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The Finite Element Method for Elliptic Equations with Discontinuous Coefficients[1]

By

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Dedicated to Professor Dr. L. Collatz on the occasion of his 60th birthday

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Summary — Zusammenfassung

The Finite Element Method for Elliptic Equations with Discontinuous Coefficients. Numerical solutions of boundary value problems for elliptic equations with discontinuous coefficients are of special interest. In the case when the interface (i.e. the surface of the discontinuity of the coefficients) is smooth enough, then also the solution is usually very smooth (except on the interface). To obtain a high order of accuracy presents some difficulty, especially if the interface does not fit with the elements (in the finite element method). In this case the norm of the error in the space W_2^1 is of the order $h^{1/2}$ (see e.g. [1]) and on one dimensional case it is easy to see that the accuracy cannot be improved. In this paper we shall show an approach which avoids this difficulty. The idea is similar to [2]. We shall show the proposed approach on a model problem — the Dirichlet problem with an interface for Laplace equation; this will avoid pure technical difficulties. The boundary of the domain and the interface will be assumed smooth enough. The sufficient condition for the smoothnees can be determined.

Die Methode der finiten Elemente für elliptische Gleichungen mit diskontinuierlichen Koeffizienten. Numerische Lösungen von Randwertproblemen elliptischer Gleichungen mit diskontinuierlichen Koeffizienten sind von besonderem Interesse. In jenem Fall, wo die „Sprungfläche" (d. h. die Fläche der Sprungstelle der Koeffizienten) genügend glatt ist, ist auch die Lösung normalerweise glatt (außer auf der „Sprungfläche" selbst). Es bereitet einige Schwierigkeiten, einen hohen Grad von Genauigkeit zu erzielen, speziell, wenn die „Sprungstelle" nicht mit den Elementen zusammenfällt (in der Methode der finiten Elemente). In diesem Fall liegt die Norm des Fehlers in dem Raum W_2^1 in der Größenordnung von $h^{1/2}$ (siehe z. B. [1]) und im eindimensionalen Fall kann man leicht erkennen, daß die Genauigkeit nicht verbessert werden kann. In dieser Arbeit wird ein Weg (ähnlich [2]) gezeigt, welcher diese Schwierigkeit vermeidet. Der vorgeschlagene Weg wird an einem Modellfall erläutert — das Dirichlet-Problem mit einer Sprungfläche für die Laplace-Gleichung; dadurch werden rein technische Schwierigkeiten vermieden. Die Randfläche und die „Sprungfläche" werden glatt genug angenommen. Eine hinreichende Bedingung für die Glattheit kann angegeben werden.

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1. The Principle Notions

Let E_n be the n-dimensional Euclidian space. Put $\underline{x} = (x_1, \ldots, x_n)$. Let Ω_0 and Ω_1 ($\Omega_1 \subset \Omega_0$) be bounded domains with smooth boundaries $\dot{\Omega_0}$ and $\dot{\Omega_1}$[1]. Further let us assume that $\dot{\Omega_1} \subset \Omega_0$, and let $\Omega_2 = \Omega_0 - \Omega_1$. Let the spaces $W_2^k(\Omega_j)$, $j = 1, 2$, (resp. ($W_2^k(\dot{\Omega_j})$, $j = 0, 1$) for $k \geqslant 0$, $k \leqslant 0$, will be the fractional spaces on Ω_j (resp. $\dot{\Omega_j}$), see e.g. [3]. Further let $W_2^k(\Omega_0) = W_2^k(\Omega_1) \dotplus W_2^k(\Omega_2)$ ($\dotplus$ means direct sum). Every function $u \in W_2^k(\Omega)_0$ can be written in the form

$$u = (u_1, u_2), \tag{1.1}$$

where $u_1 \in W_2^k(\Omega_1)$ and $u_2 \in W_2^k(\Omega_2)$ [see e.g. [3]][2] and let

$$\| u \|^2_{W_2^k(\Omega_0)} = \| u_1 \|^2_{W_2^k(\Omega_1)} + \| u_2 \|^2_{W_2^k(\Omega_2)}.$$

Further let $a_i > 0$, $i = 1, 2$ be given. Let us define $W_2^k(\Omega_0, a_1, a_2) \subset W_2^k(\Omega_0)$ for $k \geqslant 1$ the space of all functions $u = (u_1, u_2)$ such that

$$\begin{aligned} u_1 &= u_2 \text{ in } W_2^{1/2}(\dot{\Omega_1}), \\ a_1 \frac{\partial u_1}{\partial n} &= a_2 \frac{\partial u_2}{\partial n} \text{ in } W_2^{-1/2}(\dot{\Omega_1}), \end{aligned} \tag{1.2}$$

where n is the outside normal to $\dot{\Omega_1}$.

Definition 1.1. *Let there be given number $k \geqslant 0$, $k' \geqslant 0$. A system of linear manifolds $M_h^k(\Omega_0) \subset W_2^{k'}(\Omega_0)$, $0 \leqslant h \leqslant 1$ will be said to be k-proper if $M_h^k(\Omega_0)$, has the following property. For every $f \in W_2^l(\Omega_0)$, $l \geqslant 0$, it is possible to find a $v_h(f) \in M_h^k$, such that for every $l \geqslant p$, $p \leqslant k'$, $p \geqslant 0$*

$$\| f - v_h(f) \|_{W_2^p(\Omega_0)} \leqslant C(p, l)\, h^\mu \| f \|_{W_2^l(\Omega_0)} \tag{1.3}$$

where

$$\mu = \min(k - p, l - p).$$

Such spaces are discussed in [4]. If $f \in W_2^k(\Omega_0)$ then we place $f = (f_1, f_2)$. By the continuation theorem we may continue $f_i \in W_2^k(\Omega_i)$ in $W_2^k(E_n)$ while preserving the norm.

By [4] we may approximate f_1 by a linear combination of "hill" functions with the desired property in E_n and therefore also in $W_2^k(\Omega_1)$ and similar to $W_2^k(\Omega_2)$.

Now let us formulate our problem. Let $f \in W_2^k(\Omega_0)$, $g \in W_2^s(\dot{\Omega_0})$ and let a_i, $i = 1, 2$, $\underline{a} = (a_1, a_2)$ be positive numbers. Then the problem $P(\underline{a}, f, g)$ will be to find a weak solution u such that

$$-\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\, a(x) \frac{\partial u}{\partial x_i} = f \text{ on } \Omega_0, \tag{1.4}$$

[1] For simplicity, let us assume that the boundary $\dot{\Omega_0}$ and $\dot{\Omega_1}$ are C^∞.

[2] In [3] the spaces are denoted by $H^s(\Omega)$ (resp. $H^s(\Gamma)$).

$$u = g \text{ on } \Omega_0^{\cdot}, \tag{1.5}$$

and $a(x) = a_1$ on Ω_1 and equals a_2 on Ω_2.

The following result holds.

Theorem 1.1. *Let $k > -1$, $s > 1/2$. Then there exists exactly one solution u of the problem $P(\underline{a}, f, g)$ in $W_2^1(\Omega_0, a_1, a_2)$. Moreover we have*

$$\| u \|_{W_2^q(\Omega_0)} \leqslant C(q, k, s) \| f, g \|_{k,s} \tag{1.6}$$

where

$$\| f, g \|_{k,s} = \| f \|_{W_2^k(\Omega_0)} + \| g \|_{W_2^s(\Omega_0^{\cdot})} \tag{1.7}$$

and

$$q = \min(k + 2, s + 1/2)$$

see e.g. [5], [6], [7][3].

2. Numerical Solution of the Problem P (a, f, g)

Let $\psi_i(h)$, $i = 1, 2$, be a function defined for $h \in (0, 1)$. Further let $v_h^\psi \in M_h^r$ (M_h^r be r-proper and the closure is taken in $W_2^1(\Omega_0)$) be such that $v_h^\psi = (v_{h,1}, v_{h,2})$ minimize on $\overline{M_h^r}$ the quadratic functional

$$F_\psi(\underline{v}) = \sum_{j=1,2} \int_{\Omega_j} a_i \left(\sum \frac{\partial v_j}{\partial x_i} \right)^2 d\underline{x} + a_2 \psi_1(h) \oint_{\Omega_0^{\cdot}} (v_2 - g)^2 \, d\underline{s} + \\ + \psi_2(h) \oint_{\Omega_1^{\cdot}} (v_1 - v_2)^2 \, d\underline{s} - 2 \sum_{j=1,2} \int_{\Omega_j} f v_i \, d\underline{x}, \tag{2.1}$$

$$v = (v_1, v_2).$$

Now the following theorem holds.

Theorem 2.1. *Let $k > -1/2$, $s > 1$, $r \geqslant 1$, $r' = 1$ and $u = (u_1, u_2)$ the solution of the problem $P(\underline{a}, f, g)$. Further let $\psi_i(h) = C h^{-\sigma_i}$, $C > 0$, $\sigma_i > 0$, then*

$$\| u - v_h^\psi \|_{W_2^1(\Omega_0)} \leqslant C(\varepsilon) h^\mu \| f, g \|_{k,s} \tag{2.2}$$

with

$$\mu = \min \Big[k + 1, \; s - \frac{1}{2}, \; k + \frac{3}{2} - \varepsilon - \frac{\sigma_1}{2}, \; k + \frac{3}{2} - \varepsilon - \frac{\sigma_2}{2}, \\ s - \varepsilon - \frac{\sigma_1}{2}, \; s - \varepsilon - \frac{\sigma_2}{2}, \; \frac{\sigma_1}{2}, \; \frac{\sigma_2}{2}, \; r - 1, \\ r - \frac{1}{2} - \varepsilon - \frac{\sigma_1}{2}, \; r - \frac{1}{2} - \varepsilon - \frac{\sigma_2}{2} \Big], \tag{2.3}$$

with $\varepsilon > 0$, arbitrary.

Proof. Let us define the following quadratic functional

[3] In this paper C is a general constant different from place to place.

$$R_\psi(v) = \sum_{j=1,2} \int_{\Omega_j} a_j \left(\sum_i \frac{\partial(v_j - u_j)}{\partial x_i}\right)^2 d\underline{x} + \\ + a_2 \psi_1(h) \oint_{\Omega_0^{\cdot}} \left(\frac{\partial u_2}{\partial n} \frac{1}{\psi_1(h)} + (v_2 - g)\right)^2 d\underline{s} + \\ + \psi_2(h) \oint_{\Omega_1^{\cdot}} \left[\left(a_1 \frac{\partial u_1}{\partial n} + a_2 \frac{\partial u_2}{\partial n}\right) \frac{1}{2\psi_2(h)} + (v_1 - v_2)\right]^2 d\underline{s}. \tag{2.4}$$

If $k > -1/2$, $s > 1$, then using the *theorem 1.1* and SOBOLEV's embedding theorems for fractional spaces we get on integrating by parts

$$R_\psi(v) = \sum_{j=1,2} \int_{\Omega_j} a_j \left(\sum_i \left(\frac{\partial u_j}{\partial x_i}\right)\right)^2 d\underline{x} + \\ + \sum_{j=1,2} \int_{\Omega_j} a_j \left(\sum \left(\frac{\partial u_j}{\partial x_i}\right)^2\right) d\underline{x} + \\ + 2 \sum_{j=1,2} \int_{\Omega_j} a_j (\Delta u_i) v_i \, d\underline{x} - \\ - 2 a_2 \oint_{\Omega_0^{\cdot}} \frac{\partial u_2}{\partial n} v_2 \, d\underline{s} + 2 a_2 \oint_{\Omega_0^{\cdot}} \frac{\partial u_2}{\partial n} v_2 \, d\underline{s} - \\ - 2 a_2 \oint_{\Omega_0^{\cdot}} \frac{\partial u_2}{\partial n} g \, d\underline{s} + \frac{a_2}{\psi_1(h)} \oint_{\Omega_0^{\cdot}} \left(\frac{\partial u_2}{\partial n}\right)^2 d\underline{s} + \\ + a_2 \psi_1(h) \oint_{\Omega_0^{\cdot}} (v_2 - g)^2 d\underline{s} + 2 \oint_{\Omega_1^{\cdot}} \left(a_2 \frac{\partial u_2}{\partial n} v_2 - a_1 \frac{\partial u_1}{\partial n} v_1\right) d\underline{s} + \\ + \oint_{\Omega_1^{\cdot}} \left(a_1 \frac{\partial u_1}{\partial n} + a_2 \frac{\partial u_2}{\partial n}\right)(v_1 - v_2) \, d\underline{s} + \\ + \frac{1}{4\psi_2(h)} \int_{\Omega_1^{\cdot}} \left(a_1 \frac{\partial u_1}{\partial n} + a_2 \frac{\partial u_2}{\partial n}\right)^2 d\underline{s} + \psi_2(h) \oint_{\Omega_1^{\cdot}} (v_1 - v_2)^2 d\underline{s}.$$

Now because (u_1, u_2) is the solution in $W_2^1(\Omega_0, a_1, a_2)$ we have for every $v \in W_2^{1/2}(\Omega_1^{\cdot})$

$$\oint_{\Omega_1^{\cdot}} a_2 \frac{\partial u_2}{\partial n} v \, d\underline{s} = \oint_{\Omega_1^{\cdot}} a_1 \frac{\partial u_1}{\partial n} v \, d\underline{s},$$

and therefore

$$2 \oint_{\Omega_1^{\cdot}} \left(a_2 \frac{\partial u_2}{\partial n} v_2 - a_2 \frac{\partial u_2}{\partial n} v_1\right) d\underline{s} = \\ = \oint_{\Omega^{\cdot}} \left(a_1 \frac{\partial u_1}{\partial n} + a_2 \frac{\partial u_2}{\partial n}\right)(v_2 - v_1) \, d\underline{s}.$$

So we see that

$$R_\psi(v) = K_\psi(v) + F_\psi(v) \tag{2.5}$$

where

$$K_\psi(v) = \sum_{j=1,2} \int_{\Omega_j} a_j \left(\sum_i \left(\frac{u_j}{x_i} \right)^2 \right) d\underline{x} \tag{2.6}$$

$$- 2 a_2 \oint_{\Omega_0^\cdot} \frac{\partial u_2}{\partial n} g \, d\underline{s} + \frac{a_2}{\psi_1(h)} \oint_{\Omega_0^\cdot} \left(\frac{\partial u_2}{\partial n} \right)^2 d\underline{s} +$$

$$+ \frac{1}{4\psi_2(h)} \oint_{\Omega_1^\cdot} \left(a_1 \frac{\partial u_1}{\partial n} + a_2 \frac{\partial u_2}{\partial n} \right)^2 d\underline{s}.$$

Thus the minimization of (2.1) and (2.4) in $\overline{M}_h^r$ leads to the same element v_h^ψ.

Let us assume now that there exists $\hat{v}_h \in \overline{M}_h^r$ such that

$$R_\psi(\hat{v}_h) \leqslant C h^{2\nu} \| f, g \|_{k,s}^2. \tag{2.7}$$

Then obviously

$$R_\psi(v_h^\psi) \leqslant C h^{2\nu} \| f, g \|_{k,s}^2 \tag{2.8}$$

so we have

$$\sum_{j=1,2} \int_{\Omega_j} \left(\sum \frac{\partial (v_{h,j}^\psi - u_j)}{\partial x_i} \right)^2 d\underline{x} +$$

$$+ \psi_1(h) \oint_{\Omega_0^\cdot} (v_{h,2}^\psi - u_2)^2 d\underline{s} + \psi_2(h) \oint_{\Omega_1^\cdot} (v_{h,1}^\psi - v_{h,2}^\psi)^2 d\underline{s} \leqslant$$

$$\leqslant C \left[h^{2\nu} \| f, g \|_{k,s}^2 + \frac{1}{\psi_1(h)} \oint_{\Omega_0^\cdot} \left(\frac{\partial u_2}{\partial n} \right)^2 d\underline{s} + \right. \tag{2.9}$$

$$\left. + \frac{1}{\psi_2(h)} \oint_{\Omega_1^\cdot} \left(a_1 \frac{\partial u_1}{\partial n} + a_2 \frac{\partial u_2}{\partial n} \right)^2 d\underline{s} \right].$$

Because of the embedding theorem and the theorem 1.2 the right hand side of (2.9) is majorized by

$$C \left[h^{2\nu} + \frac{1}{\psi_1(h)} + \frac{1}{\psi_2(h)} \right] \| f, g \|_{k,s}^2. \tag{2.10}$$

We have

$$\| u - v_h^\psi \|_{W_2^1(\Omega_0)} \leqslant C \left[h^\nu + \left(\frac{1}{\psi_1(h)} \right)^{1/2} + \left(\frac{1}{\psi_2(h)} \right)^{1/2} \right] \| f, g \|_{k,s}. \tag{2.11}$$

By the basic property of $M_h^r(\Omega_0)$. (See *definition 1.1*) and *theorem 1.1* there exists $\hat{v}_h$ such that

$$\| u - \hat{v}_h \|_{W_2^p(\Omega_0)} \leqslant C h^\mu \| f, g \|_{k,s} \tag{2.12}$$

where

$$\mu = \min(r - p,\ q - p) \tag{2.13}$$

and

$$q = \min(k + 2, s + 1/2). \tag{2.14}$$

So we have

$$R_\psi(\hat{v}_h) \leqslant C\Big[\| u - \hat{v}_h \|^2_{W_2^1(\Omega_0)} + \psi_1(h) \oint_{\Omega_0^\cdot} (\hat{v}_{h,2} - u_2)^2 d\underline{s} + + \frac{1}{\psi_1(h)} \oint_{\Omega_0^\cdot} \left(\frac{\partial u_2}{\partial n}\right)^2 ds + \psi_2(h) \oint_{\Omega_1^\cdot} (v_{h,1} - v_{h,2})^2 ds + + \frac{1}{\psi_2(h)} \oint_{\Omega_1^\cdot} \left(\frac{\partial u_1}{\partial n}\right)^2 ds\Big]. \tag{2.15}$$

Using (2.12) for $p = 1/2 + \varepsilon$ and $p = 1$ and the well known embedding theorems we have

$$R_\psi(\hat{v}_h) \leqslant C\Big[h^{(q'-1)2} + (\psi_1(h) + \psi_2(h)\, h^{(q'-1/2-\varepsilon)2} + + \frac{1}{\psi_1(h)} + \frac{1}{\psi_2(h)}\Big] \| f, g \|^2_{k,s} \tag{2.16}$$

where

$$q' = \min(r, q).$$

Putting $\psi_i(h) = C_i\, h^{-\sigma_i}$ we have

$$R_\psi(\hat{v}_h) \leqslant C\, h^{2\nu} \| f, g \|^2_{k,s} \tag{2.17}$$

with

$$\nu = \min\Big[k + 1,\ s - \frac{1}{2},\ k + \frac{3}{2} - \varepsilon - \frac{\sigma_1}{2},\ k + \frac{3}{2} - \varepsilon - \frac{\sigma_2}{2},\ s - \varepsilon - \frac{\sigma_1}{2},\ s - \varepsilon - \frac{\sigma_2}{2},\ \frac{\sigma_1}{2},\ \frac{\sigma_2}{2},\ r - 1,\ r - \frac{1}{2} - \varepsilon - \frac{\sigma_1}{2},\ r - \frac{1}{2} - \varepsilon - \frac{\sigma_2}{2}\Big]. \tag{2.18}$$

By using (2.10) and (2.11) we have

$$\| u - v_h^\psi \|_{W_2^1(\Omega_0)} \leqslant C\, h^\nu \| f, g \|_{k,s}$$

where ν is given by (2.18).

3. Conclusions and Remarks

The approach that we have shown can be used more generally. Let us mention:

(a) The case of general elliptic equations,

(b) the case of piecewise smooth right hand side; here we formally put $a_1 = a_2$,

(c) eigenvalue problem, etc.

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