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COHOMOLOGY OF COMPACT HYPERKÄHLER MANIFOLDS AND ITS APPLICATIONS

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Abstract

We announce the structure theorem for the $H^2(M)$-generated part of cohomology of a compact hyperkähler manifold. This computation uses an action of the Lie algebra $\mathfrak{so}(4, n-2)$ where $n = \dim H^2(M)$ on the total cohomology space of M. We also prove that every two points of the connected component of the moduli space of holomorphically symplectic manifolds can be connected with so-called "twistor lines" – projective lines holomorphically embedded in the moduli space and corresponding to the hyperkähler structures. This has interesting implications for the geometry of compact hyperkähler manifolds and of holomorphic vector bundles over such manifolds.

1. Lie Algebra Action

This article is an announcement of results which are to be published in [V1]. We also give an outline of the proof, and use results of [V1] to prove some startling implications. A hyperkähler manifold is a Riemannian manifold M equipped with three complex structures I, J and K, such that $I \circ J = -J \circ I = K$ and M is Kähler with respect to I, J and K. Relations between I, J and K imply that there is an action of quaternions in its tangent space. Consequently, there is a multiplicative action of $SU(2)$ on the algebra of differential forms. This action commutes with the Laplacian. Hence, there is a canonical action of $SU(2)$ on the cohomology of M.

Let M be a complex manifold which admits a hyperkähler structure. A simple linear-algebraic argument implies that M is equipped with a holomorphic symplectic form. The Calabi-Yau theorem shows that, conversely, every compact holomorphically symplectic Kähler manifold admits a hyperkähler structure, which is uniquely defined by these data. Further on, we consider only holomorphically symplectic manifolds which are compact and of Kähler type. For simplicity of statements, we assume also that

$$\dim H^{2,0}(M) = 1 \quad \text{and} \quad H^1(M) = 0\ ,$$

though these assumptions are not necessary for most results.

The algebraic structure on $H^*(M)$ is studied using the general theory of Lefschetz-Frobenius algebras, introduced in [LLu].

Let $A = \bigoplus_{i=0}^{2d} A_i$ be a graded commutative associative algebra over a field of characteristic zero. Let $H \in End(A)$ be a linear endomorphism of A such that for all $\eta \in A_i$, $H(\eta) = (i-d)\eta$.

For all $a \in A_2$, denote by $L_a : A \to A$ the linear map which associates with $x \in A$ the element $ax \in A$. The triple $(L_a, H, \Lambda_a) \in End(A)$ is called a *Lefschetz triple* if

$$[L_a, \Lambda_a] = H \ , \quad [H, L_a] = 2L_a \ , \quad [H, \Lambda_a] = -2\Lambda_a \ .$$

A Lefschetz triple establishes a representation of the Lie algebra $\mathfrak{sl}(2)$ in the space A. For cohomology algebras, this representation arises as a part of Lefschetz's theory. In a Lefschetz triple, the endomorphism Λ_a is uniquely defined by the element $a \in A_2$ ([Bou], VIII §3). For arbitrary $a \in A_2$, a is called of *Lefschetz type* if the Lefschetz triple (L_a, H, Λ_a) exists. If $A = H^*(X)$ where X is a compact complex manifold of Kähler type, then all Kähler classes $\omega \in H^2(M)$ are elements of Lefschetz type. As one can easily check, the set $S \subset A_2$ of all elements of Lefschetz type is Zariski open in A_2.

DEFINITION 1.1: A Lefschetz-Frobenius algebra is a Frobenius graded supercommutative algebra which admits a Lefschetz triple.

DEFINITION 1.2: Let A be a Lefschetz-Frobenius algebra. The structure Lie algebra $\mathfrak{g}(A) \subset End(A)$ is a Lie subalgebra of $End(A)$ generated by L_a, Λ_a, for all elements of Lefschetz type $a \in S$.

Let M be a compact hyperkähler manifold with the complex structures I, J, K. Consider the Kähler forms $\omega_I, \omega_J, \omega_K$ associated with these complex structures. Let $\rho_I : \mathfrak{sl}(2) \to End(H^*(M))$, $\rho_J : \mathfrak{sl}(2) \to End(H^*(M))$, $\rho_K : \mathfrak{sl}(2) \to End(H^*(M))$ be the corresponding Lefschetz homomorphisms. Let $\mathfrak{a} \subset End(H^*(M))$ be the minimal Lie subalgebra which contains images of ρ_I, ρ_J, ρ_K. The algebra $\mathfrak{a}$ was computed explicitly in [V2].

Theorem 1.3 [V2]. *The Lie algebra $\mathfrak{a}$ is naturally isomorphic to $\mathfrak{so}(4,1)$.*

This statement can be regarded as a "hyperkähler Lefschetz theorem". Indeed, its proof parallels the proof of Lefschetz theorem.

Using Theorem 1.3, we compute the structure Lie algebra of $H^*(M)$.

Theorem 1.4. *Let M be a compact holomorphically symplectic manifold. Assume that $\dim H^{2,0}(M) = 1$. Let $n = \dim(H^2(M))$. Let $\mathfrak{g}(A)$ be a structure Lie algebra for $A = H^*(M)$. Then $\mathfrak{g}(A)$ is isomorphic to $\mathfrak{so}(4, n-2)$.*[1]

[1] This isomorphism can be made canonical. The Lie algebra $\mathfrak{g}(A)$ is isomorphic to $\mathfrak{so}(V \oplus \mathcal{H})$ where V is the linear space $H^2(M, \mathbb{R})$ equipped with the natural pairing of a signature $(3, n-3)$ ([B, Remarques, p. 775]; see also Theorem 2.1), and $\mathcal{H}$ is a 2-dimensional vector space with hyperbolic quadratic form.

Proof. See [V1, Theorem 11.1]. □

Let $H_r^*(M)$ be a sub-algebra of $H^*(M)$ generated by $H^2(M)$. It is easy to see that $\mathfrak{g}(A)$ acts on $H_r^*(M)$, and $H_r^*(M)$ is an irreducible representation of $\mathfrak{g}(A)$. Moreover, multiplicative structure in $H_r^*(M)$ is easily recovered from an action of $\mathfrak{g}(A)$. Using the general knowledge of representations of $\mathfrak{so}(n)$, we obtain exact knowledge of the multiplicative structure of $H_r^*(M)$. In particular, we obtain the following theorem (see [V1, Theorem 15.2]).

Theorem 1.5. *Let* $\dim_{\mathbb{C}} M = 2n$. *Then*

$$\begin{cases} H_r^{2i}(M) \cong S^i H^2(M) & \text{for } i \leq n, \text{ and} \\ H_r^{2i}(M) \cong S^{2n-i} H^2(M) & \text{for } i \geq n \ . \end{cases}$$

2. The Riemann-Hodge Pairing

Let M be a compact holomorphically symplectic manifold of Kähler type, satisfying

$$\dim H^{2,0}(M) = 1 \ .$$

In [B, Remarques, p. 775], Beauville introduces canonical 2-form on $H^2(M)$, of signature $(n-3,3)$, where $n = \dim H^2(M)$. In [V1], this form was described via the action of $SU(2)$ on $H^2(M)$.

Let ω be a Kähler class on M such that

$$\int_M \omega^{\dim_{\mathbb{C}} M} = 1 \ ,$$

and $(I, J, K, (\cdot,\cdot))$ be the corresponding hyperkähler structure. Let

$$(\cdot,\cdot)_{Her} : H^2(M,\mathbb{C}) \times H^2(M,\mathbb{C}) \to \mathbb{C}$$

be a positively Hermitian form on the second cohomology of M which corresponds to the Riemannian structure $(\cdot,\cdot)$. Let $H^2(M) = H^{inv}(M) \oplus H^+(M)$ be a decomposition such that $H^{inv}(M)$ consists of all $SU(2)$-invariant 2-forms, and $H^+(M)$ is the complementary $SU(2)$-invariant subspace. Let $(\cdot,\cdot)_{\mathcal{H}}$ be the form which is equal to $(\cdot,\cdot)_{Her}$ on $H^+(M)$ and $-(\cdot,\cdot)_{Her}$ on $H^{inv}(M)$.

REMARK: In the paper [V1] the form $(\cdot,\cdot)_{\mathcal{H}}$ is called the *Hodge-Riemann pairing* and is defined independently of [B].

Theorem 2.1 [B, Remarques, p. 775]. *The form $(\cdot,\cdot)_{\mathcal{H}}$ is independent from the choice of the complex and Kähler structure on M.*

The form $(\cdot,\cdot)_{\mathcal{H}}$ is used in the proof of Theorem 1.4.

Let $\rho_I : \mathfrak{u}(1) \to End(H^*(M))$ be a map for which $z \in \mathfrak{u}(1)$ acts on $H^{p,q}(M)$ by $(p-q)z$. Clearly, the action of $\mathfrak{u}(1)$ on $H^2(M)$ respects the form

$(\cdot,\cdot)_{\mathcal{H}}$. Let $\mathfrak{g}_M \subset End(H^*(M))$ be a Lie algebra generated by the images of ρ_I for all complex structures I on M. Let V denote the linear space $H^2(M)$ equipped with bilinear form $(\cdot,\cdot)_{\mathcal{H}}$. By Theorem 2.1, the action of $\mathfrak{g}_M$ on V preserves $(\cdot,\cdot)_{\mathcal{H}}$. This defines a Lie algebra homomorphism $\Gamma : G_M \to \mathfrak{so}(V)$. The following theorem is the chief tool in proving the Mirror Conjecture for a compact holomorphically symplectic manifold.

Theorem 2.2. *The map $\Gamma : \mathfrak{g}_M \to \mathfrak{so}(V)$ is an isomorphism.*

Proof. [V1, Theorem 13.1, 13.2]. □

The Lie algebra $\mathfrak{g}(A) \subset End(H^*(M))$ is equipped with a natural grading, induced by the grading on $H^*(M) = \bigoplus H^i(M)$. Let k be the one-dimensional Lie subalgebra of $End(H^*(M))$ spanned by Id.

Theorem 2.3. *The Lie subalgebra*

$$\mathfrak{g}_M \oplus k \subset End\big(H^*(M)\big)$$

coincides with the grading-zero part of $\mathfrak{g}(A)$. □

Let *Comp* be a (coarse, marked) moduli space of M. We have a *period map* $P_c : Comp \to \mathbb{P}H^2(M,\mathbb{C})$ associating a line $H_I^{2,0}(M) \subset H^2(M,\mathbb{C})$ to a complex structure I. Complexifying $H^2(M,\mathbb{R})$, we can consider $(\cdot,\cdot)_{\mathcal{H}}$ as a complex-linear, complex-valued form on $H^2(M,\mathbb{R})$. For all $I \in Comp$, $P_c(I)$ belongs to a conic hypersurface $C \subset \mathbb{P}H^2(M,\mathbb{C})$,

$$C = \{l \mid (l,l)_{\mathcal{H}} = 0\} \ .$$

The Torelli principle (proved by Bogomolov in the case of holomorphically symplectic manifolds, [Bo]) implies that $P_c : Comp \to C$ is etale.

Let $\underline{\mathcal{H}} = \bigoplus H^{p,q}(M)$ be a variation of Hodge structures (VHS) on Comp associated with the total cohomology space of M. Theorem 2.2 implies that there exist a VHS $\mathcal{H}$ on C, such that $\underline{\mathcal{H}}$ is a pullback of a variation of Hodge structures $\mathcal{H}$: $\underline{\mathcal{H}} = P_c^*(\mathcal{H})$. Let G_M be the Lie group associated with $\mathfrak{g}_M$, $G_M = Spin(H^2(M,\mathbb{R}), (\cdot,\cdot)_{\mathcal{H}})$. The set C is equipped with a natural action of a group G_M. This group also acts in the total cohomology space $H^*(M)$ of M. This defines an equivariant structure in the bundle $\mathcal{H}$. The chief idea used in the proof of Mirror Symmetry is the following theorem:

Theorem 2.4. *The VHS $\mathcal{H}$ is G_M-equivariant, under the natural action of G_M on C and $\mathcal{H}$.*

Proof. See [V4, Theorem 2.2]. □

To make this statement more explicit, we recall that the variation of Hodge structures is a flat bundle, equipped with a real structure and a holomorphic filtration (Hodge filtration), which is complementary to its complex adjoint filtration. Then, Theorem 2.4 says that the action of G_M on $\mathcal{H}$ maps flat sections to flat sections, and preserves the real structure and the Hodge filtration.

3. The Twistor Lines

The main technical tool used in the proof of above results is the knowledge about the (coarse, marked) moduli space *Comp* of complex structures on a holomorphically symplectic manifold M. Let ω be a Kähler class on M and $\mathcal{H} = (I, J, K, (\cdot,\cdot))$ be the corresponding hyperkähler structure. Then, for every triple of real numbers (a, b, c), $a^2 + b^2 + c^2 = 1$, the operator $aI + bJ + cK$ defines an integrable complex structure[2] on M. Identifying the set of such triples with $\mathbb{C}P^1$, we obtain a map $\mathbb{C}P^1 \overset{i_{\mathcal{H}}}{\hookrightarrow} Comp$ where *Comp* is a connected component of the coarse moduli space of M.

CLAIM 3.1. *The map $i_{\mathcal{H}}$ is a holomorphic embedding of complex analytic varieties.*

Proof. Well known (see for example [T]).

Let $P : Comp \to C$ be the period map assigning a line $H^{2,0}(M, I)$ to a complex structure I. Let $C \subset \mathbb{P}^1(H^2(M,\mathbb{C}) = P(Comp)$. According to [B], P is etale. The projective line $i_{\mathcal{H}}(\mathbb{C}P^1) \subset Comp$ is called *a twistor line*, and is denoted by $R_{\mathcal{H}}$. Twistor lines were extensively studied by Todorov ([T]).

Theorem 3.2. *Let $I_1, I_2 \in Comp$. Then there exist a sequence of intersecting twistor lines which connect I_1 with I_2.*

Proof. To prove Theorem 3.2, we have to show that a set $\widetilde{\mathcal{L}_0}$ of all twistor lines $i_{\mathcal{H}_0}(\mathbb{C}P^1)$ which are connected to $i_{\mathcal{H}}(\mathbb{C}P^1)$ with intersecting twistor lines is open. Since $P : Comp \to C$ is etale, it suffices to show that I_1, I_2 can be connected with twistor lines l_i such that $P(l_i)$ intersect $P(l_{i+1})$.

With every twistor line $R_{\mathcal{H}}$, we associate a 3-dimensional plane $\ell_{\mathcal{H}} \subset H^2(M,\mathbb{R})$ which is spanned by the Kähler classes $\omega_I, \omega_J, \omega_K$. A linear algebraic argument shows that the twistor lines $R_{\mathcal{H}_1}$ and $R_{\mathcal{H}_2}$ intersect if and only if $\dim(\ell_{\mathcal{H}_1} \cap \ell_{\mathcal{H}_1}) \geq 2$. Hence we need to show that

Theorem 3.2′. *Each pair of twistor lines $R_{\mathcal{H}}, R_{\mathcal{H}'}$ can be connected with a sequence of twistor lines $R_{\mathcal{H}} = R_{\mathcal{H}_1}, \ldots, R_{\mathcal{H}_n} = R_{\mathcal{H}'}$ such that $\dim(\ell_{\mathcal{H}_i} \cap \ell_{\mathcal{H}_{i+1}}) \geq 2$.*

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[2]This complex structure is called *a complex structure induced by a hyperkähler structure.*

LEMMA 3.3. *Let H be a hyperkähler structure on M, $i_{\mathcal{H}}(\mathbb{C}P^1) \subset Comp$ be the set of all induced complex structures, and $Kah(\mathcal{H})$ be the set of all Kähler classes corresponding to $L \in i_{\mathcal{H}}(\mathbb{C}P^1)$. Then $Kah(\mathcal{H})$ is open in $H^2(M, \mathbb{R})$.*

Proof. [V1, Claim 6.6]. □

Let $\mathcal{L}$ be the space of all triples $\omega_I, \omega_J, \omega_K$ in $H^2(M)$ which are orthonormal with respect to the pairing $(\cdot,\cdot)_{\mathcal{H}}$ of Theorem 2.1, and Hyp be the connected component of the set of all hyperkähler structures. Let $P_h : Hyp \to \mathcal{L}$ be the natural period map. Comparing dimensions and using Calabi-Yau, we observe that P_h is etale. Let $\mathcal{L}_0$ be the space of twistor lines corresponding to $\widetilde{L_0}$. Using Lemma 3.3, we find that the differential of $P_h\big|_{\mathcal{L}_0}$ is surjective. Therefore, $\mathcal{L}_0$ is open in $\mathcal{L}$, and $\tilde{\mathcal{L}}_0$ is open in the set of all twistor lines. This proves Theorem 3.2. □

4. An Outline of Proofs

Let $(I, J, K, (\cdot,\cdot))$ be a hyperkähler structure on M. One can check that the cohomology classes $\omega_I, \omega_J, \omega_K \in H^2(M, \mathbb{R})$ are orthogonal with respect to the pairing $(\cdot,\cdot)$. Let Hyp be the classifying space of the hyperkähler structures on M. Let $P_{hyp} : Hyp \to H^2(M) \times H^2(M) \times H^2(M)$ be the map which associates with the hyperkähler structure $\mathcal{H} = (I, J, K, (\cdot,\cdot))$ the triple $(\omega_I, \omega_J, \omega_K)$. Then the image of P_{hyp} in $H^2(M) \times H^2(M) \times H^2(M)$ satisfies

$$\forall\, (x,y,z) \in imP_{hyp} \quad \left|\quad \begin{array}{l} (x,y)_{\mathcal{H}} = (x,z)_{\mathcal{H}} = (y,z)_{\mathcal{H}} = 0\ , \\ (x,x)_{\mathcal{H}} = (y,y)_{\mathcal{H}} = (z,z)_{\mathcal{H}}\ , \end{array}\right. \tag{4.1}$$

where $(\cdot,\cdot)_{\mathcal{H}}$ is the canonical pairing defined above. Let $D \subset H^2(M) \times H^2(M) \times H^2(M)$ be the set defined by the equations (4.1). Using Torelli's theorem and Calabi-Yau, we prove the following statement:

Theorem 4.1. *The image of P_{hyp} is Zariski dense in D.*

Theorem 4.1 shows that all algebraic relations which are true for

$$(x,y,z) \in P_{hyp}(Hyp)$$

are true for all $(x,y,z) \in D$. Computing the Lie algebra $\mathfrak{a}$ as in Theorem 1.3, we obtain a number of relations between $x, y, z \in H^2(M)$ which hold for all $(x,y,z) \in Im(P_{hyp})$. Using the density argument, we obtain that these relations are universally true. This idea leads to the proof of Theorem 1.4.

The proof of Theorem 2.1 is deduced from the standard period argument and Theorem 3.2. Let $\mathcal{H}$ be a hyperkähler structure corresponding to I and ω. Clearly from the definition, the form $(\cdot,\cdot)_{\mathcal{H}}$ depends only on the twistor line $\mathcal{H}$, and not on the choice of particular I and ω. A computation shows that $(\cdot,\cdot)_{\mathcal{H}}$ depends on $P(I)$ and not on ω. Using the fact that C is completely connected with twistor lines (Theorem 3.2), we prove that $(\cdot,\cdot)_{\mathcal{H}}$ is independent of $\mathcal{H}$.

5. Implications

This section contains implications of our results.

5.1 Mirror symmetry ([V4]). Using Theorem 2.2 and Theorem 2.3, we compute the variation of Hodge structures corresponding to the universal VHS over the moduli space *Comp*. In [V5], it is proven that for "sufficiently generic" deformation W of a given compact holomorphically symplectic manifold M, the manifold W admits no closed holomorphic curves. Therefore, using the definition of quantum cohomology from [KM], we can easily compute the quantum variation of Frobenius algebras. Comparing these computations, we find that Mirror Conjecture is true for holomorphically symplectic manifolds, which are Mirror self-dual.

In the proof of Mirror Symmetry, we use the fact that the tangent bundle TM of a holomorphically symplectic manifold is isomorphic to its cotangent bundle $\Omega^1(M)$. For every Calabi-Yau manifold M, $\dim M = n$, the Serre's duality induces an isomorphism[3]

$$H^p(\Omega^q(M)) \cong H^p(\Lambda^{n-q}(TM)) \tag{5.1}$$

between cohomology of the holomorphic differential forms and cohomology of exterior powers of the holomorphic tangent bundle. Using the isomorphism $TM \cong \Omega^1(M)$, we interpret the isomorphism (5.1) as a map η from the total cohomology space $H^*(M)$ to itself. A linear-algebraic check ensures that this map is involutive. A slightly less elementary consideration shows that $\eta : H^*(M) \to H^*(M)$ belongs to the Lie group $G \subset End(H^*(M))$ corresponding to the Lie algebra $\mathfrak{g}(A)$ from Theorem 1.4. Clearly, the Yukawa multiplication is equal to the cup-product in cohomology twisted by η. This gives a way to describe the Yukawa product explicitly in terms of Lie algebra action.

5.2 Twistor paths.

Definition 5.2.a: Let M be a holomorphically symplectic manifold, *Comp* be its moduli space, and $P_0, \dots, P_n \subset Comp$ be a sequence of twistor lines supplied with an intersection point $x_{i+1} \in P_i \cap P_{i+1}$ for each i. We say

[3]Canonical up to a choice of a non-degenerate section of $\Omega^n(M)$.

that $\gamma = P_0, \ldots, P_n, x_1, \ldots, x_n$ is a *twistor path*. Let $I, I' \in Comp$. We say that γ is *a twistor path connecting* I *to* I' if $I \in P_0$ and $I' \in P_n$. The lines P_i are called *the edges*, and the points x_i *the vertices* of a twistor path.

Theorem 3.2 proves that every two points I, I' in $Comp$ are connected by a twistor path. Clearly, each twistor path induces a diffeomorphism $\mu_\gamma : (M, I) \to (M, I')$. We are interested in the algebro-geometrical properties of this diffeomorphism.

For every hyperkähler structure $\mathcal{H}$ on M, let $\mathfrak{g}_\mathcal{H} \subset End(H^*(M))$ be the corresponding $\mathfrak{su}(2)$ embedded to $End(H^*(M))$. Let $H^*(M)^{\mathfrak{g}_\mathcal{H}}$ be the $\mathfrak{g}_\mathcal{H}$-invariant part of $H^*(M)$. Let $I \in Comp$ and $\mathcal{H}$ be a hyperkähler structure which induces I. We say that I is *of general type with respect to* $\mathcal{H}$ if

$$H^*(M)^{\mathfrak{g}_\mathcal{H}} \cap H^*(M, \mathbb{Z}) = \bigoplus H^{p,p} \cap H^*(M, \mathbb{Z}) \ .$$

In [V5], we prove that for every hyperkähler structure, all induced complex structures are of general type, except perhaps a countable number of them. Results of [V3] and [V5] can be compressed down to the following statement.

Theorem 5.2.b. *Let $\mathcal{H}$ be a hyperkähler structure on M and I be an induced complex structure of general type.*

(i) ([V5]) *Let N be a closed complex analytic subset of (M, I). Then N is complex analytic with respect to J, for all induced complex structures*[4] *J.*

(ii) ([V3]) *Let Bun_I be the tensor category of polystable*[5] *holomorphic vector bundles of slope 0 over (M, I). For an arbitrary induced complex structure J, there exists a natural injective tensor functor $\Phi_{I\to J} : Bun_I \to Bun_J$, which is an equivalence of J is of general type with respect to $\mathcal{H}$. For I, J, J' being induced complex structures and I, J of general type, we have*

$$\Phi_{I\to J} \circ \Phi_{J\to J'} = \Phi_{I\to J'} \ .$$

Remark on proof of Theorem 5.2.b(ii): Theorem 5.2.b (ii) is an implication of the following result from [V3]. Let B be a polystable bundle on a holomorphically symplectic Kähler manifold M. We associate with the Kähler structure on M a canonical hyperkähler structure $\mathcal{H}$ as in Calabi-Yau theorem. Assume that the first and second Chern classes of stable summands of B are invariant under the natural action of $SU(2)$ in cohomology. Then there exist a unique holomorphic connection on B which is holomorphic under each of complex structures induced by $\mathcal{H}$. This lets one

[4] In [V5], such subsets are called *trianalytic*.

[5] Polystable means direct sum of stable. Stability understood in the sense of Takemoto–Mumford.

identify the categories of polystable bundles for different complex structures L induced by $\mathcal{H}$, provided that L is of general type with respect to $\mathcal{H}$.

DEFINITION 5.2.c: Let $I, J \in Comp$ and $\gamma = P_0, \ldots, P_n$ be a twistor path from I to J, which corresponds to the hyperkähler structures $\mathcal{H}_0, \ldots, \mathcal{H}_n$. We say that γ is admissible if I is of general type with respect to P_0, J to P_n, and all vertices of γ are of general type with respect to the corresponding edges.

COROLLARY 5.2.d. *Let $I, J \in Comp$, and γ be admissible twistor path from I to J.*

(i) *Let $\mu_\gamma : (M, I) \to (M, J)$ be the corresponding diffeomorphism. Then, for every complex analytic subset $N \subset (M, I)$, $\mu_\gamma(N)$ is complex analytic with respect to J, for all induced complex structures.*

(ii) *There exist a natural isomorphism of tensor categories*

$$\Phi_\gamma : Bun_I \to Bun_J \ .$$

Proof. Follows from Theorem 5.2.b. □

To sum up, whenever we can connect two complex structures by an admissible twistor path, these complex structures are quite similar from the algebro-geometrical point of view. There is a cohomological criterion of existence of an admissible twistor path, which is proven in the similar fashion to Theorem 3.2.

For $I \in Comp$, denote by $\mathrm{NS}(I, \mathbb{Q})$ the space $H^{1,1}(M, I) \cap H^2(M, \mathbb{Q}) \subset H^2(M)$. Let $Q \subset H^2(M, \mathbb{Q})$ be a subspace of $H^2(M, \mathbb{Q})$. Let

$$Comp_Q := \left\{ I \in Comp \,\middle|\, \mathrm{NS}(I, \mathbb{Q}) = Q \right\} \ .$$

Theorem 5.2.e. *Let $\mathcal{H}, \mathcal{H}'$ be hyperkähler structures, and I, I' be complex structures of general type to and induced by $\mathcal{H}, \mathcal{H}'$. Assume that $NS(I, \mathbb{Q}) = NS(I', \mathbb{Q}) = Q$, and I, I' lie in the same connected component of $Comp_Q$. Then I, I' can be connected by an admissible path.*

Proof. Follows the proof of Theorem 3.2. □

For general Q, we have no control over the number of connected components of $Comp_Q$ (unless the global Torelli theorem is proven), and therefore we cannot directly apply Theorem 5.2.e to obtain results from algebraic geometry.[6] However, when $Q = \emptyset$, $Comp_Q$ is clearly connected and open in $Comp$, assuming that $Comp$ is connected, which we assumed. On the other hand, for $I \in Comp_\emptyset$, and every $\mathcal{H}$ inducing I, I is of general type

[6] An exception is a K3 surface, where Torelli holds. For K3, $Comp_Q$ is connected for all $Q \subset H^2(M, \mathbb{Q})$.

with respect to $\mathcal{H}$ (this is essentially an implication of Theorem 2.2). This proves the following corollary.

COROLLARY 5.2.f. *Let $I, I' \in Comp_\emptyset$. Then I can be connected to I' by an admissible twistor path.* □

REMARK: We obtain that for all $I \in Comp_\emptyset$, the closed complex analytic subsets of (M, I) have the same real analytic structure, and categories of polystable holomorphic vector bundles are isomorphic. There are non-trivial polystable holomorphic vector bundles over such manifolds (tangent bundle and its tensor powers come to mind). It is not completely clear if manifolds (M, I) with $I \in Comp_\emptyset$ have any closed complex analytic subvarieties, except points.

5.3 Generalization of $(\cdot,\cdot)_\mathcal{H}$. Unlike the (otherwise clearly superior) approach used by Beauville and Bogomolov, our way of constructing the form $(\cdot,\cdot)_\mathcal{H}$ lends itself to an immediate generalization. Let $\mathfrak{g}_0(A)$ be the grading-zero part of $\mathfrak{g}(A)$ computed in Theorem 2.3, and $H^*(M)^{\mathfrak{g}_0(A)}$ be the space of all vectors invariant under $\mathfrak{g}_0(A)$. Let $H^*_\mathbf{r}(M)$ be a subalgebra of cohomology generated by $H^2(M)$ and $H^*(M)^{\mathfrak{g}_0(A)}$.[7] Let $\mathcal{H}$ be a hyperkähler structure on M. Consider the corresponding action of $SU(2)$ on $H^*(M)$. Let $H^i(M) = \bigoplus_w H^i_w(M)$ be an isotypic decomposition of $H^i(M)$ corresponding to this action. By definition, $H^i_w(M)$ is a direct sum of isomorphic $SU(2)$-representation of weight w, where w, $0 \leq w \leq i$ runs through the natural numbers of the same parity as i. Let $(\cdot,\cdot)_{Her}$ be the Hermitian metrics on cohomology induced by the Riemannian structure on M, and $(\cdot,\cdot)_\mathcal{H}$ be the pairing which is equal to $(-1)^{\frac{i-w}{2}}(\cdot,\cdot)_{Her}$ on $H^i_w(M)$.

Theorem 5.3.a. *Consider the restriction of $(\cdot,\cdot)_\mathcal{H}$ to $H^*_\mathbf{r}(M)$. This restriction $(\cdot,\cdot)_\mathcal{H}$ is non-degenerate and independent of $\mathcal{H}$ (up to a constant multiplier).*

Proof. For $i = 2$, this statement coincides with the statement of Theorem 2.1. For general i, the proof is essentially linear-algebraic and identical to the proof of [V1, Theorem 6.1]. □

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[7]There are only two known series of compact hyperkähler manifolds: Hilbert schemes of Artinian sheaves on K3 surfaces, and Hilbert schemes of Artinian sheaves on compact 2-dimensional tori, factorized by free action of a compact torus. In both cases, the cohomology algebra is computed by Nakajima ([N]). It seems reasonable to conjecture that, in either of these cases, $H^*(M) = H^*_\mathbf{r}(M)$.

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