB_3, then Out $\mathfrak{g}$ = 1. In the remaining cases Out G is defined from the following table

G	$\mathfrak{sl}(n/m)$, $n \neq m$	$\mathfrak{osp}(2n \mid 2m)$	$\mathfrak{psq}(n)$	$D((-1+\sqrt{3})/2)$
Out G	$Z_2 = (-st)$	$Z_2 = (\operatorname{Ad} J_{n,m})$	$Z_4 = (q)$	$Z_3 = (d_3)$
G	$S\pi(n)$	$S(0 \mid n)$		$SH(2n+1)$
Out G	$C^* = \{\delta_\lambda \mid \lambda^{2n} \neq 1\}$	$C^* = \{\delta_\lambda \mid \lambda^{2n} \neq 1\}$		$C^* = \{\delta_\lambda \mid \lambda \neq \pm 1\}$

and the exact sequences (1)-(3):

$$1 \to \mathbf{C} \to \mathrm{Out}\, SH(0\,|\,2n) \to \mathbf{Z}_2 \oplus \mathbf{C}^* \to 1, \tag{1}$$

where C is the one-parameter subgroup generated by the automorphism B, $Z_2 = (A)$, and $C^* = \{\delta_\lambda \,|\, \lambda \in C^*,\ \lambda^n \neq 1\}$;

$$1 \to \mathbf{C}^* \to \mathrm{Out}\, \mathfrak{psl}(n\,|\,n) \to \mathbf{Z}_2 \oplus \mathbf{Z}_2 \to 1, \tag{2}$$

where $n > 2$, $\mathbf{C}^* \cong \{\delta_\lambda \,|\, \lambda \in \mathbf{C}^*,\ \lambda^n \neq 1\}$, and $\mathbf{Z}_2 \oplus \mathbf{Z}_2 \cong (\pi) \oplus (-\mathrm{st})$;

$$1 \to SL(2)/(-1_2) \to \mathrm{Out}\, \mathfrak{psl}(2\,|\,2) \to \mathbf{Z}_2 \to 1, \tag{3}$$

where $Z_2 \cong (\pi)$, and the action of the Lie algebra $\mathfrak{sl}(2)$ of the group SL)2) on $\mathfrak{psl}(2|2)$ is defined as follows. We realize the superalgebra $\mathfrak{psl}(2|2)$ by matrices of order (2, 2) with bracket $[X, Y] = XY - (-1)^{p(X)p(Y)}YX - 2\,\mathrm{str}\, XY$. Then

$$\begin{pmatrix} a & b \\ c & -a \end{pmatrix}\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}\right) = \begin{pmatrix} 0 & aB + bJ_1C^tJ_1^{-1} \\ cJ_1B^tJ_1^{-1} + aC & 0 \end{pmatrix}.$$

We note that the order of the automorphism st in the group $\mathrm{Out}\,\mathfrak{psl}(n|n)$ is equal to 2 (respectively, 4) if n = 2k (respectively, n = 2k + 1).

THEOREM. Let $\mathfrak{g}$ be a simple, finite-dimensional Lie superalgebra. If $\varphi \in \mathrm{Aut}_0\,\mathfrak{g}$ (the connected component of the identity, then $\mathfrak{g}_\varphi^{(m)} = \mathfrak{g}^{(1)}$. Moreover, $\mathfrak{psg}(n)_q^{(4)} = \mathfrak{psg}(n)_{a^s}^{(4)}$. The remaining Lie superalgebras $\mathfrak{g}_\varphi^{(m)}$ are the following (in row 3 for n = m we take the projection of the automorphism indicated in column 2, while $(m, n) \neq (1, 2), (2, 2)$; in row 5 n > 2 and in the last row of the $\mathfrak{g}_0$-module $\mathfrak{g}_1$ (respectively, $\mathfrak{g}_2$) is irreducible with leading (odd) even weight 2 (respectively 3). $\langle 1 \rangle$ denotes the trivial module; we set

$$S = \{(A, B) \in S^2(\mathrm{id} \otimes 1) \oplus S^2(1 \otimes \mathrm{id}) \,|\, \mathrm{tr}\, A - \mathrm{tr}\, B = 0\}):$$

$\mathfrak{g}_\varphi^{(m)}$	$\mathfrak{g}_0$	$\mathfrak{g}_1$	$\mathfrak{g}_2$	$\mathfrak{g}_3$
$\mathfrak{osp}(2m\,|\,2n)^{(2)}$	$\mathfrak{osp}(2m-1\,|\,2n)$	id	—	—
$(\mathfrak{p})\mathfrak{sl}(m\,|\,n)^{(2)}_{-\mathrm{st}}$	$\mathfrak{o}(m) \oplus \mathfrak{o}(n)$	$\pi(\mathrm{id} \otimes \mathrm{id})$	S	$\pi(\mathrm{id} \otimes \mathrm{id})$
$\mathfrak{psl}(n\,|\,n)^{(2)}_\pi$	$\mathfrak{pg}(n)$	ad*	—	—
$\mathfrak{psl}(n\,|\,n)^{(2)}_{\pi_o - \mathrm{st}}$	$S\pi(n)$	ad*	—	—
$\mathfrak{psq}(n)^{(2)}$	$\mathfrak{sl}(n)$	$\pi(\mathrm{ad})$	—	—
$\mathfrak{psq}(n)^{(4)}_q$	$\mathfrak{o}(n)$	$\pi(\Lambda^2 \mathrm{id})$	$S^2\mathrm{id}/\langle 1 \rangle$	$S^2\mathrm{id}/\langle 1 \rangle$
$SH(0\,|\,2n)^{(2)}_A$	$H(0\,|\,2n-1)$	$T^0(0)$	—	—
$D((-1 + i\sqrt{3})/2)^{(3)}$	$\mathfrak{osp}(1, 2)$	$L(2)$	$\pi(L(3))$	—

These Lie superalgebras $\mathfrak{g}_\varphi^{(m)}$ are simple.

Exercise. 1) Which of the Lie superalgebras of this section are simple? For an indication see [1, 103, 104].

2) List the Lie superalgebras of differentiations of simple Lie superalgebras using [36, 57, 73, 104].

The Kac–Moody superalgebras associated with simple, finite-dimensional Lie superalgebras have a much more interesting structure than Kac–Moody algebras. In particular, if a simple Lie superalgebra $\mathfrak{g}$ has a nontrivial central extension $^\omega\mathfrak{g}$ given by the cocycle ω, then $\mathfrak{g}_\varphi^{(m)}$ has infinitely many central extensions with cocycles

$$c(\omega)_i : (X, Y) \mapsto \mathrm{Res}\, \omega(X, Y)\, t^i, \text{ where } i \in \mathbf{Z},\ X, Y \in \mathfrak{g}_\varphi^{(m)}.$$

If B is an invariant, nondegenerate, symmetric form on $\mathfrak{g}$, then an extension of the Lie superalgebra $\mathfrak{g}_\varphi^{(m)}$ is given as for Lie algebras by the cocycle

$$c(B) : (X, Y) \mapsto \mathrm{Res}\, B\left(X, \frac{dY}{dt}\right), \text{ где } X, Y \in \mathfrak{g}_\varphi^{(m)}.$$

Therefore, to describe Kac–Moody superalgebras we make the corresponding digression: we describe invariant bilinear forms on simple, finite-dimensional Lie superalgebras and their central extensions, and all central extensions of Kac superalgebras can be constructed by one of the two methods indicated.

THEOREM. The Killing form (x, y) = str ad x·ad y is nondegenerate on $\mathfrak{sl}(n|m)$ for $n \neq m$, on $\mathfrak{osp}(n|2m)$ for $n \neq 2m + 2$, and on AG_2 and AB_3. The even form str xy (respectively, $\int x(\xi)y(\xi)v_\xi$) is nondegenerate on $\mathfrak{pgl}(n|n)$ and $\mathfrak{osp}(2n+2|2n)$ (respectively, on Po(0|2n)), while the odd form otr xy (respectively, $\int x(\xi)y(\xi)v_\xi$) is nondegenerate on $\mathfrak{q}(n)$ (respectively, Po(0|2n + 1)). On $\mathfrak{psl}$, $\mathfrak{psq}$ and SH the forms induced by those described are nondegenerate. All the forms enumerated above are symmetric and invariant. On Lie superalgebras of the series SП, W, S, S' there are no nonzero invariant forms.

There are invariant skew-symmetric forms only on the following simple Lie superalgebras

$\mathfrak{g}$	Cocycle ω	Extension $\omega_\mathfrak{g}$
$\mathfrak{psl}(2\|2)$	$\begin{pmatrix} A & B \\ C & D \end{pmatrix}, \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \mapsto \begin{cases} \operatorname{tr} CC' \\ \operatorname{tr}(CB'+BC') \\ \operatorname{tr} BB' \end{cases}$	SH(4) $\mathfrak{sl}(2\|2)$ SH(4)
$\mathfrak{psl}(n\|n)$ for $n>2$	$\begin{pmatrix} A & B \\ C & D \end{pmatrix}, \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \mapsto \operatorname{tr}(CB'+BC')$	$\mathfrak{sl}(n\|n)$
$\mathfrak{psg}(n)$	$(A, B), (A', B') \mapsto \operatorname{tr} BB'$	$\mathfrak{sg}(n)$
SП(4)	$\begin{pmatrix} A & B \\ C & -A^t \end{pmatrix}, \begin{pmatrix} A' & B' \\ C' & -A'^{+} \end{pmatrix} \mapsto \operatorname{tr} CC'$	AS
SH(n) for $n>4$	$H_f, H_{f'} \mapsto \sum \frac{\partial f}{\partial \xi_i}(0) \frac{\partial f'}{\partial \xi_i}(0)$	SPo(n)

The extension AS is named in honor of its discoverer A. Sergeev.

A description of contragradient Lie superalgebras and systems of simple roots is presented in [42] where the Coxeter automorphisms are also listed in those cases where it was possible to assign a meaning to superpositions of them.

7. Real Forms. Any real Lie superalgebra is either a complex Lie superalgebra $\mathfrak{g}$ considered as a real algebra $\mathfrak{g}^R$ or a real form $\mathfrak{h}$ of a complex Lie superalgebra $\mathfrak{g}$, i.e., $\mathfrak{h} \otimes_R C = \mathfrak{g}$. Each real form $\mathfrak{h}$ of a complex Lie superalgebra $\mathfrak{g}$ has the form $\mathfrak{g}^\varphi = \{g \in \mathfrak{g} \mid \varphi(g) = g$, where φ is an antilinear automorphism of order 2.

In the finite-dimensional case Serganova [56], correcting [104, 125], described all automorphisms for simple Lie superalgebras. Moreover, she described automorphisms of order 2 of real Lie superalgebras (to such automorphisms there correspond symmetric superspaces) and automorphisms of order 4 involutive on $\mathfrak{g}_{\bar{0}}$; physicists investigate such structures. There was not room in [55] for a description of the automorphisms for AG_2 and AB_3 which has also been completely carried out (see [39]).

V. V. Serganova also described real forms and automorphisms for Kac superalgebras and, apparently for the first time, for Kac algebras [55] and Lie superalgebras of string theories [57] and also automorphisms of order 2 (and 4) of real forms of the Lie superalgebras of string theories — infinite-dimensional (semi) symmetric spaces (cf. [116-118]).

Real forms of simple Lie superalgebras of formal vector fields, their automorphism, and Lie superalgebras of their differentiations are described in [36].

2. Modules over Lie Superalgebras of Vector Fields

1. Two Categories. We consider two categories of modules over infinite-dimensional Lie superalgebras $\mathscr{L}$ of vector fields with the Weisfeiler filtration: discrete and topological. A discrete $\mathscr{L}$-module I is a module such that dim $U(\mathscr{L}_0)_i < \infty$ for any vector $i \in I$. To each discrete module I there corresponds a topological module $I^* = \operatorname{Hom}_k(I, k)$ with a topology having a basis of neighborhoods of zero formed by the annihilators of finite-dimensional subsuperspaces of the module I, and $I = \operatorname{Hom}_k^c(I^*, k)$, where Hom^c is the superspace of continuous homomorphisms. It is obvious that the categories of discrete and topological $\mathscr{L}$-modules are dual. If $\mathscr{L}$ is a finite-dimensional Lie superalgebra with the standard gradation, $\mathscr{L}_i = \bigoplus_{j>i} L_i$, then both these categories coincide with the category of finite-dimensional $\mathscr{L}$-modules. This indicates that these categories are natural.

The superspace T(V) of formal tensor fields of type V constructed on the basis of a finite-dimensional L_0-module V is an example of a topological $\mathscr{L}$-module: we set $\mathscr{L}_1 V=0$ and $T(V) = \mathrm{Hom}_{U(\mathscr{L}_0)}(U(\mathscr{L}), V)$. For the Poisson and Butane superalgebras and their regradations it is natural to alter the definition slightly so that the center goes over into the operator of multiplication by a scalar. Generalizing what is called prequantization or geometric quantization, we describe also irreducible modules over the Poisson superalgebra in the modules $T_h(V)=\mathrm{Hom}_{U(\mathscr{L}_0)}(U(\mathscr{L})/(K_1-h), V)$ and over the Butane superalgebra in the modules $T_h(V) = \mathrm{Hom}_{U(\mathscr{L}_0)[\tau]}(U(\mathscr{L})[\tau](M_1\tau-h), V)$, where $h\in k$, $p(\tau)=1$, $\tau^2=0$.

For the Lie superalgebras of string theories (the Kac superalgebra $\mathfrak{g}_\varphi^{(m)}$) we define the series of modules $\mathscr{T}_{(\mu)}(V)$ (respectively, $V^{(1)}$). We denote by $\mathscr{T}(V)$ the module $\mathbf{C}[t^{-1}]T(V)$. We set $\mathscr{T}_\mu(V)=t^\mu\mathscr{T}(V)$ (respectively, $V^{(1)}=V\otimes\mathbf{C}[t^{-1}, t]$, where V is a $\mathfrak{g}$-module).

2. Integrodifferential Forms. Together with the variables x, we introduce variables 'x such that $p('x_i) = p(x_i) + 1$, and we call elements of the commutative superalgebra $\Omega = \mathbf{C}[[x]]$ (formal) differential forms. We define a Z-gradation on the superalgebra Ω by setting $\deg x_i = 0$, $\deg 'x_i = 1$. It is obvious that $\Omega=\bigoplus_{i\geq 0}\Omega^i$.

We define the exterior differential $d:\Omega^i \to \Omega^{i+1}$ by setting $d(x_i) = 'x_i$, $d'x_i = 0$, $d(\omega_1\omega_2) = d\omega_1\omega_2 + (-1)^{p(\omega_1)}\omega_1 d\omega_2$. For any vector field $D = \Sigma f_i\partial/\partial x_i$ we define interior multiplication i_D: $\Omega^i \to \Omega^{i-1}$ on the field D by setting $i_D = (-1)^{p(D)}\Sigma f_i\partial/\partial 'x_i$. We define the Lie derivative L_D along the field D by setting $L_D = [d, i_D]$. In the coordinates x, 'x we have $d = \Sigma 'x_i(\partial/\partial x_i)$.

Remark. In some works using differential forms another definition is given of differential forms in which Ω coincides with $\mathbf{C}[[x]]['x]$ as a space, where $\deg 'x_i = 1$, $\deg x_i = 0$, but $p('x_i) = p(x_i)$, and multiplication satisfies the condition

$$\varphi_1 * \varphi_2 = (-1)^{\deg\varphi_1 \deg\varphi_2 + p(\varphi_1)p(\varphi_2)}\varphi_2 * \varphi_1,$$

where $\varphi_1, \varphi_2 \in \Omega$. We go over from the multiplication * to the new multiplication

$$\varphi_1\varphi_2 = (-1)^{\deg\varphi_1 p(\varphi_2)}\varphi_1 * \varphi_2$$

by setting simultaneously $p_{\mathrm{new}}(\varphi)=\deg\varphi \pmod 2 + p_{\mathrm{old}}(\varphi)$; we find that Ω is a commutative superalgebra, which is considerably more convenient (cf. [39, 129]).

Let J be the commutative superalgebra consisting of operators generated by operators of multiplication by series $f \in \mathbf{C}[[x]]$ and by interior multiplications i_D where $D\in\mathscr{L}$. Since $i_{\frac{\partial}{\partial x_i}} = (-1)^{p(x_i)}\frac{\partial}{\partial' x_i}$, it follows that $J = \mathbf{C}[[x]]\left[\frac{\partial}{\partial' x}\right]$.

On the superalgebra J we introduce a Z-gradation by setting $\deg x_i = 0$, $\deg(\partial/\partial 'x_i) = 1$.

On J we define the structure of a Ω-module by setting

$$'x_i\left(\frac{\partial}{\partial' x_j}\right) = (-1)^{(p(x_i)+1)(p(x_j)+1)}\delta_{ij}.$$

In other words,

$$'x_i = (-1)^{p(x_i)+1}\frac{\partial}{\partial(\partial/\partial' x_i)}.$$

We denote by $\Sigma = \bigoplus_{i<n-m}\Sigma_i$ the Z-graded J-module with generator v_x, where $\deg v_x = n - m$ and $p(v_x) = (n - m) \bmod 2$. We call the elements $\sigma \in \Sigma_i$ integral forms of degree $n - m - i$. (This name is occasioned by the fact that elements of the superspace Σ are formal analogues of forms which can be integrated; see [8].)

On Σ we define the structure of a Ω-module by setting $'x_i(P\Delta_x) = ('x_i P)_x$, where $P \in J$. It is easy to see that this structure is consistent with the action of the operators d, i_D, and L_D according to Leibniz's rule.

Special cases: a) m = 0. Then $\Omega^i = 0$ for $i > n$, while $\Sigma_i = 0$ for $i < 0$. Moreover, the mapping $v_x \mapsto 'x_1 \ldots 'x_n$ gives an isomorphism of the superspace Ω^i with the superspace Σ_i which preserves all structures.

b) n = 0. In this case there is a homomorphism of the $\mathscr{L}$-modules $\int : \Sigma_m \to \mathbf{C}$ which we call the Berezin integral in honor of its discoverer F. A. Berezin. The integral is defined by the formula $\int \xi_1 \ldots \xi_m v_\xi = 1$, and $\int \xi_1^{\nu_1} \ldots \xi_m^{\nu_m} v_\xi = 0$, if $\prod_{i>1} \nu_i = 0$. We denote by $\int$ also the composition of the Berezin integral with the natural imbedding $\Sigma_{-m} \xrightarrow{\int} \mathbf{C} \hookrightarrow \Omega^0$.

c) m = 1. We generalize Ω^i and Σ_j to superspaces Φ^λ, where $\lambda \in \mathbf{C}$. Let $x = (u_1, \ldots, u_n, \xi)$. We define the Ω-module $\Phi = \bigoplus_{\lambda \in \mathbf{C}} \Phi^\lambda$ (we assume that $\deg' x_i = 1 \in \mathbf{C}$) as a module generated by by the generators $'\xi^\lambda$, where $\lambda \in \mathbf{C}$ and $\deg' \xi^\lambda = \lambda$, $p('\xi^\lambda) = \bar{0}$, while $'\xi' \xi^\lambda = '\xi^{\lambda+1}$. Setting $\frac{\partial}{\partial' \xi} '\xi^\lambda = \lambda' \xi^{\lambda-1}$ $\left(\text{and } \frac{\partial}{\partial' u_i} '\xi^\lambda = \frac{\partial}{\partial' \xi} 'u_i = 0\right)$, we define the action on Φ of the operators d, i_D and L_D. It is not hard to see that Φ is a commutative superalgebra, while the action of the operators d, i_D and L_D on Φ satisfies the Leibniz rule.

We define homomorphisms $\alpha : \Omega \to \bigoplus_{r \in \mathbf{Z}} \Phi^r$ and $\beta : \bigoplus_{r \in \mathbf{Z}} \Phi^r \to \Sigma$ by setting

$$\alpha(\omega) = \omega' \xi^0, \quad \beta(u_1 \ldots 'u_n' \xi^{-1}) = v_x.$$

It is easy to see that the sequence

$$0 \to \Omega \xrightarrow{\alpha} \bigoplus_{r \in \mathbf{Z}} \Phi^r \xrightarrow{\beta} \Sigma \to 0$$

is exact, while the homomorphisms α and β are consistent with the structures of the Ω-module and the actions of the operators d, i_D and L_D.

The construction of $\mathscr{L}$-modules Φ admits the following generalization for $m \neq 1$. (Formal) pseudodifferential forms of degree λ, where $\lambda \in \mathbf{C}$, are elements of the superspace $\Omega^{(\lambda)} = T(E^\lambda(\mathrm{id}))$, where $E^\lambda(\mathrm{id})$ is the λ-th exterior power of the standard (n, m)-dimensional L_0-module id. They have the form

$$f(x)' u_1^{\alpha_1} \ldots 'u_n^{\alpha_n} ' \xi_1^{\beta_1} \ldots '\xi_m^{\beta_m},$$

where $\alpha_i = 0, 1$ and $\beta_j \in \mathrm{C}$, and $|\alpha| + |\beta| = \lambda$ (as usual, $|\alpha| = \alpha_1 + \ldots + \alpha_n$, and $|\beta| = \beta_1 + \ldots + \beta_m$).

Similarly, (formal) pseudointegral forms of degree $n - m - \lambda$, where $\lambda \in \mathbf{C}$, are elements of the superspace $\Sigma_{(n-m-\lambda)} = T(E^\lambda(\mathrm{id}^*) \otimes \mathrm{str})$. They have the form

$$f(x) \left(\frac{\partial}{\partial' u_1}\right)^{\alpha_1} \ldots \left(\frac{\partial}{\partial' u_n}\right)^{\alpha_n} \left(\frac{\partial}{\partial' \xi_1}\right)^{\beta_1} \ldots \left(\frac{\partial}{\partial' \xi_m}\right)^{\beta_m} v_x,$$

where $\alpha_i = 0, 1$ and $\beta_j \in \mathbf{C}$, and $|\alpha| + |\beta| = \lambda$.

We set $\Omega^{(*)} = \bigoplus_{\lambda \in \mathbf{C}} \Omega^{(\lambda)}$ and $\Sigma_{(*)} = \bigoplus_{\lambda \in \mathbf{C}} \Sigma_{(n-m-\lambda)}$.

Remark. It is also convenient to consider pseudoforms such that $\beta_i \in \mathrm{C}$ and $\beta_j \in \mathrm{Z}$ for $j \neq i$. Such pseudoforms correspond to L_0-modules induced from parabolic subalgebras larger than $(L_0)_{\bar{0}}$ (see [69, 70]).

The L_0-modules $E^\lambda(\mathrm{id})$ are infinite-dimensional with the exception of the case m = 1, so that the pseudoforms are not tensor fields. However, on restricting to a Lie subsuperalgebra, the superspace of pseudoforms can be expanded in an infinite sum of superspaces of tensor fields (with finite-dimensional fiber).

We note that $\Sigma_{(*)}$ is a $\Omega^{(*)}$-module with a consistent action of the operators d, i_D and L_D.

3. Connections. The formal constructions we introduce imitate the following geometric picture: $\mathscr{F}$ is a space of functions on an open domain U, Der $\mathscr{F}$ is the Lie algebra of vector fields on U, M is the $\mathscr{F}$-module of sections of a vector bundle on U with finite-dimensional fiber, and Ω is the algebra of exterior forms on U; see [46].

Suppose now that $\mathscr{F} = \mathbf{C}[[x]]$ (or $\mathscr{F}$ is any commutative superalgebra with a 1), $\mathscr{L} \subset \mathrm{Der}\mathscr{F}$ is a Lie subsuperalgebra, M is a free $\mathscr{F}$-module of finite rank, and Ω is a commutative superalgebra of differential forms.

A connection on M is an odd C-linear mapping $\nabla : M \to \Omega^1 \otimes_{\mathscr{F}} M$, satisfying the condition $\nabla(fm) = df \otimes m + (-1)^{p(f)} f \nabla(m)$, where $f \in \mathscr{F}$, $m \in M$.

Since as a superspace $\mathrm{Der}\,\mathcal{F}=\Pi(\mathrm{Hom}_{\mathcal{F}}(\Omega^1,\mathcal{F}))$, then dualizing and applying the functor Π we arrive at the following equivalent definition: a connection on M is an even mapping

$$\nabla:\mathcal{L}\times M\to M,\quad (D,m)\mapsto\nabla(D)m$$

which is $\mathcal{F}$-linear in the first argument and additive in the second and is such that

$$\nabla(D)(fm)=D(f)m+(-1)^{p(f)p(D)}f\nabla(D)(m),$$

where $f\in\mathcal{F}$, $m\in M$, $D\in\mathcal{L}$. We call the operator $\nabla(D)$ the covariant derivative along the vector field D. We call an element ∇-horizontal if $\nabla(D)m = 0$ for all $D\in\mathcal{L}$. It is easy to see that if M is a free $\mathcal{F}$-module, then on M there exists at least one connection (we set $\nabla(f_i m_i) = df_i \otimes m_i$ where $\{m_i\}$ is a basis of the module M and $f_i \in\mathcal{F}$). The set of connections of a module M forms an affine superspace isomorphic to $\Omega^1\otimes_{\mathcal{F}}\mathrm{End}_{\mathcal{F}}M$. We fix some connection ∇_0 (in a free module it is convenient to fix a flat connection, i.e., a connection relative to which all elements of some basis of this module are horizontal). Then any connection has the form $\nabla = \nabla_0 + \alpha$, where $\alpha\in\Omega^1\otimes_{\mathcal{F}}\mathrm{End}_{\mathcal{F}}M$. In a fixed basis of a module M of dimension $n + m\varepsilon$ a form α can be considered an element of $\mathrm{Mat}(n|m;\Omega)$. We call this matrix α the form of the connection ∇.

We extend ∇ to mappings

$$\nabla:\Omega^i\otimes_{\mathcal{F}}M\to\Omega^{i+1}\otimes_{\mathcal{F}}M \text{ and } \nabla:M\otimes_{\mathcal{F}}\Sigma_i\to M\otimes_{\mathcal{F}}\Sigma_{i+1}$$

by setting

$$\nabla(\omega\otimes m)=d\omega\otimes m+(-1)^{p(\omega)}\omega\otimes\nabla(m)$$

and

$$\nabla(m\otimes\sigma)=T(\nabla(m))\otimes\sigma+(-1)^{p(m)}m\otimes d\sigma,$$

where $T:\Omega^1\otimes_{\mathcal{F}}M\cong M\otimes_{\mathcal{F}}\Omega^1$ is the twisting isomorphism. We call the curvature form of the connection ∇ the operator $F_\nabla=\frac{1}{2}[\nabla,\nabla]=\nabla^2$.

Obviously, a connection extends also to pseudodifferential and pseudointegral forms.

Suppose that a module M is "linear," i.e., has one generator. Suppose that the parity of the generator is $\bar{0}$. It then follows from the Darboux theorem for supermanifolds that a connection form with a nondegenerate curvature form has in some coordinates the form

$$\alpha_1'=\sum_{1<i<n}(p_i dq_i-q_i dp_i)+\sum_{1<j<m}\xi_j d\xi_j.$$

If the generator of the module M is odd, then it follows from the Darboux theorem that the connection form α_0' has the form

$$\alpha_0'=\theta\sum_{1<i<n}(\xi_i dq_i+q_i d\xi_i),$$

where θ is an odd auxiliary coordinate. Usually it is convenient in place of α_1' and α_0' to consider the forms $\alpha_1 = dt + \alpha_1'$ and $\alpha_0 = (\partial/\partial\theta)(\theta d\tau + \alpha_0')$.

4. Primitive Forms. The space of i-forms (differential or integral) with constant coefficients is irreducible over L_0 only if L_0 is of the series $\mathfrak{sl}$ or $\mathfrak{gl}$. In other cases it is reducible. The decomposition of this space into irreducible components, for example, on a symplectic manifold, is achieved due to the fact that the Lie algebra $\mathfrak{sl}(2)$ acts on forms and commutes with the action of Hamiltonian vector fields. The vectors of the $\mathfrak{sl}(2)$-action of lowest order are called primitive forms. We shall describe an analogous structure in the supercase.

a) Suppose there is given a canonical, even, closed 2-form ω_0 in 2n even and m odd variables. The following operators act on the superspaces Ω, $\Omega^{(*)}$, Σ and $\Sigma_{(*)}$: X_+ — multiplication by ω_0, X_- — interior multiplication by the bivector dual to ω_0, $H = [X_+, X_-]$, d, and $\delta = [X_-, d]$. We note that as $H(2n|m)$-modules the superspaces Ω and Σ as well as $\Omega^{(*)}$ and $\Sigma_{(*)}$ are isomorphic; therefore, it suffices to consider, for example, only (pseudo) differential forms.

THEOREM. The operators X_+, X_- and H define on Ω and $\Omega^{(*)}$ structures of $\mathfrak{sl}(2)$-modules invariant relative to $H(2n|m)$.

Proof. Invariance of the operators X_+, X_- and H follows from the definitions. In canonical coordinates the operators X_+ and X_- have the form

$$X_+=\Sigma\hat{q}_i'p_i+\Sigma'\xi_j^2 1/2,\quad X_-=-\left[\Sigma\frac{\partial}{\partial' q_i}\frac{\partial}{\partial' p_i}+\Sigma\left(\frac{\partial}{\partial'\xi_j}\right)^2 1/2\right].$$

We now note that it suffices to carry out all computations in the (2, 0)- and (0, 1)-dimensional cases separately and immediately find that

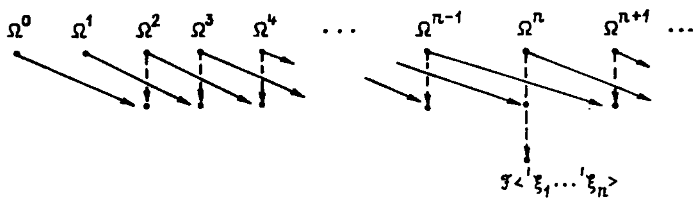
$$H=X_+X_--X_-X_+=(m-2n)/2-\Sigma' x_i\partial/\partial' x_i,$$

where $'x = ('p, 'q, '\xi)$. Hence, $[H, X_\pm] = \pm 2X_\pm$ which was required to prove.

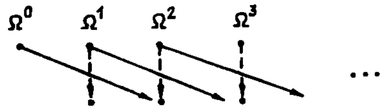
We call the vectors of lowest order relative to the $\mathfrak{sl}(2)$-action in Ω primitive forms. It is not clear what to consider primitive forms in $\Omega^{(*)}$. The structure of the $\mathfrak{sl}(2)$-module Ω for $(2n-m)/2\in\mathbb{N}$ is quite complex (see [65-71], where it is completely described).

b) Suppose there is given a canonical, odd, nondegenerate, closed 2-form ω_1 in n even and n odd variables.

It is easy to compute the homologies of the operator ω_1 in forms with constant coefficients: these are $\langle '\xi_1,\ldots,'\xi_n\rangle$. From this we obtain a description of irreducible L_0-submodules in integrodifferential forms with constant coefficients. Indicating the L_0-submodule by a dashed arrow and the action of the operator ω_1 by a solid arrow, we represent the module Ω as a graph



for $n > 1$, while for $n = 1$ it has the form



The graphs with reversed arrows describe the structure of the Jordan–Holder series in the space of integral forms.

Elements of modules corresponding to vertices of the upper (lower) row in the graph of Ω (respectively, Σ) we call primitive forms. As in a), it is not clear what to consider the analogue of this concept for pseudoforms.

c) Primitive forms with values in a linear bundle with a connection. Let

$$\frac{h}{2}\alpha_1'=\frac{h}{2}\left[\sum(p_i dq_i-q_i dp_i)+\frac{1}{2}\sum\xi_j d\xi_j\right]$$

be the connection form $\nabla_+ = d + (h/2)\alpha_1'$ in a (1, 0)-dimensional $\mathcal{F}$-module. Let $T_h(0) = \Omega_h^0$ be the superspace of sections of this $\mathcal{F}$-module. Since $\Omega_h=\Omega\otimes_{\mathcal{F}}\Omega_h^0$, it follows that the operators X_+, X_- and H extend in an obvious manner from Ω to Ω_h. We set $\nabla_- = [X_-, \nabla_+]$.

THEOREM (Bernshtein; see [35]). The operators X_+, X_-, H, ∇_+ and ∇_- define on Ω_h and $\Omega_h^{(*)}$ for $h\neq 0$ structures of $\mathfrak{osp}(1|2)$-modules invariant relative to $Po(2n|m)$.

Proof. Invariance follows from the definitions. In canonical coordinates we have

$$\nabla_+=\sum{}' x_i\frac{\partial}{\partial x_i}+\frac{h}{2}\left[\sum(p_i'q_i-q_i'p_i)+\frac{1}{2}\sum\xi_j'\xi_j\right].$$

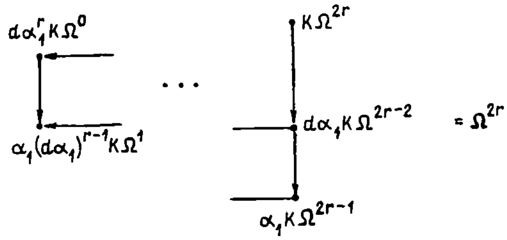
As above, it suffices to carry out all computations in the (2, 0)- and (0, 1)-dimensional cases separately. Elements of the superspace $\sqrt{P_h^r}=\operatorname{Ker}\nabla_-\cap P_h^r$ we call ∇_+-primitive r-forms with torsion h.

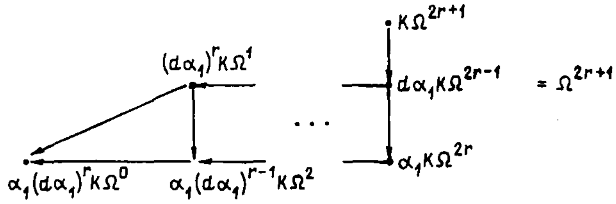
The case of a bundle with a connection whose curvature form is $h\theta\omega_1$ can be considered in a similar but more intricate way (see [35]).

d) The contact case. Suppose there is given a (0, 1)-dimensional $\mathscr{F}$-module with generator which is the 1-form

$$\alpha_1 = dt + \sum (p_i dq_i - q_i dp_i) + \frac{1}{2} \sum \xi_j d\xi_j.$$

For m = 0 we identify Ω^{2n+1-r} and Σ_{2n+1-r}. We set $K\Omega^r = \Omega^r/(\alpha_1\Omega^{r-1} + d\alpha_1\Omega^{r-2})$ for $r \le n$ if m = 0 and for all r if $m \neq 0$, while $K\Sigma_{2n+1-m-r} = \operatorname{Ker}\alpha_1 \cap \operatorname{Ker} d\alpha_1 \cap \Sigma_{2n+1-m-2}$ for r > n if m = 0 and for all r if $m \neq 0$. We represent the structure of submodules in Ω^r and Σ_{2n+1-r} by means of a graph whose vertices are irreducible $\mathscr{L}$-modules, while the arrows indicate the submodules. By induction it is easy to prove the following:





Proposition. The graphs $\Gamma(\Omega^r)$ for $m \neq 0$, $(2n - m)/2 \notin N$ and m = 0, $r \leqslant n$ are shown above. The superspace $K\Omega^r$ is equal to 0 if m = 1, r > 2n. The graphs for integral (2n + 1 − m − r)-forms are obtained from the graphs $\Gamma(\Omega^r)$ by replacing $K\Omega$ by $K\Sigma$ and reversing the arrows.

We call elements of the superspaces $K\Omega$ and $K\Sigma$ primitive forms. The structure of the Jordan–Holder series in the superspaces Ω^r and Σ_r for $(2n - m)/2 \notin \mathbf{N}$ can be derived from results of G. S. Shmelev; the case of the form α_0 is analogous to part b).

3. Irreducible Representations, Invariant Operators, and Analogues of the Poincare Lemma

We shall show that each irreducible topological module over a Lie superalgebra $\mathscr{L}$ of vector fields can be realized as a submodule in some T(V). Our purpose is to describe this submodule as explicitly as possible. For this we study $\mathscr{L}$-homomorphisms $T(V_1) \to T(V_2)$ and find that our submodule is the kernel of one such homomorphism. We present the description of irreducible modules in three steps.

1) We describe all $\mathscr{L}$-invariant differential operators c: $T(V_1) \to T(V_2)$ where V_1 and V_2 are irreducible L_0-modules. The conceptual part of the description is the basic lemma formulating the problem in terms of special vectors. It is found that all invariant operators arrange the spaces of tensor fields on which they are defined into complexes generalizing the de Rham complex.

2) We prove that these complexes are exact, that is, we generalize the Poincare lemma.

3) On the basis of a resolution of modules of the type T(V) for any irreducible $\mathscr{L}$-module it is not difficult to compute its character. If the Lie superalgebra $\mathscr{L}$ is contragradient, then it is possible to describe the character by a single formula generalizing the formula of H. Weyl. For noncontragradient (super) Lie algebras there is no single formula for the character.

Remark. For Lie superalgebras of vector fields also, of course, it is possible to define modules M(V) which are analogues of Verma modules. Nothing is known about such modules.

1. The Main Lemma. It is more convenient to obtain a description of invariant operators in terms of induced modules $I(V) = U(\mathscr{L}) \otimes_{U(\mathscr{L}_0)} V$, where V is an irreducible $\mathscr{L}_0$-module

such that $\mathscr{L}_1 V = 0$. It follows immediately from the definitions that $I(V) = T(V^*)^*$. It is easy to see that a description of homogeneous differential operators $c: T(V_1) \to T(V_2)$ of order i is equivalent to a description of homogeneous vectors of degree $-i$ in the module $I(V_1^*)$ such that $\mathscr{L}_1 f = 0$: such vectors are the images of elements of the space V_2^* under the mapping c^*.

LEMMA. 1) $T(V)^* = T(V^* \otimes \mathrm{str})$, where str is the representation of the Lie superalgebra $\mathfrak{gl}(\dim L_{-1})$ in the 1-dimensional superspace denoted by str; the representation is given by the supertrace.

2) If M is a nontrivial $\mathscr{L}$-module, then $M^{\mathscr{L}_1} \neq 0$ and $M^{\mathscr{L}}$ is an L_0-module.

3) $\mathrm{Hom}_{\mathscr{L}}(I(V), M) = \mathrm{Hom}_{L_0}(V, M^{\mathscr{L}_1})$.

4) The mapping $\varphi: I(M^{\mathscr{L}_1}) \to M$, given by the formula $\varphi: u \otimes m \mapsto um$ where $u \in U(\mathscr{L})$ and $m \in M^{\mathscr{L}_1}$, is an epimorphism if M is an irreducible $\mathscr{L}$-module.

For a proof see [11]. Thus, the description of invariant operators has been reduced to the description of the superspaces $I(V)^{\mathscr{L}_1}$. Elements of this superspace A. N. Rudakov called special vectors. It follows from Shur's lemma and the theory of the leading weight that it suffices to describe the leading vectors of the superspace $I(V)^{\mathscr{L}_1}$ relative to L_0. We shall formulate the description of invariant differential operators which follows from such a description.

2. The Case $W(n|m)$. THEOREM. The sequences

$$0 \to \mathbf{C} \to \Omega_0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \dots, \tag{1}$$

$$\dots \xrightarrow{d} \Sigma_{-1-m} \xrightarrow{d} \Sigma_{-m} \xrightarrow{d} \dots \xrightarrow{d} \Sigma_{n-m} \xrightarrow{d} 0 \quad \text{for} \quad m \neq 0 \tag{2}$$

are exact everywhere except at Σ_{-m}. In this term the space Ker d/Im d is generated over $\mathbf{C}$ by the element $\xi_1 \dots \xi_m \frac{\partial}{\partial' u_1} \dots \frac{\partial}{\partial' u_n} v_x$.

For m = 1 the sequence

$$\dots \xrightarrow{d} \Phi^{\lambda-1} \xrightarrow{d} \Phi^{\lambda} \xrightarrow{d} \Phi^{\lambda+1} \xrightarrow{d} \dots \tag{3}$$

is exact for any $\lambda \notin \mathbf{Z}$. For $\lambda \in \mathbf{Z}$ there is the exact sequence $0 \to \Omega \to \bigoplus_{\lambda \in \mathbf{Z}} \Phi^\lambda \to \Sigma \to 0$.

For n = 0 the sequence

$$\dots \xrightarrow{d} \Sigma_{-m-1} \xrightarrow{d} \Sigma_{-m} \xrightarrow{\int} \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \dots \tag{4}$$

is exact, where $\int = v_\xi^* \frac{\partial}{\partial \xi_1} \dots \frac{\partial}{\partial \xi_m}$ is the Berezin integral.

If $c: T(V_1) \to T(V_2)$ is a $W(n|m)$-invariant differential operator and V_1 and V_2 are irreducible $\mathfrak{gl}(n|m)$-modules, then up to application of the functor Π either the operator c is a scalar operator or $T(V_1)$ and $T(V_2)$ are neighboring terms in the sequences (1)-(4) and c is proportional to the corresponding operator.

If V is an irreducible $\mathfrak{gl}(n|m)$-module, then in the $W(n|m)$-module $T(V)$ there is the irreducible submodule $\mathrm{irr}\, T(V) = T(V)$ if $T(V)$ is not contained in the sequences (1)-(4), while in the other cases $\mathrm{irr}\,\Omega^i = \mathrm{Ker}\, d \cap \Omega^i$, $\mathrm{irr}\,\Phi^\lambda = \mathrm{Ker}\, d \cap \Phi^\lambda$ for $\lambda \notin \mathbf{Z}$, $\mathrm{irr}\,\Sigma_i = \mathrm{Im}\, d \cap \Sigma_i$.

The modules $\mathrm{irr}\, T(V)$ and $\Pi\, \mathrm{irr}\, T(V)$ are pairwise inequivalent and exhaust all irreducible topological $W(n|m)$-modules.

The character of the induced module $I(V)$ can be computed very simply:

$$\mathrm{ch}\, \mathbf{C}\left[\left[\frac{\partial}{\partial x}\right]\right] \cdot \mathrm{ch}\, V = N\, \mathrm{ch}\, V,$$

where $N = \prod_{i,j} (1 + \varepsilon e^{\beta_j})/(1 - e^{\alpha_i})$, α_i is the weight of the vector $\partial/\partial u_i$, and β_j is the weight of the vector $\partial/\partial \xi_j$. The resolutions (1)-(4) make it possible to compute the character of any discrete irreducible module. In particular, in the finite-dimensional case we obtain three formulas: for irr T(V) = T(V) and the two

$$\operatorname{ch} d\Omega^{r-1} = \frac{N}{D} \sum_{w \in W} \operatorname{sgn} w e^{w(\rho + r\beta_1)} (1 + \varepsilon e^{w\beta_1})^{-1},$$

$$\operatorname{ch} d\Sigma_r = \frac{N\Sigma^{m+1}}{D} \sum_{w \in W} \operatorname{sgn} w e^{w(\rho + \Phi - (r+1)\beta_m)} (1 + \varepsilon e^{w\beta_m})^{-1},$$

where W is the Weyl group of the Lie algebra $L_0 = \mathfrak{gl}(m)$, ρ is defined for L_0, and $\Phi = \Sigma\beta_i$.

3. Case $S(n|m)$. One further operator is added to the invariant operators: $dv_x^{-1}d: \Sigma_{n-m-1} \to \Omega^1$. The description of irreducible $S(n|m)$-modules is obtained from part 2 by a straightforward modification which we leave to the reader; see [38].

4. Series H and Po. In this case the space of integrodifferential forms of given degree with constant coefficients is not an irreducible L_0-module. The irreducible components correspond to primitive forms (see Sec. 2) and to something unknown for spaces of pseudoforms.

We note that for the series Po the degree of a special vector is defined only modulo 2, and the graded Po-module gr $I_h(V)$ is isomorphic to the H-module I(V); hence, there are no more than I(V) special vectors in $I_h(V)$. The special vectors of degree −1 are the same, while those of degree −2 differ by the leading vector of degree 0. The fact that a vector is special does not mean for the series Po that it generates a proper submodule. For the corresponding investigation and a description of the vectors for the series H see [65-71].

The only H-invariant operator is d (and in the finite-dimensional case $\int$), to which there is added a scalar operator — multiplication by the conserved form ω_0. Since $\Omega^i \cong \Sigma_{n-m-i}$, the differential in the space Σ can be considered to act in Ω; in this case it is called a codifferential.

Remark. In [68] a description is obtained of the irreducible representations of the Lie superalgebra $H(2|2;\lambda)$ — a deformation of the Lie superalgebra $H(2|2)$ which preserves the exotic form $d\xi_1^{(1-2\lambda)/\lambda} \times (dp_1dq_1 + d\xi_1d\xi_2)$.

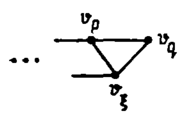
5. We present a description of the special vectors in the cases $H(2n|1)$ and $K(2n+1|m)$ for $m \neq 0$ that remain unpublished.

Let V be an $\mathfrak{osp}(1|2n)$-module, let $f_i = \sum H_{x_{j_1}} \ldots H_{x_{j_i}} v_{j_1 \ldots j_i}$, where $v_{j_1 \ldots j_i} \in V$, and let $x = (q, p)$ be a special vector of degree $-i$ in the $H(2n|1)$-module T(V).

1) n = 1. Let $f_1 = H_q v_q + H_p v_p + H_\xi v_\xi$. The condition that f_1 be a special leading vector has the form $H_{q^2} f_1 = H_{p\xi} f_1 = 0$, whence

$$H_{p\xi} v_q = 0, \quad H_{p\xi} v_\xi = -v_q, \quad H_{p\xi} v_p = -v_\xi, \quad H_{q^2} v_p = 0.$$

We represent the L_0-module V as a graph whose vertices are weight vectors, $\mathfrak{sl}(s)$ acts along horizontal lines, and the weight increases to the right. We obtain



From this it is evident that the following cases are possible: 1) $v_q \neq 0$; 2) $v_q = 0$, $v_\xi \neq 0$; 3) $v_q = v_\xi = 0$, $v_p \neq 0$, in which ($\chi(v)$ denotes the weight of the vector v) 1) $\chi(v_q) = 1$, $\chi(f_1) = 0$, 2) $\chi(v_\xi) = \chi(f_1) = 1$, 3) $\chi(v_p) = 0$, $\chi(f_1) = 1$.

Let $f_2 = H_q^2 v_{q^2} + H_q H_p v_{qp} + H_p^2 v_{p^2} + H_q H_\xi v_q + H_p H_\xi v_p$. Restricting the representation in the module T(V) to $H(2|0)$, we find, considering [54], that $v_{q^2} = 0$. The conditions that the vector f_2 be special and a leading vector give

$$H_{p\xi} v_q = 0, \quad H_{p\xi} v_{qp} = -v_q, \quad H_{p\xi} v_p = -v_{qp}, \quad H_{p\xi} v_{p^2} = -v_p,$$
$$H_{q^2} v_p = 0, \quad H_{q^2} v_{p^2} = 0, \quad H_{q^2} v_{qp} = -2v_{p^2}.$$

From the description of $\mathfrak{osp}(1|2)$-modules (which is very similar to the description of $\mathfrak{sl}(2)$-modules) it follows that $v_q = 0$. Therefore, only the case $v_{pq} \neq 0$ is possible and then $\chi(v_{qp}) = \chi(f_2) = 1$.

2) We find by induction that the special vectors for $H(2n|1)$ are

a) $f_1=\sum_{i<n} H_{q_i}v_i+H_\xi u+\sum_{i<n} H_{p_i}w_i$, where $\chi(v_s)=(1,\ldots,1,0,\ldots,0)$ (s ones), while $\chi(f_1)=(1,\ldots,1,0,\ldots,0)$ (s − 1 ones);

b) $f_1=H_\xi u+\sum_{i<n} H_{p_i}w_i$, where $\chi(u)=\chi(f_1)=(1,\ldots,1)$;

c) $f_1=\sum_{i<r} H_{p_i}w_i$, where $\chi(w_r)=(1,\ldots,1,0,\ldots,0)$, $\chi(f_1)=(1,\ldots,1,0,\ldots,0)$ (r − 1 and r ones, respectively);

d) $f_2=\sum_{r<s} H_{p_r}\left(\sum_{r<i<n} H_{p_i}w_i+\sum_{s+1<j<n} H_{q_j}v_j+H_{q_r}v+H_\xi u\right)$, where $\chi(v)=\chi(f_2)=(1,\ldots,1,0,\ldots,0)$ (s ones).

Suppose now that V is a $\mathfrak{cosp}(m|2n)$-module and $f_{2s+r}=\sum K_1{}^sK_{x_{j_1}}\ldots K_{x_{j_r}}v_{j_1\ldots j_r}$ is a special vector of the $K(2n+1|m)$-module $T(V)$.

We note first of all that there is the imbedding $W(1|0)\to K(2n+1|m)$ given by the formula $f(t)\frac{d}{dt}\mapsto K_f$. It therefore follows from the description of $W(1|0)$-modules that the special vectors have the form

$$\sum K_1K_{x_{j_1}}\ldots K_{x_{j_r}}v_{j_1\ldots j_r}+\sum K_{x_{i_1}}\ldots K_{x_{i_{r+2}}}v_{i_1\ldots i_{r+2}}.$$

Below $\chi=(c;\chi_0)$, where c is the value on K_t, and χ_0 is the label of the weight relative to the standard basis of the Cartan subalgebra in $\mathfrak{osp}$.

a) $m=1$, $n=0$. Let $f_1=K_\xi v$. Then $K_{t\xi}f_1=K_tv$. Hence, $\chi(v)=0$, $\chi(f_1)=-1$. Let $f_2=K_1v$. Then $K_{t\xi}f_2=-K_\xi v$. Hence, $v=0$. Let $f_3=K_1K_\xi v$. Then $K_{t\xi}f_1=(K_1K_t-2k_\xi^2)v=K_1(K_t-1)v$ and $K_{t^2\xi}f_3=-4K_{t\xi}K_\xi v=-4K_tv$. Hence, $v=0$.

b) $m=2$, $n=0$. Let $f_1=aK_\xi+bK_\eta v$, where $ab=0$ (otherwise f_1 is not a weight vector). Then $K_{t\xi}f_1=b(K_t-K_{\xi\eta})v$, $K_{t\eta}f_1=a(K_1+K_{\xi\eta})v$. Hence, one of the following two cases is possible:

$$\begin{aligned}&1)\ a=0,\ \chi(v)=(\lambda,\lambda),\ \chi(f_1)=(\lambda-1,\lambda-1);\\&2)\ b=0,\ \chi(v)=(-\lambda,\lambda),\ \chi(f_1)=(-\lambda-1,\lambda+1).\end{aligned}$$

Let $f_2=aK_\xi K_\eta v+bK_1v$. Then

$$\begin{aligned}K_{t\xi}f_2&=-aK_\xi(K_t-K_{\xi\eta})v-2bK_\xi v,\\K_{t\eta}f_2&=aK_\eta(K_t-K_{\xi\eta}-2)v-2bK_\eta v,\\K_{t\xi\eta}f_2&=a(K_{t\xi}K_\eta-K_\xi K_{t\eta})v-2bK_{\xi\eta}v=a(K_t-K_{\xi\eta})v-2bK_{\xi\eta}v.\end{aligned}$$

Let $K_tv=Ev$, $K_{\xi\eta}v=Hv$. Then $2b=(H-E)a$, $2b=(H+E-2)a$, $2bH=(E-H)a$. Hence, the following three cases are possible:

1) $a=0$ and hence $b=0$;

2) $b=0$, and hence $\chi(v)=(1,1)$, $\chi(f_2)=(-1,1)$;

3) $ab\neq 0$, and hence $a=-b$ and $\chi(v)=(1,-1)$, $\chi(f_2)=(-1,-1)$. It is not hard to see that to the operator $dv_x^{-1}d:\Sigma_{-2}\to\Omega^1$ there correspond precisely two special vectors — one on each subsuperspace of type $T(\chi)$.

Let $f_3=aK_1K_\xi v+bK_1K_\eta v$ where $ab=0$ (otherwise f_3 is not a weight vector). Then $K_{t\xi}f_3=-2aK_\xi^2v+\ldots=aK_1v+\ldots$, where the terms ... are not proportional to K_1. Hence, $a=0$. Applying $K_{t\eta}$, we obtain $b=0$.

Let $f_4=K_1K_\xi K_\eta v$. Then

$$K_{t\xi}f_4 = -K_1K_\xi(K_t - K_{\xi\eta})v,$$
$$K_{t\eta}f_4 = K_1K_\eta(K_t - K_{\xi\eta} - 2)v,$$
$$K_{t\xi\eta}f_4 = -[2K_\xi K_\eta K_{\xi\eta} + K_1(K_t - K_{\xi\eta})]v.$$

Hence, $v = 0$. Thus, there are no singular vectors of degree ≤ -3.

c) $m = 3$, $n = 0$. Let $f_1 = K_\xi v_\xi + K_\eta v_\eta + K_\theta v_\theta$. Then

$$K_{\xi\theta}f_1 = K_\xi K_{\xi\theta}v_\xi + (-K_\theta + K_\eta K_{\xi\theta})v_\eta + (k_\xi + k_\theta k_{\xi\theta})v_\theta,$$
$$K_{t\eta}f_1 = (K_{\xi\eta} + K_t)v_\xi - 2K_{\eta\theta}v_\theta,$$
$$K_{\xi\eta\theta}f_1 = -K_{\xi\theta}v_\xi + K_{\eta\theta}v_\eta + K_{\xi\eta}v_\theta.$$

Thus, the following three cases are possible:

1) $\chi(v_\eta) = (n + 2, n)$, $\chi(f_1) = (n + 1, n - 1)$;

2) $\chi(v_\theta) = (1, -1)$, $\chi(f_1) = (0, -1)$, but the leading weight does not satisfy the necessary conditions: in this case V is an infinite-dimensional superspace;

3) $\chi(v_\xi) = (-n, n)$, $\chi(f_1) = (-n - 1, n + 1)$.

Let $f_2 = K_1v + K_\xi K_\eta v_{\xi\eta} + K_\xi K_\eta v_{\xi\theta} + K_\eta K_\theta v_{\eta\theta}$. Then

$$K_{\xi\theta}f_2 = K_1K_{\xi\theta}v + K_\xi(-K_\theta + K_\eta K_{\xi\theta})v_{\xi\eta} + K_\xi K_\theta K_{\xi\theta}v_{\xi\theta} + (-K_\theta^2 + 2K_\eta K_\xi + 2K_\eta K_\theta K_{\xi\theta})v_{\eta\theta},$$

$$K_{\xi\theta}v + \frac{3}{2}v_{\eta\theta} = 0, \quad K_{\xi\theta}v_{\xi\eta} = 2v_{\eta\theta}, \quad K_{\xi\theta}v_{\xi\theta} = v_{\xi\eta}, \quad K_{\xi\theta}v_{\eta\theta} = 0, \tag{1}$$

i.e., $v = aw$, $v_{\xi\eta} = bw$.

$$K_{t\eta}f_2 = K_\eta[-2a + (K_t + K_{\xi\eta} - 2)b]w + 2K_\eta K_{\eta\theta}v_{\eta\theta} + 2K_\xi K_{\eta\theta}v_{\xi\theta} + K_\theta(K_t + K_{\xi\eta} - 1)v_{\xi\theta},$$

$$K_{\xi\eta\theta}f_2 = (K_\theta - K_\eta K_{\xi\theta} - K_\xi K_{\eta\theta})v_{\xi\eta} - (K_\xi + K_\theta K_{\xi\theta} + K_\xi K_{\xi\eta})v_{\xi\theta} + (K_\eta - K_\theta K_{\eta\theta} - K_\eta K_{\xi\eta})v_{\eta\theta}.$$

Let E and H be the same as in part b). Then

$$[-2a + (E + H - 2)b]w - 2K_{\eta\theta}v_{\eta\theta} = 0, \tag{2}$$
$$(E + H - 1)v_{\xi\theta} = 0, \tag{3}$$
$$K_{\eta\theta}v_{\xi\theta} = 0, \tag{4}$$
$$(H - 1)v_{\xi\theta} - K_{\eta\theta}bw = 0, \tag{5}$$
$$K_{\xi\theta}v_{\xi\theta} + K_{\eta\theta}v_{\eta\theta} - bw = 0, \tag{6}$$
$$(1 - H)v_{\eta\theta} - K_{\xi\theta}bw = 0. \tag{7}$$

It follows from (7) and (1) that $(1 - H)v_{\eta\theta} - 2v_{\eta\theta} = 0$. Hence, $Hv_{\eta\theta} = -v_{\eta\theta}$. It follows from (3) that $E = 2$. We thus obtain three cases:

1) $\chi(v_{\eta\theta}) = (2, -1)$, $\chi(f_2) = (0, -2)$. If $v_{\eta\theta} = 0$, then

$$K_{\xi\theta}[(1 + H)v_{\xi\theta} + K_{\eta\theta}bw] = (2 + H)K_{\xi\theta}v_{\xi\theta} + Hbw = 2(1 + H)bw.$$

If $b = 0$, then it follows from (2) that $a = 0$. Thus, considering (2), we obtain

2) $\chi(w) = (2, -1)$, $b = 2a$, $\chi(f_2) = (0, -1)$.

If $v_{\eta\theta} = w = 0$, then

3) $\chi(v_{\xi\theta}) = (2, -1)$, $\chi(f_2) = (0, 0)$. In all cases the fiber in the bundle is infinite-dimensional.

d) $n = m = 1$. The special vectors relative to $H(2|1)$ and $Po(2|1)$ of degree -1 have the same form. The L_0-module in L_1, where $\mathscr{L} = K(3|1)$, which is "extra" as compared with $Po(2|1)$, is generated by the operator K_{tq}; therefore, the condition that f_1 be special is $K_{tq}f_1 = 0$. From it we immediately find that there are three possible cases:

1) $v_q \neq 0$, $\chi(v_q) = (-2, 1)$, $\chi(f_1) = (-3, 0)$;
2) $v_q = 0$, $v_\xi \neq 0$, $\chi(v_\xi) = (-1, 1)$, $\chi(f_1) = (-2, 1)$;
3) $v_q = v_\xi = 0$, $v_p \neq 0$, $\chi(v_p) = (0, 0)$, $\chi(f_1) = (-1, 1)$.

Imbedding $K(1|1)$ in $K(3|1)$ in the obvious manner, we see that there are no special vectors of degree -2.

We find by induction that the special vectors for $K(2n+1|1)$ are in form and weight relative to $\mathfrak{osp}(1|2n)$ the same as for $H(2n|1)$, and their weights on the center are

a) $z(v_s)=-2n+s-1,\ z(f_1)=-2n+s-2$;

b) $z(u)=-n,\ z(f_1)=-n-1$;

c) $z(w_r)=-r+1,\ z(f_1)=-r$.

e) $n=1$, $m=2$. We use the description of the special vectors for $H(2|2)$ of [69, 70]:

1) $f_1=H_q w^1+H_p w^2+H_\xi v_1+H_\eta v_2$, where $\chi(v_2)=(\mu,0)$, $\chi(f_1)=(\mu-1,0)$;

2) $f_1=H_\xi v_1$, where $\chi(v_1)=(\mu,0)$, $\chi(f_1)=(\mu+1,0)$;

3) $f_1=H_q w^1+H_p w^2+H_\xi v_1$, where $\chi(w^1)=(-1,1)$, $\chi(f_1)=(-1,0)$;

4) $f_1=H_p w^2+H_\xi v_1$, where $\chi(w^2)=(-1,0)$, $\chi(f_1)=(-1,1)$.

Then the condition $K_{tq}f_1=0$, where H_f is replaced by K_f, and the conditions $K_{t\eta}f_1=K_{t\xi}f_1=0$ following from it give in these four cases

1) $z(v_2)=\mu,\ z(f_1)=\mu-1$;

2) $z(v_1)=\mu,\ z(f_1)=\mu+1$;

3) $z(w^1)=-2,\ z(f_1)=-3$;

4) $z(w^2)=0,\ z(f_1)=-1$.

Noting that the special vectors f_2 relative to $K(1|2)$ have the form $K_\xi K_\eta v$ or $(K_\xi K_\eta v-K_1)v$, that relative to $K(3|0)$ the special vector f_2 has the form $K_1 v+K_p(K_q v_{12}+K_p v_{22})$, and observing the description of the special vectors f_2 relative to $H(2|2)$ in [69, 70], we find that only $K_1 v-K_p(K_q v_{12}+K_p v_{22})-K_\xi K_\eta v$ can be a special vector. However, the operations $K_{q\xi}$ and $K_{q\eta}$ take this vector into a vector which on restriction to $K(3|0)$ is not special. We find by induction that there are no special vectors of degree $\leqslant -2$ for $K(2n+1|2)$, while the vectors of degree -1 have the same form as for $H(2n|2)$ (see [69, 70]).

f) $m\geqslant 3$. As shown in [69, 70], the special vectors relative to $H(2n|m)$ for m 3 have the form a)-c):

a) $f_1=\sum H_{q_i}w_i+\sum H_{p_i}w^i+\sum H_{\xi_j}v_j$, where $\chi(v_m)=(\lambda,0,\ldots,0)$, and $\chi(f_1)=(\lambda-1,0,\ldots,0)$;

b) $f_1=H_{\xi_1}v_1$, where $\chi(v_1)=(\lambda,0,\ldots,0)$, and $\chi(f_1)=(\lambda+1,0,\ldots,0)$;

c) $f_2=H_{\eta_1}\left(\sum H_{q_i}w_i+\sum H_{p_i}w^i+\sum_{2<j<m}H_{\xi_j}v_j\right)$, where $\chi(v_m)=\chi(f_2)=(\lambda,0,\ldots,0)$.

From the theorem on the leading weight it follows that the fiber is finite-dimensional if λ is an integer, while $\lambda\geqslant 1$ in case a) and $\lambda\geqslant 0$ in cases b) and c). Replacing H_f by K_f, we find that

$$K_{tq_1}f_1=-\sum K_{q_1q_i}w_i+(K_{p_1q_1}-K_1)w^1-\sum_{i\geqslant 2}K_{q_1p_i}w^i-\sum K_{q_1\xi_j}v_j$$

in case a), while in case b) $K_{tq_1}f_1=-K_{q\xi_1}v_1$. From the conditions on the vectors a) and b) in [69, 70] it follows that they are special. In order to find their weights, we compute $K_{t\xi_m}f_1$ in case a) and $K_{t\xi_1}f_1$ in case b). As for $m=2$, we find that $z(V)=\lambda$.

Induction completes the description of the special vectors of degree -2; they are possible only for $K(2n+1|2n+2)$ (since $K(2n+1|2n+2)$ preserves volume).

6. <u>$\mathscr{L}$ of the Series $\mathfrak{gl}$, $\mathfrak{sl}$, Π, $S\Pi$ and $\mathfrak{osp}(2|2n)$.</u> For these cases see [10, 33, 34, 37].

7. For a proof of exactness of the sequences (1)-(4) in the formal case see [11] and in the smooth case [39]. The proof of [11] goes through on the "horizontal" portions of the

sequences of $\mathscr{L}$-modules for $\mathscr{L} \neq W(n|m)$ (see 34), i.e., where the modules are connected by operators of order 1. At other sites we consider the Cartan subalgebra $\mathfrak{h} \subset L_0$. The elements of $\mathfrak{h}$ act trivially on the homologies of the complexes considered. Since the elements of $\mathfrak{h}$ have all eigenvalues equal to 0 only for the sequence (4) considered as a sequence of $\mathscr{L}$-modules relative to the imbedding $\mathscr{L} \subset W(0|N)$, where $N = \dim L_{-1}$, the assertion regarding exactness of the sequence of modules has been proved.

4. Characters of Irreducible Modules

We note first of all that there exist two irreducible $\mathscr{L}$-modules with leading weight χ: if one of them (with even leading vector) is denoted by V_χ, then the other is $\Pi(V_\chi)$ and $\operatorname{ch} \Pi(V_\chi) = \varepsilon \operatorname{ch} V_\chi$.

1. The characters of $W(0|m)$-modules are described in Sec. 3; for the finite-dimensional cases of the series S, SH, and their derivatives see [38, 66, 67]. We observe that by carrying N in the formulas of 3.2 under the summation sign we obtain the more symmetric expressions

$$\operatorname{ch} \operatorname{irr} \Omega^r = \frac{1}{D} \sum_{w \in W} \operatorname{sgn} w \cdot w \left[e^{\rho + r\beta_1} \prod_{\beta_i \neq \beta_1} \left(1 + \varepsilon e^{\beta_i}\right) \right], \qquad \operatorname{ch} \operatorname{irr} \Sigma_{-r-1} = \frac{\varepsilon^{n+1}}{D} \sum_{w \in W} \operatorname{sgn} w \cdot w \left[e^{\rho - r\beta_1 - \Phi} \prod_{\beta_i \neq \beta_1} \left(1 + \varepsilon e^{\beta_i}\right) \right].$$

2. For induced $\mathscr{L}$-modules ($\mathscr{L}$ of the series $\mathfrak{sl}$, $\mathfrak{gl}$, $\mathfrak{osp}(2|2n)$) we obtain by carrying N under the summation sign in the numerator and introducing P = ρ — (the half sum of positive roots corresponding to odd vectors)

$$\operatorname{ch} I(L_\chi) = \frac{\sum_{w \in W} \operatorname{sgn} w \cdot w \left[e^{\chi + P} \prod \left(1 + \varepsilon e^{\beta_i}\right) \right]}{\sum_{w \in W} \operatorname{sgn} w \cdot e^{wP}}.$$

For ε = 1 (respectively, −1) this formula becomes a "character" (respectively, "supercharacter") in the sense of Kac [105, 106]. Using the resolutions for noninduced $\mathfrak{sl}(1|n)$- and $\mathfrak{osp}(2|n)$-modules, we obtain the formula

$$\operatorname{ch} \operatorname{irr}(L_\chi) = \frac{\sum_{w \in W} \operatorname{sgn} w \cdot e^{w(\chi + P)} \prod_{(\chi + P, w\beta_i) \neq 0} \left(1 + \varepsilon e^{\beta_i}\right)}{\sum_{w \in W} \operatorname{sgn} w e^{wP}},$$

where (,) is a nondegenerate invariant form on $\mathscr{L}$.

Conjecture. After appropriate modification in the definition of P this formula holds for modules over Kac–Moody superalgebras having a Cartan matrix. This conjecture has been proved when all odd roots are nonisotropic (see [105]).

3. We shall show how to derive the formulas of parts 1, 2. From the exactness of the sequence (4) and the fact that $p(d) = \bar{1}$, we have

$$\operatorname{ch} \operatorname{irr} \Omega^r = \sum_{i > 0} (-\varepsilon)^i \operatorname{ch} \Omega^{r+i} = \frac{N}{D} \sum_{i > 0} \sum_{w \in W} (-\varepsilon)^i \operatorname{sgn} w e^{w(\rho + (r+i)\beta_1)}$$

$$= \frac{N}{D} \sum_{w \in W} \operatorname{sgn} w e^{w(\rho + r\beta_1)} \sum_{i > 0} \left(-\varepsilon e^{w\beta_1 i}\right) = \frac{N}{D} \sum_{w \in W} \operatorname{sgn} w \frac{e^{w(\rho + r\beta_1)}}{1 + \varepsilon e^{w\beta_1}}.$$

Similarly, after shifting by P, the sequences for the series $\mathfrak{sl}$, $\mathfrak{osp}(2|2n)$ form geometric progressions with denominators e^{β_i} with different i for different horizontal lines, but after averaging relative to the Weyl group W it is possible to compute the sequence for some one denominator. Together with the computations of the number $(\chi + P, \beta_i)$ carried out in [106], we obtain the required result.

4. Theorem on the Leading Weight. For Lie superalgebras $\mathfrak{g} = \mathfrak{osp}(m|2n)$ for m ≠ 1, 2 the formula for the characters has not been proved as it has for AG_2 and AB_3. The theorem on the leading weight has been proved. Surprisingly enough, the conditions which must be imposed on the leading weight do not coincide with the conditions for $\mathfrak{g}_{\bar{0}}$! These conditions, which are quite intricate, were found by Kac [103, 104].

THEOREM. Let $\mathfrak{g}$ be one of the simple, finite-dimensional, contragradient Lie superalgebras, and let V be a finite-dimensional, irreducible $\mathfrak{g}$-module. Then there exists a unique leading weight vector χ, i.e., one that is annihilated by operators corresponding to positive roots, and if $\chi_1 = \chi_2$, then $V_1 \cong V_2$ or $V_1 \cong \pi(V_2)$. We normalize the Cartan matrix so that $a_{s,s+1} = 1$ if $a_{ss} = 0$, and we let $\{h_i\}$ be the standard basis in the Cartan subalgebra of the Lie algebra $\mathfrak{g}_{\bar{0}}$ (see [12]). Then the labels of the leading weight $\chi_i = \chi(h_i)$ satisfy the following conditions: $a_i \in Z^+$ if $i \neq s$, and for numbers k and b of the table

$\mathfrak{g}$	k	b
$\mathfrak{osp}(1\mid 2n)$	$a_n 1/2$	0
$\mathfrak{osp}(2m+1\mid 2n)$	$a_n - u_{n+1} - a_{m+n-1} - a_{m+n} 1/2$	$m>1$
$\mathfrak{osp}(2m\mid 2n)$	$a_n - a_{n+1} - a_{m+n} - (a_{m+n-1} - - a_{m+n})/2$	$m>2$
$D(\alpha)$	$(1-\alpha)^{-1}(2a_1 - a_2 - \alpha a_3)$	2
AG_2	$1/2(a_1 - 2a_2 - 3a_3)$	3
AB_3	$(2a_1 - 3a_2 - 4a_3 - 2a_4)/3$	4

the following conditions are satisfied: $k \in Z^+$ and $k \neq 1$ for AG_2 and AB_3, while if $k < b$, then

For	The following relations hold
$\mathfrak{osp}(2m+1\mid 2n)$	$a_{n+k+1} = \ldots = a_{m+n} = 0$
$\mathfrak{osp}(2m\mid 2n)$ $m>2$	$a_{n+k+1} = \ldots = a_{m+n}$ for $k \leqslant m-2$ $a_{m+n-1} = a_{m+n}$ for $k = m-1$
$D(\alpha)$	$a_l = 0$ for $k=0$ $(a_3+1)\alpha = \pm(a_2+1)$ for $k=1$
AB_3	$a_l = 0$ for $k=0$ $a_2 = a_4 = 0$ for $k=2$ $a_2 = 2a_4 + 1$ for $k=3$
AG_2	$a_l = 0$ for $k=0$ $a_2 = 0$ for $k=2$

If $\mathfrak{g} = \mathfrak{psq}(n-1)$ then the labels of the leading weight relative to the standard basis of the Cartan subalgebra in $\mathfrak{g}_{\bar{0}}$ are such that $a_i \in Z^+$, while if $a_i = 0$, then $a_1 + 2a_2 + \ldots + (i-1)a_{i-1} = a_n + 2a_{n-1} + \ldots (n-i)a_{i+1}$.

For other simple, finite-dimensional $\mathfrak{g}$ the restrictions on the labels of the leading weight are the same as for the Lie algebra $\mathfrak{g}_0$ in the standard gradation of the Lie superalgebra $\mathfrak{g}$.

5. Other Results on Lie Superalgebras

Structure. Kac [18-20] classified the simple, finite-dimensional Lie superalgebras over C. Part of his results were obtained independently by many others (see [104, 112, 127, 130]). We indicate without proofs that the relations between the generators of a simple Lie superalgebra with a Cartan matrix are analogous to the relations (DR) of Sec. 0 for a Lie algebra if $\mathfrak{g} \neq \mathfrak{psq}$ or $\mathfrak{psq}^{(1)}$ For the noncontragradient Kac superalgebras $\mathfrak{psq}$ and $\mathfrak{psq}^{(1)}$ the relations have not been computed.

For simple infinite-dimensional Lie superalgebras of formal vector fields and string theories the relations in $\mathfrak{n}_+$ have degree 2 for $n + m \geqslant 10$ (this estimate can sometimes be reduced) where (n, m) is the dimension of the space on which the superalgebra is realized by vector fields.

The classification of Lie superalgebras (infinite-dimensional, Z-graded algebras of finite growth) is apparently close to the stage of the classification of the analogous Lie algebras, and, other than the deformations and contractions of the examples of Sec. 1, there are no other simple Lie superalgebras.

What is a semisimple Lie superalgebra? Let $\mathscr{E}$ be a principal bundle over a (0, q)-dimensional base $\mathscr{B}$ with gauge supergroup $\mathscr{G}$ whose Lie superalgebra $\mathfrak{g}$ is simple. The Lie superalgebra of the supergroup of diffeomorphisms of the bundle $\mathscr{E}$ is $S = \operatorname{Der}\mathfrak{g}\otimes\Lambda(q) ⊕ W(0|q)$. Let S_i be a Lie superalgebra such that $\mathfrak{g}\otimes\Lambda(q)\subset S_i\cap S\subset \operatorname{der}\mathfrak{g}\otimes\Lambda(q)$, and $S_i \pmod{\mathfrak{g}\otimes\Lambda(q)}\supset \langle \frac{\partial}{\partial\xi_1},\dots,\frac{\partial}{\partial\xi_q}\rangle$. Then a semisimple Lie superalgebra is the sum of superalgebras of the form S_i (see [104]).

What conditions must be imposed in order that at least part of the semisimple Lie superalgebras can be described explicitly? Is it possible to require that $\operatorname{Aut}\mathscr{B}\subset W(0|q)$ be simple? The possibility suggested in [5] reduces to the direct sum of simple algebras, which is uninteresting.

An Analogue of the Lie Theorem [104]. Let $\mathfrak{g}$ be a solvable Lie superalgebra. A linear form $l\in\mathfrak{g}^*$ is called distinguished if $l(\mathfrak{g}_{\bar 1})=l([\mathfrak{g}_{\bar 0}, \mathfrak{g}_{\bar 0}])=0$. Let D be the space of distinguished forms, let $D_0=\{l\in D\mid l([\mathfrak{g},\mathfrak{g}])=0\}$, and let D_1 be the subgroup of D_0 generated by forms which are given by 1-dimensional factors in the adjoint representation of the Lie superalgebra $\mathfrak{g}$.

Let $d\subset D_0$ be a subgroup with $\lambda\in d$, and let ρ be a representation of the Lie superalgebra $\mathfrak{g}$ in the superspace V. The representations ρ and $\rho_\lambda = \rho\otimes 1_\lambda$, where 1_λ is the 1-dimensional representation given by the character λ, are called d-equivalent.

For $l\in D/D_0$ we set $\mathfrak{g}_l=\{g\in\mathfrak{g}\mid l([g, g_1])=0 \text{ for } g_1\in\mathfrak{g}\}$. It is obvious that $\mathfrak{g}_{\bar 0}\subset\mathfrak{g}_l\subset\mathfrak{g}$, $\mathfrak{g}_l$ is a Lie superalgebra, and $l([\mathfrak{g}_l, \mathfrak{g}_l])=0$. The Lie superalgebra $\mathfrak{p}\subset\mathfrak{g}$ is called subordinate to the functional l if $l([\mathfrak{p},\mathfrak{p}])=0$ and $\mathfrak{g}_l\subset\mathfrak{p}$.

A solvable Lie superalgebra for which all irreducible factors in the adjoint representation are 1-dimensional is called completely solvable.

We denote by $\{\mathfrak{g}, l\}$ the 1-dimensional $\mathfrak{g}$-module defined on the basis of the form $l\in D$ by the formula $g(v) = l(g)v$ where $v\in\{\mathfrak{g}, l\}$.

THEOREM. If V is an irreducible, finite-dimensional $\mathfrak{g}$-module, then all irreducible factors of the module V considered as a $\mathfrak{g}_{\bar 0}$-module are one-dimensional, while the linear forms on $\mathfrak{g}$ (extended by zero to $\mathfrak{g}_{\bar 1}$) corresponding to them belong to one class $l_V\in D/D_0$.

Let $[l]\in D/D_0$, and let $\mathfrak{p}$ be a maximal Lie subsuperalgebra subordinate to the class $[l]$. Then the $\mathfrak{g}$-module $\operatorname{Ind}^{\mathfrak{g}}_{\{\mathfrak{p},l\}}\{\mathfrak{p}, l\}$, where $l\in[l]$, is irreducible. Two such $\mathfrak{g}$-modules are D_0-equivalent if $[l_1] = [l_2]$.

Any finite-dimensional, irreducible $\mathfrak{g}$-module V is isomorphic to a module of the form $\operatorname{Ind}^{\mathfrak{g}}_{\mathfrak{p}}\{\mathfrak{p}, l\}$, where $l\in[l_V]$, while $\mathfrak{p}$ is a maximal Lie subsuperalgebra subordinate to the functional l.

If $\mathfrak{g}$ is completely solvable, then everywhere above in the theorem D_0 can be replaced by D_1.

Nonsolvable Finite-Dimensional Representations. They are described for Lie superalgebras of the series $\mathfrak{gl}(1|n)$, $\mathfrak{sl}(1|n)$ and $\mathfrak{osp}(2|2n)$ (see [37, 72]). This can be done by reduction to [17] where nonsolvable representations of the superalgebra $\Lambda(2)$ are described.

Invariant Operators. Unary operators invariant relative to Lie superalgebras of vector fields have practically not been described in the nonstandard representation (see only [35]); almost all such operators are new, and it would be very interesting to give an interpretation of them.

As we have seen, with unary operators there are associated complexes of free $\Lambda(n)$-modules. It can be shown that to such complexes there correspond bundles on P^n (see [2, 3, 7]). A large number of natural examples of such complexes were presented above. What kinds of bundles correspond to them?

P. Ya. Grozman described binary invariant differential operators in tensor fields. Lie superalgebras also crept into his list (see [14]). We shall consider them in more detail. Let $\Omega^i_\mu=\Omega^i\otimes_{\mathscr{F}}T(\mu,\dots,\mu)$, and let $\Omega^*_* = \bigoplus_{i>0,\mu\in C}\Omega^i_\mu$; similarly, let $L^*_* = \bigoplus_{i>0,\mu\in C} L^i_\mu$ where $L^i_\mu=E^i_{\mathscr{F}}(W(n|0))\otimes_{\mathscr{F}}T(\mu,\dots,\mu)$. Then in local coordinates the Grozman operators P_6 and P_8 are given by the formulas

$$P_6(\omega_1\delta^\mu, \omega_2\delta^\nu)=[\nu(d\omega_1)\omega_2-(-1)^{p(\omega_1)}\mu\omega_1 d\omega_2]\,\delta^{\mu+\nu},$$

$$P_8(X\delta^\mu, Y\delta^\nu)=[(\nu-1)(\mu+\nu-1)\operatorname{div}X\cdot Y+(-1)^{p(X)}(\mu-1)\quad(\mu+\nu-1)\cdot\operatorname{div}Y-(\mu-1)(\nu-1)\operatorname{div}XY]\,\delta^{\mu+\nu};$$

where $\omega_1, \omega_2 \in \Omega^*$, $X, Y \in L^*$, δ is the volume element, and

$$\operatorname{div} \prod D_i = \sum (-1)^i \operatorname{div} D_i \prod_{j \neq i} D_j.$$

By multiplying the operator P_6 (respectively, P_8) by a suitable function of the degree of the volume form, we define the structure of a Lie superalgebra on part of the superspace $\Pi(\Omega^*_*)$ (respectively, $\Pi(L^*_*)$). For example, $P_6' = \frac{\mu - \nu}{\mu\nu} P_6$ gives such a structure on $\Pi(\Omega^*_*/d\Omega^*)$, $P_6'' = \frac{\mu-\nu}{\mu+\nu} P_6$ on $\Pi\left(d\Omega^* \oplus \left(\underset{\mu \neq 0}{\oplus} \Omega^*_\mu\right)\right)$.

Problem. Describe the domain of the operator $P_8' = (1/(\mu + \nu - 1))P_8$. Compute other functions $f(\mu, \nu)$ such that $f(\mu, \nu)P_8$ defines the structure of a Lie superalgebra on the part of the superspace L^*_* and describe this part.

We note that by considering twisted forms depending on μ in such a manner that they decay rapidly as $|\mu| \to \infty$ and passing to their Fourier transforms on μ it is possible to define new operators P_6' and P_6'' on the entire space obtained (the same goes for P_8), for example,

$$P_6'(f, g) = d(fg) - (-1)^{p(f)} dIf \cdot \frac{\partial g}{\partial t} - \frac{\partial f}{\partial t} dIg,$$

where f, g are functions of x, dx, and t, and $If = -i \int_{-\infty}^{t} f(x, dx, t_0)dt_0$, i.e., $I = (\text{"}d^{-1}\text{"})/dt$.

Problem. Give an interpretation of the Lie superalgebras defined by operators obtained from P_6 on Ω^*_* and its Fourier image (and the same for P_8).

Grozman also described binary operators invariant relative to Lie algebras of the series S, H, and K. With several exceptions they reduce to W-invariant operators. Binary operators invariant under $W(n|m)$ are obtained from Grozman's list if, aside from Ω, we also consider the spaces Φ and Σ.

Lie superalgebras of supersymmetry groups (the analogues of Poincare algebras) are contractions of the real forms of simple Lie superalgebras. Their extensions and representations (including infinite-dimensional representations) are studied mainly by physicists. Various special results have so far been obtained [64, 119-124].

Cohomologies. For the definition see [31, 63]. Some cohomologies with trivial coefficients for simple finite-dimensional Lie superalgebras have been computed in [63]. The cohomologies of nilpotent subalgebras of simple Lie superalgebras, which have a number of applications, have only partially been computed. The corresponding answers are presented above in the form of resolutions of the induced modules and in [52, 109].

Some cohomologies are computed in [39] in connection with the description in cohomological terms of important differential-geometric concepts such as the Riemann and Weyl tensors, etc.

Lie Superalgebras and Differential Equations. With the help of the point functor the scheme of Adler–Kostant can be carried over to Lie superalgebras without difficulty, but it aquires an unexpected feature: the mechanics can be described not only by the Poisson bracket but also by the Butane bracket [25, 41, 42]. Complete integrability of such equations is guaranteed by the theorems of V. N. Shander (see [39]). Multidimensionalization has partially been considered in [30].

Characteristic p. Practically nothing has been done here. The definition of Lie superalgebras in a simple characterization requires additional conditions (see [39]). In characteristic 2 it is possible to not observe the difference between algebras and superalgebras, and some examples of such algebras were presented in [113].

Some Isolated Results. Invariant functions on Lie superalgebras are described in [40, 58, 131]. Instantons are described in [16] in terms of the Grassmann superalgebra. In [24, 39] a connection is indicated between Hill equations and "part" of KdV, orbits of the group corresponding to the Virasoro algebra V, and the superalgebra $\mathscr{K}(1)$. The description of irreducible representations of finite-dimensional Lie superalgebras [34] was later repeatedly rediscovered in partial form [76-78, 82-86, 88, 96, 99-102, 119-124]. Another approach to the description of representations by means of analogues of Young diagrams was successfully applied in [79, 81]. In [115] a rather strange condition was distinguished — congruence of representations of Lie superalgebras. Finally, the result of the notes [52, 115, 128] is clear from their title.

Very recently in a series of papers in Dokl. Akad. Nauk D. P. Zhelobenko obtained an explicit description of homomorphisms between Verma modules for simple, finite-dimensional Lie algebras which carries over to Lie superalgebras with a Cartan matrix and to some superalgebras of string theories, while I. Penkov and I. Skornyakov supered the results of [80, 90] regarding $\mathscr{D}$-modules by obtaining an analogue of the Borel—Weyl—Bott theorem. We note that another analogue of this theorem in the form of a result regarding the cohomologies of a maximal nilpotent subalgebra has so far been obtained (B. L. Feigin and D. A. Leites) only for $\mathfrak{sl}(1|n)$ and $\mathfrak{osp}(2|2n)$ and the simplest root system in them — with one gray circle [104, 42] at the very end of the diagram. For subalgebras corresponding to other systems the answer is very complicated. Apparently, these difficulties are connected with the fact that it is so far unclear what serves as an analogue of the Weyl group for superalgebras. In the formula for characters over $\mathfrak{sl}$, for example, the terms are numbered by integral points of positive cones $(Z^+)^{\dim \mathfrak{h}}$ numbered by the Weyl group of the even part. Some isolated results have been obtained in [39, 51, 92].

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