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ANALYTIC AND REIDEMEISTER TORSION FOR REPRESENTATIONS IN FINITE TYPE HILBERT MODULES

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Abstract

For a closed Riemannian manifold (M, g) we extend the definition of analytic and Reidemeister torsion associated to a unitary representation of $\pi_1(M)$ on a finite dimensional vector space to a representation on a $\mathcal{A}$-Hilbert module $\mathcal{W}$ of finite type where $\mathcal{A}$ is a finite von Neumann algebra. If $(M, \mathcal{W})$ is of determinant class we prove, generalizing the Cheeger-Müller theorem, that the analytic and Reidemeister torsion are equal. In particular, this proves the conjecture that for closed Riemannian manifolds with positive Novikov-Shubin invariants, the L_2-analytic and L_2-Reidemeister torsions are equal.

Table of Contents

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0. Introduction

The purpose of this paper is to prove the equality of L_2-analytic and L_2-Reidemeister torsion. Both torsions are numerical invariants defined for closed manifolds of determinant class, in particular for closed manifolds with positive Novikov-Shubin invariants. For these manifolds their equality has been conjectured by Carey, Mathai, Lott, Lück, Rothenberg and others (cf. e.g. [LoLü, conjecture 9.7]). The interest of the conjecture comes, among other issues, from the geometric significance of the L_2-analytic torsion and the fact that sometimes the L_2-Reidemeister torsion can be computed numerically in an efficient way. Indeed, if M is a closed hyperbolic manifold of dimension 3, the L_2-analytic torsion coincides, up to a factor $-1/3\pi$, with the hyperbolic volume (cf. [Lo]).

We establish the conjecture by proving a more general result. Given a closed Riemannian manifold (M,g), we extend the notion of analytic and Reidemeister torsions to unitary representations of the fundamental group $\pi_1(M)$ on a $\mathcal{A}$-Hilbert module $\mathcal{W}$ of finite type (cf. section 4) where $\mathcal{A}$ is a finite von Neumann algebra, and prove the equality of the two torsions when $(M,\mathcal{W})$ is of determinant class. We point out that in the case where $\mathcal{A}$ is $\mathbf{C}$, we obtain a new proof of the well known result due to, independently, Cheeger [Ch] and Müller [Mü].

From the analytic point of view, the additional complexity and difficulty comes from the fact that the Laplacians associated to such representations, may have continuous spectrum and 0 might be in the essential spectrum.

In order to formulate our results more precisely, we introduce the following notation. Let M be a closed smooth manifold. A generalized triangulation of M is a pair $\tau=(h,g')$ with the following properties:

(T1) $h: M \to \mathbf{R}$ is a smooth Morse function which is self-indexing ($h(x) = \text{index}(x)$ for any critical point x of h);

(T2) g' is a Riemannian metric so that $-\text{grad}_{g'} h$ satisfies the Morse-Smale condition (for any two critical points x and y of h, the stable manifold W_x^+ and the unstable manifold W_y^-, with respect to $-\text{grad}_{g'} h$, intersect transversely);

(T3) in a neighborhood of any critical point of h one can introduce local coordinates such that, with q denoting the index of this critical point,

$$h(x) = q - (x_1^2 + \dots + x_q^2)/2 + (x_{q+1}^2 + \dots + x_d^2)/2$$

and the metric g' is Euclidean in these coordinates.

The unstable manifolds W_x^- provide a partition of M into open cells where W_x^- is an open cell of dimension equal to the index of x. The name "generalized triangulation" for the pair (h, g') is justified as a generalized triangulation can be viewed as a generalization of a simplicial triangulation.[1]

Let (M, g) be a closed Riemannian manifold with infinite fundamental group $\Gamma = \pi_1(M)$ and let $\tau = (h, g')$ be a generalized triangulation. Note that Γ is countable. Let $p : \tilde{M} \to M$ be the universal covering of M and denote by $\tilde{g}$ and $\tilde{\tau} = (\tilde{h}, \tilde{g}')$ the lifts of g and τ on $\tilde{M}$. The Laplace operator Δ_q acting on compactly supported, smooth q-forms on $\tilde{M}$ is essentially selfadjoint. Its closure, also denoted by Δ_q, is therefore selfadjoint; it is defined on a dense subspace of the L_2-completion of the space of smooth forms with compact support with respect to the scalar product induced by the metric $\tilde{g}$. Observe that Δ_q is Γ-equivariant and nonnegative. We can therefore define the spectral projectors $Q_q(\lambda)$ of Δ_q corresponding to the interval $(-\infty, \lambda]$. They are Γ-equivariant and admit a Γ-trace which we denote by $N_q(\lambda)$.

Let $\mathcal{C}^q(\tilde{\tau}) := l^2(\mathrm{Cr}_q(\tilde{h}))$ where $l^2(\mathrm{Cr}_q(\tilde{h}))$ denotes the Hilbert space of l_2-summable, complex-valued sequences indexed by the countable set $\mathrm{Cr}_q(\tilde{h})$ of critical points of $\tilde{h}$ of index q. The left action of Γ on $\mathrm{Cr}_q(\tilde{h})$ makes $l^2(\mathrm{Cr}_q(\tilde{h}))$ the underlying Hilbert space of a unitary Γ-representation. The intersections of the stable and the unstable manifolds of $-\mathrm{grad}_{\tilde{g}'} \tilde{h}$ induce a bounded, Γ-equivariant, linear map

$$\delta_q : \mathcal{C}^q(\tilde{\tau}) \to \mathcal{C}^{q+1}(\tilde{\tau}) \ .$$

Let δ_q^* be the adjoint of δ_q and introduce

$$\Delta_q^{\mathrm{comb}} := \delta_q^* \cdot \delta_q + \delta_{q-1} \cdot \delta_{q-1}^* \ .$$

Observe that Δ_q^{comb} is a Γ-equivariant, bounded, nonnegative, selfadjoint operator on $\mathcal{C}^q(\tilde{\tau})$. We can therefore define the spectral projectors $Q_q^{\mathrm{comb}}(\lambda)$ of Δ_q^{comb} corresponding to the interval $(-\infty, \lambda]$. These projectors are Γ-equivariant and thus admit a Γ-trace, which we denote by $N_q^{\mathrm{comb}}(\lambda)$ (cf. section 1).

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[1] Given a smooth simplicial triangulation τ_{sim}, one can construct a generalized triangulation $\tau = (h, g')$ so that the unstable manifolds W_x^- corresponding to $\mathrm{grad}_{g'} h$, with x a critical point of h, are the open simplexes of τ_{sim} (cf. [P]).

We say that
(1) (M, g) is of *a-determinant* class if $-\infty < \int_{0+}^{1} (\log \lambda) dN_q(\lambda)$ for all q.
(2) (M, τ) is of *c-determinant* class if $-\infty < \int_{0+}^{1} (\log \lambda) dN_q^{\text{comb}}(\lambda)$ for all q.

Here $\int_{0+}^{1}$ denotes the Stieltjes integral on the half open interval $(0, 1]$. The following result can be derived from work of Gromov-Shubin [GrSh, Theorem 2.1] (cf. also [E1,2]).

PROPOSITION 1. *Let M be a closed manifold equipped with a Riemannian metric g and a generalized triangulation τ. Then:*
(1) (M, g) is of a-determinant class iff (M, τ) is of c-determinant class.
(2) Let (M', τ') be another manifold with generalized triangulation τ'. If M and M' are homotopy equivalent, then (M, τ) is of c-determinant class iff (M', τ') is.

A more general statement, Proposition 5.6, will be proven in section 5.

DEFINITION. *A compact manifold M is of determinant class if for some generalized triangulation τ (and then for any), (M, τ) is of c-determinant class.*

If M is of determinant class the logarithm of the ζ-regularized determinant, $\log \det_{\mathcal{N}} \Delta_q$, is a finite real number for all q and one can introduce the L_2-analytic torsion T_{an} :

$$\log T_{\text{an}} := \tfrac{1}{2} \sum_q (-1)^{q+1} q \log \det_{\mathcal{N}} \Delta_q \ .$$

Similarly, if M is of determinant class, $\log \det_{\mathcal{N}} \Delta_q^{\text{comb}}$ is a finite real number for all q and one can define the combinatorial torsion:

$$\log T_{\text{comb}} := \tfrac{1}{2} \sum_q (-1)^{q+1} q \log \det_{\mathcal{N}} \Delta_q^{\text{comb}} \ .$$

To define the L_2-Reidemeister torsion, T_{Re}, it remains to introduce an additional number T_{met}. Notice that $\text{Null}(\Delta_q)$ consists of smooth forms and that integration of smooth q-forms over a smooth q-chain induces (cf. a theorem by Dodziuk [Do]) an isomorphism $\theta_q^{-1} : \text{Null}(\Delta_q) \to \overline{H}^q(\mathcal{C}^*(\tilde{\tau}), \delta_*) = \text{Null}(\Delta_q^{\text{comb}})$ of $\mathcal{N}(\Gamma)$-Hilbert modules where $\mathcal{N}(\Gamma)$ is the von Neumann algebra associated to Γ. Define

$$\log \text{Vol}_{\mathcal{N}}(\theta_q) := \tfrac{1}{2} \log \det_{\mathcal{N}} (\theta_q^* \theta_q)$$

where we used that $\det_{\mathcal{N}}(\theta_q^* \theta_q) > 0$ as $(\theta_q^* \theta_q)$ is a selfadjoint, positive, bounded, Γ-equivariant operator on the Γ- Hilbert space $\text{Null}(\Delta_q^{\text{comb}})$ whose spectrum is bounded away from 0. As a consequence (cf. section 1)

$\log \det_N(\theta_q^*\theta_q)$ is a well defined real number and one introduces

$$\log T_{\text{met}} := \frac{1}{2}\sum_q (-1)^q \log \det_N(\theta_q^*\theta_q) \ .$$

Combining the above definitions we define the L_2-Reidemeister torsion T_{Re}

$$\log T_{\text{Re}} = \log T_{\text{comb}} + \log T_{\text{met}} \ .$$

The concepts of L_2-analytic and L_2-Reidemeister torsion were considered by Novikov-Shubin in 1986 [NSh1] (cf. also later work by Lott [Lo], Lück-Rothenberg [LüRo], Mathai [M] and Carey-Mathai [CM]). The main objective of this paper is to prove the following:

Theorem 1. *Let M be a closed manifold of determinant class of odd dimension d. Then, for any Riemannian metric g and for any generalized triangulation τ, both T_{an} and T_{Re} are positive real numbers and*

$$T_{\text{an}} = T_{\text{Re}} \ .$$

Rather than viewing Theorem 1 as an L_2-version of the Cheeger-Müller theorem, we derive it as a particular case of a generalization of the Cheeger-Müller theorem (cf. Theorem 2 below). This generalization concerns the extension of the analytic and Reidemeister torsion associated to a closed Riemannian manifold and a finite dimensional unitary representation of Γ to a unitary representation of Γ on a $\mathcal{A}$-Hilbert module of finite type. A representation of this type is called an $(\mathcal{A}, \Gamma^{\text{op}})$-Hilbert module of finite type. Here $\mathcal{A}$ is a finite von Neumann algebra. A similar approach was used by Singer (cf. [Sin]) for the proof of the L_2-index theorem.

In order to formulate this generalization we must introduce (cf. section 2) a calculus of elliptic pseudodifferential $\mathcal{A}$-operators acting on sections of a bundle of $\mathcal{A}$-Hilbert modules of finite type over a compact manifold and develop a theory of regularized determinants for (nonnegative) elliptic pseudodifferential $\mathcal{A}$-operators of positive order. Typically the spectrum of such an operator is no longer discrete (cf. section 2).

Let us now describe Theorem 2 in more detail.

Assume that A is an elliptic operator in the new calculus. For an angle θ and $\epsilon > 0$ introduce the solid angle

$$V_{\theta,\epsilon} := \{z \in \mathbb{C} : |z| < \epsilon\} \cup \{z \in \mathbb{C} \setminus 0 : arg(z) \in (\theta - \epsilon, \theta + \epsilon)\} \ .$$

DEFINITION. (1) *θ is an Agmon angle for A, if there exists $\epsilon > 0$ so that*

$$\text{spec}(A) \cap V_{\theta,\epsilon} = \emptyset \ .$$

(2) *θ is a principal angle for A if there exists $\epsilon > 0$ so that*

$$\text{spec}\left(\sigma_A(x,\xi)\right) \cap V_{\theta,\epsilon} = \emptyset$$

for all $(x,\xi) \in S_x^*M$ *where* S^*M *denotes the cosphere bundle and* $\sigma_A(x,\xi)$ *is the principal symbol of* A.

It is well known that (1) implies (2) but not conversely. If, in addition, A is of order $m > 0$ and admits an Agmon angle θ, one can define the regularized determinant, $\det_{\theta,N} A \in \mathbb{C}$. In the sequel, θ will be chosen to be π and we will drop the subscript π in $\det_{\pi,N}$. If A is of order $m > 0$, nonnegative and if $0 \in \operatorname{spec}(A)$ then the ellipticity of A implies that the nullspace, $\operatorname{Null}(A)$, is an $\mathcal{A}$-Hilbert module of finite von Neumann dimension, $\dim_N \operatorname{Null}(A)$. Consider the 1-parameter family, $A + \lambda$, λ being the spectral parameter. For $\lambda > 0$, introduce the function $\log\det_N(A+\lambda) - \dim_N \operatorname{Null}(A) \log \lambda$. We can view this function as an element in the vector space $\mathbf{D}$ consisting of equivalence classes $[f]$ of real analytic functions $f : (0,\infty) \to \mathbb{R}$ with $f \sim g$ iff $\lim_{\lambda\to 0}(f(\lambda) - g(\lambda)) = 0$. The elements of $\mathbf{D}$ represented by the constant functions form a subspace of $\mathbf{D}$ which can be identified with $\mathbf{R}$, the space of real numbers.

Given a closed Riemannian manifold (M,g), an arbitrary $(\mathcal{A},\Gamma^{\mathrm{op}})$-Hilbert module of finite type, $\mathcal{W}$, and a generalized triangulation τ we define (cf. section 4) $\log T_{\mathrm{an}}(M,g,\mathcal{W})$ and $\log T_{\mathrm{Re}}(M,g,\tau,\mathcal{W})$ as elements of $\mathbf{D}$. As above we consider the analytic resp. combinatorial Laplacians associated to $(M,g,\mathcal{W})$ resp. $(M,\tau,\mathcal{W})$ and introduce the notion of a triple $(M,g,\mathcal{W})$ resp. $(M,\tau,\mathcal{W})$, of *a-determinant*, resp. of *c-determinant* class. Proposition 1 can be generalized to say that these two notions are equivalent and homotopy invariant (Proposition 5.6, section 5). This allows us to introduce the notion of a pair $(M,\mathcal{W})$ to be of determinant class. We point out that for $\mathcal{A} = \mathbb{C}$ any pair $(M,\mathcal{W})$ is of determinant class.

Theorem 2. *Let* M *be a closed manifold of odd dimension* d *and* $\mathcal{W}$ *an* $(\mathcal{A},\Gamma^{\mathrm{op}})$*-Hilbert module of finite type. If the pair* $(M,\mathcal{W})$ *is of determinant class then, for any Riemannian metric* g *and any generalized triangulation* τ *of* M, $\log T_{\mathrm{an}}(M,g,\mathcal{W})$ *and* $\log T_{\mathrm{Re}}(M,\tau,g,\mathcal{W})$ *are both finite, real numbers and*

$$\log T_{\mathrm{an}} = \log T_{\mathrm{Re}} \ .$$

Let us make a few comments concerning Theorem 2 and Theorem 1:

(1) If M is of even dimension, then both torsions are equal to 1 (cf. formulas 4.8).

(2) If $\mathcal{A} = \mathbb{C}$, the $(\mathbb{C},\Gamma^{\mathrm{op}})$-Hilbert modules of finite type are precisely the unitary Γ-representations and Theorem 2 reduces to the Cheeger-Müller Theorem ([Ch], [Mü]) and, when specialized to this situation, we thus obtain a new proof of their theorem.

(3) If $\mathcal{A} = \mathcal{N}(\Gamma)$, the von Neumann algebra associated to Γ, and $\mathcal{W} = l_2(\Gamma)$

is viewed as a $(\mathcal{N}(\Gamma), \Gamma^{op})$-Hilbert module of finite type (cf. section 1.4), then Theorem 2 reduces to Theorem 1.

(4) Lott-Lück have conjectured (cf. [LoLü, Conjecture 9.2]) that all compact manifolds have positive Novikov-Shubin invariants and, therefore, are of determinant class. The conjecture has been verified for many compact manifolds and in particular for all compact manifolds whose fundamental group is free or free abelian. A weaker conjecture is that all compact manifolds are of determinant class. This conjecture has been verified for manifolds whose fundamental group is residually finite (cf. [BuFrKa3]).

(5) Assign to each compact Riemannian manifold (M, g) with M of determinant class the L_2-(analytic=Reidemeister) torsion if the fundamental group $\pi_1(M)$ is infinite and the $\#(\pi_1(M))$-th root[2] of the (analytic=Reidemeister) torsion of the universal cover of (M, g) with the one dimensional trivial representation of $\pi_1(M)$ if $\pi_1(M)$ is finite. In this way one obtains a numerical invariant $T(M, g)$, which satisfies the product formula

$$\log T(M_1 \times M_2, g_1 \times g_2) = \chi(M_2) \log T(M_1, g_1) + \chi(M_1) \log T(M_2, g_2) ,$$

and has the property that for any n-sheeted covering $(\tilde{M}, \tilde{g})$ of (M, g)

$$\log T(\tilde{M}, \tilde{g}) = n \log T(M, g) .$$

Here $\chi(M)$ denotes the Euler-Poincaré characteristic of M. For compact manifolds with trivial L_2 Betti numbers, in particular for manifolds of the homotopy type of a mapping torus, this invariant is independent of the metric (and is in fact a homotopy type invariant as will be shown in a subsequent paper). This invariant was calculated for a large class of 3-dimensional manifolds; its logarithm is zero for Seifert manifolds (cf. [LüRo]) and $(-1/3\pi)\,\mathrm{Vol}(M, g)$ for a hyperbolic manifold (M, g), (cf. [Lo]). The calculation in [LüRo] was done for the Reidemeister torsion and in [Lo] for the analytic torsion.

(6) W. Lück ([Lü2]) found an algorithm to calculate the L_2-Reidemeister torsion of a 3-dimensional hyperbolic manifold in terms of a balanced presentation of its fundamental group. By Theorem 1 and by remark (5) above the algorithm also calculates the hyperbolic volume.

Theorem 2 is derived from Corollary C (section 6.2), a relative version of Theorem 2, using product formulas for the analytic torsion and the Reidemeister torsion (section 4) and the metric anomaly (Lemma 6.11). To state Corollary C let M and M' be two closed manifolds of the same dimension with fundamental groups isomorphic to Γ, and assume that they

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[2] $\#(\pi_1(M))$ denotes the cardinality of $\pi_1(M)$

are equipped with generalized triangulations $\tau = (g, h)$ and $\tau' = (g', h')$ such that the functions h and h' have the same number of critical points for each index. Then, for an arbitrary $(\mathcal{A}, \Gamma^{\mathrm{op}})$-Hilbert module of finite type $\mathcal{W}$, with $(M, \mathcal{W})$ and $(M', \mathcal{W})$ of determinant class

$$\log T_{\mathrm{an}} - \log T'_{\mathrm{an}} = \log T_{\mathrm{Re}} - \log T'_{\mathrm{Re}} \ .$$

In order to prove Corollary C we use the Witten deformation of the de Rham complex associated with a generalized triangulation $\tau = (g, h)$ (cf. section 5). The Witten deformation permits us to define smooth functions $\log T_{\mathrm{an}}(h, t)$, $\log T_{\mathrm{sm}}(h, t)$ and $\log T_{\mathrm{la}}(h, t)$ with $\log T_{\mathrm{an}}(h, 0) = \log T_{\mathrm{an}}$ where $\log T_{\mathrm{an}}(h, t) = \log T_{\mathrm{sm}}(h, t) + \log T_{\mathrm{la}}(h, t)$ is a decomposition of $\log T_{\mathrm{an}}(h, t)$ into a part $\log T_{\mathrm{sm}}(h, t)$ which corresponds to the small spectrum of the Laplacians $\Delta_q(t)$ and a complimentary part $\log T_{\mathrm{la}}(h, t)$. The results presented in sections 2 and 3 lead to the conclusion that these three functions have asymptotic expansion when $t \to \infty$. The free term of such an expansion refers to the 0'th order coefficient of the expansion as $t \to \infty$. The results from sections 3.2 and the extension of Helffer-Sjöstrand's analysis of the Witten complex ([HSj1]) to the analogous complex constructed for differential forms on M with coefficients in $\mathcal{W}$ (cf. section 5), permit us to show that the free term of

$$\log T_{\mathrm{an}}(h, t) - \log T_{\mathrm{sm}}(h, t) - \big(\log T_{\mathrm{an}}(h', t) - \log T_{\mathrm{sm}}(h', t)\big)$$

is equal to

$$\log T_{\mathrm{an}} - \log T_{\mathrm{Re}} - \big(\log T'_{\mathrm{an}} - \log T'_{\mathrm{Re}}\big) \ .$$

Finally, using the Mayer Vietoris type formula and the asymptotic expansion of the logarithm of the determinant (cf. section 3) we show that the free term of $\log T_{\mathrm{la}}(h, t) - \log T_{\mathrm{la}}(h', t)$ is equal to zero and thus conclude Corollary C.

The paper is organized as follows:

In section 1 we recall, for the convenience of the reader, the concepts of a finite von Neumann algebra $\mathcal{A}$, an $\mathcal{A}$-Hilbert module of finite type, a finite (von Neumann) dimensional representation of a group, determinants in the von Neumann sense and the torsion of a finite complex of $\mathcal{A}$-Hilbert modules of finite type. This section is entirely expository.

In section 2 and 3 we describe the theory of pseudodifferential operators acting on sections of a given bundle $\mathcal{E} \to M$ of $\mathcal{A}$-Hilbert modules of finite type. In particular, we extend Seeley's result on zeta-functions for elliptic pseudodifferential operators and the corresponding regularized determinants, as well as the results of [BuFrKa2], to the extent needed in this paper, for this new class of operators. The calculus of such operators is not new, but we failed to find a reference suited to our needs (cf. e.g. [FMi], [Le], [Mo] and [Lu] for related work).

In section 4 we define the analytic torsion and Reidemeister torsion and we prove a product formula for each of them. These product formulas are slight generalizations of the product formulas presented in [Lo] and [CM], but for the convenience of the reader we include the proofs.

In section 5, we discuss the Witten deformation of the de Rham complex of M with coefficients in a $(\mathcal{A}, \Gamma^{op})$-Hilbert module of finite type $\mathcal{W}$ and show that the work of Witten-Helffer-Sjöstrand can be extended to this more general situation where the spectrum of the Laplacian $\Delta_q(t)$ is typically not discrete. The main new result is Proposition 5.2 (separation of the spectrum of $\Delta_q(t)$) which permits us to decompose the deformed de Rham complex into the direct sum of a 'small' and a 'large' subcomplex where the small subcomplex is a complex of $\mathcal{A}$-Hilbert modules of finite type. The very same arguments as in the case $\mathcal{A} = \mathbb{C}$ (cf. [HSj1,2], [BZ1,2]) can now be used to conclude that the small subcomplex is, up to normalization, asymptotically isometric to the combinatorial complex associated with τ.

In section 6 we present the proof of Theorem 2.

One can generalize the analytic and Reidemeister torsions associated to $(M, g, \mathcal{W})$ to include additional data, for example a finite dimensional hermitian vector bundle on M equipped with a flat connection. By the same methods as presented in this paper one can prove a result which compares these two generalized torsions. In the case $\mathcal{A} = \mathbb{C}$ results of this type were first established in [BZ1] (cf. also [BuFrKa1]).

Using the same arguments as in [Lü1], one can extend Theorem 2 to compact manifolds with boundary. Both extensions are useful for the calculations of the L_2 torsions; together with some applications they will be presented in a forthcoming paper.

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1. Linear Algebra in the von Neumann Sense

In this section we collect for the convenience of the reader a number of definitions and results concerning linear algebra in the von Neumann sense (cf. e.g. [CM], [Co], [D], [GrSh], [LüRo] for reference).

1.1 $\mathcal{A}$-Hilbert modules.

DEFINITION 1.1. *A finite von Neumann algebra $\mathcal{A}$ is a unital $\mathbb{C}$-algebra with a $*$-operation and a faithful trace $\mathrm{tr}_N : \mathcal{A} \to \mathbb{C}$ which satisfies the following properties:*

(Tr1) *$\langle .,. \rangle : \mathcal{A} \times \mathcal{A} \to \mathbb{C}$, defined by $\langle a, b \rangle = \mathrm{tr}_N(ab^*)$, is a scalar product and the completion $\mathcal{A}_2$ of $\mathcal{A}$ with respect to this scalar product is a separable Hilbert space.*

(Tr2) *$\mathcal{A}$ is weakly closed, when viewed as a subalgebra of $\mathcal{L}(\mathcal{A}_2) := \mathcal{L}(\mathcal{A}_2, \mathcal{A}_2)$, the space of linear, bounded operators on $\mathcal{A}_2$, where elements of $\mathcal{A}$ are identified with the corresponding left translations in $\mathcal{L}(\mathcal{A}_2)$ (a sequence $\{a_n\}_{n \geq 1}$ in $\mathcal{A}$ converges weakly to $a \in \mathcal{A}_2$ if $\lim_{n\to\infty} \langle a_n x, y \rangle = \langle ax, y \rangle$ for all $x, y \in \mathcal{A}_2$).*

(Tr3) *The trace is normal, i.e. for any monotone increasing net, $(a_i)_{i \in I}$, such that $a_i \geq 0$ and $a = \sup_{i \in I} a_i$ exists in $\mathcal{A}$, one has $\mathrm{tr}_N\, a = \sup_{i \in I} \mathrm{tr}_N\, a_i$. Here $a_i \geq 0$ means that $a_i = a_i^*$ and $\langle a_i x, x \rangle \geq 0$ for all $x \in \mathcal{A}_2$.*

In the sequel, $\mathcal{A}$ will always denote a finite von Neumann algebra. Introduce the opposite algebra $\mathcal{A}^{\mathrm{op}}$ of $\mathcal{A}$, where $\mathcal{A}^{\mathrm{op}}$ has the same underlying vector space, $|\mathcal{A}^{\mathrm{op}}| = |\mathcal{A}|$, $*$-operation, trace and unit element as $\mathcal{A}$, but the multiplication "$\cdot_{\mathrm{op}}$" of the elements $a, b \in |\mathcal{A}^{\mathrm{op}}|$ is defined by $a \cdot_{\mathrm{op}} b = b \cdot a$. Note that $\mathcal{A}^{\mathrm{op}}$ is a finite von Neumann algebra as well. The right translation by elements of $\mathcal{A}$ induces an embedding $r : \mathcal{A}^{\mathrm{op}} \to \mathcal{L}(\mathcal{A}_2)$ which identifies $\mathcal{A}^{\mathrm{op}}$ with the subalgebra $\mathcal{L}_{\mathcal{A}}(\mathcal{A}_2) \subset \mathcal{L}(\mathcal{A}_2)$ of bounded $\mathcal{A}$-linear maps (with respect to the $\mathcal{A}$-module structure of $\mathcal{A}_2$ induced by left multiplication). Therefore we can introduce a trace on $\mathcal{L}_{\mathcal{A}}(\mathcal{A}_2)$, also denoted by tr_N, defined for $f \in \mathcal{L}_{\mathcal{A}}(\mathcal{A}_2)$ by

$$\mathrm{tr}_N(f) := \mathrm{tr}_N\left(r^{-1}(f)\right) .$$

DEFINITION 1.2. (1) *$\mathcal{W}$ is an $\mathcal{A}$-Hilbert module if*

(HM1) *$\mathcal{W}$ is a Hilbert space with inner product denoted by $\langle .,. \rangle$.*

(HM2) *$\mathcal{W}$ is a left $\mathcal{A}$-module so that $\langle a^* v, w \rangle = \langle v, aw \rangle$ ($a \in \mathcal{A}$; $v, w \in \mathcal{W}$).*

(HM3) *$\mathcal{W}$ is isometric to a closed submodule of $\mathcal{A}_2 \otimes V$ where V is a separable Hilbert space and the tensor product $\mathcal{A}_2 \otimes V$ is taken in the category of Hilbert spaces.*

(2) *$\mathcal{W}$ is an $\mathcal{A}$-Hilbert module of finite type if $\mathcal{W}$ is an $\mathcal{A}$-Hilbert module and*

(HM4) *$\mathcal{W}$ is isometric to a closed submodule $\mathcal{A}_2 \otimes V$ where V is a finite dimensional vector space.*

(3) *A morphism $f : \mathcal{W}_1 \to \mathcal{W}_2$ between $\mathcal{A}$-Hilbert modules of finite type, $\mathcal{W}_1$ and $\mathcal{W}_2$, is a bounded, $\mathcal{A}$-linear operator; f is an isomorphism if it is bijective and both f and f^{-1} are morphisms.*

Let $\mathcal{W}$ be an $\mathcal{A}$-Hilbert module. An element v in $\mathcal{W}$ is called regular if the map $i_v : \mathcal{A} \to \mathcal{W}$, defined by $i_v(a) = av$, extends to an $\mathcal{A}$-linear bounded map $\mathcal{A}_2 \to \mathcal{W}$. If $\mathcal{W} = \mathcal{A}_2$ then the set of regular elements of $\mathcal{W}$ can be identified to $\mathcal{A}^{\mathrm{op}}$.

DEFINITION 1.3. *A collection $(e_j)_{j\in J}$ $(J \subset \mathbb{N})$, of regular elements of $\mathcal{W}$ is called a base of $\mathcal{W}$ if*

$$i : \oplus_{\nu\in J}(\mathcal{A}_2)_\nu \to \mathcal{W} \tag{1.1}$$

is an isomorphism where each $(\mathcal{A}_2)_\nu$ is a copy of $\mathcal{A}_2$ and $i = \sum_{\nu\in J} i_{e_\nu}$. The base is called orthonormal if, in addition, i is an isometry. A Hilbert module is free if it has a base.

If J is infinite, the direct sum in (1.1) is meant to be a direct sum in the category of Hilbert spaces. The base $(e_j)_{j\in J}$ is orthonormal iff for any $i, j \in J$ and $a, b \in \mathcal{A}$, $\langle ae_i, be_j\rangle = \langle a, b\rangle\delta_{ij}$. If $(e_j)_{j\in J}$ is a base of $\mathcal{W}$, then $(f_j)_{j\in J}$ with $f_\nu = i(i^*i)^{-\frac{1}{2}}e_\nu$ is an orthonormal base of $\mathcal{W}$. This method of constructing an orthonormal base is used in subsection 5.2. Let $\mathcal{W}$ be an $\mathcal{A}$-Hilbert module of finite type. The algebra $\mathcal{L}_\mathcal{A}(\mathcal{W}) := \mathcal{L}_\mathcal{A}(\mathcal{W}, \mathcal{W})$ of bounded $\mathcal{A}$-linear operators on $\mathcal{W}$ is a finite von Neumann algebra, whose trace is defined in the following way. First assume that the module $\mathcal{W}$ is free. Choose a basis $\{e_1, \ldots, e_l\}$ $(l < \infty)$. With respect to this basis an operator $A \in \mathcal{L}_\mathcal{A}(\mathcal{W})$ has a matrix representation $(a_{ij})_{1\le i,j\le l}$, $i, j = 1, \ldots, l$, with entries a_{ij} in $\mathcal{L}_\mathcal{A}(\mathcal{A}_2) = \mathcal{A}^{\mathrm{op}}$. Define $\mathrm{tr}_\mathcal{N}(A) = \sum_{i=1}^{l} tr_\mathcal{N} a_{ii}$. One shows that $\mathrm{tr}_\mathcal{N}(A)$ is independent of the chosen basis and therefore well defined. In the general case $\mathcal{W}$ is a closed invariant subspace of a free $\mathcal{A}$-Hilbert module $\mathcal{V}$ of finite type. We write $\mathcal{V} = \mathcal{W} \oplus \mathcal{W}^\perp$ and consider $\tilde{A} = A \oplus 0 \in \mathcal{L}_\mathcal{A}(\mathcal{V}, \mathcal{V})$. Define $\mathrm{tr}_\mathcal{N}(A) := \mathrm{tr}_\mathcal{N}(\tilde{A})$. One shows that $\mathrm{tr}_\mathcal{N}(A)$ is independent of the choice of $\mathcal{V}$.

For an $\mathcal{A}$-Hilbert module $\mathcal{W}$ of finite type one defines the dimension $\dim_\mathcal{N}(\mathcal{W})$ in the von Neumann sense by $\dim_\mathcal{N} \mathcal{W} := \mathrm{tr}_\mathcal{N} \mathrm{Id}_\mathcal{W}$. If $\mathcal{W}$ is not of finite type one sets $\dim_\mathcal{N} \mathcal{W} := \sup\{\dim_\mathcal{N} \mathcal{W}'; \mathcal{W}'$ closed submodule of finite type$\}$. The von Neumann dimension is always a nonnegative real number or $+\infty$.

REMARK 1.4: Assume $\mathcal{W}_1$ and $\mathcal{W}_2$ are $\mathcal{A}$-Hilbert modules, such that $\mathcal{W}_1$ is a closed invariant subspace (i.e. a $\mathcal{A}$-submodule) of $\mathcal{W}_2$ and $\dim_\mathcal{N}(\mathcal{W}_1) = \dim_\mathcal{N}(\mathcal{W}_2) < \infty$, then $\mathcal{W}_1 = \mathcal{W}_2$. The von Neumann dimension of a Hilbert direct sum is the sum (possibly infinite) of the von Neumann dimension of the summands.

REMARK 1.5: Assume that $\mathcal{W}_1$ and $\mathcal{W}_2$ are $\mathcal{A}$-Hilbert modules of finite type.

(1) If $f \in \mathcal{L}_{\mathcal{A}}(\mathcal{W}_1, \mathcal{W}_2)$ and $g \in \mathcal{L}_{\mathcal{A}}(\mathcal{W}_2, \mathcal{W}_1)$ then $\mathrm{tr}_N(fg)^n = \mathrm{tr}_N(gf)^n$ for any $n \geq 1$.
(2) If $f : \mathcal{W}_1 \to \mathcal{W}_2$ is an isomorphism and $\alpha_i \in \mathcal{L}_{\mathcal{A}}(\mathcal{W}_i)$, $i = 1, 2$, so that $f \cdot \alpha_1 = \alpha_2 \cdot f$ then $\mathrm{tr}_N \alpha_1 = \mathrm{tr}_N \alpha_2$.

If $\mathcal{A}'$ and $\mathcal{A}''$ are two finite von Neumann algebras the tensor product $\mathcal{A}' \otimes \mathcal{A}''$ is defined as the weak closure of the image of the algebraic tensor product of $\mathcal{A}'$ and $\mathcal{A}''$ in $\mathcal{L}(\mathcal{A}'_2 \otimes \mathcal{A}''_2)$ (cf. [D, p. 25]). The algebra $\mathcal{A}' \otimes \mathcal{A}''$ is again a finite von Neumann algebra whose trace has the property that $\mathrm{tr}_N(a' \otimes a'') = \mathrm{tr}_N a' \, \mathrm{tr}_N a''$. If $\mathcal{W}'$ and $\mathcal{W}''$ are $\mathcal{A}'$- resp. $\mathcal{A}''$-Hilbert modules of finite type then $\mathcal{W}' \otimes \mathcal{W}''$ is an $\mathcal{A}' \otimes \mathcal{A}''$-Hilbert module of finite type; moreover, given $f' \in \mathcal{L}_{\mathcal{A}'}(\mathcal{W}')$ and $f'' \in \mathcal{L}_{\mathcal{A}''}(\mathcal{W}'')$, $\mathrm{tr}_N(f' \otimes f'') = \mathrm{tr}_N f' \, \mathrm{tr}_N f''$.

DEFINITION 1.6. *A morphism* $f : \mathcal{W}_1 \to \mathcal{W}_2$ *is a weak isomorphism iff* $\mathrm{Null}(f) = 0$ *and* $\overline{\mathrm{Range}(f)} = \mathcal{W}_2$.

REMARK 1.7 (Polar decomposition): A weak isomorphism $f : \mathcal{W}_1 \to \mathcal{W}_2$ can be factored as $f = gf'$ where $f' : \mathcal{W}_1 \to \mathcal{W}_1$ is a weak isomorphism and $g : \mathcal{W}_1 \to \mathcal{W}_2$ is an isometric isomorphism given by $f' = (f^*f)^{1/2}$, $g = f \cdot (f^*f)^{-1/2}$. If f is a weak isomorphism, then $\dim_N \mathcal{W}_1 = \dim_N \mathcal{W}_2$.

1.2 Determinant in the von Neumann sense. Throughout this subsection we consider only $\mathcal{A}$-Hilbert modules of finite type. In this subsection we define two spectral invariants for an element $f \in \mathcal{L}_{\mathcal{A}}(\mathcal{W})$. The first one, $\det_N(f)$, is defined for f having π as a weak Agmon angle (cf. Definition 1.8) where as the second one, $\mathrm{Vol}_N(f)$ is defined for arbitrary f.

DEFINITION 1.8. (1) *π is an Agmon angle for $f \in \mathcal{L}_{\mathcal{A}}(\mathcal{W})$ iff there exists $\epsilon > 0$ so that* $\mathrm{spec}(f) \cap V_{\pi,\epsilon} = \emptyset$ *with $V_{\pi,\epsilon}$ defined as in the introduction.*

(2) *π is a weak Agmon angle for $f \in \mathcal{L}_{\mathcal{A}}(\mathcal{W})$ iff π is an Agmon angle for $f + \lambda$ for any $\lambda > 0$.*

First we consider the case where π is an Agmon angle for f. In this case f is an isomorphism. Define the complex powers of f, $f^s \in \mathcal{L}_{\mathcal{A}}(\mathcal{W})$, $s \in \mathbb{C}$, by the formula

$$f^s = \frac{1}{2\pi i} \int_\gamma \lambda^s (\lambda - f)^{-1} d\lambda \, , \tag{1.2}$$

where λ^s is a branch of the complex power s defined on $\mathbb{C}_\pi = \mathbb{C} \setminus \{z = \rho e^{i\pi}; \rho \in [0, \infty)\}$ and γ is a closed contour in $\mathbb{C}_\pi$ which surrounds the compact set $\mathrm{spec}\, f$ in $\mathbb{C}_\pi$ with counterclockwise orientation. Notice that for $\Re s < 0$, by Cauchy's theorem, the contour γ in (1.2) can be replaced by the contour $\gamma_{\pi,\epsilon} = \gamma_1 \cup \gamma_2 \cup \gamma_3$ where $\gamma_1 := \{z = \rho e^{i\pi}; \infty \leq \rho \leq \epsilon/2\}$, $\gamma_2 := \{z = \frac{\epsilon}{2} e^{i\alpha}; \pi \geq \alpha \geq -\pi\}$ and $\gamma_3 := \{z = \rho e^{i(-\pi)}; \frac{\epsilon}{2} \leq \rho \leq \infty\}$.

Notice that f^s is an entire function in $s \in \mathbb{C}$ with values in $\mathcal{L}_{\mathcal{A}}(\mathcal{W})$ and $\operatorname{tr}_N(f^s)$ is an entire function on $\mathbb{C}$. Define the determinant $\det_N f$ in the von Neumann sense by

$$\log \det_N f = \frac{d}{ds}\bigg|_{s=0} \operatorname{tr}_N(f^s) \ . \tag{1.3}$$

We remark that this notion of determinant, in the case when f is positive and selfadjoint, coincides with the one introduced by Fuglede and Kadison [FuK].

If $f \in \mathcal{L}_{\mathcal{A}}(\mathcal{W}_1, \mathcal{W}_2)$ is an isomorphism then f^*f is a selfadjoint positive isomorphism and one shows that $\det_N(f^*f) > 0$. Define

$$\operatorname{Vol}_N f := \left(\det{}_N(f^*f)\right)^{1/2} .$$

PROPOSITION 1.9. (1) *Suppose* $f_t \in \mathcal{L}_{\mathcal{A}}(\mathcal{W})$, *with* t *in an interval* $I \subset \mathbf{R}$, *is a family of class* C^1 *(in norm sense) of morphisms and* π *is an Agmon angle for all of them. Then* $\log\det_N(f_t)$ *is of class* C^1 *and*

$$\frac{d}{dt} \log \det_N(f_t) = \operatorname{tr}_N \left(\left(\frac{d}{dt} f_t \right) f_t^{-1} \right) \ . \tag{1.4}$$

(2) *Suppose* $f_i \in \mathcal{L}_{\mathcal{A}}(\mathcal{W}_i)$, $i = 1,2$ *with* $\mathcal{W}_1$ *and* $\mathcal{W}_2$ $\mathcal{A}$*-Hilbert modules of finite type and* $\alpha : \mathcal{W}_1 \to \mathcal{W}_2$ *is an isomorphism so that* $\alpha f_1 = f_2 \alpha$. *Then the following statements hold:*

(a) $\operatorname{spec} f_1 = \operatorname{spec} f_2$ *and therefore* π *is an Agmon angle for* f_1 *iff it is an Agmon angle for* f_2. *In this case* $\log\det_N f_1 = \log\det_N f_2$.

(b) f_1 *is an isomorphism iff* f_2 *is an isomorphism. In this case* $\operatorname{Vol}_N f_1 = \operatorname{Vol}_N f_2$.

(3) *Suppose* $f \in \mathcal{L}_{\mathcal{A}}(\mathcal{W}_1 \oplus \mathcal{W}_2)$ *is of the form*

$$f = \begin{pmatrix} f_1 & 0 \\ g & f_2 \end{pmatrix} .$$

Then the following statements hold:

(a) $\operatorname{spec} f = \operatorname{spec} f_1 \cup \operatorname{spec} f_2$ *and therefore* π *is an Agmon angle for* f *iff it is an Agmon angle for both* f_1 *and* f_2. *In this case*

$$\log \det_N f = \log \det_N f_1 + \log \det_N f_2 \ . \tag{1.5A}$$

(b) f *is an isomorphism iff* f_1 *and* f_2 *are both isomorphisms. In this case*

$$\log \operatorname{Vol}_N f = \log \operatorname{Vol}_N f_1 + \log \operatorname{Vol}_N f_2 \ . \tag{1.5B}$$

(4) *Suppose* $\mathcal{W}_1, \mathcal{W}_2$ *and* $\mathcal{W}_3$ *are* $\mathcal{A}$*-Hilbert modules of finite type. If* $f_1 \in \mathcal{L}_{\mathcal{A}}(\mathcal{W}_1, \mathcal{W}_2)$ *and* $f_2 \in \mathcal{L}_{\mathcal{A}}(\mathcal{W}_2, \mathcal{W}_3)$ *are isomorphisms then* $f_2 \cdot f_1 \in \mathcal{L}_{\mathcal{A}}(\mathcal{W}_1, \mathcal{W}_3)$ *is an isomorphism and*

$$\log \operatorname{Vol}_N(f_2 \cdot f_1) = \log \operatorname{Vol}_N f_1 + \log \operatorname{Vol}_N f_2 \ . \tag{1.6}$$

(5) *If $\alpha_i \in \mathcal{L}_{\mathcal{A}}(\mathcal{W}_i)$, $i = 1,2$, are isometries and $f : \mathcal{W}_1 \to \mathcal{W}_2$ is an isomorphism then $\alpha_2 f \alpha_1 \in \mathcal{L}_{\mathcal{A}}(\mathcal{W}_1, \mathcal{W}_2)$ is an isomorphism and*

$$\log \mathrm{Vol}_N(\alpha_2 f \alpha_1) = \log \mathrm{Vol}_N f \ . \tag{1.7}$$

Proof. All these statements can be proved in an elementary way. For the convenience of the reader we give the proof in Appendix 1.

Next we consider the case where π is a weak Agmon angle for $f \in \mathcal{L}_{\mathcal{A}}(\mathcal{W})$. In this case π is an Agmon angle for $f + \lambda$ with any $\lambda > 0$. One verifies that $\log \det_N(f+\lambda)$ is a real analytic function in $\lambda \in (0,\infty)$. We define $\log \det_N f$ as the element in $\mathbf{D}$ (cf. Introduction), represented by the real analytic function

$$\log \det_N(f+\lambda) - \log\lambda \dim_N \mathrm{Null}(f) \ . \tag{1.8}$$

We note that parts (2) and (3) of Proposition 1.9 extend to this case as well.

Let $f \in \mathcal{L}_{\mathcal{A}}(\mathcal{W}_1, \mathcal{W}_2)$ be a weak isomorphism. For $\lambda \geq 0$ denote by $\mathcal{P}_f(\lambda)$ the set of all $\mathcal{A}$-invariant closed subspaces $\mathcal{L} \subset \mathcal{W}_1$ such that, for $x \in \mathcal{L}$, $\|f(x)\| \leq \lambda\|x\|$. Following Gromov-Shubin ([GrSh, formula 3.8, p. 386]) introduce the function $F_f : [0,\infty) \to [0,\infty)$ defined by

$$F_f(\lambda) := \sup\big\{ \dim_N \mathcal{L}; \mathcal{L} \in \mathcal{P}_f(\lambda)\big\} \ . \tag{1.9}$$

Observe that the function $F_f(\lambda)$ is nondecreasing, left continuous, $F_f(0) = 0$ and $F_f(\lambda) = \dim_N(\mathcal{W})$ for $\lambda \geq \|f\|$. Note that f is an isomorphism iff there exists $\lambda_0 > 0$ such that $F_f(\lambda) = 0$ for $\lambda < \lambda_0$. The Novikov-Shubin invariant $\alpha(f)$ associated to a weak isomorphism $f \in \mathcal{L}_{\mathcal{A}}(\mathcal{W}_1, \mathcal{W}_2)$ is defined by

$$\alpha(f) := \liminf_{\lambda \to 0} \frac{\log F_f(\lambda)}{\log \lambda} \in [0,\infty] \ . \tag{1.10}$$

Note that $\alpha(f) = \infty$ if f is an isomorphism.

If $f \in \mathcal{L}_{\mathcal{A}}(\mathcal{W}_1, \mathcal{W}_2)$ is an arbitrary morphism let

$$\overline{f} : \mathcal{W}_1' = \mathcal{W}_1/\mathrm{Null}(f) \to \overline{\mathrm{Range}}\, f = \mathcal{W}_2' \ . \tag{1.11}$$

Note that $\overline{f}$ is a weak isomorphism and define $\alpha(f)$ and $F_f(\lambda)$ by

$$\alpha(f) := \alpha(\overline{f}) \ ; \qquad F_f(\lambda) := F_{\overline{f}}(\lambda) \ . \tag{1.12}$$

PROPOSITION 1.10. (1) *For any weak isomorphism $f \in \mathcal{L}_{\mathcal{A}}(\mathcal{W}_1, \mathcal{W}_2)$*

$$F_f(\lambda) = F_{(f^* f)^{1/2}}(\lambda) = F_{f^*}(\lambda) \ .$$

(2) *If $f : \mathcal{W}_1 \oplus \mathcal{W}_2 \to \mathcal{W}_1 \oplus \mathcal{W}_2$ is a weak isomorphism of the form*

$$f = \begin{pmatrix} f_1 & 0 \\ g & f_2 \end{pmatrix} ,$$

then f_1 and f_2 are both weak isomorphisms and

$$\max\big\{F_{f_1}(\lambda), F_{f_2}(\lambda)\big\} \leq F_f(\lambda) \leq F_{f_1}(\lambda) + F_{f_2}(\lambda) \ .$$

(3) *If* $f \in \mathcal{L}_{\mathcal{A}}(\mathcal{W})$ *is nonnegative and selfadjoint, define the spectral projectors* $Q_f(\lambda) \in \mathcal{L}_{\mathcal{A}}(\mathcal{W})$ *corresponding to the interval* $(-\infty, \lambda]$ *and* $N_f(\lambda) := \operatorname{tr}_N Q_f(\lambda)$. *For* $\lambda \geq 0$, *and* $\overline{f}$ *given by* (1.11)

$$N_f(\lambda) = \dim_N \left(\operatorname{Null}(f)\right) + F_{\overline{f}}(\lambda) . \tag{1.13}$$

The verification of these statements is straightforward using the definition of $F_f(\lambda)$ (cf. Appendix 1).

The function $N_f(\lambda)$ is called the spectral distribution function of f. Note that $F_{\overline{f}}(\lambda)$ is nondecreasing and $F_{\overline{f}}(0) = 0$. $F_{\overline{f}}(\lambda)$ can be used to represent $\log \det_N(\overline{f}^*\overline{f})^{1/2}$ in **D** as the function given by the Stieltjes integral $\int_{0+}^{\infty} \log(\mu + \lambda) dF_{\overline{f}}(\mu)$.

Denote by **F** the set of functions $F : [0, \infty) \to [0, \infty)$ satisfying

(F1) $F(0) = 0$;
(F2) $F(\lambda)$ is nondecreasing;
(F3) F is continuous to the left,

and recall the following definitions of Gromov-Shubin (cf. [GrSh]).

DEFINITION 1.11. (1) *Functions* $F, G \in \mathbf{F}$ *are said to be dilational equivalent, denoted* $F \overset{d}{\sim} G$, *iff there exists* $C > 0$ *such that for* $\lambda \geq 0$

$$G(C^{-1}\lambda) \leq F(\lambda) \leq G(C\lambda) . \tag{1.14}$$

(2) *Functions* $F, G \in \mathbf{F}$ *are said to be dilational equivalent near zero, denoted* $F \underset{0}{\overset{d}{\sim}} G$ *iff there exist* $C > 0$ *and* $\lambda_0 > 0$ *such that* (1.14) *holds for* $\lambda < \lambda_0$.

We end this subsection with the following observation. Suppose that $\psi : \mathcal{A}' \to \mathcal{A}''$ is a homomorphism of finite von Neumann algebras which is injective and makes $\mathcal{A}''_2$ an $\mathcal{A}'$-Hilbert module of finite type. Then it makes any $\mathcal{A}''$-Hilbert module of finite type $\mathcal{W}$ an $\mathcal{A}'$-Hilbert module of finite type and we have $\mathcal{L}_{\mathcal{A}''}(\mathcal{W}) \subseteq \mathcal{L}_{\mathcal{A}'}(\mathcal{W})$.

REMARK 1.12: Assume that for any $f \in \mathcal{A}'$, $\operatorname{tr}_{N,\mathcal{A}'}(\psi(f) : \mathcal{A}''_2 \to \mathcal{A}''_2) = r \operatorname{tr}_{N,\mathcal{A}''}(\psi(f))$. Then:

(1) $\dim_{N,\mathcal{A}'}(\mathcal{W}) = r \dim_{N,\mathcal{A}''}(\mathcal{W})$.
(2) If $f \in \mathcal{L}_{\mathcal{A}''}(\mathcal{W})$ then $\operatorname{tr}_{N,\mathcal{A}'}(f) = r \operatorname{tr}_{N,\mathcal{A}''}(f)$.
(3) If π is a weak Agmon angle for f then $\log \det_{N,\mathcal{A}'}(f) = r \log \det_{N,\mathcal{A}''}(f)$.

1.3 Cochain complexes of finite type and torsion in the von Neumann sense.

DEFINITION 1.13. *A cochain complex in the category of* $\mathcal{A}$*-Hilbert modules of finite type,* $\mathcal{C} = (\mathcal{C}_i, d_i)$, *consists of a collection of Hilbert modules of finite type* $\mathcal{C}_i$, *all but finitely many zero, and a collection of morphisms*

$d_i : \mathcal{C}_i \to \mathcal{C}_{i+1}$ which satisfy $d_i d_{i-1} = 0$. In the sequel we always assume that $\mathcal{C}_i = 0$ for $i < 0$ and refer to such a complex as a cochain complex of finite type over $\mathcal{A}$, or simply as a cochain complex of finite type.

The reduced cohomology of $\mathcal{C}$, $\overline{H}^i(\mathcal{C})$, is defined by

$$\overline{H}^i(\mathcal{C}) = \mathrm{Null}(d_i)/\overline{\mathrm{Range}(d_{i-1})} .$$

Define the Betti numbers and Euler-Poincaré characteristic of $\mathcal{C}$ by

$$\beta_i(\mathcal{C}) := \dim_N \overline{H}^i(\mathcal{C}) ; \qquad \chi(\mathcal{C}) := \sum_i (-1)^i \beta_i(\mathcal{C}) , \tag{1.15}$$

and introduce a weighted version of the Euler-Poincaré characteristic,

$$\psi(\mathcal{C}) := \sum_i (-1)^i i \beta_i(\mathcal{C}) . \tag{1.16}$$

Denote by $d_i^* : \mathcal{C}_{i+1} \to \mathcal{C}_i$ the adjoint of d_i, and consider $\Delta_i = d_i^* d_i + d_{i-1} d_{i-1}^*$. The operator Δ_i is a selfadjoint and nonnegative morphism.

DEFINITION 1.14. (1) *Given two cochain complexes of finite type over $\mathcal{A}, \mathcal{C}'$ and $\mathcal{C}''$, a morphism $\mathbf{f} : \mathcal{C}' \to \mathcal{C}''$ is given by a collection of morphisms $f_i : \mathcal{C}_i' \to \mathcal{C}_i''$ which commute with the differentials d_j.*

(2) *A homotopy $\underline{\mathbf{t}}$ between morphisms $\mathbf{f}$ and $\mathbf{g}$ is given by a collection of morphism $t_i : \mathcal{C}_i' \to \mathcal{C}_{i-1}''$ satisfying*

$$f_i - g_i = d_{i-1}'' t_i + t_{i+1} d_i' . \tag{1.17}$$

(3) *Two cochain complexes $\mathcal{C}'$ and $\mathcal{C}''$ are homotopy equivalent if there exist morphisms (in the category of complexes) $\mathbf{f} : \mathcal{C}' \to \mathcal{C}''$ and $\mathbf{g} : \mathcal{C}'' \to \mathcal{C}'$ so that $\mathbf{g} \cdot \mathbf{f}$ resp. $\mathbf{f} \cdot \mathbf{g}$ is homotopic to $\mathbf{id}_{\mathcal{C}'}$ resp. $\mathbf{id}_{\mathcal{C}''}$.*

Given a morphism $\mathbf{f} : \mathcal{C}' \to \mathcal{C}''$, denote by $\overline{H}(\mathbf{f})^i$ the induced morphisms of $\mathcal{A}$-Hilbert modules $\overline{H}(\mathbf{f})^i : \overline{H}^i(\mathcal{C}') \to \overline{H}^i(\mathcal{C}'')$. Note that if $\mathbf{f}_r : \mathcal{C}' \to \mathcal{C}''$, $r = 1, 2$, are two homotopic morphisms then $\overline{H}(\mathbf{f}_1)^i = \overline{H}(\mathbf{f}_2)^i$ for all i. Given a finite type cochain complex $\mathcal{C} = (\mathcal{C}_i, d_i)$, each $\mathcal{C}_i$ can be decomposed as a direct sum of mutually orthogonal subspaces $\mathcal{C}_i = \mathcal{H}_i \oplus \mathcal{C}_i^+ \oplus \mathcal{C}_i^-$ with

$$\mathcal{H}_i = \mathrm{Null}\ \Delta_i ; \qquad \mathcal{C}_i^+ = \overline{d_{i-1}(\mathcal{C}_{i-1})} , \qquad \mathcal{C}_i^- = \overline{d_i^*(\mathcal{C}_{i+1})} . \tag{1.18}$$

This decomposition is called the Hodge decomposition. The map d_i can then be described by a 3×3 matrix of the form

$$d_i = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \underline{d}_i \\ 0 & 0 & 0 \end{pmatrix} , \tag{1.19}$$

where $\underline{d}_i : \mathcal{C}_i^- \to \mathcal{C}_{i+1}^+$ is a weak isomorphism and the combinatorial Laplacian $\Delta_i = d_{i-1} d_{i-1}^* + d_i^* d_i$ then takes the form of the diagonal matrix $\mathrm{diag}(0, \underline{d}_{i-1} \underline{d}_{i-1}^*, \underline{d}_i^* \underline{d}_i)$.

Let $\mathbf{f} : \mathcal{C}^1 \to \mathcal{C}^2$ be a morphism. With respect to the Hodge decompositions of $\mathcal{C}_i^1$ and $\mathcal{C}_i^2$, the morphism $f_i : \mathcal{C}_i^1 \to \mathcal{C}_i^2$ can be written as a 3×3-matrix of the form

$$f_i = \begin{pmatrix} f_{i,11} & 0 & f_{i,13} \\ f_{i,21} & f_{i,22} & f_{i,23} \\ 0 & 0 & f_{i,33} \end{pmatrix} \tag{1.20}$$

where $f_{i,11} \in \mathcal{L}_{\mathcal{A}}(\mathcal{H}_i^1, \mathcal{H}_i^2)$, $f_{i,22} \in \mathcal{L}_{\mathcal{A}}(\mathcal{C}_i^{1,+}, \mathcal{C}_i^{2,+})$, $f_{i,33} \in \mathcal{L}_{\mathcal{A}}(\mathcal{C}_i^{1,-}, \mathcal{C}_i^{2,-})$ and $\underline{d}_i^2 \cdot f_{i,33} = f_{i+1,22} \cdot \underline{d}_i^1$.

Definition 1.15. *A cochain complex $\mathcal{C}$ is called perfect if, for any i, $\underline{d}_i$ is an isomorphism.*

For a perfect cochain complex $\operatorname{spec} \Delta_i \setminus \{0\}$ is bounded away from zero $(1 \le i \le d)$.

Lemma 1.16. (1) *Given a cochain complex $\mathcal{C} = (\mathcal{C}_i, d_i)$ one can find modifications $\tilde{d}_i$'s of d_i's so that $\tilde{\mathcal{C}} = (\mathcal{C}_i, \tilde{d}_i)$ is perfect and has the same Hodge decomposition as $\mathcal{C}$.*

(2) *Given an isomorphism $\mathbf{f} : \mathcal{C}^1 \to \mathcal{C}^2$ of cochain complexes $\mathcal{C}^k = (\mathcal{C}_i^k, d_i)$, $k = 1, 2$, one can find modifications $\tilde{d}_i^k$ of d_i^k so that*

$$f_{i+1}\tilde{d}_i^1 = \tilde{d}_i^2 f_i \tag{1.21}$$

and the cochain complexes $\tilde{\mathcal{C}}^k = (\mathcal{C}_i^k, \tilde{d}_i^k)$ are perfect and have the same Hodge decompositions as $\mathcal{C}^k$ $(1 \le k \le 2)$.

Proof. Statement (1) follows by choosing $\tilde{d}_i$ of the form

$$\tilde{d}_i = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \tilde{\underline{d}}_i \\ 0 & 0 & 0 \end{pmatrix} . \tag{1.22}$$

where $\tilde{\underline{d}}_i$ is the isometry in the polar decomposition of $\underline{d}_i$ given by $\tilde{\underline{d}}_i = \underline{d}_i(\underline{d}_i^* \underline{d}_i)^{-\frac{1}{2}}$.

(2) With respect to the Hodge decomposition of $\mathcal{C}_i^1$ and $\mathcal{C}_i^2$ define $\tilde{d}_i^1$ as in (1) and choose $\tilde{d}_i^2$ to be of the form

$$\tilde{d}_i^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \tilde{\underline{d}}_i^2 \\ 0 & 0 & 0 \end{pmatrix}$$

with $\tilde{\underline{d}}_i^2 := f_{i+1,22} \cdot \tilde{\underline{d}}_i^1 \cdot f_{i,33}^{-1}$. □

In section 6.1 we will need the following:

Proposition 1.17. *Suppose $\mathcal{C}(t) = (\mathcal{C}_i(t), d_i(t))$ is a family of cochain complexes of finite type depending on a parameter $t \ge 0$, and $\mathbf{f}(t) : \mathcal{C}(t) \to \mathcal{C}$*

is an isomorphism of cochain complexes for any t. Introduce $\log V(t) := \sum_{q=0}^{d}(-1)^q \log \mathrm{Vol}_N \overline{H}(\mathbf{f}(t))^q$. *Assume that $\mathcal{C}_i(t)$ and $\mathcal{C}_i$ are free modules and that there exist orthonormal bases $e_{i,1}(t), \dots, e_{i,l_i}(t)$ for $\mathcal{C}_i(t)$ and $e_{i,1}, \dots, e_{i,l_i}$ for $\mathcal{C}_i$ so that $f_i(t)$, when expressed with respect to these bases, is an $l_i \times l_i$-matrix with entries in $\mathcal{A}^{\mathrm{op}}$ of the form $Id + O(1/t)$. Then* $\log V(t) = O(1/t)$.

Proof. In view of Lemma 1.16(1) it suffices to prove the result for the case where $\underline{d}_i(t)$ and $\underline{d}_i$ are isometries. In view of (1.20) and Proposition 1.9(3)

$$\log \mathrm{Vol}_N \overline{H}(\mathbf{f}(t))^i = \log \mathrm{Vol}_N (f_i(t)) - \log \mathrm{Vol}_N (f_{i,22}(t)) - \log \mathrm{Vol}_N (f_{i,33}(t)) \ . \tag{1.23}$$

As $\underline{d}_i f_{i,33}(t) = f_{i+1,22}(t)\underline{d}_i(t)$, Proposition 1.9(5) implies that

$$\log \mathrm{Vol}_N (f_{i,33}(t)) = \log \det{}_N (f_{i+1,22}(t)) \ . \tag{1.24}$$

Taking the alternating sum of (1.23) therefore leads to

$$\sum_{i=0}^{d}(-1)^i \log \mathrm{Vol}_N (\overline{H}(f(t))^i) = \sum_{i=0}^{d}(-1)^i \log \mathrm{Vol}_N (f_i(t)) \ . \tag{1.25}$$

The claim now follows from the assumptions made on the asymptotics of $f_i(t)$. □

Given a cochain complex $\mathcal{C}$ of finite type, introduce, following Gromov-Shubin ([GrSh], cf. also end of section 1.2), the functions $F_{\mathcal{C},i}(\lambda) \in \mathbf{F}$ defined by $F_{\mathcal{C},i}(\lambda) := F_{\underline{d}_i^* \underline{d}_i}(\lambda)$ and the numbers α_i defined by $\alpha_i := \alpha(\underline{d}_i)$ (cf. (1.10)). The following result is due to Gromov-Shubin ([GrSh, Proof of Proposition 4.1]).

PROPOSITION 1.18. *Suppose $\mathbf{f} : \mathcal{C}' \to \mathcal{C}''$ and $\mathbf{g} : \mathcal{C}'' \to \mathcal{C}'$ are two morphisms of cochain complexes so that $\mathbf{id}_{\mathcal{C}'}$ is homotopic to $\mathbf{gf}$ by a homotopy $\underline{\mathbf{t}} = \{t_i\}$. Then*

$$F_{\mathcal{C}',i}(\lambda) \le F_{\mathcal{C}'',i}(4\|f_{i+1}\|^2\|g_i\|^2\lambda) \quad \textit{for} \quad 0 < \lambda < \frac{1}{4\|t_{i+1}\|^2} \ . \tag{1.26}$$

In particular if $\mathbf{f} : \mathcal{C}' \to \mathcal{C}''$ is an isomorphism then $F_{\mathcal{C}',i}(\lambda) \overset{d}{\sim} F_{\mathcal{C}'',i}(\lambda)$ (choose $\underline{\mathbf{t}} = 0$); if $\mathcal{C}'$ and $\mathcal{C}''$ are homotopy equivalent, then $F_{\mathcal{C}',i}(\lambda) \underset{0}{\overset{d}{\sim}} F_{\mathcal{C}'',i}(\lambda)$, and therefore $\alpha'_i = \alpha''_i$.

The torsion $\log T(\mathcal{C})$ is the element in $\mathbf{D}$ defined by

$$\log T(\mathcal{C}) = \tfrac{1}{2}\sum_i (-1)^{i+1} i \log \det{}_N \Delta_i \ . \tag{1.27}$$

The spectral distribution functions $N_i(\lambda) := N_{\Delta_i}(\lambda)$ satisfy (cf. [GrSh, (3.6)])

$$N_i(\lambda) = \beta_i + F_{i-1}(\lambda) + F_i(\lambda) \tag{1.28}$$

where $F_i = F_{\mathcal{C},i}$. Therefore $\log T(\mathcal{C})$ can be represented by the real analytic function in λ

$$\tfrac{1}{2}\sum_i(-1)^i\int_{0+}^{\infty}\log(\mu+\lambda)dF_i(\mu)\ . \tag{1.29}$$

DEFINITION 1.19. *A cochain complex $\mathcal{C}$ of finite type is of determinant class iff $\int_{0+}^{1}\log(\lambda)dN_i(\lambda) > -\infty$ for all i, or equivalently $\int_1^\infty \frac{1}{x}(tr_N e^{-x\Delta_i} - \beta_i)dx < \infty$ for all i.*

We point out that if $\mathcal{C}$ is of determinant class, then $\log T(\mathcal{C})$ is in $\mathbf{R} \subset \mathbf{D}$, and a sufficient condition for $\mathcal{C}$ to be of determinant class is that $\alpha_k > 0$ for $0 \le k \le d$.

The following Lemma 1.20 can be used to deduce from Proposition 1.18 that a cochain complex of finite type which is homotopy equivalent to a cochain complex of determinant class is of determinant class as well.

LEMMA 1.20. *Assume that $N_1(\mu)$ and $N_2(\mu)$ are nonnegative, increasing function on $(0,1]$ such that $N_1(0+) = N_2(0+) = 0$ and $N_1(\mu) \le N_2(\mu)$ for $0 < \mu \le 1$. Let $f(\mu)$ be a nonnegative decreasing continuous function on $(0,1]$. Then*

$$\int_{0+}^{1+} f(\mu)dN_1(\mu) \le \int_{0+}^{1+} f(\mu)dN_2(\mu)\ .$$

Proof. Without loss of generality assume that $\int_{0+}^{1+} f(\mu)dN_2(\mu) < \infty$. Use that $N_2(0+) = 0$ and f is decreasing to conclude that $\lim_{\mu\to 0+} f(\mu)N_2(\mu) = 0$ and so $\lim_{\mu\to 0+} f(\mu)N_1(\mu) = 0$ as well. Using the integration by parts formula for the Stieltjes integral one concludes that

$$\int_{\epsilon+}^{1+} f(\mu)dN_1(\mu) \le \int_{\epsilon+}^{1+} f(\mu)dN_2(\mu) + f(\epsilon)\big(N_2(\epsilon) - N_1(\epsilon)\big)\ .$$

Taking the limit $\epsilon \to 0$ leads to the conclusion. □

It will be convenient for the proof of the product formula below to introduce for $\lambda > 0$ and $s \in \mathbf{C}$ with $\Re s > 0$ the zeta function associated with the complex $\mathcal{C}$,

$$\zeta_{\mathcal{C}}(\lambda, s) = \tfrac{1}{2}\sum_i(-1)^i i\, \mathrm{tr}_N\left((\Delta_i + \lambda)^{-s}\right)\ . \tag{1.30}$$

This function is real analytic in λ, complex analytic in s for $\Re s > 0$ and admits an analytic continuation to the entire s-plane so that $s = 0$ is a regular point. Notice that $\log T(\mathcal{C})$ is also represented by the real analytic function in λ given by

$$\frac{d}{ds}\Big|_{s=0}\zeta_{\mathcal{C}}(\lambda, s) - \psi(\mathcal{C})\log\lambda \tag{1.31}$$

where $\psi(\mathcal{C})$ denotes the weighted Euler-Poincaré characteristic (1.16) (cf. also [Go] for (1.31)). Moreover, by the spectral theorem (cf. Appendix 1)

$$\zeta_{\mathcal{C}}(\lambda, s) = \frac{1}{2}\sum_i (-1)^i i \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{(-t\lambda)} \operatorname{tr}_N e^{-t\Delta_i} dt \, . \tag{1.32}$$

Suppose that $\mathcal{A}'$ resp. $\mathcal{A}''$ are finite von Neumann algebras. Note that $\mathcal{A} = \mathcal{A}' \otimes \mathcal{A}''$ (tensor product in the category of finite von Neumann algebras) is also a finite von Neumann algebra. If $\mathcal{W}'$ resp. $\mathcal{W}''$ are $\mathcal{A}'$ resp. $\mathcal{A}''$-Hilbert modules of finite type then the tensor product $\mathcal{W}' \otimes \mathcal{W}''$ (tensor product in the category of Hilbert spaces) is an $\mathcal{A}$-Hilbert module of finite type and

$$\dim_N(\mathcal{W}' \otimes \mathcal{W}'') = \dim_N \mathcal{W}' \cdot \dim_N \mathcal{W}'' \, .$$

Let $\mathcal{C}'$ resp. $\mathcal{C}''$, be two cochain complexes of finite type over $\mathcal{A}'$, resp. $\mathcal{A}''$. Denote by $\mathcal{C} = \mathcal{C}' \otimes \mathcal{C}''$ the tensor product of these cochain complexes,

$$\mathcal{C}_i = \sum_{p+r=i} \mathcal{C}'_p \otimes \mathcal{C}''_r \, , \quad d_i = \sum_{p+r=i} d'_p \otimes id + (-1)^p id \otimes d''_r \, .$$

Then $\mathcal{C}$ is a cochain complex of finite type over $\mathcal{A}$.

PROPOSITION 1.21 (cf. [CM], [LüRo]). *Let $\mathcal{C}'$, resp. $\mathcal{C}''$ be two finite type cochain complexes over $\mathcal{A}'$ resp. $\mathcal{A}''$. Then, with $\mathcal{C} = \mathcal{C}' \otimes \mathcal{C}''$,*

(1) $\overline{H}^i(\mathcal{C}) = \sum_{p+r=i} \overline{H}^p(\mathcal{C}') \otimes \overline{H}^r(\mathcal{C}'')$

(2) $\zeta_{\mathcal{C}}(\lambda, s) = \zeta_{\mathcal{C}'}(\lambda, s)\chi(\mathcal{C}'') + \zeta_{\mathcal{C}''}(\lambda, s)\chi(\mathcal{C}')$

(3) $\psi(\mathcal{C}) = \psi(\mathcal{C}')\chi(\mathcal{C}'') + \psi(\mathcal{C}'')\chi(\mathcal{C}')$.

Proof. The proof of (1) can be found, e.g. in [LüRo, Theorem 3.16] and (3) follows from (1). To prove (2) (cf. [CM]) let $\mathcal{C}$ be a cochain complex of finite type over $\mathcal{A}'$, and $\mathcal{V}$ be a $\mathcal{A}''$-Hilbert module of finite type. We will show below that

$$\sum_q (-1)^q \operatorname{tr}_N(e^{-t\Delta_q}) = \chi(\mathcal{C}) \, . \tag{1.33}$$

Therefore if $\beta : \mathcal{V} \to \mathcal{V}$ is a morphism, then

$$\sum_q (-1)^q \operatorname{tr}_N(e^{-t\Delta_q} \otimes \beta) = \operatorname{tr}_N(\beta)\chi(\mathcal{C}) \, . \tag{1.34}$$

To prove (1.33) we use the matrix representation of Δ_q with respect to the Hodge decomposition, $\operatorname{diag}(0, \underline{d}_{q-1}\underline{d}^*_{q-1}, \underline{d}^*_q\underline{d}_q)$ and Proposition 1.9 (1); they give

$$\operatorname{tr}_N e^{-t\Delta_{q-1}}|_{\mathcal{C}^-_{q-1}} = \operatorname{tr}_N e^{-t\Delta_q}|_{\mathcal{C}^+_q}$$

and consequently

$$\operatorname{tr}_N e^{-t\Delta_q} = \operatorname{tr}_N(e^{-t\Delta_q|_{\mathcal{C}^+_q}}) + \operatorname{tr}_N(e^{-t\Delta_{q+1}|_{\mathcal{C}^+_{q+1}}}) + \dim_N \operatorname{Null}(\Delta_q)$$

which leads to (1.33). Next, decompose $\Delta_q = \bigoplus_{p+r=q} \Delta_{p,r}$, where $\Delta_{p,r} = \Delta'_p \otimes id + id \otimes \Delta''_r$ to obtain

$$2\zeta_C(\lambda,s) = \sum_{p,r}(-1)^{(p+r)}(p+r)\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-t\lambda}\operatorname{tr}_N(e^{-t\Delta'_p}\otimes e^{-t\Delta''_r})dt$$
$$= \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-t\lambda}\operatorname{tr}_N\left(\left\{\bigoplus_p(-1)^p p e^{-t\Delta'_p}\right\}\otimes\left\{\bigoplus_r(-1)^r e^{-t\Delta''_r}\right\}\right)dt$$
$$+ \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-t\lambda}\operatorname{tr}_N\left(\left\{\bigoplus_p(-1)^p e^{-t\Delta'_p}\right\}\otimes\left\{\bigoplus_r(-1)^r r e^{-t\Delta''_r}\right\}\right)dt$$

which, in view of (1.34), is equal to

$$\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-t\lambda}\Big(\chi(C'')\sum_p(-1)^p p\operatorname{tr}_N e^{-t\Delta'_p}+\chi(C')\sum_r(-1)^r r\operatorname{tr}_N e^{-t\Delta''_r}\Big)dt$$
$$= 2\zeta_{C'}(\lambda,s)\cdot\chi(C'') + 2\zeta_{C''}(\lambda,s)\cdot\chi(C')\ . \qquad \square$$

Corollary 1.22 (cf. [CM], [LüRo]). *With the same assumptions as in Proposition 1.21, the following identity, viewed in the vector space* $\mathbf{D}$, *holds:*

$$\log T(C) = \chi(C'')\log T(C') + \chi(C')\log T(C'')\ .$$

1.4 $(\mathcal{A},\Gamma^{\mathrm{op}})$-Hilbert modules and bundles of $\mathcal{A}$-Hilbert modules .

Definition 1.23. *$\mathcal{W}$ is an $(\mathcal{A},\Gamma^{\mathrm{op}})$-Hilbert module of finite type if*
(BM1) *$\mathcal{W}$ is an $\mathcal{A}$-Hilbert module of finite type;*
(BM2) *$\mathcal{W}$ is a Γ^{op}-Hilbert module, defined by a unitary representation of Γ;*
(BM3) *the action of $\mathcal{A}$ and Γ^{op} commute.*

Let X be a countable set. Denote by $l_2(X)$ the Hilbert space obtained by completion of $\mathbb{C}(X) = \{f : X \to \mathbb{C};\ \underline{\mathrm{supp}}(f)$ is finite$\}$ with respect to the scalar product $\langle f_1, f_2\rangle := \sum_{x\in X} f(x)\overline{g(x)}$.

Example 1.24: Let Γ be a countable group and $\mathbb{C}(\Gamma)$ denote the unital $\mathbb{C}$-algebra with multiplication defined by convolution and $*$-operation induced by the map, $g \mapsto g^{-1}$. The algebra $\mathbb{C}(\Gamma)$ has a finite trace given by $tr(f) := f(e)$ where e denotes the unit element in Γ, and acts from the left by convolutions on $l_2(\Gamma)$. This algebra can be viewed as a $*$-subalgebra of $\mathcal{L}_\Gamma(l_2(\Gamma), l_2(\Gamma))$. Denote by $\mathcal{N}(\Gamma)$ its weak closure in $\mathcal{L}_\Gamma(l_2(\Gamma), l_2(\Gamma))$. Then $\mathcal{N}(\Gamma)$ is a finite von Neumann algebra referred to as the von Neumann algebra associated to Γ.

Example 1.25: Let $\rho : \Gamma \times X \to X$ be a *left* action of Γ on the set X with finite isotropy groups. ρ induces a left action of Γ by isometries which makes $l_2(X)$ an $\mathcal{N}(\Gamma)$-Hilbert module; if the quotient set $\Gamma\backslash X$ is finite, then this module is a Hilbert module of finite type. Suppose, in addition, that Γ' is another countable group and $\rho' : X \times \Gamma' \to X$ is a *right* action of Γ' on X so that ρ and ρ' commute. Γ' induces an action by isometries on $l_2(X)$ which makes $l_2(X)$ an $(\mathcal{N}(\Gamma), \Gamma'^{\mathrm{op}})$-Hilbert module of finite type. As an example, consider the case $X = |\Gamma|$, the underlying set of Γ, $\Gamma = \Gamma'$ and ρ and ρ' given by $\rho(g_1, g_2) = g_1 g_2$, and $\rho'(g_2, g_1) = g_2 g_1^{-1}$. Then $l_2(\Gamma)$ is an $(\mathcal{N}(\Gamma), \Gamma^{\mathrm{op}})$-Hilbert module of finite type, referred to as the *regular birepresentation.*

DEFINITION 1.26. *A smooth bundle $p : \mathcal{E} \to M$ over a smooth manifold M is a bundle of $\mathcal{A}$-Hilbert modules of finite type with fiber $\mathcal{W}$ if*

(B1) *$p : \mathcal{E} \to M$ is a smooth bundle of topological vector spaces, equipped with a Hermitian structure μ which makes each fiber $p^{-1}(x)$, $x \in M$, into a separable Hilbert space;*

(B2) *$\mathcal{E}$ is equipped with a smooth fiberwise action $\rho : \mathcal{A} \times \mathcal{E} \to \mathcal{E}$ which makes each fiber $p^{-1}(x)$ an $\mathcal{A}$-Hilbert module of finite type.*

(B3) *$\mathcal{W}$ is an $\mathcal{A}$-Hilbert module of finite type and $p : \mathcal{E} \to M$ is locally isomorphic to $p_o : \mathcal{W} \times M \to M$ where the local isomorphism intertwines p, p_o, the Hermitian structures and the $\mathcal{A}$-actions.*

EXAMPLE 1.27: Let M be a closed smooth manifold with fundamental group $\Gamma := \pi_1(M)$ and let $\mathcal{W}$ be an $(\mathcal{A}, \Gamma^{\mathrm{op}})$-Hilbert module of finite type. Let $\tilde{p} : \mathcal{W} \times \tilde{M} \to \tilde{M}$ be the trivial smooth bundle of $\mathcal{A}$-Hilbert modules; $\tilde{p}$ is Γ-equivariant with respect to the diagonal action of Γ on $\mathcal{W} \times \tilde{M}$ and the left action of Γ on $\tilde{M}$. Therefore $\tilde{p}$ induces $p : \mathcal{E} = \mathcal{W} \times_\Gamma \tilde{M} \to M$ which is a smooth bundle of $\mathcal{A}$-Hilbert modules of finite type. This bundle is the canonical bundle over M, associated to $\mathcal{W}$.

2. Calculus of Pseudodifferential Operators Acting on $\mathcal{A}$-Hilbert Bundles of Finite Type

In this section we construct a calculus of pseudodifferential operators, called pseudodifferential $\mathcal{A}$-operators, on a compact manifold, where $\mathcal{A}$ is a finite von Neumann algebra (cf. e.g. [FMi],[Le],[Lu] and [Mo] for related work).

2.1 Sobolev spaces, symbols and kernels. Let B be a Banach space. For $u \in \mathcal{S}(\mathbb{R}^d, B)$, the space of functions $u : \mathbb{R}^d \to B$ of Schwartz class, $\|u\|_s$ denotes the Sobolev s-norm given by

$$\|u\|_s^2 := \int_{\mathbb{R}^d} \left(1 + |\xi|^2\right)^s \left\|\hat{u}(\xi)\right\|^2 d\xi$$

where $\hat{u}(\xi)$ denotes the Fourier transform of u.

DEFINITION 2.1. (1) *The Sobolev space* $H_s(\mathbb{R}^d, B)$ *is the completion of* $\mathcal{S}(\mathbb{R}^d, B)$ *with respect to the Sobolev s-norm; equivalently, it can be defined as the space of all distributions* $u \in \mathcal{S}'(\mathbb{R}^d, B)$ *with*

$$\left(1 + |\xi|^2\right)^{s/2} \hat{u} \in L_2(\mathbb{R}^d; B) .$$

(2) *The space* $H_s^{\mathrm{loc}}(\mathbb{R}^d, B)$ *is the space of all distributions* $u \in \mathcal{D}'(\mathbb{R}^d, B)$ *such that* $\phi u \in H_s(\mathbb{R}^d, B)$ *for any* $\phi \in C_0^\infty(\mathbb{R}^d)$.

Most of the properties of the Sobolev spaces $H_s(\mathbb{R}^d, B)$ are the same as of the usual Sobolev spaces for functions with values in finite dimensional vector spaces (cf. [Le]). Let $\mathcal{W}$ be an $\mathcal{A}$-Hilbert module. The space $H_s(\mathbb{R}^d, \mathcal{W})$ is an $\mathcal{A}$-Hilbert module whose dual can be identified with $H_{-s}(\mathbb{R}^d, \mathcal{W})$. Note also that $H_s^{\mathrm{loc}}(\mathbb{R}^d, \mathcal{W})$ is an $\mathcal{A}$-module. Extending the classical case $\mathcal{A} = \mathbb{C}$, symbols are defined as follows:

DEFINITION 2.2. (1) *A function* $a \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d, \mathcal{L}_{\mathcal{A}}(\mathcal{W}, \mathcal{W}))$ *is a symbol of order* $m \in \mathbb{R}$, *denoted by* $a \in S_{\mathcal{W}}^m = S_{\mathcal{W}}^m(\mathbb{R}^d \times \mathbb{R}^d)$, *if the following conditions hold:*

(Sy1) $a(x, \xi)$ *has compact support in* x;

(Sy2) *for any multiindices,* α *and* β, *there exists a constant,* $C_{\alpha\beta}$, *such that*

$$\left\|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)\right\| \leq C_{\alpha\beta}\left(1 + |\xi|\right)^{m - |\beta|} . \tag{2.1}$$

(2) (cf. [Sh1, p. 29]) *A symbol* $a \in S_{\mathcal{W}}^m$ *is classical if it admits an expansion of classical type* $\sum_{j \geq 0} \psi(\xi) a_{m-j}(x, \xi)$ *with* $\psi \in C^\infty(\mathbb{R}^d)$ *given by*

$$\psi(\xi) = \begin{cases} 0 & \text{for } |\xi| \leq \frac{1}{2} \\ 1 & \text{for } |\xi| \geq 1 . \end{cases}$$

This means that:

(Sy3) $a_{m-j} \in C^\infty(\mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\}), \mathcal{L}_{\mathcal{A}}(\mathcal{W}, \mathcal{W}))$ *has compact* x*-support and is positively homogeneous of degree* $m - j$;

(Sy4) $a(x, \xi) - \sum_{j=0}^{N-1} \psi(\xi) a_{m-j}(x, \xi) \in S_{\mathcal{W}}^{m-N}$ *for all* $N \geq 0$.

Subsequently, we always assume that all symbols are classical.

Given $a \in S_{\mathcal{W}}^m$, define a linear operator $A : C_0^\infty(\mathbb{R}^d, \mathcal{W}) \to C_0^\infty(\mathbb{R}^d, \mathcal{W})$ by

$$Au(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} d\xi \int_{\mathbb{R}^d} dx\, e^{i(x-y,\xi)} a(x, \xi) u(y) . \tag{2.2}$$

The principal symbol of a classical pseudodifferential operator A, $\sigma_A(x, \xi) = a_m(x, \xi)$, is invariantly defined as a smooth function on $T^*\mathbb{R}^d \setminus \{0\}$ with values in $\mathcal{L}_{\mathcal{A}}(\mathcal{W}, \mathcal{W})$.

The operator A is said to be a pseudodifferential $\mathcal{A}$-operator of order m, denoted $A \in \Psi DO_{\mathcal{A}}^m(\mathbb{R}^d \times \mathcal{W})$, and can be extended to a bounded, linear operator (any $s \in \mathbf{R}$)

$$A : H_s(\mathbf{R}^d, \mathcal{W}) \to H_{s-m}(\mathbf{R}^d, \mathcal{W}) .$$

The Schwartz kernel of A, $K_A(x,y)$, is given formally as an oscillatory integral

$$K_A(x,y) = \int_{\mathbf{R}^d} e^{i(x-y,\xi)} a(x,\xi) d\xi . \tag{2.3}$$

We note that if $m < -d$, the kernel $K_A(x,y)$ is continuous.

DEFINITION 2.3. *A pseudodifferential operator A in $\Psi DO^m_{\mathcal{A}}(\mathbf{R}^d \times \mathcal{W})$ is said to be uniformly elliptic on $U \subset \mathbf{R}^d$ if the principal symbol $a_m(x,\xi)$ is invertible for $\xi \in \mathbf{R}^d \setminus \{0\}$ for all $x \in U$ and*

$$\left\| a_m(x,\xi)^{-1} \right\| \leq C_1 \left(1 + |\xi|\right)^{-m} \quad \text{for } x \in U , \quad |\xi| \geq 1 . \tag{2.4}$$

Note that as $a_m(x,\xi) \in \mathcal{L}_{\mathcal{A}}(\mathcal{W},\mathcal{W})$ the inverse satisfies $a_m(x,\xi)^{-1} \in \mathcal{L}_{\mathcal{A}}(\mathcal{W},\mathcal{W})$.

2.2 Pseudodifferential operators on bundles of Hilbert modules. Throughout this section let (M,g) denote a compact Riemannian manifold of dimension d, possibly with boundary, $\mathcal{A}$ a finite von Neumann algebra, $\mathcal{W}$ an $\mathcal{A}$-Hilbert module of finite type and $p : \mathcal{E} \to M$ a bundle of $\mathcal{A}$-Hilbert modules with fiber $\mathcal{W}$.

Introduce the Banach bundles of bounded linear operators $\mathcal{L} \to M \times M$ and $\mathcal{B} = \mathcal{L}_{\mathcal{A}} \to M \times M$ whose fibers at $(x,y) \in M \times M$ are given by

$$\mathcal{L}_{xy} = \mathcal{L}(\mathcal{E}_y, \mathcal{E}_x) ; \qquad \mathcal{B}_{xy} = \mathcal{B}(\mathcal{E}_y, \mathcal{E}_x)$$

where $\mathcal{L}(\mathcal{E}_y, \mathcal{E}_x)$ denotes the Banach space of all bounded linear operators from the fiber $\mathcal{E}_y$ to the fiber $\mathcal{E}_x$ and

$$\mathcal{B}_{xy} = \{ f \in \mathcal{L}_{xy} ; f \text{ is } \mathcal{A}\text{-linear} \} .$$

The Banach bundle $\omega : \mathcal{B} \to M \times M$ has the following properties:

(Bu1) $\mathcal{B}_{xy}$ is a weakly closed linear subspace of $\mathcal{L}_{xy}$;

(Bu2) if $b \in \mathcal{B}_{xy}$, then $b^* \in \mathcal{B}_{yx}$;

(Bu3) if $b \in \mathcal{B}_{xy}$, $b' \in \mathcal{B}_{yz}$, then $bb' \in \mathcal{B}_{xz}$;

(Bu4) Id $\in \mathcal{B}_{xx}$;

(Bu5) if $a \in \mathcal{B}_{xx}$ is invertible then $a^{-1} \in \mathcal{B}_{xx}$.

Denote by U an open connected subset of M and let $X = \mathbf{R}^d$ or, in case U is a neighborhood of a boundary point of M, $X = \mathbf{R}^d_+ := \{(x_1, \ldots, x_d) ; x_d \geq 0\}$.

DEFINITION 2.4. *A pair (ϕ, Φ) of smooth diffeomorphisms $\phi : U \to X$ and $\Phi : \mathcal{E}|U \to X \times \mathcal{W}$ is said to be a coordinate chart of $(M, \mathcal{E} \to M)$ if ϕ is a chart of M and Φ is an $\mathcal{A}$-trivialization of $\mathcal{E} \to M$ over U.*

In particular, $\Phi_x := \Phi|_{p^{-1}(x)} : p^{-1}(x) \to \mathcal{W}$ is an isometry.

In order to define the Sobolev spaces $H_s(\mathcal{E}) = H_s(M, \mathcal{E})$ we proceed as follows: If M is a closed manifold, define the Sobolev space $H_s(\mathcal{E}) = H_s(M, \mathcal{E})$ by a standard localizing procedure, using the definition of section 2.1 and a smooth partition of unity subordinate to an open cover of M which comes from an atlas of $\mathcal{E} \to M$. If M is a compact manifold with boundary we first consider the double $\tilde{\mathcal{E}} \to \tilde{M}$ of $\mathcal{E} \to M$, identify M with M_+, one of the two copies of M in $\tilde{M}$, and denote by M_- the closure of $M \backslash M_+$. Then define

$$H_s(\mathcal{E}) := H_s(\tilde{M}, \tilde{\mathcal{E}}) / \{u \in H_s(\tilde{M}, \tilde{\mathcal{E}}) : \operatorname{supp} u \subset M_-\} .$$

(Equivalently, the Sobolev norms with non-negative integer indices can be defined using a Riemannian metric on M and a connection on $\mathcal{E}$.) We will also use $L_2(\mathcal{E})$ for $H_0(\mathcal{E})$. The inner product in $H_s(\mathcal{E})$ will depend on the particular choice of the partition of unity of M; however a different choice of partition of unity will lead to an equivalent inner product. The Sobolev s-norm of an element $u \in H_s(\mathcal{E})$ will be denoted by $\|u\|_s$. We point out that for $s \geq t$, $H_s(\mathcal{E})$ embeds into $H_t(\mathcal{E})$. (This embedding, however, is not compact if $\mathcal{W}$ is of infinite dimension, i.e. Rellich's lemma does not hold).

Definition 2.5. (1) *A linear operator $A : C^\infty(\mathcal{E}) \to C^\infty(\mathcal{E})$ is an $\mathcal{A}$-smoothing operator, if A is of the form*

$$(Au)(x) = \int_M K_A(x, y) u(y) dy$$

where the Schwartz kernel K_A of A is a smooth section of the bundle $\mathcal{B} \to \mathcal{M} \times \mathcal{M}$. The set of these operators is denoted by $\Psi DO_{\mathcal{A}}^{-\infty}(\mathcal{E})$.

(2) *A linear operator $A : C^\infty(\mathcal{E}) \to C^\infty(\mathcal{E})$ is a pseudodifferential $\mathcal{A}$-operator of order m if for some atlas $(\phi_j, \Phi_j)_{j \in J}$ of $\mathcal{E} \to M$, $A = \sum_j A_j + T$ where T is an $\mathcal{A}$-smoothing operator and the operators A_j are operators with support in the domain of ϕ_j and, when expressed in local coordinates, pseudodifferential $\mathcal{A}$-operators of order m. The set of these operators is denoted by $\Psi DO_{\mathcal{A}}^m(\mathcal{E})$.*

In the case when M is a manifold with boundary, we will always assume that pseudodifferential operators have the transmission property (cf. e.g. [Bo, section 2]).[3] One shows that $A \in \Psi DO_{\mathcal{A}}^m(\mathcal{E})$ can be extended, for any $s \in \mathbb{R}$, to a bounded linear operator

$$A : H_s(\mathcal{E}) \to H_{s-m}(\mathcal{E}) .$$

[3] The transmission property of a pseudodifferential operator can be defined in terms of the symbol of this operator: Let $x = (x', x_d)$ be arbitrary local coordinates in a neighborhood of the boundary ∂M with the boundary defined by $x_d = 0$, and denote by $\xi = (\xi', \xi_d)$ the dual coordinates. A pseudodifferential operator A with a classical symbol is said to have

The principal symbol σ_A of A can be defined invariantly as a smooth function $\sigma_A(x,\cdot) : T^*M \setminus \{0\} \to \mathcal{L}_{\mathcal{A}}(\mathcal{E}_x)$.

Note that $\cap_m \Psi DO^m_{\mathcal{A}}(\mathcal{E})$ identifies to $\Psi DO^{-\infty}_{\mathcal{A}}(\mathcal{E})$ and, in general, $\mathcal{A}$-smoothing operators are not compact.

As in the classical theory one develops a calculus for these pseudodifferential operators. In particular, one shows that the composition $A \circ B$ of two pseudodifferential operators A and B as well as the adjoint A^* (with respect to the Hermitian structure on $\mathcal{E} \to M$) are pseudodifferential operators of the expected order.

2.3 Elliptic pseudodifferential operators.

DEFINITION 2.6. *An operator $A \in \Psi DO^m_{\mathcal{A}}(\mathcal{E})$ is said to be elliptic if the principal symbol of A, $\sigma_A(x,\xi)$, is invertible for all $x \in M$ and all $\xi \in T^*_x M \setminus \{0\}$.*

To simplify the exposition we assume that M is closed. As in the classical case one can construct a parametrix, $R(\mu)$, for the operator $(\mu - A)$ when A is elliptic and $\mu \in \mathbb{C} \setminus \bigcup_{(x,\xi)\in T^*M\setminus\{0\}} \operatorname{spec}(\sigma_A(x,\xi))$.

The operator $R(\mu)$ is an element of $\Psi DO^{-m}_{\mathcal{A}}(\mathcal{E})$ and represents an inverse of $(\mu - A)$ up to smoothing operators. Let U be a chart of M which belongs to an atlas of $\mathcal{E} \to M$. Denote by ϕ and Φ the diffeomorphisms

$$\phi : \mathbb{R}^d \to U \subset M$$
$$\Phi : \mathbb{R}^d \times \mathcal{W} \to \mathcal{E}_{|_U}$$

where U is an open subset of M and Φ trivializes the bundle $p : \mathcal{E} \to M$ over U such that $p\Phi = \phi p_1$ with $p_1 : \mathbb{R}^d \times \mathcal{W} \to \mathbb{R}^d$.

The symbol of $R(\mu)$ in the chart U has an asymptotic expansion determined inductively as follows:

$$r_{-m}(x,\xi,\mu) = \big(\mu - a_m(x,\xi)\big)^{-1}$$

and, for $j \geq 1$,

$$r_{-m-j}(x,\xi,\mu) = r_{-m}(x,\xi,\mu)\left(\sum_{k=0}^{j-1} \sum_{|\alpha|+l+k=j} \frac{1}{\alpha!}\partial^\alpha_\xi a_{m-l}(x,\xi) D^\alpha_x r_{-m-k}(x,\xi,\mu)\right) \tag{2.5}$$

---

the transmission property if the homogeneous components $a_{m-j}(x', x_d; \xi', \xi_d)$ of the symbol expansion of A (cf. Definition 2.2) satisfy

$$D^k_{x_d} D^\alpha_{\xi'} a_{m-j}(x', 0; 0, 1) = e^{i\pi(m-j-|\alpha|)} D^k_{x_d} D^\alpha_{\xi'} a_{m-j}(x', 0; 0, -1) .$$

All pseudodifferential operators that arise from differential elliptic boundary value problems have the transmission property.

where α is a multiindex, $\alpha = (\alpha_1, \ldots, \alpha_d)$, $\alpha! = \alpha_1!\alpha_2!\cdots\alpha_d!$, and $D_x^\alpha = (\frac{1}{i}\partial_x)^\alpha$. The term $r_{-m-j}(x,\xi,\mu)$ is an element of $\mathcal{L}_{\mathcal{A}}(\mathcal{W},\mathcal{W})$ and is positively homogeneous of degree $-m-j$ in $(\xi, \mu^{\frac{1}{m}})$:

$$r_{-m-j}(x, \lambda\xi, \lambda^m \mu) = \lambda^{-m-j} r_{-m-j}(x,\xi,\mu) \tag{2.6}$$

for any $\xi \in \mathbb{R}^d \setminus \{0\}$ and $\lambda > 0$.

As in the classical case the parametrix of an invertible elliptic pseudodifferential operator is readily used to show that its inverse is also pseudodifferential.

PROPOSITION 2.7. *Assume that M is closed and that $A \in \Psi DO^m_{\mathcal{A}}(\mathcal{E})$ is elliptic. If A considered as a bounded linear operator, $A : H_m(\mathcal{E}) \to L_2(\mathcal{E})$, is one-to-one and onto, then $A^{-1} \in \Psi DO^{-m}_{\mathcal{A}}(\mathcal{E})$.*

Proof (cf. [Sh2]). Denote by $B \in \Psi DO^{-m}_{\mathcal{A}}(\mathcal{E})$ a parametrix for A. The operators $T_1 := AB - \mathrm{Id}$ and $T_2 = BA - \mathrm{Id}$ are in $\Psi DO^{-\infty}_{\mathcal{A}}(\mathcal{E})$. From this we conclude that $A^{-1} = B - A^{-1}T_1$. The statement follows once we prove that $A^{-1}T_1 \in \Psi DO^{-\infty}_{\mathcal{A}}(\mathcal{E})$. First notice that $A^{-1}T_1$ has a smooth Schwartz kernel, $K_{A^{-1}T_1}(x,y) \in \mathcal{L}(\mathcal{E}_y, \mathcal{E}_x)$. This is a consequence from the fact that A^{-1} maps $C^\infty(\mathcal{E})$ into itself which can be verified as follows: Let $u \in C^\infty(\mathcal{E})$ and set $v := A^{-1}u$. As B is a pseudodifferential operator, it follows from above that $\mathrm{singsupp}(Av) \supset \mathrm{singsupp}(BAv) = \mathrm{singsupp}(v + T_2 v) = \mathrm{singsupp}(v)$.

The converse inclusion is also true as A is a pseudodifferential operator. Therefore, $\mathrm{singsupp}(Av) = \mathrm{singsupp}(v)$ and $\emptyset = \mathrm{singsupp}(u) = \mathrm{singsupp}(Av) = \mathrm{singsupp}(v)$, i.e. $v \in C^\infty(\mathcal{E})$.

Having established that A^{-1} is a pseudodifferential operator one verifies that A^{-1} is $\mathcal{A}$-linear. □

As in the classical theory one proves the following estimates for the resolvent:

PROPOSITION 2.8 (cf. [S1],[Sh1, Theorem 9.2]). *Assume that $A \in \Psi DO^m_{\mathcal{A}}(\mathcal{E})$ is an elliptic operator of order $m \geq 0$ such that $A : H_m(\mathcal{E}) \to L_2(\mathcal{E})$ is one-to-one and onto. Further, assume that π is an Agmon angle for A.*

Then for $\lambda < 0$ with $|\lambda|$ sufficiently large and for $0 \leq m' \leq m$,

$$\left\|(\lambda - A)^{-1}\right\|_{0 \to m'} \leq C_{m'} |\lambda|^{-1+\frac{m'}{m}} \tag{2.7}$$

for some constants $C_{m'} > 0$.

2.4 Zeta-function and regularized determinant of an invertible elliptic operator. Let (M, g) be a closed Riemannian manifold. Assume that $A \in \Psi DO^m_{\mathcal{A}}(\mathcal{E})$ is elliptic and of positive order, $m > 0$, with π as an Agmon angle; i.e. there exists $\epsilon > 0$ such that

(1) $V_{\pi,\epsilon} \cap \operatorname{spec} A = \emptyset$;
As a consequence π is also a principal angle
(2) $V_{\pi,\epsilon} \cap \left(\cup_{x \in M, (x,\xi) \in S_x^* M} \operatorname{spec}(\sigma_A(x,\xi)) \right) = \emptyset$.

The solid angle $V_{\pi,\epsilon}$ is defined as in the introduction. Note that (1) implies the invertibility of A viewed as a bounded linear operator, $A : H_m(\mathcal{E}) \to L_2(\mathcal{E})$. Moreover, for $\Re s < 0$, one can define the complex powers of A by

$$A^s := \frac{1}{2\pi i} \int_{\gamma_{\pi,\epsilon}} \mu^s (\mu - A)^{-1} d\mu \tag{2.8}$$

where $\gamma_{\pi,\epsilon}$ is a path in $\mathbb{C}$ as defined in section 1.2. For s satisfying $0 \leq k - 1 \leq \Re s < k \in \mathbb{N}$ one defines

$$A^s = A^k A^{s-k} \ .$$

It follows from Proposition 2.7 using arguments due to Seeley [S1], that $A^s \in \Psi DO_{\mathcal{A}}^s(\mathcal{E})$ (after suitably generalizing the concept of order to complex numbers $s \in \mathbb{C}$), depending holomorphically on s. Moreover, for $\Re s < -\frac{d}{m}$, A^s has a von Neumann trace

$$\operatorname{tr}_{\mathrm{N}}(A^s) := \int_M \operatorname{tr}_{\mathrm{N}} K_{A^s}(x,x) dx$$

where $K_{A^s}(x,y) \in \mathcal{L}(\mathcal{E}_y, \mathcal{E}_x)$ denotes the Schwartz kernel of A^s.

For $\alpha \in C^\infty(M, \mathbb{C})$ and $\Re s > \frac{d}{m}$ one defines the generalized zeta-function

$$\zeta_{\alpha,A}(s) = \operatorname{tr}_{\mathrm{N}}(\alpha A^{-s}) \ .$$

If A is selfadjoint and strictly positive one can derive, as in the classical case, the following heat trace representation of the generalized zeta function

$$\zeta_{\alpha,A}(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{(-t\lambda)} \operatorname{tr}_N \alpha e^{-tA} dt \ . \tag{2.9}$$

As in [S1] (cf. also [G, Lemma 1.7.7]) one shows

Theorem 2.9 (cf. [S1]). (1) *Assume $A \in \Psi DO_{\mathcal{A}}^m(\mathcal{E})$ where $m > 0$ and A is elliptic with π as an Agmon angle. If $\alpha \in C^\infty(M, \mathbb{C})$, then $\zeta_{\alpha,A}(s)$ admits a meromorphic extension to the entire s-plane. The extension has at most simple poles and $s = 0$ is a regular point. The value of $\zeta_{\alpha,A}(s)$ at $s = 0$ is given by*

$$\zeta_{\alpha,A}(0) = \int_M \alpha(x) I_d(x) \tag{2.10}$$

where $I_d(x)$ is a density on M. In an appropriate coordinate chart, $I_d(x)$ is given by

$$I_d(x) = \frac{1}{m} \frac{1}{(2\pi)^d} \int_{|\xi|=1} d\xi \int_0^\infty \operatorname{tr}_{\mathrm{N}} \left(r_{-m-d}(x, \xi, -\mu) \right) d\mu \ . \tag{2.11}$$

If A is a differential operator and $d = \dim M$ is odd, then $I_d(x) \equiv 0$.

(2) *Assume that $A(t) : H_m(\mathcal{E}) \to L_2(\mathcal{E})$ is a family of classical pseudodifferential operators of order m depending in a C^r-fashion on a parameter t varying in an open set of $\mathbf{R}$. Assume that $A(t)$ is elliptic and that there exists $\epsilon' > 0$ such that for all $t \in \Lambda_{0,\epsilon'}$, $\operatorname{spec} A(t) \cap V_{\pi,\epsilon'} = \emptyset$. Then $\zeta_{A(t)}(s)$ is a family of functions holomorphic in s in a t-independent neighborhood of $s = 0$ which depends in a C^r-fashion on t.*

Theorem 2.9 above allows us to introduce the ζ-regularized determinant of an elliptic operator $A \in \Psi DO^m_{\mathcal{A}}(\mathcal{E})$ of order $m > 0$ with π as an Agmon angle:

$$\det A := \exp\left\{ -\frac{d}{ds}\bigg|_{s=0} \zeta_A(s) \right\} .$$

To treat the case where A is not invertible, first note the following:

LEMMA 2.10. *Assume that $A \in \Psi DO^m_{\mathcal{A}}(\mathcal{E})$, $m > 0$, is elliptic. Then the nullspace of A,* Null(A), *is an $\mathcal{A}$-Hilbert module of finite von Neumann dimension,* $\dim_{\mathcal{N}}(\mathrm{Null}(A))$.

Sketch of the proof. It is sufficient to prove the statement in the case when A is self-adjoint and non-negative because $\mathrm{Null}(A) = \mathrm{Null}(A^* A)$. Let k be a positive integer such that $km > d$. Then, by Proposition 2.7, $(I + A)^{-k}$ is a pseudodifferential operator of order $-km$, and its Schwartz kernel is continuous. Therefore,

$$\mathrm{tr}_{\mathrm{N}}(I + A)^{-k} < \infty .$$

On the other hand,

$$\mathrm{tr}_{\mathrm{N}}(I + A)^{-k} \geq \dim_{\mathcal{N}} \left(\mathrm{Null}(A) \right) ,$$

hence the result. □

Assume that A is an elliptic operator, $A \in \Psi DO^m_{\mathcal{A}}(\mathcal{E})$, of order $m > 0$ with π as a weak Agmon angle (cf. Definition 1.8). Then the operator $A + \lambda$ with $\lambda > 0$ has π as an Agmon angle and $\log \det_{\mathcal{N}}(A + \lambda)$ is a real analytic function in λ. Define $\log \det_{\mathcal{N}}(A)$ to be the element in $\mathbf{D}$ represented by the analytic function

$$\log \det_{\mathcal{N}}(A) := \log \det_{\mathcal{N}}(A + \lambda) - \dim_{\mathrm{N}} \left(\mathrm{Null}(A) \right) \log \lambda . \qquad (2.12)$$

DEFINITION 2.11. *A is of determinant class if*

$$\lim_{\lambda \downarrow 0} \left(\log \det_{\mathcal{N}}(A + \lambda) - \dim_{\mathrm{N}}(\mathrm{Null}(A)) \log \lambda \right) \qquad (2.13)$$

exists. In this case, $\log \det_{\mathcal{N}} A$ *is a real number.*

If A is selfadjoint and nonnegative, there is a functional calculus for A. In particular, one can introduce the spectral projections $Q(\lambda)$ corresponding to the intervals $(-\infty, \lambda]$. Using Proposition 2.8 and the assumption that A is nonnegative, one verifies that $Q(\lambda)$ is in $\Psi DO_A^{-\infty}(\mathcal{E})$ for any value of $\lambda \in \mathbb{R}$. Denote the distribution kernel of $Q(\lambda)$ by K_λ and define the spectral distribution function

$$N_A(\lambda) := \int_M \mathrm{tr}_N K_\lambda(x,x)dx \ . \tag{2.14}$$

Note that $N_A(\lambda)$ is nonnegative, right continuous and monotone increasing as a function of λ. Moreover, $N_A(\lambda) = 0$ for $\lambda < 0$ and there exists a constant $C > 0$, so that

$$N_A(\lambda) \leq C|\lambda|^{\frac{d}{m}} \ , \qquad \lambda \geq 1 \ . \tag{2.15}$$

The asymptotic estimate (2.15) is obtained by observing that $(\Delta+1)^{-\frac{m}{4}}(A+1)(\Delta+1)^{-\frac{m}{4}}$ is an elliptic, selfadjoint, positive operator in $\Psi DO_A^0(\mathcal{E})$. (Δ is the Laplacian acting on $C^\infty(\mathcal{E})$ induced by the canonical flat connection of $\mathcal{E} \to M$ and a Riemannian metric g.) Therefore there exists $C' > 0$, so that $C'Id \leq (\Delta+1)^{-m/4}(A+1)(\Delta+1)^{-m/4}$. This implies that $C'(\Delta+1)^{-m/2} \leq (A+1)$ or $N_{A+1}(\lambda) \leq N_{C'(\Delta+1)^{m/2}}(\lambda) = N_{\Delta+1}((\frac{\lambda}{C'})^{2/m})$. By the variational characterization of $N_{\Delta+1}(\lambda)$, $N_{\Delta+1}(\lambda) \leq C''\lambda^{d/2}$ for some constant $C'' > 0$ and all $\lambda \geq 1$.

PROPOSITION 2.12. *Assume that $A \in \Psi DO_A^m(\mathcal{E})$ is an elliptic, selfadjoint, nonnegative operator of order $m \geq 1$ with π as a principal angle. Then the following statements are equivalent:*

(1) *A is of determinant class.*

(2) $\int_{0+}^1 \log \lambda dN_A(\lambda) > -\infty$.

Here the integral $\int_{0+}^1$ denotes the Stieltjes integral on the half open interval $(0,1]$.

(3) $\int_1^\infty \frac{1}{x}(tr_N e^{-xA} - \dim_N(\mathrm{Null}\, A))dx < \infty$.

The proof of the proposition uses the heat kernel representation of the determinant which we briefly discuss (cf. [G, section 1.6]). Let γ be a path in $\mathbb{C}$ defined by the composition $\gamma_- \circ \gamma_+$ of two straight half lines:

$$\gamma_+ := \{x + i(x+1);\ -1 \leq x \leq \infty\}$$
$$\gamma_- := \{x - i(x+1);\ -1 \leq x \leq \infty\}$$

where γ_+ starts at infinity and γ_- starts at $x = -1$. Using Proposition 2.8 we may define the following bounded linear operator on $L_2(\mathcal{E})$:

$$e^{-tA} := \frac{1}{2\pi i}\int_\gamma e^{-t\lambda}(\lambda - A)^{-1}d\lambda \ .$$

One verifies that $e^{-tA} \in \Psi DO_A^{-\infty}(\mathcal{E})$ for $t > 0$. Hence, e^{-tA} has a smooth kernel, denoted $K_A(x,y,t)$, with values in $\mathcal{B}$ and admits a finite von Neumann trace, $\mathrm{tr}_{\mathrm{N}} e^{-tA}$, given by

$$\mathrm{tr}_{\mathrm{N}} e^{-tA} = \int_0^\infty e^{-t\lambda} dN_A(\lambda) \ .$$

As in the classical case one shows that for $t \to 0$, the kernel $K_A(x,y,t)$ has an expansion on the diagonal $x = y$ of the form ($N \geq 1$ arbitrary)

$$K(x,x,t) = \sum_{j=0}^{N-1} t^{\frac{j-d}{m}} l_j(x) + \sum_{j=1}^{N} (t^j \log t) m_j(x) + O(t^{\frac{N-d}{m}}) \ . \qquad (2.16)$$

When A is elliptic nonnegative and selfadjoint then $l_d = I_d$; this can be verified as in [G, p. 79].

Proof of Proposition 2.12. Using the representation (2.9) of the zeta-function we deduce that

$$\log \det_N (A+\lambda) = -\frac{d}{ds}\Big|_{s=0} \frac{1}{\Gamma(s)} \int_0^1 t^{s-1} \left(\mathrm{tr}_{\mathrm{N}} e^{-(A+\lambda)t} dt\right)$$
$$- \int_1^\infty t^{-1} \left(\mathrm{tr}_{\mathrm{N}} e^{-(A+\lambda)t} dt\right) \ . \qquad (2.17)$$

The expansion (2.16) is used to show that

$$-\frac{d}{ds}\Big|_{s=0} \frac{1}{\Gamma(s)} \int_0^1 t^{s-1} \left(\mathrm{tr}_{\mathrm{N}} e^{-(A+\lambda)t} - \dim_{\mathrm{N}}(\mathrm{Null}(A)) e^{-\lambda t}\right) dt$$

is a continuous function of λ for $\lambda \geq 0$. To analyze

$$G(\lambda) = \int_1^\infty t^{-1} \left(\mathrm{tr}_{\mathrm{N}} e^{-(A+\lambda)t} - \dim_{\mathrm{N}}(\mathrm{Null}(A)) e^{-\lambda t}\right) dt$$

we write, applying Fubini's theorem together with $\mathrm{tr}_{\mathrm{N}} e^{-At} = \int_{-\infty}^{\infty} e^{-\mu t} dN_A(\mu)$ and $\dim_{\mathrm{N}}(\mathrm{Null}(A)) = N_A(0)$,

$$G(\lambda) = \int_{0+}^\infty dN_A(\mu) \int_1^\infty t^{-1} e^{-(\mu+\lambda)t} dt$$
$$= \int_{\lambda+}^\infty dN_{A+\lambda}(\mu) \int_1^\infty t^{-1} e^{-\mu t} dt \ .$$

For $0 < \lambda \leq 1$, write $G(\lambda) = G_1(\lambda) + G_2(\lambda)$ where $G_1(\lambda)$ and $G_2(\lambda)$ are given by

$$G_1(\lambda) = \int_{1+}^\infty dN_{A+\lambda}(\mu) \int_1^\infty t^{-1} e^{-\mu t} dt$$
$$G_2(\lambda) = \int_{\lambda+}^1 dN_{A+\lambda}(\mu) \int_1^\infty t^{-1} e^{-\mu t} dt \ .$$

The function $G_1(\lambda)$ is estimated in a straightforward way. Concerning $G_2(\lambda)$, note that

$$\int_1^\infty t^{-1}e^{-\mu t}dt = -\log\mu + (1-e^{-\mu})\log\mu + \int_\mu^\infty e^{-s}\log s ds$$

and that the function $(1-e^{-\mu})\log\mu + \int_\mu^\infty e^{-s}\log s ds$ is bounded for $\mu \in [0,1]$. This proves the equivalence of (1) and (2). The equivalence of (2) and (3) follows by the same estimates. □

2.5 Elliptic boundary value problems. The previous discussion on the zeta function and the determinants of elliptic differential operators on closed manifolds can be extended to elliptic boundary values problems on compact manifolds with boundary. In the case $\mathcal{A} = \mathbb{C}$ this was done by Seeley (cf. [S2]). In this paper we will consider only Dirichlet type boundary value problems for elliptic differential operators of Laplace-Beltrami type of even order. An elliptic operator B of order $2k$, $k > 0$ is of Laplace-Beltrami type if the principal symbol is of the form $\sigma_B(x,\xi) = \|\xi\|^{2k}Id_{\mathcal{E}_x}$.

Let (M,g) be a compact Riemannian manifold with boundary $\partial M \neq \emptyset$, A a second order positive selfadjoint differential operator of Laplace-Beltrami type and let B be an elliptic differential operator of Laplace-Beltrami type of order $2k$. Denote by $J : C^\infty(\mathcal{E}) \to C^\infty(\mathcal{E}|_{\partial M})$ the trace operator which associates to $u \in C^\infty(\mathcal{E})$ its restriction, $u|_{\partial M}$. Introduce the restriction $B_D : C^\infty_D(\mathcal{E}) \to C^\infty(\mathcal{E})$ of B to $C^\infty_D(\mathcal{E}) := \{u \in C^\infty(\mathcal{E}) : u|_{\partial M} = 0, Au|_{\partial M} = 0, \cdots, A^{k-1}u|_{\partial M} = 0\}$. Notice that $C^\infty_D(\mathcal{E})$ depends on A and on the order of B.

Following [S2] one constructs a parametrix, $R(\mu)$, for $\mu - B_D$ in a similar fashion as in the case $\partial M = \emptyset$, describing inductively the asymptotic expansion of the symbol of the parametrix. The constructions differ, in the case of a manifold with boundary, in that each term in the symbol expansion includes a term arising from the boundary condition. These added terms arising from the boundary conditions only depend on the symbol expansion of B, A and its derivatives along the boundary ∂M.

The trace operator $J : C^\infty(\mathcal{E}) \to C^\infty(\mathcal{E}|_{\partial M})$ induces bounded operators $JA^r : H_m(\mathcal{E}) \to H_{m-2r+\frac{1}{2}}(\mathcal{E}|_{\partial M})$ ($m \in \mathbb{Z}, r \in \mathbb{N}$). Propositions 2.7 and 2.8 remain true if one replaces $H_m(\mathcal{E})$ with $\{u \in H_m(\mathcal{E}) \mid Ju=0, \ldots, JA^{k-1}u=0\}$.

One can introduce complex powers of B_D and, for $\Re s > \frac{d}{2}$, the zeta-function $\zeta_{B_D}(s)$ and its generalized version $\zeta_{\alpha,B_D}(s)$ (cf. section 2.4). Following Seeley's arguments one obtains the analog of Theorem 2.9.

Theorem 2.9′. *Let (M,g) be a compact Riemannian manifold with boundary $\partial M \neq \emptyset$. Assume that A is an $\mathcal{A}$-linear selfadjoint, positive, differential* *operator of order 2 of Laplace-Beltrami type and B is an $\mathcal{A}$-linear differential operator of order $2k$ and of Laplace-Beltrami type. (Note that π is an Agmon angle for A and therefore a principal angle as well). Suppose that π is an Agmon angle for B_D. The function $\zeta_{\alpha,B_D}(s)$ admits a meromorphic continuation to the entire s-plane. The continuation has at most simple poles and $s=0$ is a regular point. The value of $\zeta_{\alpha,B_D}(s)$ at $s=0$ is given by*

$$\zeta_{\alpha,B_D}(0) = \int_M \alpha(x) I_d(x) + \int_{\partial M} \alpha(x)\beta_d(x) \tag{2.18}$$

where in a coordinate chart of $(M, \mathcal{E} \to M)$, $I_d(x)$ is defined as in (2.11). In a coordinate chart of $(\partial M, \mathcal{E}|_{\partial M} \to \partial M)$, $\beta_d(x)$ is given by a formula similar to that found in [S2] involving at most the first d terms of the symbol expansion of A and its derivatives up to order d.

Theorem 2.9′ allows us to introduce the ζ-regularized determinant of B_D by

$$\log \det_N B_D = -\frac{d}{ds}\bigg|_{s=0} \zeta_{B_D}(s) \,. \tag{2.19}$$

3. Asymptotic Expansion and the Mayer-Vietoris Type Formula for Determinants

3.1 Parametric pseudodifferential operators. Let $\Lambda_{0,\epsilon}$ denote the solid angle in $\mathbb{C}$ given by $\Lambda_{0,\epsilon} = \{re^{i\theta}; r \geq 0,\ 2\pi|\theta| \leq \epsilon\}$. Consider a family of pseudodifferential operators $A(t)$, $t \in \Lambda_{0,\epsilon}$ with $A(t) \in \Psi DO^m_{\mathcal{A}}(\mathcal{E})$.

Definition 3.1. *Let M be a closed manifold. $A(t)$ is a 1-parameter family of weight χ in $\Psi DO^m_{\mathcal{A}}(\mathcal{E})$ if for any chart $\phi : X \to U$ of an atlas of $\mathcal{E} \to M$ (where $X = \mathbf{R}^d$, or in case U is a neighborhood of a boundary point, $X = \mathbf{R}^d_+$) and for all h, $h' \in C^\infty_0(U)$, the operator $h'Ah$, when expressed in local coordinates, has an $\mathcal{L}_{\mathcal{A}}(\mathcal{W},\mathcal{W})$-valued symbol $a = a_{h,h';U}$ satisfying the following properties:*

(1) *for any multiindices α, β there is a constant $C_{\alpha,\beta} > 0$ such that*

$$\left\|\partial^\alpha_x \partial^\beta_\xi a(x,\xi,t)\right\| \leq C_{\alpha\beta}\left(1 + |\xi| + |t|^{\frac{1}{\chi}}\right)^{m-|\beta|}$$

where $x \in X$, $\xi \in \mathbf{R}^d$, and $t \in \Lambda_{0,\epsilon}$;

(2) *a has an asymptotic expansion*

$$a \sim \sum_{j\geq 0} \psi(\xi,t) a_{m-j}(x,\xi,t) \tag{3.1}$$

with $\psi \in C^\infty(\mathbf{R}^d \times \Lambda_{0,\epsilon})$ *satisfying*

$$\psi(\xi,t) = \begin{cases} 0 & \text{if } |(\xi,t)| \le \frac{1}{2} \\ 1 & \text{if } |(\xi,t)| \ge 1 \end{cases}$$

and $a_{m-j} \in C^\infty(X, \mathbf{R}^d \setminus \{0\}, \Lambda_{0,\epsilon}; \mathcal{L}_{\mathcal{A}}(\mathcal{W},\mathcal{W}))$ *depending in a* C^1*-fashion on the parameter* t, *has compact* x*-support and is positive homogeneous of degree* $m-j$ *in* $\xi, t^{\frac{1}{\chi}}$, *i.e.*

$$a_{m-j}(x, \tau\xi, \tau^{\frac{1}{\chi}} t) = \tau^{m-j} a_{m-j}(x,\xi,t)$$

for all $\tau > 0$.

By (3.1) *we mean that for any multiindices* α, β *and any* $N \ge 0$ *there exists constants* $C_{\alpha,\beta,N} > 0$, *such that for any* $t \in \Lambda_{0,\epsilon}$

$$\left\| \partial_x^\alpha \partial_\xi^\beta \left[a(x,\xi,t) - \sum_{j=0}^{N-1} \psi(\xi) a_{m-j}(x,\xi,t) \right] \right\| \le C_{\alpha\beta N} \left(1 + |\xi| + |t|^{1/\chi}\right)^{m-N-|\beta|} .$$

A similar concept is necessary for compact manifolds with boundary. For this paper it suffices to describe such a concept only for differential operators.

DEFINITION 3.1′. *Let* M *be a compact manifold with boundary.* $A(t)$ *is a 1-parameter family of* $\mathcal{A}$*-linear differential operators of order* m *and of weight* χ *if, in a local chart,*

$$A(t) = \sum_{j=0}^{m} a_{m-j}(x, D, t)$$

where $D = \frac{1}{i}\partial_x$ *and* $a_{m-j}(x,\xi,t)$ *is a polynomial in* ξ *with*

$$a_{m-j}(x, \tau\xi, \tau^{\frac{1}{\chi}} t) = \tau^{m-j} a_{m-j}(x,\xi,t)$$

for all $\tau > 0$.

In the case where M is closed one proves (cf. e.g. [Sh1, Theorem 9.1]) that for any $s \in \mathbf{R}$ and $l \ge m$, $A(t)$ is a bounded linear operator, $A(t) : H_s(\mathcal{E}) \to H_{s-l}(\mathcal{E})$. Denote by $|||A(t)|||_{s \to s-l}$ the operator norm of $A(t)$, viewed as an operator $A(t) : H_s(\mathcal{E}) \to H_{s-l}(\mathcal{E})$.

Theorem 3.2 (cf. [Sh1, Theorem 9.1]). *Let* M *be closed. The following estimates hold:*

(1) *if* $m \ge 0$ *and* $l \ge m$, *then* $\|A(t)\|_{s \to s-l} \le C_{s,l}(1 + |t|^{\frac{1}{\chi}})^m$;

(2) *if* $m \le 0$ *and* $m \le l \le 0$, *then* $\|A(t)\|_{s \to s-l} \le C_{s,l}(1 + |t|^{\frac{1}{\chi}})^{-(l-m)}$.

DEFINITION 3.3. *A 1-parameter family* $A(t)$ *in* $\Psi DO_{\mathcal{A}}^m(\mathcal{E})$ *is elliptic with parameter, if for any chart* $\phi : X \to U$ *of an atlas of* $\mathcal{E} \to M$ *(where*

$X = \mathbf{R}^d$, or in case U is a neighborhood of a boundary point, $X = \mathbf{R}^d_+$) and for all $h, h' \in C_0^\infty(U)$, the operator $h'Ah$, when expressed in local coordinates, has principal symbol $a_m(x,\xi,t)$ with values in $\mathcal{L}_{\mathcal{A}}(\mathcal{W},\mathcal{W})$ such that for all $x \in X$ with $h(\phi(x))h'(\phi(x)) \neq 0$, $a_m(x,\xi,t)$ is invertible for $(\xi,t) \in (\mathbf{R}^d \times \Lambda_{0,\epsilon}) \setminus \{(0,0)\}$.

Let M be closed. For a 1-parameter family $A(t)$ in $\Psi DO^m_{\mathcal{A}}(\mathcal{E})$ elliptic with parameter, one constructs a parametrix, $R(\mu,t)$, for $\mu - A(t)$: Given $\mu \notin \bigcup_{t\in\Lambda_{0,\epsilon}} \operatorname{spec}(A(t))$, $R(\mu,t)$ is a 1-parameter family in $\Psi DO^{-m}_{\mathcal{A}}(\mathcal{E})$ satisfying

$$R(\mu,t)\bigl(\mu - A(t)\bigr) - \mathrm{Id} \in \Psi DO^{-\infty}_{\mathcal{A}}(\mathcal{E})$$

and

$$\bigl(\mu - A(t)\bigr)R(\mu,t) - \mathrm{Id} \in \Psi DO^{-\infty}_{\mathcal{A}}(\mathcal{E})\ .$$

In local coordinates the symbol of $R(\mu,t)$ is constructed inductively:

$$\begin{aligned} r_{-m}(x,\xi,t,\mu) &= \bigl(\mu - a_m(x,\xi,t)\bigr)^{-1} \\ r_{-m-j}(x,\xi,t,\mu) & \qquad\qquad (3.2) \\ &= r_{-m}(x,\xi,t,\mu)\Bigl(\sum_{k=0}^{j-1}\ \sum_{|\alpha|+l+k=j} \frac{1}{\alpha!}\partial_\xi^\alpha a_{m-l}(x,\xi,t)D_x^\alpha r_{-m-k}(x,\xi,t,\mu)\Bigr) \end{aligned}$$

where $D_x = \frac{1}{i}\partial_x$. The term $r_{-m-j}(x,\xi,t,\mu)$ is positive homogeneous of degree $-m-j$ in $(\xi, t^{\frac{1}{\chi}}, \mu^{\frac{1}{m}})$.

In the case where M is compact with nonempty boundary, one can construct a parametrix $R(\mu,t)$ for a one parameter family $B_D(t)$ with boundary conditions defined by A (cf. comment and footnote following Definition 2.5 and section 2.5) when $B(t)$ is a family of differential operators of Laplace Beltrami type elliptic with parameter. The parametrix is given by a sum of a pseudodifferential operator and a singular Green operator. The symbol of the pseudodifferential operator is constructed in the same way as in the case of a manifold without boundary. The symbol of the singular Green operator only depends on the symbol expansion of $B(t)$ and A and their derivatives along the boundary ∂M.

3.2 Asymptotic expansion of determinants. As in [BuFrKa2, Appendix], one proves a result concerning the asymptotic expansion for the determinant of a 1-parameter family $A(t)$ in $\Psi DO^m_{\mathcal{A}}(\mathcal{E})$, $A(t)$ elliptic with parameter.

Theorem 3.4. *Let M be a closed manifold. Assume that $A(t)$ is a 1-parameter family in $\Psi DO^m_{\mathcal{A}}(\mathcal{E})$ of order $m \geq 1$, elliptic with parameter* *of weight χ. Further assume that there exists $\epsilon' > 0$ such that for all $t \in \Lambda_{0,\epsilon'}$, $\operatorname{spec}(A(t)) \cap V_{\pi,\epsilon'} = \emptyset$. Then the function $\log\det_N A(t)$ admits an asymptotic expansion for $t \to \infty$ of the form*

$$\log\det_N A(t) \sim \sum_{-\infty}^{d} \overline{a}_j |t|^{\frac{j}{\chi}} + \sum_{0}^{d} \overline{b}_j |t|^{\frac{j}{\chi}} \log|t| \tag{3.3}$$

where $\overline{a}_j = \int_M a_j(x, \frac{t}{|t|})dx$, $\overline{b}_j = \int_M b_j(x, \frac{t}{|t|})dx$, are defined by smooth densities $a_j(x, \frac{t}{|t|})$ and $b_j(x, \frac{t}{|t|})$, which can be computed by a formula involving finitely many terms in the symbol expansion of $A(t)$ and finitely many of its derivatives.

In particular, with respect to a coordinate chart, $a_0(x, \frac{t}{|t|})$ is given by

$$a_0\left(x, \frac{t}{|t|}\right) = \frac{d}{ds}\Big|_{s=0} \frac{1}{(2\pi)^d} \frac{1}{2\pi i} \int_{\mathbf{R}^d} d\xi \int_{\gamma_{\pi,\epsilon}} d\mu \mu^{-s} \operatorname{tr}_N \left(r_{-m-d}\left(x, \xi, \frac{t}{|t|}, \mu\right)\right)$$
$$= -\frac{1}{(2\pi)^d} \int_{\mathbf{R}^d} d\xi \int_0^\infty d\mu \operatorname{tr}_N \left(r_{-m-d}\left(x, \xi, \frac{t}{|t|}, -\mu\right)\right) . \tag{3.4}$$

A similar result holds in the case where M has a nonempty boundary, $\partial M \neq \emptyset$ (cf. [BuFrKa2]). With the notation introduced in section 2.5, one obtains

Theorem 3.5. *Let (M, g) be a compact Riemannian manifold with boundary $\partial M \neq \emptyset$. Assume that A is a second order selfadjoint, positive differential operator of Laplace Beltrami type and $B(t)$, $t \in \Lambda_{0,\epsilon}$, is a 1-parameter family of order $2k$ differential operators of Laplace Beltrami type, elliptic with parameter of weight χ and $k \geq 1$. Assume that there exists $\epsilon' > 0$ such that for all $t \in \Lambda_{0,\epsilon'}$, $\operatorname{spec} B_D(t) \cap V_{\pi,\epsilon'} = \emptyset$ (cf. section 2.5 for the definition of B_D). Then the function $\log\det_N B_D(t)$ admits an asymptotic expansion for $t \to \infty$ of the form*

$$\log\det_N B_D(t) \sim \sum_{j=-\infty}^{d} (\overline{a}_j + \overline{a}_j^b)|t|^{\frac{j}{\chi}} + \sum_{j=0}^{d} (\overline{b}_j + \overline{b}_j^b)|t|^{\frac{j}{\chi}} \log|t| \tag{3.5}$$

where $\overline{a}_j$ and $\overline{b}_j$ are given as in Theorem 3.4. The quantities $\overline{a}_j^b$ and $\overline{b}_j^b$ are contributions from the boundary and are of the form

$$\overline{a}_j^b = \int_{\partial M} a_j^b\left(x, \frac{t}{|t|}\right) \; ; \quad \overline{b}_j^b = \int_{\partial M} b_j^b\left(x, \frac{t}{|t|}\right) . \tag{3.6}$$

In a coordinate chart of $(\partial M, \mathcal{E}|_{\partial M} \to \partial M)$ the densities $a_j^b(x, \frac{t}{|t|})$ are given by a formula each involving only finitely many terms in the symbol expansion of $B(t)$ while the densities $b_j^b(x, \frac{t}{|t|})$ are given by a formula involving *only finitely many terms in the symbol expansions of $B(t)$ and A in a neighborhood of ∂M.*

3.3 Mayer-Vietoris type formula for determinants. We restrict ourselves to the case needed for this paper. We assume throughout this subsection that (M, g) is a closed Riemannian manifold. Let Γ be a smooth embedded hypersurface in M with trivial normal bundle. Consider an elliptic, selfadjoint, positive, differential operator A of order 2, $A : C^\infty(\mathcal{E}) \to C^\infty(\mathcal{E})$, of Laplace-Beltrami type (i.e. the principal symbol is of the form $\sigma_A(x,\xi) = \|\xi\|^2 Id_{\mathcal{E}_x}$) with $\operatorname{spec}(A) \subset [\epsilon, \infty)$ for some $\epsilon > 0$. Denote by M_Γ the manifold whose interior is $M \backslash \Gamma$, and whose boundary is $\partial M_\Gamma = \Gamma^+ \sqcup \Gamma^-$, where Γ^+ and Γ^- are both copies of Γ. Let g_Γ be the Riemannian metric on M_Γ induced by the metric g and let $\mathcal{E}_\Gamma \to M_\Gamma$ be the pullback of the bundle $\mathcal{E} \to M$ by the canonical map from M_Γ to M. The operator A induces the operator $A^\Gamma : C^\infty(\mathcal{E}_\Gamma) \to C^\infty(\mathcal{E}_\Gamma)$. Denote by $A^\Gamma_D : C^\infty_D(\mathcal{E}_\Gamma) \to C^\infty(\mathcal{E}_\Gamma)$, the restriction of A^Γ to $C^\infty_D(\mathcal{E}_\Gamma) := \{u \in C^\infty(\mathcal{E}_\Gamma) : u|_{\partial M_\Gamma} = 0\}$. Then A^Γ_D is elliptic and positive with $\operatorname{spec}(A^\Gamma_D) \subset [\epsilon, \infty)$. Thus, in particular, π is an Agmon angle for A^Γ_D.

Introduce the Dirichlet to Neumann operator, R_{DN}, associated to a unit vector field normal to Γ. This operator is defined as the composition

$$C^\infty(\mathcal{E}|_\Gamma) \xrightarrow{\Delta_{\text{ia}}} C^\infty(\mathcal{E}|_{\Gamma^+}) \oplus C^\infty(\mathcal{E}|_{\Gamma^-}) \xrightarrow{P_D} C^\infty(\mathcal{E}_\Gamma)$$
$$\xrightarrow{N} C^\infty(\mathcal{E}|_{\Gamma^+}) \oplus C^\infty(\mathcal{E}|_{\Gamma^-}) \xrightarrow{\Delta_{\text{if}}} C^\infty(\mathcal{E}|_\Gamma)\ .$$

Here Δ_{ia} is the diagonal operator, $\Delta_{\text{ia}}(f) = (f, f)$, Δ_{if} is the difference operator, $\Delta_{\text{if}}(f^+, f^-) = f^+ - f^-$, and P_D is the Poisson operator associated to A^Γ, i.e. the operator which maps $(f_+, f_-) \in C^\infty(\mathcal{E}|_{\Gamma^+}) \oplus C^\infty(\mathcal{E}|_{\Gamma^-})$ to the solution of the problem $Au = 0$, $u|_{\Gamma^\pm} = f_\pm$. Let $n(x)$, $x \in \Gamma$, be a vector field of unit covectors normal to Γ, pointing outward with respect to Γ_+. The operator N is an arbitrary first order differential operator on Γ with the principal symbol $\sigma_N(x,\xi) = i\langle n(x), \xi\rangle Id$. Note that the operator R_{DN} does not depend on the choice of N. In fact, on Γ, two different choices of N differ by a zero-order operator, and the contribution of this zero-order operator disappears after Δ_{if} has been applied. The following result is an extension of a result proven in [BuFrKa2] for $\mathcal{A} = \mathbb{C}$ (cf. also [L]).

Theorem 3.6. *Assume that (M, g) is a closed Riemannian manifold and A is an elliptic, selfadjoint, positive differential operator, $A : C^\infty(\mathcal{E}) \to C^\infty(\mathcal{E})$ of order 2 of Laplace-Beltrami type with* $\operatorname{spec}(A) \subset [\epsilon, \infty)$ *for some* $\epsilon > 0$. *Then R_{DN} is an invertible pseudodifferential operator in $\Psi DO^1_{\mathcal{B}}(\Gamma)$. The inverse R^{-1}_{DN} is given by*

$$R^{-1}_{DN} = J_\Gamma A^{-1}(\cdot \otimes \delta_\Gamma) \tag{3.7}$$

where J_Γ *is the trace operator* $J_\Gamma : H_s(\mathcal{E}) \to H_{s-1}(\mathcal{E}|_\Gamma)$ *and* δ_Γ *denotes the Dirac distribution along* Γ (cf. [BuFrKa2, (4.5)]). *As a consequence one concludes*

(1) R_{DN} *is selfadjoint and positive with* $\operatorname{spec}(R_{DN}) \subset [\epsilon', \infty)$ *for some* $\epsilon' > 0$. *In particular,* π *is an Agmon angle for* R_{DN}.

(2) *The principal symbol,* $\sigma(R_{DN}^{-1})$, *of* R_{DN}^{-1} *can be computed in terms of the principal symbol* $\sigma(A^{-1})$ *of* A^{-1} (cf. [BuFrKa2, (4.6)]):

$$\sigma(R_{DN}^{-1})(x', \xi') = \frac{1}{2\pi} \int_{\mathbb{R}} \sigma(A^{-1})(x', 0, \xi', \eta) d\eta \tag{3.8}$$

where $x = (x', w)$ *are coordinates in a collar neighborhood of* Γ *such that* x' *are coordinates of* Γ *and the normal vector field along* Γ *is represented by* $\frac{\partial}{\partial w}$. *In a coordinate chart for* Γ *which arises from a chart belonging to an atlas of* $\mathcal{E}|_\Gamma \to \Gamma$, *the symbol of* R_{DN} *consists of terms which depend only on the terms of the expansion of the symbol of* A *and its derivatives in an arbitrarily small neighborhood of* Γ.

(3)

$$\det{}_N(A) = \bar{c} \det{}_N(A_D^\Gamma) \det{}_N(R_{DN}) \tag{3.9}$$

where $\bar{c} = \exp\left(\int_\Gamma c(x)\right)$ *and the density* $c(x)$, *when expressed in a coordinate chart of* Γ *which is contained in an atlas of* $\mathcal{E}|_\Gamma \to \Gamma$, *depends only on the symbol of* A *and its derivatives in an arbitrarily small neighborhood of* Γ.

(4) *Assume that instead of the single operator* A, *there is a family* $A(\lambda) : C^\infty(\mathcal{E}) \to C^\infty(\mathcal{E})$ *of differential operators of order 2 of Laplace-Beltrami type with parameter* $\lambda \in \Lambda_{0,\epsilon'}$, $\epsilon' > 0$, *of weight* χ *such that* $A(\lambda)$ *is elliptic and selfadjoint for each* λ. *Introduce as above* $A(\lambda)_D^\Gamma$, $R_{DN}(\lambda)$ *and assume that* $\operatorname{spec}(A(\lambda)) \cap V_{\pi,\epsilon'} = \emptyset$ *for some* $\epsilon' > 0$ *and for all* $\lambda \in \Lambda_{0,\epsilon'}$. *Then* $R_{DN}(\lambda)$ *is an invertible family of pseudodifferential operators with parameter* (cf. [BuFrKa2, (3.13)]) *of order 1 and weight* χ.

In the remainder of this section we will present a proof of this theorem. We begin with some observation about the operator A_D^Γ.

LEMMA 3.7. (1i). *The operator* $A_D^\Gamma : C_D^\infty(\mathcal{E}_\Gamma) \to C^\infty(\mathcal{E}_\Gamma)$ *has a self-adjoint extension* $\bar{A}_D^\Gamma$ *with domain* $\operatorname{dom}(\bar{A}_D^\Gamma) := \{u \in H_2(\mathcal{E}_\Gamma) \mid Ju = 0\}$.

1ii. A_D^Γ *induces a quadratic form* Q_A *with domain* $\operatorname{dom}(Q_A) := \{u \in H_1(\mathcal{E}_\Gamma) \mid Ju = 0\}$.

(2) *The operator* $\bar{A}_D^\Gamma$ *is positive definite and its spectrum bounded from below by* ϵ.

(3) *The operator*

$$(A^\Gamma, J) : C^\infty(\mathcal{E}_\Gamma) \to C^\infty(\mathcal{E}_\Gamma) \oplus C^\infty(\mathcal{E}_\Gamma|_{\Gamma^+ \sqcup \Gamma^-})$$

defined by $(A^\Gamma, J)u = (A^\Gamma u, Ju)$ *can be extended to an invertible operator* $(A^\Gamma, J)^-$,

$$(A^\Gamma, J)^- : H_2(\mathcal{E}_\Gamma) \to L^2(\mathcal{E}_\Gamma) \oplus H_{2-\frac{1}{2}}(\mathcal{E}_\Gamma|_{\Gamma^+ \sqcup \Gamma^-}) .$$

Proof. (1) Using a partition of unity and integration by parts one shows that A^Γ_D is symmetric. Clearly, $\bar{A}^\Gamma_D$ is well defined and selfadjoint.

(2) To prove that $\bar{A}^\Gamma_D$ is positive definite and its spectrum bounded away from zero, one first notices that for any $u \in C^\infty(\mathcal{E}_\Gamma)$ with $u|_{\Gamma^+ \sqcup \Gamma^-} = 0$, one can find a sequence $\{\phi_n\}$ such that $\mathrm{supp}(\phi_n) \subset M \setminus \Gamma$ and ϕ_n converges to u in $H_1(\mathcal{E}_\Gamma)$. Observe that $\langle A^\Gamma_D \phi_n, \phi_n\rangle = \langle A\phi_n, \phi_n\rangle \geq \epsilon\|\phi_n\|^2$ since the spectrum of A is contained in $[\epsilon, \infty)$. Then, integrating by parts, one concludes that

$$Q_A(u) = \lim_{n\to\infty} \langle A\phi_n, \phi_n\rangle \geq \epsilon\|u\|^2 .$$

(3) As $\bar{A}^\Gamma_J$ is injective, so is the extension $(A^\Gamma_D, J)^-$. To prove that this extension is onto, consider $f \in L^2(\mathcal{E}_\Gamma)$ and $\varphi \in H_{2-\frac{1}{2}}(\mathcal{E}_\Gamma|_{\Gamma^+ \sqcup \Gamma^-})$. Choose any section $v \in H_2(\mathcal{E}_\Gamma)$ so that $Jv = \varphi$. As $\bar{A}^\Gamma_D$ is invertible, there exists $w \in H_2(\mathcal{E}_\Gamma)$ satisfying $\bar{A}^\Gamma_D w = f - \bar{A}^\Gamma_D v$ and the boundary conditions $Jw = 0$. Therefore $u = w + v$ is an element in $H_2(\mathcal{E}_\Gamma)$ with $(A_\Gamma, J)^- u = (f, \varphi)$. Altogether one concludes that $(A^\Gamma, J)^-$ is an isomorphism of Hilbert modules. □

In the proof of Theorem 3.6 we will use the operator $A - \alpha_k t$, $\alpha_k = e^{i\frac{\pi + 2k\pi}{d}}$ for $0 \leq k \leq d-1$, where $d = \dim(M)$. Note that we have

Lemma 3.8. *The following operators are invertible* ($0 \leq k \leq d-1$, and $t \geq 0$)

$$(A^\Gamma - \alpha_k t, J) : C^\infty(\mathcal{E}_\Gamma) \to C^\infty(\mathcal{E}_\Gamma) \oplus C^\infty(\mathcal{E}_\Gamma|_{\Gamma^+ \sqcup \Gamma^-}) .$$

Proof. As $\alpha_k \in \mathbf{C} \setminus \mathbf{R}^+$ and thus, for $t \geq 0$, $\alpha_k t \notin \mathrm{spec}(A^\Gamma_D)$, the operator $(A^\Gamma - \alpha_k t, J)$ is injective. To prove that this operator is onto and then invertible one argues as in the proof of Lemma 3.7 (3). □

Since $(A^\Gamma - \alpha_k t, J)$ is invertible, we can define the Poisson operator $P(\alpha_k t)$ associated to $(A^\Gamma - \alpha_k t, J)$, $P(\alpha_k t) : C^\infty(\mathcal{E}|_{\Gamma^+ \sqcup \Gamma^-}) \to C^\infty(\mathcal{E}_\Gamma)$, i.e. for $\varphi \in C^\infty(\mathcal{E}|_{\Gamma^+ \sqcup \Gamma^-})$, $u = P(\alpha_k t)(\varphi)$ is the solution in $C^\infty(\mathcal{E}_\Gamma)$ of $(A^\Gamma - \alpha_k t)u = 0$ with boundary conditions $u|_{\Gamma^+ \sqcup \Gamma^-} = \varphi$.

Let $R(\alpha_k t) : C^\infty(\mathcal{E}|_\Gamma) \to C^\infty(\mathcal{E}|_\Gamma)$ be the Dirichlet to Neumann operator corresponding to $A_\Gamma - \alpha_k t$. Then the following result holds.

Lemma 3.9. *For* $0 \leq k \leq d-1$, *and* $t \geq 0$, $R(\alpha_k t)$ *is an invertible classical* ΨDO *in* $\Psi DO^1_A(\mathcal{E}|_\Gamma)$, *which is elliptic with parameter* t *of weight* 1.

Proof. In a sufficiently small collar neighborhood U of Γ, choose coordinates $x = (x', s)$ such that $(x', 0) \in \Gamma$ and $\frac{\partial}{\partial s}\big|_{(x',0)} = n_{(x',0)}$. Let $\xi = (\xi', \eta)$ be coordinates in the cotangent space corresponding to the coordinates (x', s). Write $(A - \alpha_k t) = -A_2 \frac{\partial^2}{\partial s^2} + \frac{1}{i} A_1 \frac{\partial u}{\partial s} + A_0$, where the A_j's are differential operators of order at most $2-j$. The A_j's induce on Γ differential operators, again denoted by A_j, $A_j : C^\infty(\mathcal{E}|_\Gamma) \to C^\infty(\mathcal{E}|_\Gamma)$. Since the leading symbol of A is given by $\sigma(A)(x, (\xi', \eta)) = \|(\xi', \eta)\|^2$ and since $n_{(x',0)}$ is the unit normal to Γ at $(x', 0)$, one has $A_2(x) = Id_x \in End_x(\mathcal{E}_x, \mathcal{E}_x)$ on Γ.

For any $\varphi \in C^\infty(\mathcal{E}|_\Gamma)$ and $t \geq 0$ we can choose $u \in C^\infty(\mathcal{E}_\Gamma) \cap C(\mathcal{E})$ such that $(A - \alpha_k t)u = 0$ on $M \setminus \Gamma$ and $u|_\Gamma = \varphi$. Then $\frac{\partial u}{\partial s}(x', s)$ has a jump across Γ, which is $-R(\alpha_k t)(\varphi)(x')$. Hence

$$\frac{\partial u}{\partial s}(x', s) = -R(\alpha_k t)(\varphi)(x') H(s) + v(x', s) \ ,$$

where $v(x', s) \in C^\infty(\mathcal{E}_\Gamma|_U) \cap C(\mathcal{E}|_U)$ and $H(s)$ is the Heavyside function. Therefore, on U,

$$(A - \alpha_k t)u = A_2 R(\alpha_k t)(\varphi) \otimes \delta_\Gamma - A_2 \frac{\partial v}{\partial s} + \frac{1}{i} A_1 \frac{\partial u}{\partial s} + A_0 u \ .$$

Since $(A - \alpha_k t)u = 0$ on $M \setminus \Gamma$, we conclude that, on $U \cap (M \setminus \Gamma)$,

$$-A_2 \frac{\partial v}{\partial s} + \frac{1}{i} A_1 \frac{\partial u}{\partial s} + A_0 u = 0 \ .$$

As $-\frac{\partial v}{\partial s} + \frac{1}{i} A_1 \frac{\partial u}{\partial s} + A_0 u \in L^2(\mathcal{E}|_U)$, it follows that

$$(A - \alpha_k t)u = A_2\big(R(\alpha_k t)\varphi \otimes \delta_\Gamma\big) = R(\alpha_k t)\varphi \otimes \delta_\Gamma$$

as $A_2 = Id$ on Γ. Note that $(A - \alpha_k t) : L^2(\mathcal{E}) \to H_{-2}(\mathcal{E})$ is invertible and therefore one obtains for $\varphi \in C^\infty(\mathcal{E}|_\Gamma)$

$$\varphi = J_\Gamma \cdot (A - \alpha_k t)^{-1} \big(R(\alpha_k t)\varphi \otimes \delta_\Gamma\big)$$

where J_Γ is the restriction operator to Γ. From this identity it follows that $R(\alpha_k t)$ is invertible. Moreover, setting $\phi = R(\alpha_k t)\varphi$,

$$\begin{aligned} R(\alpha_k t)^{-1}\phi &= J_\Gamma \cdot (A - \alpha_k t)^{-1} \cdot (\phi \otimes \delta_\Gamma) \\ &= J_\Gamma \cdot \int_{\mathbf{R}^{d-1}} \int_{\mathbf{R}} e^{i(x',s)\cdot(\xi',\eta)} \big[(A - \alpha_k t)^{-1}(\phi \otimes \delta_\Gamma)\big]^\wedge(\xi', \eta) d\eta d\xi' \\ &= \int_{\mathbf{R}^{d-1}} e^{ix'\cdot\xi'} \int_{\mathbf{R}} \sigma((A - \alpha_k t)^{-1})(x', 0, \xi', \eta) \hat{\phi}(\xi') \cdot \frac{1}{\sqrt{2\pi}} d\eta d\xi' \ . \end{aligned}$$

Hence $R(\alpha_k t)^{-1}$ is in $\Psi\mathrm{DO}_{\mathcal{A}}^{-1}(\mathcal{E}|_\Gamma)$ with full symbol $\sigma_{R(\alpha_k t)^{-1}}(x', \xi')$ given by

$$\sigma_{R(\alpha_k t)^{-1}}(x', \xi') = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} \sigma_{(A - \alpha_k t)^{-1}}(x', 0, \xi', \eta) d\eta \ . \tag{3.10}$$

Therefore $R(\alpha_k t)$ is an operator in $\Psi\mathrm{DO}^1_{\mathcal{A}}(\mathcal{E}|_\Gamma)$ with parameter t of weight 1. The ellipticity with parameter of $R(\alpha_k t)$ follows from the formula (3.10) of the symbol and the fact that $(A - \alpha_k t)$ is elliptic with parameter. □

LEMMA 3.10. *For ϵ' sufficiently small and $0 \leq k \leq d-1$, the operator $R(\alpha_k t)$ does not have any eigenvalues in $V_{\pi,\epsilon'}$, where $V_{\pi,\epsilon'} = \{z \in \mathbb{C} \mid \pi - \epsilon' < arg(z) < \pi + \epsilon' \text{ or } |z| < \epsilon'\}$. Hence $R(\alpha_k t)$ has π as an Agmon angle.*

Proof. By assumption, $A : C^\infty(\mathcal{E}) \to C^\infty(\mathcal{E})$ is self-adjoint and positive definite and its spectrum is bounded from below by $\epsilon > 0$. Consider the spectral projections $Q(\lambda)$ corresponding to the operator A, defined in section 2. Notice that $Q(\lambda)$ are smoothing operators and, for $\lambda < \epsilon$, $Q(\lambda) = 0$. Moreover, for any $\varphi \in C^\infty(\mathcal{E}|_\Gamma), \varphi \otimes \delta_\Gamma$ is an element in $H^{-1}(\mathcal{E})$, thus $Q(\lambda)(\varphi \otimes \delta_\Gamma) \in C^\infty(\mathcal{E})$ and $(A - \alpha_k t)^{-1}(\varphi \otimes \delta_\Gamma) \in L^2(\mathcal{E})$. Since $R(\alpha_k t)^{-1} = J_\Gamma(A - \alpha_k t)^{-1}(\cdot \otimes \delta_\Gamma)$, by the spectral representation theorem for A, one obtains

$$\langle R(\alpha_k t)^{-1}\varphi_1, \varphi_2\rangle = \int_0^\infty (\lambda - \alpha_k t)^{-1} d\langle Q(\lambda)(\varphi_1 \otimes \delta_\Gamma), Q(\lambda)(\varphi_2 \otimes \delta_\Gamma)\rangle \ ,$$

$\varphi_1, \varphi_2 \in C^\infty(\mathcal{E}|_\Gamma)$. Together with Lemma 3.9 this implies that $V_{\pi,\epsilon'}$ has an empty intersection with $\operatorname{spec} R(\alpha_k t)$ for $0 < \epsilon'$ sufficiently small. □

Using the above formula together with formula (3.10), one obtains as an immediate consequence the following:

COROLLARY 3.11. *The operator $R_{DN} = R(0)$ is essentially self-adjoint and positive definite. The principal symbol of R_{DN}^{-1} is given by (3.8) and its symbol has the property described in part (2) of Theorem 3.6.*

Having thus established parts (1) and (2) in Theorem 3.6 our next task is to prove formula (3.9).

Consider the families of operators $A^d + t^d$ and $(A^\Gamma)^d + t^d$ for nonnegative real numbers t ($d = \dim(M)$). Then $A^d + t^d$ and $(A^\Gamma)^d + t^d$ are elliptic differential operators with parameter, where the weight of t is 2. The reason for considering these operators comes from the fact that the inverse of $A^d + t^d$ is of trace class while the inverse of $A + t$ is not if $d = \dim(M) \geq 2$. If the operator A^{-1} were of trace class, the proof of formula (3.9) would be considerably simpler. Our strategy will be to first prove a version of formula (3.9) for A^d, using the fact that $(A^d)^{-1}$ is of trace class and derive (3.9) from it.

Note that

$$(A^\Gamma)^d + t^d = (A^\Gamma - te^{i\frac{\pi}{d}})(A^\Gamma - te^{i\frac{3\pi}{d}}) \cdots (A^\Gamma - te^{i\frac{\pi + 2\pi(d-1)}{d}}) \ .$$

Let us introduce the boundary operators

$$J_d(t), N_d(t) : C^\infty(\mathcal{E}_\Gamma) \to \big(\oplus_d C^\infty(\mathcal{E}_\Gamma|_{\Gamma^+ \sqcup \Gamma^-})\big)$$

by setting

$$J_d(t) = \big(J, J(A^\Gamma - \alpha_0 t), J(A^\Gamma - \alpha_1 t)(A^\Gamma - \alpha_0 t), \\ \cdots, J(A^\Gamma - \alpha_{d-2} t)\cdots(A^\Gamma - \alpha_0 t)\big) ,$$

and

$$N_d(t) = \big(N, N(A^\Gamma - \alpha_0 t), N(A^\Gamma - \alpha_1 t)(A^\Gamma - \alpha_0 t), \\ \cdots, N(A^\Gamma - \alpha_{d-2} t)\cdots(A^\Gamma - \alpha_0 t)\big) .$$

By Lemma 3.8

$$\big((A^\Gamma)^d + t^d, J_d(t)\big) : C^\infty(\mathcal{E}_\Gamma) \to C^\infty(\mathcal{E}_\Gamma) \oplus \big(\oplus_d C^\infty(\mathcal{E}_\Gamma|_{\Gamma^+ \sqcup \Gamma^-})\big)$$

is invertible. Therefore the corresponding Poisson operator $\tilde{P}_d(t)$: $\oplus_d C^\infty(\mathcal{E}|_{\Gamma^+ \sqcup \Gamma^-}) \to C^\infty(\mathcal{E}_\Gamma)$ is well defined.

LEMMA 3.12. *The Poisson operator $\tilde{P}_d(t)$ associated to $((A^\Gamma)^d + t^d, J_d(t))$ is given by*

$$\tilde{P}_d(t)(\varphi_0, \cdots, \varphi_{d-1}) = P(\alpha_0 t)\varphi_0 + (A^\Gamma - \alpha_0 t)_D^{-1} P(\alpha_1 t)\varphi_1 + \\ \cdots + \big(A^\Gamma - \alpha_D^{-1}(A^\Gamma - \alpha_1 t)_D^{-1} \cdots (A^\Gamma - \alpha_{d-2} t)_D^{-1} P(\alpha_{d-1} t)\varphi_{d-1}\big) ,$$

where $(A^\Gamma - \alpha_k t)_D$ is the restriction of $A^\Gamma - \alpha_k t$ to $\{u \in C^\infty(\mathcal{E}_\Gamma) \mid Ju = 0\}$.

Proof. Denoting the right-hand side of the claimed identity by $L_d(t)(\varphi_0, \cdots, \varphi_{d-1})$ one obtains

$$\big((A^\Gamma)^d + t^d\big) \cdot L_d(t)(\varphi_0, \cdots, \varphi_{d-1}) = 0 .$$

Moreover, for $0 \leq k \leq d-2$, $L_d(t)(\varphi_0, \cdots, \varphi_{d-1})$ satisfies the boundary conditions

$$\big(J(A^\Gamma - \alpha_{k-1} t)(A^\Gamma - \alpha_{k-2} t)\cdots(A^\Gamma - \alpha_0 t)\big) L_d(t)(\varphi_0, \cdots, \varphi_{d-1}) = \\ J(A^\Gamma - \alpha_{k-1} t)\cdots(A^\Gamma - \alpha_0 t)P(\alpha_0 t)\varphi_0 + \cdots + \\ J(A^\Gamma - \alpha_{k-1} t)\cdots(A^\Gamma - \alpha_0 t)(A^\Gamma - \alpha_0 t)_D^{-1} \cdots (A^\Gamma - \alpha_{k-1} t)_D^{-1} P(\alpha_k t)\varphi_k + \cdots + \\ J(A^\Gamma - \alpha_{k-1} t)\cdots(A^\Gamma - \alpha_0 t)(A^\Gamma - \alpha_0 t)_D^{-1} \cdots (A^\Gamma - \alpha_{d-2} t)_D^{-1} P(\alpha_{d-1} t)\varphi_{d-1} \\ = \varphi_k ,$$

since $(A^\Gamma - \alpha_j t)P(\alpha_j t) = 0$ and $J(A^\Gamma - \alpha_j t)_D^{-1} = 0$. These two properties of $L_d(t)$ establish the claimed identity. □

Further, let us consider the boundary conditions $J_d(t)$ and $N_d(t)$ for $t = 0$. Note that

$$J_d(0) = \big(J, JA^\Gamma, \cdots, J(A^\Gamma)^{d-1}\big) ; \quad N_d(0) = \big(N, NA^\Gamma, \cdots, N(A^\Gamma)^{d-1}\big) .$$

Let $\Omega(t)$ be the following lower-triangular $d \times d$ matrix

$$\Omega(t) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \alpha_0 t & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_0^{d-1} t^{d-1} & t^{d-2} \sum_{k=0}^{d-2} \alpha_0^{d-2-k} \alpha_1^k & \cdots & 1 \end{pmatrix} .$$

Then $J_d(0) = \Omega(t) J_d(t)$ as well as $N_d(0) = \Omega(t) N_d(t)$. Let $P_d(t) := \tilde{P_d}(t)\Omega(t)^{-1}$ and notice that $P_d(t)$ is the Poisson operator corresponding to $((A^\Gamma)^d + t^d, J_d(0))$.

Consider the operator $\tilde{R}_d(t) = \tilde{\Delta_{if}} \cdot N_d(t) \cdot \tilde{P}_d(t) \cdot \tilde{\Delta_{ia}}$ with $\tilde{\Delta_{if}} := \oplus_d \Delta_{if}$ and $\tilde{\Delta_{ia}} := \oplus_d \Delta_{ia}$. Then

$$\begin{gathered} \tilde{R}_d(t)(\varphi_0, \cdots, \varphi_{d-1}) = \\ \tilde{\Delta_{if}} \cdot \{N, N(A^\Gamma - \alpha_0 t), \cdots, N(A^\Gamma - \alpha_{d-2} t) \cdots (A^\Gamma - \alpha_0 t)\} \cdot \\ \{P(\alpha_0 t)\, \Delta_{ia}\, \varphi_0 + (A^\Gamma - \alpha_0 t)_D^{-1} P(\alpha_1 t)\, \Delta_{ia}\, \varphi_1 + \cdots + \\ (A^\Gamma - \alpha_0 t)_D^{-1} (A^\Gamma - \alpha_1 t)_D^{-1} \cdots (A^\Gamma - \alpha_{d-2} t)_D^{-1} P(\alpha_{d-1} t)\, \Delta_{ia}\, \varphi_{d-1}\} . \end{gathered}$$

Thus $\tilde{R}_d(t) : \oplus_d C^\infty(\mathcal{E}|_\Gamma) \to \oplus_d C^\infty(\mathcal{E}|_\Gamma)$ can be represented by a $d \times d$ matrix of upper triangular form,

$$\begin{pmatrix} R(\alpha_0 t) & \cdots & \cdots & \cdots \\ 0 & R(\alpha_1 t) & \cdots & \cdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & R(\alpha_{d-1} t) \end{pmatrix} ,$$

where $R(\alpha_k t)$ is the Dirichlet to Neumann operator corresponding to $A^\Gamma - \alpha_k t$ defined earlier. In particular, in view of Lemma 3.10, one concludes that $\tilde{R}_d(t)$ is invertible and has π as an Agmon angle.

Finally introduce the operator $R_d(t)$ associated to $(A^\Gamma)^d + t^d$, $J_d(0)$ and $N_d(0)$. Then

$$\begin{aligned} \tilde{R}_d(t) &= \tilde{\Delta_{if}} \cdot N_d(t) \cdot \tilde{P}_d(t) \cdot \tilde{\Delta_{ia}} = \tilde{\Delta_{if}} \cdot \Omega(t)^{-1} \cdot N_d(0) \cdot P_d(t) \cdot \Omega(t) \cdot \tilde{\Delta_{ia}} \\ &= \Omega(t)^{-1} \cdot \tilde{\Delta_{if}} \cdot N_d(0) \cdot P_d(t) \cdot \tilde{\Delta_{ia}} \cdot \Omega(t) = \Omega(t)^{-1} \cdot R_d(t) \cdot \Omega(t) . \end{aligned}$$

As a consequence, $R_d(t)$ has the same spectrum as $\tilde{R}_d(t)$ and therefore, $R_d(t)$ is invertible, has π as an Agmon angle and satisfies $\log \det_N(R_d(t)) = \log \det_N(\tilde{R}_d(t))$. Since $\tilde{R}_d(t)$ is of upper-triangular form one has

$$\log \det_N \left(\tilde{R}_d(t)\right) = \sum_{k=0}^{d-1} \log \det_N \left(R(\alpha_k t)\right) .$$

As A is positive and selfadjoint, the operator $A^d + t^d : C^\infty(\mathcal{E}) \to C^\infty(\mathcal{E})$ is invertible for $t \geq 0$. Using the kernel $k_t(x, y)$ of $(A^d + t^d)^{-1}$ this operator can be extended to $C^\infty(\mathcal{E}_\Gamma)$ by setting ($u \in C^\infty(\mathcal{E}_\Gamma)$)

$$\left((A^d + t^d)^{-1}\right)^\Gamma u(x) = \int_{M_\Gamma} k_t(x,y)u(y)dy \ .$$

As already noticed Lemma 3.8 implies that

$$\left((A^\Gamma)^d + t^d, J_d(t)\right) : C^\infty(\mathcal{E}_\Gamma) \to C^\infty(\mathcal{E}_\Gamma) \oplus_d C^\infty(\mathcal{E}_\Gamma|_{\Gamma^+ \sqcup \Gamma^-})$$

is invertible. Thus, since $J_d(0) = \Omega(t)J_d(t)$, one concludes that $(A^d + t^d, J_d(0))$ is invertible as well. Denote by $((A^\Gamma)^d + t^d)_D$ the restriction of $(A^\Gamma)^d + t^d$ to $\{u \in C^\infty(\mathcal{E}_\Gamma) \mid J_d(0)u = 0\}$ and let $((A^\Gamma)^d + t^d)_D^{-1}$ be its inverse.

Lemma 3.13. $((A^\Gamma)^d + t^d)_D^{-1} = ((A^d + t^d)^{-1})^\Gamma - P_d(t) \cdot J_d(0) \cdot ((A^d + t^d)^{-1})^\Gamma$.

Proof. Denote by $L(t)$ the right-hand side of the claimed identity. One verifies that for $u \in C^\infty(\mathcal{E}_\Gamma)$

$$\left((A^\Gamma)^d + t^d\right)L(t)u = u$$

and

$$J_d(0)L(t)u = J_d(0) \cdot \left((A^\Gamma)^d + t^d\right)^{-1} u - J_d(0) \cdot \left((A^\Gamma)^d + t^d\right)^{-1} u = 0 \ .$$

These two identities imply that $L(t) = ((A^\Gamma)^d + t^d)_D^{-1}$. □

Lemma 3.14. (i) $\frac{d}{dt}P_d(t) = -dt^{d-1}((A^\Gamma)^d + t^d)_D^{-1} \cdot P_d(t)$

(ii) $R_d(t)^{-1} \cdot \frac{d}{dt}R_d(t) = -dt^{d-1}R_d(t)^{-1} \cdot \tilde{\Delta}_{if} \cdot N_d(0) \cdot ((A^\Gamma)^d + t^d)_D^{-1} \cdot P_d(t) \cdot \tilde{\Delta}_{ia}$.

In particular, d being the dimension of M, $R_d(t)^{-1} \cdot \frac{d}{dt}R_d(t)$ is of trace class.

Proof. (i) Differentiate $((A^\Gamma)^d + t^d) \cdot P_d(t) = 0$ with respect to t to obtain

$$\left((A^\Gamma)^d + t^d\right) \cdot \frac{d}{dt}P_d(t) = -\frac{d}{dt}\left((A^\Gamma)^d + t^d\right) \cdot P_d(t) = -dt^{d-1}P_d(t) \ .$$

Similarly, differentiating $J_d(0) \cdot P_d(t) = Id$ with respect to t yields $J_d(0)\frac{d}{dt}P_d(t) = 0$. Hence

$$\left((A^\Gamma)^d + t^d\right)_D \cdot \frac{d}{dt}P_d(t) = -dt^{d-1}P_d(t)$$

and therefore

$$\frac{d}{dt}P_d(t) = -dt^{d-1}\left((A^\Gamma)^d + t^d\right)_D^{-1} \cdot P_d(t) \ .$$

(ii) follows from the definition of $R_d(t)$ and (i). □

Taking into account that $\tilde{\Delta}_{ia}(\oplus_d C^\infty(\mathcal{E}|_\Gamma)) = \{(\varphi, \varphi) \mid \varphi \in \oplus_d C^\infty(\mathcal{E}|_\Gamma)\}$ define $Pr_\Gamma : \tilde{\Delta}_{ia}(\oplus_d C^\infty(\mathcal{E}|_\Gamma)) \to \oplus_d C^\infty(\mathcal{E}|_\Gamma)$ by $Pr_\Gamma(\varphi, \varphi) = \varphi$.

Corollary 3.15.

$$R_d(t)^{-1} \cdot \frac{d}{dt}R_d(t) = dt^{d-1}Pr_\Gamma \cdot J_d(0) \cdot \left((A^d + t^d)^{-1}\right)^\Gamma \cdot P_d(t) \cdot \tilde{\Delta}_{ia} \ .$$

Proof. By Lemma 3.13 and 3.14

$$R_d(t)^{-1} \cdot \frac{d}{dt} R_d(t) = -dt^{d-1} \big(\tilde{\Delta_{if}} \cdot N_d(0) \cdot P_d(t) \cdot \tilde{\Delta_{ia}}\big)^{-1} \cdot$$
$$\tilde{\Delta_{if}} \cdot N_d(0) \cdot \big(((A^d + t^d)^{-1})^\Gamma - P_d(t) \cdot J_d(0) \cdot ((A^d + t^d)^{-1})^\Gamma\big) \cdot P_d(t) \cdot \tilde{\Delta_{ia}} \ .$$

Clearly $\tilde{\Delta_{if}} \cdot N_d(0) \cdot ((A^d + t^d)^{-1})^\Gamma(u) = 0$ for $u \in C^\infty(\mathcal{E})$ and thus $\tilde{\Delta_{if}} \cdot N_d(0) \cdot ((A^d + t^d)^{-1})^\Gamma \cdot P_d(t) \cdot \tilde{\Delta_{ia}} = 0$. Therefore $R_d(t)^{-1} \cdot \frac{d}{dt} R_d(t) = dt^{d-1}(\tilde{\Delta_{if}} \cdot N_d(0) \cdot P_d(t) \cdot \tilde{\Delta_{ia}})^{-1} \cdot \tilde{\Delta_{if}} \cdot N_d(0) \cdot P_d(t) \cdot B_d(0) \cdot ((A^d + t^d)^{-1})^\Gamma \cdot P_d(t) \cdot \tilde{\Delta_{ia}}$.

Note that for any $u \in C^\infty(\mathcal{E}_\Gamma)$, the boundary values of $((A^d + t^d)^{-1})^\Gamma u$ on Γ^+ and Γ^- are the same, i.e.

$$J_d(0)\big((A^d + t^d)^{-1}\big)^\Gamma u|_{\Gamma^+} = J_d(0)\big((A^d + t^d)^{-1}\big)^\Gamma u|_{\Gamma^-} \ .$$

Hence

$$J_d(0) \cdot \big((A^d+t^d)^{-1}\big)^\Gamma \cdot P_d(t) \cdot \tilde{\Delta_{ia}} = \tilde{\Delta_{ia}} \cdot Pr_\Gamma \cdot J_d(0) \cdot \big((A^d+t^d)^{-1}\big)^\Gamma \cdot P_d(t) \cdot \tilde{\Delta_{ia}} \ .$$

□

As $(A^d + t^d)^{-1}, ((A^d + t^d)^{-1})^\Gamma$ and $R_d(t)^{-1} \frac{d}{dt} R_d(t)$ are of trace class we can apply the variational formula for regularized determinants:

LEMMA 3.16. *Let $L(t)$ denote any of the operators $A^d + t^d$, $(A^d + t^d)_D$ or $R_d(t)$. Then, for any $t \geq 0$,*

$$\frac{d}{dt} \log \det{}_N L(t) = tr_N \left(Q(t)^{-1} \frac{d}{dt} L(t) \right) \ .$$

Lemmas 3.13-16 and Corollary 3.15 lead to the following result:

LEMMA 3.17. *Let $A^d + t^d$ and $((A^\Gamma)^d + t^d)_D$ be as above. Then, for $t \geq 0$,*

$$\frac{d}{dt}\big(\log \det{}_N(A^d + t^d) - \log \det{}_N((A^\Gamma)^d + t^d)_D\big) = \frac{d}{dt} \log \det{}_N R_d(t) \ .$$

Proof. Define $w(t) := \frac{d}{dt}(\log \det_N(A^d + t^d) - \log \det_N(((A^\Gamma)^d + t^d), J_d(0)))$.

By Lemma 3.16 and Lemma 3.13

$$\begin{aligned} w(t) &= tr_N \left(\frac{d}{dt}(A^d+t^d) \cdot (A^d+t^d)^{-1} - \frac{d}{dt}\big((A^\Gamma)^d+t^d\big)_D \cdot \big((A^\Gamma)^d+t^d\big)_D^{-1} \right) \\ &= dt^{d-1} tr_N\big((A^d + t^d)^{-1} - ((A^\Gamma)^d + t^d)_D^{-1}\big) \\ &= dt^{d-1} tr_N\big(((A^d + t^d)^{-1})^\Gamma - ((A^\Gamma)^d + t^d)_D^{-1}\big) \\ &= dt^{d-1} tr_N\big(P_d(t) \cdot J_d(0) \cdot ((A^d + t^d)^{-1})^\Gamma\big) \ . \end{aligned}$$

On the other hand, by Lemma 3.16 and Corollary 3.15 and the commutativity of the trace,

$$\begin{aligned}\frac{d}{dt}\log\det_N R_d(t) &= tr_N\left(\frac{d}{dt}R_d(t)\cdot R_d(t)^{-1}\right)\\ &= dt^{d-1}tr_N\big(Pr_\Gamma\cdot J_d(0)\cdot((A^d+t^d)^{-1})^\Gamma\cdot P_d(t)\cdot\tilde{\Delta_{ia}}\big)\\ &= dt^{d-1}tr_N\big(P_d(t)\cdot\tilde{\Delta_{ia}}\cdot Pr_\Gamma\cdot J_d(0)\cdot((A^d+t^d)^{-1})^\Gamma\big)\\ &= dt^{d-1}tr_N\big(P_d(t)\cdot J_d(0)\cdot((A^d+t^d)^{-1})^\Gamma\big)\ .\end{aligned}$$

Comparing the above two identities one obtains

$$w(t)=\frac{d}{dt}\log\det_N R_d(t)\ . \qquad \square$$

Since $\log\det_N R_d(t) = \sum_{k=0}^{d-1}\log\det_N R(\alpha_k t)$, one concludes from Lemma 3.17 that

$$\log\det_N(A^d+t^d)-\log\det_N\left((A^\Gamma)^d+t^d\right)_D=\tilde{c}+\sum_{k=0}^{d-1}\log\det_N R(\alpha_k t)\ ,$$

where $\tilde{c}$ is independent of t.

Note that $\log\det_N(A^d+t^d)$, $\log\det_N((A^\Gamma)^d+t^d, J_d(0))$ and $\log\det_N R(\alpha_k t)$ $(0\le k\le d-1)$ have asymptotic expansions as $t\to+\infty$. Following Voros [V] or Friedlander [Fr], or using Theorems 3.4 and 3.5, the constant terms in the asymptotic expansions of $\log\det_N(A^d+t^d)$ and $\log\det_N((A^\Gamma)^d+t^d)_D$ are zero. Let $\pi_0(R(\alpha_k t))$ be the constant term in the asymptotic expansion of $\log\det_N(R(\alpha_k t))$. Then $\tilde{c}=-\sum_{k=0}^{d-1}\pi_0(R(\alpha_k t))$, which is computable in terms of the symbol of $R_k(t)$ (cf. Theorem 3.4).

LEMMA 3.18. (i) $\det_N(A^\Gamma)^d_D=(\det_N A^\Gamma_D)^d$;
(ii) $\det_N(A^d)=(\det_N A)^d$.

Proof. (i) follows from the following identity for the spectral distribution function $(\lambda\ge 0)$,

$$N(A^\Gamma_D,\lambda)=N\big((A^\Gamma)^d_D,\lambda^d\big)\ .$$

(ii) is proved in the same way. $\square$

Proof of Theorem 3.6. Statements (1) and (2) are contained in Corollary 3.11. Concerning statement (3) set $t=0$ in Lemma 3.17 to obtain

$$\log\det_N A^d-\log\det_N(A^\Gamma)^d_D=\tilde{c}+\log\det_N R_d(0)\ .$$

By Lemma 3.18, $\log(\det_N A)^d-\log(\det_N A^\Gamma_D)^d=\tilde{c}+\log(\det_N R_{DN})^d$. Hence

$$\log\det_N A=\log(\bar{c})+\log\det_N A^\Gamma_D+\log\det_N R_{DN}\ ,$$

where $\log(\bar{c})=-\frac{1}{d}\sum_{k=0}^{d-1}\pi_0(R(\alpha_k t))$. Using Theorem 3.4 the result follows. Part (4) of Theorem 3.6 follows from formula (3.10) applied to $t=0$ and the family $A(\lambda)$. $\square$

4. Torsions and Witten Deformation of the Analytic Torsion

4.1 Reidemeister and analytic torsion in the von Neumann sense. Let M be a connected, closed manifold of dimension d and $\mathcal{W}$ an $(\mathcal{A}, \Gamma^{op})$-Hilbert module of finite type with $\Gamma = \pi_1(M)$ the fundamental group of M. Let $p : \mathcal{E} \to M$ be the bundle of $\mathcal{A}$-Hilbert modules over M associated to $\mathcal{W}$ as described in section 1.4. The fiber of this bundle is isomorphic to the $\mathcal{A}$-Hilbert module $\mathcal{W}$. The smooth bundle $p : \mathcal{E} \to M$ is equipped with a flat *canonical* connection. Both its Hermitian structure μ and fiberwise $\mathcal{A}$-action ρ are left invariant by the parallel transport induced by the canonical connection.

Let $h : M \to \mathbf{R}$ be a smooth Morse function. For convenience we assume that h is self-indexing, i.e. $h(x) = \text{index}(x)$ for any critical point x of h (cf. [Mi2]). Let g' be a Riemannian metric so that $\tau = (h, g')$ is a generalized triangulation. This means that for any two critical points x and y of h, the unstable manifold W_x^- and the stable manifold W_y^+, associated to the vector field $-\text{grad}_{g'} h$, intersect transversally and, in a neighborhood of any critical point x of h, there exist coordinates $y_1, \dots, y_d$, with respect to which h is of the form $h(y) = k - (y_1^2 + \dots + y_k^2)/2 + (y_{k+1}^2 + \dots + y_d^2)/2$ with $k = \text{index}(x)$ and the metric g' is Euclidean (cf. Introduction). Let $\tilde{M} \to M$ be the universal covering of M and $\tilde{h}$ and $\tilde{g}'$ be the lifts of h and g' on $\tilde{M}$. Denote by $\text{Cr}_q(h) \subset M$, resp. $\text{Cr}_q(\tilde{h}) \subset \tilde{M}$, the set of critical points of index q of h, resp. $\tilde{h}$, and let $\text{Cr}(\tilde{h}) = \cup_q \text{Cr}_q(\tilde{h})$. Clearly the group Γ acts freely on $\text{Cr}_q(\tilde{h})$, for any q, and the quotient set can be identified with $\text{Cr}_q(h)$.

For each $\tilde{x} \in \text{Cr}(\tilde{h})$ choose orientations $O_{\tilde{x}} = (O_{\tilde{x}}^+, O_{\tilde{x}}^-)$ for the stable and the unstable manifolds $W_{\tilde{x}}^+$ and $W_{\tilde{x}}^-$, so that they are Γ-invariant and denote

$$O_h := \{O_{\tilde{x}}; \tilde{x} \in \text{Cr}(\tilde{h})\} .$$

To the quadruple $(M, \tau, O_h, \mathcal{W})$ we associate a cochain complex of finite type over the von Neumann algebra $\mathcal{A}$, $\mathcal{C}(M, \tau, O_h) = \{\mathcal{C}^q, \delta_q\}$. The components $\mathcal{C}^q$ are the $\mathcal{A}$-Hilbert module of finite type, $\mathcal{C}^q := \Gamma(\mathcal{E}|_{\text{Cr}_q(h)}) = \bigoplus_{x \in \text{Cr}_q(h)} \mathcal{E}_x$ which can be identified with the module of Γ-equivariant maps $f : \text{Cr}_q(\tilde{h}) \to \mathcal{W}$. To define the maps δ_q a few remarks are in order. The orientations O_h permit us to define the functions $\mu_q : \text{Cr}_q(\tilde{h}) \times \text{Cr}_{q-1}(\tilde{h}) \to \mathbf{Z}$, $\mu_q(\tilde{x}, \tilde{y}) :=$ intersection number $(W_{\tilde{x}}^- \cap V, W_{\tilde{y}}^+ \cap V)$, where $V := \tilde{h}^{-1}(q - 1/2)$. Notice that the functions μ_q have the following properties:

(In1) $\mu_q(\tilde{x}, \tilde{y}) = \mu_q(g\tilde{x}, g\tilde{y})$, for all $g \in \pi_1(M)$;

(In2) $\{\tilde{x} \in \text{Cr}_q(\tilde{h}); \mu_q(\tilde{x}, \tilde{y}) \neq 0\}$ is finite for any $\tilde{y} \in \text{Cr}_{q-1}(\tilde{h})$;

(In3) $\{\tilde{y} \in \text{Cr}_{q-1}(\tilde{h}); \mu_q(\tilde{x}, \tilde{y}) \neq 0\}$ is finite for any $\tilde{x} \in \text{Cr}_q(\tilde{h})$;

(In4) $\sum_{\tilde{y}\in \mathrm{Cr}_{q-1}(\tilde{h})} \mu_q(\tilde{x},\tilde{y}) \cdot \mu_{q-1}(\tilde{y},\tilde{z}) = 0$ for any $\tilde{x} \in \mathrm{Cr}_q(\tilde{h})$ and any $\tilde{z} \in \mathrm{Cr}_{q-2}(\tilde{h})$.

Properties (In1)-(In3) imply that for any Γ-equivariant map $f : \mathrm{Cr}_{q-1}(\tilde{h}) \to \mathcal{W}$, we can define the Γ-equivariant map $\delta_{q-1}(f) : \mathrm{Cr}_q(\tilde{h}) \to \mathcal{W}$ by the formula

$$\delta_{q-1}(f)(\tilde{x}) = \sum_{\tilde{y}\in \mathrm{Cr}_{q-1}(\tilde{h})} \mu_q(\tilde{x},\tilde{y}) f(\tilde{y}) \ . \tag{4.1}$$

By property (In4), $\delta_q \cdot \delta_{q-1} = 0$.

One defines $\log T_{\mathrm{comb}}(M,\tau) \in \mathbf{D}$ by (cf. section 1)

$$\log T_{\mathrm{comb}}(M,\tau) := \log T\big(\mathcal{C}(M,\tau,O_h,\mathcal{W})\big) \ . \tag{4.2}$$

One can show that $\log T_{\mathrm{comb}}$ is independent of the choice of the orientations O_h.

Let (M,g) be a Riemannian manifold and $\mathcal{W}$ a $(\mathcal{A},\Gamma^{\mathrm{op}})$-Hilbert module of finite type. Let $\Lambda^q(M;\mathcal{E}) = C^\infty(\mathcal{E}\otimes\Lambda^q(T^*M))$ be the space of smooth q-forms with values in $\mathcal{W}$ where T^*M denotes the cotangent bundle of M and $p : \mathcal{E} \to M$ is a smooth bundle of $\mathcal{A}$-Hilbert modules of finite type with fiber $\mathcal{W}$. The Riemannian metric g induces the Hodge $*$ operators $R_q : \Lambda^q(T^*M) \to \Lambda^{d-q}(T^*M)$ and the Hermitian structure μ on $\mathcal{E}$ together with the Hodge operators R_q induce a Hermitian structure on $\mathcal{E}\otimes\Lambda^q(T^*M)$ given by ($s, s' \in C^\infty(\mathcal{E}); w, w' \in C^\infty(\Lambda^q(T^*M))$)

$$(s\otimes w, s'\otimes w')(x) = \mu_x\big(s(x), s'(x)\big) R_d\big(w(x)\wedge R_q w'(x)\big) \ .$$

As a consequence, $\mathcal{E}\otimes\Lambda^q(T^*M)$ is a smooth bundle of $\mathcal{A}$-Hilbert modules. The canonical connection in $p : \mathcal{E} \to M$ can be interpreted as a first order differential operator ${}_{\mathcal{W}}d_q : \Lambda^q(M;\mathcal{E}) \to \Lambda^{q+1}(M;\mathcal{E})$. As the canonical connection is flat, ${}_{\mathcal{W}}d_{q+1}\cdot {}_{\mathcal{W}}d_q = 0$ for any q. Notice that ${}_{\mathcal{W}}d_q$ is an $\mathcal{A}$-linear, differential operator. If the action of Γ on $\mathcal{W}$ is trivial, then ${}_{\mathcal{W}}d$ is the usual exterior differential $\mathrm{Id}\otimes d$. In case there is no risk of ambiguity we will write d instead of ${}_{\mathcal{W}}d$ and continue to call it exterior differential.

The formal adjoint of ${}_{\mathcal{W}}d_q$ with respect to the above defined Hermitian structure is a first order differential operator ${}_{\mathcal{W}}d_q^* : \Lambda^{q+1}(M;\mathcal{E}) \to \Lambda^q(M;\mathcal{E})$ and is again $\mathcal{A}$-linear. Introduce the Laplacians, acting on q-forms,

$$\Delta_q = d_q^* d_q + d_{q-1} d_{q-1}^* \ .$$

The operators Δ_q are essentially selfadjoint, nonnegative, elliptic and $\mathcal{A}$-linear. The space $\Lambda^q(M;\mathcal{E})$ can be equipped with the scalar product

$$\langle u_1, u_2\rangle_r = \big\langle (\mathrm{Id}+\Delta_q)^{r/2}(u_1), (\mathrm{Id}+\Delta_q)^{r/2}(u_2)\big\rangle \tag{4.3}$$

where

$$\begin{aligned}\big\langle(\mathrm{Id}+\Delta_q)^{r/2}(u_1),(\mathrm{Id}+\Delta_q)^{r/2}(u_2)\big\rangle\\ = \int_M \big((\mathrm{Id}+\Delta_q)^{r/2}(u_1),(\mathrm{Id}+\Delta_q)^{r/2}(u_2)\big)(x)d\,\mathrm{vol}_g\ .\end{aligned}$$

The completion of $\Lambda^q(M,\mathcal{E})$ with respect to the scalar product $\langle .\,,.\rangle_r$ is an $\mathcal{A}$-Hilbert module $H_r(\Lambda^q(M;\mathcal{E}))$, the space of forms of degree q and of Sobolev class of order r. It is well known that the Sobolev norm is equivalent to the norm defined by the scalar product $\langle .\,,.\rangle_r$. In the case where $r=0$, we write also $L_2(\Lambda^q(M;\mathcal{E}))$. Obviously, these Hilbert modules are not of finite type. Note that the operators $(\mathrm{Id}+\Delta_q)^{r/2}$ define isometries between $H_{r'}(\Lambda^q(M;\mathcal{E}))$ and $H_{(r'-r)}(\Lambda^q(M;\mathcal{E}))$. Let $\mathcal{H}_q$ be the $\mathcal{A}$-Hilbert module of harmonic q-forms

$$\mathcal{H}_q=\big\{\omega\in L_2(\Lambda^q(M;\mathcal{E}));\Delta_q(\omega)=0\big\}\ .$$

Since Δ_q is elliptic, $\mathcal{H}_q\subset\Lambda^q(M;\mathcal{E})$. The integration on the q-cells of the generalized triangulation τ, which are given by the unstable manifolds of $-\mathrm{grad}_{g'}h$, defines an $\mathcal{A}$-linear map

$$\mathrm{Int}^{(q)}:\Lambda^q(M;\mathcal{E})\to\mathcal{C}^q$$

so that $\delta_q\,\mathrm{Int}^{(q)}=\mathrm{Int}^{(q+1)}d_q$ (cf. Appendix by F. Laudenbach in [BZ1]). Denote by π_q the canonical projection $\pi_q:\mathcal{C}^q\to\mathrm{Null}(\Delta_q^{\mathrm{comb}})$. By a theorem of Dodziuk [Do] of de Rham type, the map

$$\overline{\mathrm{Int}}^{(q)}:\mathcal{H}_q\to\mathrm{Null}(\Delta_q^{\mathrm{comb}})\ ,$$

defined by the restriction of $\pi_q\,\mathrm{Int}^{(q)}$ to $\mathcal{H}_q$, is an isomorphism of Hilbert modules. Denote its inverse by θ_q. Since $\mathrm{Null}(\Delta_q^{\mathrm{comb}})$ is an $\mathcal{A}$-Hilbert module of finite type so is $\mathcal{H}_q$. Define T_{met} as the positive real number, viewed as an element in $\mathbf{D}$ (cf. Introduction) by

$$\log T_{\mathrm{met}}(M,g,\mathcal{W},\tau):=\tfrac{1}{2}\sum_q(-1)^q\log\det{}_N(\theta_q^*\theta_q)\ . \qquad (4.4)$$

The Reidemeister torsion $T_{\mathrm{Re}}(M,g,\mathcal{W},\tau)\in\mathbf{D}$ is defined (cf. [CM], [LüRo]) by

$$\log T_{\mathrm{Re}}(M,g,\mathcal{W},\tau)=\log T_{\mathrm{comb}}(M,\mathcal{W},\tau)+\log T_{\mathrm{met}}(M,g,\mathcal{W},\tau) \qquad (4.5)$$

and the analytic torsion $T_{\mathrm{an}}(M,g,\mathcal{W})\in\mathbf{D}$ (cf. [Lo], [M] and section 2.4 for the definition of log det) by

$$\log T_{\mathrm{an}}(M,g,\mathcal{W})=\tfrac{1}{2}\sum_q(-1)^{q+1}q\log\det{}_N(\Delta_q)\ . \qquad (4.6)$$

Introduce for $\lambda\geq0$ the functions $F_q(\lambda):=F_{\underline{d}_q^*\underline{d}_q}(\lambda)=\sup\{\dim_N\mathcal{L};\,\mathcal{L}\in\mathcal{P}_q(\lambda)\}$

where $\mathcal{P}_q(\lambda)$ consists of all $\mathcal{A}$-invariant closed subspaces $\mathcal{L} \subset \overline{d_q^*(\Lambda^{q+1}(M;\mathcal{E}))} \subset L_2(\Lambda^q(M;\mathcal{E}))$, so that for any $\omega \in \mathcal{L}$, ω is in the domain of definition of d_q and

$$\|d_q\omega\| \leq \lambda^{1/2}\|\omega\| \ . \tag{4.7}$$

The functions $F_q(\lambda)$ are elements in the space $\mathbf{F}$ (cf. section 1). By arguments of Gromov-Shubin (cf. section 1.2) the spectral distribution functions $N_k(\lambda)=N_{\Delta_k}(\lambda)$ of the Laplace operator Δ_k are given by $\beta_k+F_{k-1}(\lambda)+F_k(\lambda)$.

DEFINITION 4.1. (1) *The system $(M,\tau,\mathcal{W})$ is said to be of c-determinant class iff for $0 \leq k \leq d$,*

$$\int_{0+}^1 \log\lambda dN_{\Delta_k^{\text{comb}}}(\lambda) > -\infty \ .$$

(2) *The system $(M,g,\mathcal{W})$ is said to be of a-determinant class iff for $0 \leq k \leq d$,*

$$\int_{0+}^1 \log\lambda dN_{\Delta_k}(\lambda) > -\infty \ ,$$

or, equivalently, for $0 \leq k \leq d$,

$$\int_{0}^1 \log\lambda dF_k(\lambda) > -\infty \ .$$

It will be shown in Proposition 5.6 that conditions (1) and (2) are equivalent.

We finish this subsection with the following observations concerning torsion and Poincaré duality. First note that, using the Hodge $*$ operators it follows that Δ_q and Δ_{d-q} are isospectral and therefore

$$\log T_{\text{an}}(M,g,\mathcal{W}) = (-1)^{d+1}\log T_{\text{an}}(M,g,\mathcal{W}) \ . \tag{4.8'}$$

The same identity holds for T_{comb}. Let $\tau=(h,g)$ be a generalized triangulation of the closed manifold M^d. Then $\tau_D=(d-h,g)$ is also a generalized triangulation. The critical points of index q and the corresponding stable manifolds of $-\text{grad}_g h$ are the same as the critical points of index $d-q$ and the corresponding unstable manifolds of $-\text{grad}_g(d-h)$. The orientations $O_h := \{O_{\tilde{x}}; \tilde{x} \in \text{Cr}(\tilde{h})\} \cup \{\tilde{O}\}$ induce the orientations O_{d-h}. The above identification of the critical points of h and $d-h$ can be used to obtain isometries $PD_q : \mathcal{C}_\tau^q \to \mathcal{C}_{\tau_D}^{d-q}$ where $\mathcal{C}_\tau^q := \mathcal{C}^q(M,\tau,O_h)$ and $\mathcal{C}_{\tau_D}^q := \mathcal{C}^q(M,\tau_D,O_{d-h})$. This leads to

$$\log T_{\text{comb}}(M,\tau,\mathcal{W}) = (-1)^{d+1}\log T_{\text{comb}}(M,\tau_D,\mathcal{W}) \ . \tag{4.8''}$$

Finally we derive the corresponding identity for T_{met}. The isometries PD_q provide a duality between $(\mathcal{C}_\tau^q, \delta_q)$ and $(\mathcal{C}_{\tau_D}^{d-q}, \delta_{d-q}^*)$ and induce isometries between $\mathrm{Null}(\Delta_{q,\tau}^{\mathrm{comb}})$ and $\mathrm{Null}(\Delta_{d-q,\tau_D}^{\mathrm{comb}})$. Here $\Delta_{q,i}^{\mathrm{comb}}$, $i=\tau,\tau_D$, denote the Laplacians in the cochain complexes $\mathcal{C}_i^q$. It is shown in Proposition 5.9 that

$$(\overline{\mathrm{Int}}_{\tau_D}^{(d-q)})^* \cdot PD_q \cdot \overline{\mathrm{Int}}_{\tau}^{(q)} = R_q|_{\mathcal{H}_q} \tag{4.9}$$

where $(\overline{\mathrm{Int}}_{\tau_D}^{(d-q)})^*$ denotes the adjoint of $\overline{\mathrm{Int}}_{\tau_D}^{(d-q)}$. Thus

$$\log T_{met}(M,g,\tau,\mathcal{W}) = (-1)^{d+1}\log T_{met}(M,g,\tau_D,\mathcal{W}) \ . \tag{4.8'''}$$

4.2 Witten's deformation of the analytic torsion. Let $\omega \in \Lambda^1(M)$ be a smooth closed 1-form on M. Introduce a perturbation $(\Lambda^q(M;\mathcal{E}),_{\mathcal{W}} d_q^\omega)$ of the de Rham complex $(\Lambda^q(M;\mathcal{E}),_{\mathcal{W}} d_q)$ with

$$d_q^\omega \equiv_{\mathcal{W}} d_q^\omega :=_{\mathcal{W}} d_q + \omega \wedge (.) \ .$$

The formal adjoint of d_q^ω with respect to the Hermitian structure on $\mathcal{E}\otimes\Lambda^q(T^*M)$, introduced in section 4.1, is a first order $\mathcal{A}$-linear, differential operator

$$(d_q^\omega)^* : \Lambda^{q+1}(M;\mathcal{E}) \to \Lambda^q(M;\mathcal{E}) \ .$$

Introduce the perturbed Laplacians, acting on q-forms,

$$\Delta_q^\omega = (d_q^\omega)^* d_q^\omega + d_{q-1}^\omega (d_{q-1}^\omega)^* \ . \tag{4.10}$$

The operators Δ_q^ω are $\mathcal{A}$-linear, elliptic operators which are positive and essentially selfadjoint. They are zero'th order perturbations of the Laplacians Δ_q defined above. The case $\omega = tdh$ where $h : M \to \mathbf{R}$ is a smooth function and $t \in \mathbf{R}$ was considered by Witten (cf. [W]). The multiplication by e^{th} defines, for any r, a linear operator on the Sobolev space $H_r(\Lambda^q(M;\mathcal{E}))$, which is an isomorphism of $\mathcal{A}$-Hilbert modules and we have $d_q(t) := d_q^{tdh} = e^{-th} d_q e^{th}$. We call the operators $\Delta_q(t) = \Delta_q^{tdh}$ the Witten Laplacians associated to h. More generally, we will refer to the complex $(\Lambda^q(M;\mathcal{E}), d_q^\omega(t))$ with $d_q^\omega(t) := d_q^{t\omega}$, depending on the parameter t, as the Witten complex. Define the perturbed analytic torsion $T_{\mathrm{an}}(M,g,\mathcal{W},\omega)$ as an element in the vector space $\mathbf{D}$

$$\log T_{\mathrm{an}}(M,g,\mathcal{W},\omega) := \tfrac{1}{2}\sum_q (-1)^{q+1} q \log \det{}_{\mathcal{N}}(\Delta_q^\omega)$$

and the Witten deformation of the analytic torsion $T_{\mathrm{an}}(M,g,\mathcal{W},\omega)$

$$\log T_{\mathrm{an}}(M,g,\mathcal{W},\omega)(t) := \log T_{\mathrm{an}}(M,g,\mathcal{W},t\omega) \ . \tag{4.11}$$

If $(M,g,\mathcal{W})$ is of *a-determinant* class and $\omega{=}dh$ then $\log T_{\mathrm{an}}(M,g,\mathcal{W},t\omega) \in \mathbf{R} \subset \mathbf{D}$, for any t. Indeed consider the functions $F_{\underline{d}_k^*(t)\underline{d}_k(t)}(\lambda)$ defined as above by replacing d_k with $d_k(t)$. As $(L_2(\Lambda^k(M;\mathcal{E})), d_k)$ and $(L_2(\Lambda^k(M;\mathcal{E})), d_k(t))$ are isomorphic, one concludes from Proposition 1.18, that $F_{\underline{d}_k^*(t)\underline{d}_k(t)}(\lambda) \overset{d}{\sim} F_k(\lambda)$ and thus, by Lemma 1.20, $\Delta_k(t)$ is of determinant class iff Δ_k is.

4.3 Product and sum formulas. For $i = 1,2$, let $\mathcal{A}_i$ be finite von Neumann algebras, (M_i, g_i, τ_i) closed Riemannian manifolds of dimension d_i (even or odd), equipped with the generalized triangulations $\tau_i = (h_i, g_i')$. Let $\mathcal{W}_i$ be $(\mathcal{A}_i, \Gamma_i^{\mathrm{op}})$-Hilbert modules of finite type $\Gamma_i = \pi_1(M_i)$, and $\omega_i \in \Lambda^1(M_i)$ closed 1-forms ($i = 1,2$). Introduce $\mathcal{A} := \mathcal{A}_1 \otimes \mathcal{A}_2$, $\mathcal{W} := \mathcal{W}_1 \otimes \mathcal{W}_2$, $M = M_1 \times M_2$, $g = g_1 \times g_2$, $\tau := (h := h_1 \circ p_1 + h_2 \circ p_2, g' = g_1' \times g_2')$ and $\omega = p_1^*(\omega_1) + p_2^*(\omega_2)$, where $p_j : M_1 \times M_2 \to M_j$ are the canonical projections. Further denote by $\mathcal{E} \to M$ and $\mathcal{E}_i \to M_i$ ($i = 1,2$) the bundles associated to $\mathcal{W}_i$ and $\mathcal{W}$.

PROPOSITION 4.2 (Product formula) (cf. [CM], [Lo], [LüRo]). *With the hypotheses above, the following identities, when viewed in* $\mathbf{D}$, *hold:*

$$\begin{aligned}\log T_{\mathrm{an}}(M, g, \mathcal{W}, \omega) &= \chi(M_1; \mathcal{W}_1) \cdot \log T_{\mathrm{an}}(M_2, g_2, \mathcal{W}_2, \omega_2)\\ &\quad + \chi(M_2; \mathcal{W}_2) \cdot \log T_{\mathrm{an}}(M_1, g_1, \mathcal{W}_1, \omega_1) \ ; \qquad (4.12)\\ \log T_{\mathrm{Re}}(M, g, \mathcal{W}, \tau) &= \chi(M_1; \mathcal{W}_1) \cdot \log T_{\mathrm{Re}}(M_2, g_2, \mathcal{W}_2, \tau_2)\\ &\quad + \chi(M_2; \mathcal{W}_2) \cdot \log T_{\mathrm{Re}}(M_1, g_1, \mathcal{W}_1, \tau_1) \ . \qquad (4.13)\end{aligned}$$

Proof. Formula (4.13) follows from Corollary 1.22 and Proposition 1.21. To prove (4.12) observe that

$$L_2(\Lambda^r(M, \mathcal{E})) = \oplus_{p+q=r} L_2(\Lambda^p(M_1, \mathcal{E}_1)) \otimes L_2(\Lambda^q(M_2, \mathcal{E}_2))$$

and note that $\Delta_q = \oplus_{p+r=q} \Delta_{(p,r)}$ with

$$\Delta_{(p,r)} = (\Delta_p' \otimes \mathrm{Id}) + (\mathrm{Id} \otimes \Delta_r'')$$

is an $\mathcal{A}$-linear, elliptic, differential operator where Δ_p' and Δ_r'' denote the Laplacians corresponding to $\mathcal{E}_1 \to M_1$, respectively, $\mathcal{E}_2 \to M_2$. Notice that $e^{-t\Delta_{p,r}} = e^{-t\Delta_p'} \otimes e^{-t\Delta_r''}$ is of trace class in the von Neumann sense. As in (1.30) introduce

$$\zeta_M(\lambda, s) := \tfrac{1}{2} \sum_{q \geq 1} (-1)^q q \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \operatorname{tr}_N e^{-t(\Delta_q + \lambda)} dt \ . \qquad (4.14)$$

First notice that it follows from a theorem of de Rham type due to Dodziuk [Do] that $\chi(M_i; \mathcal{W}_i) = \sum (-1)^q \dim_N(\overline{\mathcal{H}}_q(M_i; \mathcal{E}_i))$, $i = 1,2$. In view of Proposition 1.21(3), to prove (4.12) it then suffices to verify that for $\lambda > 0$

$$\zeta_M(\lambda, s) = \zeta_{M_1}(\lambda, s) \cdot \chi(M_2; \mathcal{W}_2) + \zeta_{M_2}(\lambda, s) \cdot \chi(M_1; \mathcal{W}_1) \ . \qquad (4.15)$$

Following the line of arguments of the proof of Proposition 1.21 it remains to show that

$$\operatorname{tr}_N e^{-t\Delta_{q+1}^+} = \operatorname{tr}_N e^{-t\Delta_q^-} \qquad (4.16)$$

where $\Delta_q^\pm$ denote the restrictions of Δ_q to $\Lambda^{\pm,q}(M;\mathcal{E})$. Here

$$\Lambda^{+,q}(M;\mathcal{E}) = closure\big(d_{q-1}\Lambda^{q-1}(M;\mathcal{E})\big) \ ,$$
$$\Lambda^{-,q}(M;\mathcal{E}) = closure\big(d_q^*\Lambda^{q+1}(M;\mathcal{E})\big) \ , \tag{4.17}$$

where the word closure refers to the closure with respect to the C^∞ topology. The operator d_q maps the space $\Lambda^{-,q}(M;\mathcal{E})$ injectively onto a dense subspace of $\Lambda^{+,q+1}(M;\mathcal{E})$. As before denote by $\underline{d}_q$ the restriction of d_q to $\Lambda^{-,q}(M,\mathcal{E})$. Then

$$A_q := \underline{d}_q(\underline{d}_q^*\underline{d}_q)^{-1/2} : L_2\big(\Lambda^{-,q}(M;\mathcal{E})\big) \to L_2\big(\Lambda^{+,q+1}(M;\mathcal{E})\big)$$

is an isometry and intertwines Δ_q^- with Δ_{q+1}^+, $\Delta_{q+1}^+ A_q = A_q\Delta_q^-$. Thus Δ_q^- and Δ_{q+1}^+ are isospectral and, therefore ($\lambda \in \mathbf{R}$)

$$N_q^-(\lambda) = N_{q+1}^+(\lambda) \tag{4.18}$$

where $N_q^\pm(\lambda)$ are the spectral distribution functions of $\Delta_q^\pm$. By the functional calculus, equation (4.16) follows from (4.18). □

Let $\mathcal{A}$ be a finite von Neumann algebra, (M,g) a closed Riemannian manifold equipped with a generalized triangulation τ, $\mathcal{W}_i$, $i = 1,2$, two $(\mathcal{A},\Gamma^{op})$-Hilbert modules of finite type ($\Gamma = \pi_1(M)$), and $\omega \in \Lambda^1(M)$ a closed 1-form. Introduce $\mathcal{W} := \mathcal{W}_1 \otimes \mathcal{W}_2$, and denote by $\mathcal{E} \to M$ and $\mathcal{E}_i \to M$ ($i = 1,2$) the bundles associated to $\mathcal{W}$ and $\mathcal{W}_i$.

PROPOSITION 4.3 (Sum formula). *With the hypotheses above the following identities, when viewed in* $\mathbf{D}$, *hold:*

$$\log T_{\text{an}}(M,g,\mathcal{W},\omega) = \log T_{\text{an}}(M,g,\mathcal{W}_1,\omega) + \log T_{\text{an}}(M,g,\mathcal{W}_2,\omega) \ ;$$
$$\log T_{\text{Re}}(M,g,\mathcal{W},\tau) = \log T_{\text{Re}}(M,g,\mathcal{W}_1,\tau) + \log T_{\text{Re}}(M,g,\mathcal{W}_2,\tau) \ .$$

Proof. Both equalities follow immediately from the fact that, for $0 \le q \le d$,

$$\big(\Lambda^q(M,\mathcal{E}),\ {}_{\mathcal{W}}d_q^\omega\big) = \big(\Lambda^q(M,\mathcal{E}_1),\ {}_{\mathcal{W}_1}d_q^\omega\big) \oplus \big(\Lambda^q(M,\mathcal{E}_2),\ {}_{\mathcal{W}_2}d_q^\omega\big)$$

and

$$\big(\mathcal{C}^q(M,\tau,O_h,\mathcal{W}),\ {}_{\mathcal{W}}\delta_q\big)$$
$$= \big(\mathcal{C}^q(M,\tau,O_h,\mathcal{W}_1),\ {}_{\mathcal{W}_1}\delta_q\big) \oplus \big(\mathcal{C}^q(M,\tau,O_h,\mathcal{W}_2),\ {}_{\mathcal{W}_2}\delta_q\big) \ . \qquad \square$$

5. Witten's Deformation of the de Rham Complex

5.1 The small subcomplex of the deformed de Rham complex. Assume that (M,g) is a closed Riemannian manifold and let $h : M \to \mathbf{R}$ be a Morse function, so that $\tau = (h,g)$ is a generalized triangulation. Let $\mathcal{W}$ be a $(\mathcal{A},\Gamma^{op})$-Hilbert module of finite type with $\Gamma = \pi_1(M)$. To simplify the exposition we assume throughout this subsection that $\mathcal{W}$ is a free

$\mathcal{A}$-Hilbert module (regarding this assumption cf. Proposition 5.6). In fact all statements can be formulated and proven by the same arguments without the free hypothesis on $\mathcal{W}$. Denote by $\mathcal{E} \to M$ the bundle of $\mathcal{A}$-Hilbert modules associated to $\mathcal{W}$. Let $x_{q;j} \in \mathrm{Cr}_q(h)$ be a critical point of index q and U_{qj} an open neighborhood of $x_{q;j}$.

DEFINITION 5.1. *U_{qj} is said to be an H-neighborhood of $x_{q;j}$ if there is a ball $B_{2\alpha} := \{x \in \mathbf{R}^d; |x| < 2\alpha\}$ and diffeomorphisms $\phi : B_{2\alpha} \to U_{qj}$ and $\Phi : B_{2\alpha} \times \mathcal{W} \to \mathcal{E}|_{U_{qj}}$ covering ϕ with the following properties:*

(i) $\phi(0) = x_{q;j}$;

(ii) *when expressed in the coordinates provided by ϕ, h is of the form*

$$h(x) = q - (x_1^2 + \ldots + x_q^2)/2 + (x_{q+1}^2 + \ldots + x_d^2)/2 \ ;$$

(iii) *the pullback $\phi^*(g)$ of the Riemannian metric g is the Euclidean metric;*

(iv) *Φ is a trivialization of $\mathcal{E}|_{U_{qj}}$. such pairs $(\phi, \Phi)$are called H-coordinates. For later use we define $U'_{qj} := \phi(B_\alpha)$.*

A collection $(U_x)_{x\in Cr(h)}$ of H-neighborhoods is called a system of H-neighborhoods if, in addition, they are pairwise disjoint.

As in section 4, denote by $\Lambda^q(M;\mathcal{E}) := C^\infty(\mathcal{E} \otimes \Lambda^q(T^*(M)))$ the $\mathcal{A}$-module of smooth q-forms with values in $\mathcal{E}$ and by $L_2(\Lambda^q(M;\mathcal{E}))$ its L_2-completion, which is an $\mathcal{A}$-Hilbert module. We write $\Lambda^q(M;\mathcal{W})$ for $\Lambda^q(M;\mathcal{E})$ when $\mathcal{E} = M \times \mathcal{W}$ is the trivial bundle and $\Lambda^q(M;\mathbf{R})$ for the space of smooth q-forms on M. Consider the Witten Laplacian $\Delta_q(t) : \Lambda^q(M;\mathcal{E}) \to \Lambda^q(M;\mathcal{E})$ associated to h and observe (cf. [HSj1] or [CyFKS, Proposition 11.13]) that

$$\Delta_q(t) = \Delta_q + t^2|\nabla h|^2 + tL_q \tag{5.1}$$

where L_q is a zero'th order differential $\mathcal{A}$-operator on $\Lambda^q(M;\mathcal{E})$, hence given by a bundle endomorphism, and where $|\nabla h|^2$, $\nabla h = \mathrm{grad}_g\, h$, is a scalar valued function on M given by $|\nabla h|^2(x) = \sum_{1\le i,j\le d} g^{ij}(x)\frac{\partial h}{\partial x_i}\frac{\partial h}{\partial x_j}$ with $(g^{ij}(x))$ denoting the inverse of the metric tensor g when expressed in local coordinates. Δ_q denotes the Laplacian on q forms with coefficients in the bundle $\mathcal{E}$ and is a nonnegative, selfadjoint, elliptic differential $\mathcal{A}$-operator. Let $\Lambda^q(M;\mathcal{E})_{\mathrm{sm}}$ be the image (which depends on t) of the spectral projector $Q_q(1,t)$ of $\Delta_q(t)$, corresponding to the interval $(-\infty, 1]$. This space consists of smooth q-forms and is an $\mathcal{A}$-Hilbert module.

The purpose of this subsection is to prove the separation of spectrum property of $\Delta_q(t)$, Proposition 5.2, and therefore obtain, for t sufficiently large, $(\Lambda^q(M;\mathcal{E})_{\mathrm{sm}}, d_q(t))$ as a smooth family of subcomplexes of $(\Lambda^q(M;\mathcal{E}), d_q(t))$ where $d_q(t) = e^{-th}d_qe^{th}$. Related results have been also obtained by Shubin [Sh3]. In subsection 5.2 we will show that a scaled version of this family of subcomplexes converges for $t \to \infty$ to the cochain complex $(\mathcal{C}^*(M,\tau,O_h),\delta_*)$, introduced in section 4. In the case $\mathcal{A} = \mathbb{C}$ and $\mathcal{W} = \mathbb{C}$ this was done by Helffer and Sjöstrand [HSj1]. Their arguments are still valid in the general case. Bismut and Zhang [BZ1] (cf. also [BZ2] for a simplified version) verified this in the case $\mathcal{A} = \mathbb{C}$. In subsection 5.2 we present a proof for an arbitrary finite von Neumann algebra $\mathcal{A}$.

Consider $h_k : \mathbb{R}^d \to \mathbb{R}$ defined by $h_k(x) = k + \frac{1}{2}(-\sum_1^k |x_i|^2 + \sum_{k+1}^d |x_i|^2)$ and denote by $\tilde{\Delta}_q : \Lambda^q(\mathbb{R}^d;\mathbb{C}) \to \Lambda^q(\mathbb{R}^d;\mathbb{C})$ the flat Laplacian on q-forms on $\mathbb{R}^d$ and by $\tilde{\Delta}_{q;k}(t) : \Lambda^q(\mathbb{R}^d;\mathbb{C}) \to \Lambda^q(\mathbb{R}^d;\mathbb{C})$ the Witten Laplacian associated to h_k. A straightforward calculation shows that

$$\tilde{\Delta}_{q;k}(t) = \tilde{\Delta}_q + t^2|x|^2 - t(d-2k) + 2t(N^+_{q;k} - N^-_{q;k}) \ , \tag{5.2}$$

where $N^+_{q;k}$ and $N^-_{q;k}$ are the number operators introduced in [HSj1] (cf. also [BZ1]), defined by

$$N^+_{q;k}(dx_{i_1} \wedge \ldots \wedge dx_{i_q}) = \#\{j | k+1 \le i_j \le d\} dx_{i_1} \wedge \ldots \wedge dx_{i_q}$$

and $N^-_{q;k} := q\mathrm{Id} - N^+_{q;k}$. Denote by $\tilde{\omega}_q(t) \in \Lambda^q(\mathbb{R}^d;\mathbb{C})$ the q-form defined by

$$\tilde{\omega}_q(x,t) := (t/\pi)^{d/4} e^{-t|x|^2/2} dx_1 \wedge \ldots \wedge dx_q \ . \tag{5.3}$$

For $\eta > 0$, let $\nu_\eta : \mathbb{R} \to [0,1]$ be a smooth map equal to 1 on the interval $(-\infty, \eta/2)$ and equal to 0 on the interval (η, ∞). For $\epsilon > 0$, which we will choose later at our convenience, define $\tilde{\psi}_q(t) \in \Lambda^q(\mathbb{R}^d;\mathbb{C})$ by

$$\tilde{\psi}_q(t) := \beta(t)^{-1} \nu_\epsilon(|x|) \tilde{\omega}_q(t) \tag{5.3'}$$

where $\beta(t) = \|\nu_\epsilon(|x|)\tilde{\omega}_q(t)\|$, with $\| \ \|$ denoting the L_2-norm. With respect to the scalar product in $\Lambda^q(\mathbb{R}^d;\mathbb{C})$ induced by the flat metric of $\mathbb{R}^d$, $\langle \tilde{\omega}_k(t), \tilde{\omega}_k(t)\rangle = 1$ and $\langle \tilde{\psi}_k(t), \tilde{\psi}_k(t)\rangle = 1$. Consider $\Delta_q = \tilde{\Delta}_q \otimes \mathrm{Id}$ and $\Delta_{q;k}(t) = \tilde{\Delta}_{q;k}(t) \otimes \mathrm{Id}$, defined on $\Lambda^q(\mathbb{R}^d;\mathcal{W})$. The operators $\Delta_{q,k}(t)$ are nonnegative, essentially selfadjoint, elliptic $\mathcal{A}$-operators with the following properties:

(HO1) $\operatorname{spec} \Delta_{q;k}(t)$ is discrete and contained in $2t\mathbb{Z}_{\geq 0}$; each eigenvalue has infinite multiplicity if $\dim_{\mathbb{C}} \mathcal{A} = \infty$.

(HO2) $\mathrm{Null}(\Delta_{q;k}(t)) = 0$ if $k \neq q$; $\mathrm{Null}(\Delta_{q;q}(t))$ is an $\mathcal{A}$-Hilbert module isometric to $\mathcal{W}$.

(HO3) Assume that $\{v_1, \ldots, v_l\}$ is an orthonormal basis of $\mathcal{W}$ (cf. Definition 1.3), i.e. a collection of regular elements which generate $\mathcal{W}$ as an $\mathcal{A}$-Hilbert module and such that for any $a, b \in \mathcal{A}$,

$$\langle av_i, bv_j\rangle = \langle a, b\rangle \delta_{ij} \ . \tag{5.4}$$

Then $\omega_{q,i}(t) := \tilde{\omega}_q(t) \otimes v_i$, $1 \le i \le l$, is a basis for $\mathrm{Null}(\tilde{\Delta}_{q;q}(t))$ and

$\psi_{q,i}(t) := \tilde{\psi}_q(t) \otimes v_i$, $1 \le i \le l$, is an orthonormal basis for the $\mathcal{A}$-Hilbert submodule generated $\psi_{q,i}(t)$.

A straightforward calculation, using (5.2), (5.3) and (5.3′), shows that there exist constants $t_0 \equiv t_0(\epsilon) > 0$, $C(\epsilon) > 0$ and $C_0(\epsilon) > 0$, so that, for $1 \le i \le l$,

$$\left|\Delta_{q;q}(t)\psi_{q,i}(x,t)\right| \le C_0(\epsilon)e^{-C(\epsilon)t} \qquad (x \in M \;\; t \ge t_0) \tag{5.5}$$

(with similar estimates for all the derivatives of $\psi_{q,i}(t)$)

$$\left\langle \Delta_{q;k}(t)\psi_{q,i}(t), \psi_{q,i}(t)\right\rangle \ge 2t|q-k| \qquad (t \ge 0) \tag{5.6}$$

and, in view of (HO1), for any $\omega \in \Lambda^q(\mathbb{R}^d; \mathcal{W})$ with compact support and $\langle \omega, \psi \rangle = 0$ for ψ in the Hilbert module generated by $\psi_{q;i}(t)$, $1 \le i \le l$,

$$\left\langle \Delta_{q;q}(t)\omega, \omega \right\rangle \ge C(\epsilon)t\|\omega\|^2 \; . \tag{5.7}$$

(Cf. Appendix 2 for a verification of (5.5)-(5.7).)

Let $x \in \mathrm{Cr}_k(h)$ and U_x be an H-neighborhood as in Definition 5.1 and denote by d the distance function on M. Choose $\epsilon > 0$ so that the balls $B(x; 4\epsilon) = \{y \in M; d(x,y) \le 4\epsilon\}$, centered at the critical points x, are pairwise disjoint, and $B(x; 3\epsilon) \subset U_x$. Choose once and for all a base point $x_0 \in M$, an orthonormal basis $e_1, \dots, e_l$ of $\mathcal{E}_{x_0}$, and for each critical point $x = x_{q;j} \in \mathrm{Cr}_q(h)$ a homotopy class $[\gamma_x]$ of paths, joining x_0 and x (choose $\gamma_{x_0} = \{x_0\}$). Denote by $e_{q;j,1}, \dots, e_{q;j,l}$ the orthonormal basis of $\mathcal{E}_x$ obtained by the parallel transport (induced by the canonical flat connection on $\mathcal{E}$) of $e_1, \dots, e_l$ along γ_x. By parallel transport, one can identify $\mathcal{E}|_{U_x}$ with $U_x \times \mathcal{W}$ and, using a system of H-neighborhoods U_x, one can identify the forms $\omega \in \Lambda^q(M; \mathcal{E})$ having support in U_x with forms in $\Lambda^q(\mathbb{R}^d; \mathcal{W})$. In this way, for any $x \in Cr_q(h)$, the element $\psi_{q,i}(t)$ identifies with an element in $\Lambda^q(M; \mathcal{E})$ denoted by $\psi_{x,i}(t)$, with compact support in U_x. Since e_i are regular elements in $\mathcal{E}_{x_0}$, the $\psi_{x,i}(t)$'s are regular elements in $L_2(\Lambda^q(M; \mathcal{E}))$. The forms $\psi_{x,i}(t)$ $(1 \le i \le l,\, x \in \mathrm{Cr}_q(h))$ satisfy (5.4), and therefore provide an orthonormal basis for the $\mathcal{A}$-Hilbert submodule which they generate.

PROPOSITION 5.2. *There exist positive constants C', C'', and t_0 so that for $t \ge t_0$ and $0 \le q \le d$,* $\mathrm{spec}(\Delta_q(t)) \cap (e^{-tC'}, C''t) = \emptyset$.

Proof. In a first step we prove that there exist $t_0 > 0$, $C' > 0$ and $C'' > 0$ so that for $t \ge t_0$ there exists a pair of orthogonal closed subspaces $W_1 = W_1(t)$, $W_2 = W_2(t)$ of $L_2(\Lambda^q(M; \mathcal{E}))$ with $W_1 \subset \Lambda^q(M; \mathcal{E})$ with the following properties (1) $W_1 \cap W_2 = \{0\}$; (2) $W_1 + W_2 = L_2(\Lambda^q(M; \mathcal{E}))$; (3) $\langle \Delta_q(t)\omega, \omega \rangle \le e^{-tC'} \langle \omega, \omega \rangle$ for $\omega \in W_1$; and (4) $\langle \Delta_q(t)\omega, \omega \rangle \ge C''t\langle \omega, \omega \rangle$ for $\omega \in W_2 \cap \Lambda^q(M; \mathcal{E})$.

In a second step we show that, using step 1, Proposition 5.2 follows. Let us prove step 2 first. We claim that Proposition 5.2 holds with the constants t_0, C' and C'' as in step 1. To prove this claim we argue by contradiction. Assume that there exist $0 \leq q \leq d$, $t \geq t_0$ and a real number $\mu \in \operatorname{spec} \Delta_q(t) \cap (e^{-tC'}, C''t)$. Then there exists a sequence $(u_j)_{j \geq 1}$ of approximate eigenfunctions u_j in $\Lambda^q(M; \mathcal{E})$, $\|u_j\| = 1$, satisfying

$$\left\|\Delta_q(t)u_j - \mu u_j\right\| \leq \frac{1}{j} \ . \tag{5.8}$$

Decomposing $u_j = v_j + w_j \in W_1(t) \oplus W_2(t)$, and using the fact that $\Delta_q(t)$ is selfadjoint, one obtains

$$\left\langle \Delta_q(t)u_j, v_j \right\rangle = \left\langle \Delta_q(t)v_j, v_j \right\rangle + \left\langle w_j, \Delta_q(t)v_j \right\rangle \ , \tag{5.9}$$

$$\left\langle \Delta_q(t)u_j, w_j \right\rangle = \left\langle \Delta_q(t)v_j, w_j \right\rangle + \left\langle w_j, \Delta_q(t)w_j \right\rangle \ . \tag{5.9'}$$

Further

$$\mu\|v_j\|^2 = \langle \mu u_j, v_j \rangle = \left\langle \Delta_q(t)u_j, v_j \right\rangle - \left\langle \Delta_q(t)u_j - \mu u_j, v_j \right\rangle \ , \tag{5.9''}$$

$$\mu\|w_j\|^2 = \langle \mu u_j, w_j \rangle = \left\langle \Delta_q(t)u_j, w_j \right\rangle - \left\langle \Delta_q(t)u_j - \mu u_j, w_j \right\rangle \ . \tag{5.9'''}$$

Combine (5.9″) and (5.9) to obtain

$$\mu\|v_j\|^2 - \left\langle \Delta_q(t)v_j, v_j \right\rangle + \left\langle \Delta_q(t)u_j - \mu u_j, v_j \right\rangle = \left\langle w_j, \Delta_q(t)v_j \right\rangle$$

and, similarly, combine (5.9‴) and (5.9′)

$$\mu\|w_j\|^2 - \left\langle \Delta_q(t)w_j, w_j \right\rangle + \left\langle \Delta_q(t)u_j - \mu u_j, w_j \right\rangle = \left\langle \Delta_q(t)v_j, w_j \right\rangle \ .$$

The right-hand sides of the two identities above have the same real part and thus,

$$\mu\|v_j\|^2 - \langle \Delta_q(t)v_j, v_j \rangle = -\,Re\langle \Delta_q(t)u_j - \mu u_j, v_j - w_j \rangle + \mu\|w_j\|^2 - \langle \Delta_q(t)w_j, w_j \rangle \ .$$

Using (3), (4) and (5.8) this leads to, with $\|w_j\|^2 = 1 - \|v_j\|^2 \leq 1$,

$$(\mu - e^{-tC'})\|v_j\|^2 \leq \frac{2}{j} + (\mu - C''t)(1 - \|v_j\|^2) \ .$$

Without loss of generality we may assume that $\lim_{j \to \infty} \|v_j\|^2 = \alpha^2$ exists. Then

$$(\mu - e^{-tC'})\alpha^2 \leq (\mu - C''t)(1 - \alpha^2)$$

which contradicts the assumption $e^{-tC'} < \mu < C''t$.

It remains to prove step 1. Define $W_1 := W_1(t)$ to be the $\mathcal{A}$-Hilbert module generated by $\psi_{x,i}(t)$ $(1 \leq i \leq l,\ x \in \mathrm{Cr}_q(h))$ and $W_2 := W_2(t)$ its orthogonal complement in $L^2(\Lambda^q(M; \mathcal{E}))$. Clearly properties (1) and (2) are satisfied. Further note that the space $V_1 \equiv V_1(t)$ of elements of the form $\omega = \sum_{1 \leq i \leq l, x \in \mathrm{Cr}_q(h)} a_{x,i} \psi_{x,i}(t)$ with $a_{x,i} \in \mathcal{A}$ is dense in $\mathcal{W}_1$. Note also that $\Delta_q(t)$, when restricted to U_x with $x \in \mathrm{Cr}_k(h)$ and expressed in local coordinates considered in Definition 5.1, coincides with $\Delta_{q;k}(t)$. Therefore, as $\Delta_q(t)$ is $\mathcal{A}$-linear and in view of (5.4) and the disjointness of the supports of $\psi_{x,i}$ and $\psi_{y,j}$ for $x \neq y \in Cr_q(h)$ we conclude from (5.5) with $C = C(\epsilon)$ and $t_0 = t_0(\epsilon)$ as in (5.5), for $t \geq t_0$ and $\omega \in V_1(t)$,

$$\begin{aligned}\big\langle \Delta_q(t)\omega, \omega\big\rangle &= \sum_{1\leq i\leq l; x\in \mathrm{Cr}_q(h)} \big\langle a_{x,i}\Delta_q(t)\psi_{x,i}(t), a_{x,i}\psi_{x,i}(t)\big\rangle \\ &\leq C\sum_{i,x} \|a_{x,i}\|^2 e^{-tC} \leq C\|\omega\|^2 e^{-tC} \ .\end{aligned}$$

By choosing t_0 bigger, if necessary, (3) follows.

It remains to check the estimate (4). Denote by $\chi_x : M \to \mathbf{R}$ the smooth nonnegative cut-off function with support in U_x defined by $\nu_{2\epsilon}$ (cf. (5.3′)) and introduce $\chi := \sum_{x\in Cr(h)} \chi_x$. For $\omega \in W_2 \cap \Lambda^q(M;\mathcal{E})$, define $\omega_1 := \chi\omega$ and $\omega_2 := (1-\chi)\omega$ and observe that the support of ω_2 is disjoint from the support of any element in W_1; therefore $\omega_2 \in W_2 \cap \Lambda^q(M;\mathcal{E})$ hence $\omega_1 \in W_2 \cap \Lambda^q(M;\mathcal{E})$. Since $\Delta_q(t)$ is essentially selfadjoint one obtains

$$\big\langle \Delta_q(t)\omega, \omega\big\rangle = \big\langle \Delta_q(t)\omega_1, \omega_1\big\rangle + 2\,\mathrm{Re}\big\langle \Delta_q(t)\omega_1, \omega_2\big\rangle + \big\langle \Delta_q(t)\omega_2, \omega_2\big\rangle \ . \quad (5.10)$$

We show that there exist positive constants t_0, C_1, C_2, C_3, C_4 depending only on the geometry of $(M, \mathcal{E} \to M)$ and the chosen ϵ, so that for any $\omega \in W_2 \cap \Lambda^q(M;\mathcal{E})$ and $t > t_0$ the following estimates, (5.11)-(5.14) hold:

$$\big\langle \Delta_q(t)\omega_2, \omega_2\big\rangle \geq \langle \Delta_q\omega_2, \omega_2\rangle + C_1t^2\|\omega_2\|^2 - C_2t\|\omega_2\|^2 \ ; \quad (5.11)$$

$$\big\langle \Delta_q(t)\omega_1, \omega_1\big\rangle \geq C_3t\|\omega_1\|^2 \ ; \quad (5.12)$$

$$\big\langle \Delta_q(t)\omega_1, \omega_1\big\rangle \geq \langle \Delta_q\omega_1, \omega_1\rangle - C_2t\|\omega_1\|^2 \ . \quad (5.13)$$

For any $\alpha > 0$,

$$\begin{aligned}\mathrm{Re}\big\langle \Delta_q(t)\omega_1, \omega_2\big\rangle \geq &-C_4(1+\alpha^{-2})\big(\|\omega_1\|^2 + \|\omega_2\|^2\big) \\ &- C_4\alpha^2\big\langle \Delta_q\omega_2, \omega_2\big\rangle - C_4\alpha^2\langle \Delta_q\omega_1, \omega_1\rangle \ .\end{aligned} \quad (5.14)$$

For any $0 \leq \delta \leq 1$, multiply (5.12) by $1-\delta$ and (5.13) by δ and take the sum to get

$$\big\langle \Delta_q(t)\omega_1, \omega_1\big\rangle \geq (1-\delta)\langle \Delta_q\omega_1, \omega_1\rangle + t\big(\delta C_3 - (1-\delta)C_2\big)\|\omega_1\|^2 \ . \quad (5.15)$$

To complete the proof of property (4) combine (5.10) with the estimates (5.15), (5.14), and (5.11) to obtain (for $0 < \delta < 1$, $\alpha > 0$)

$$\begin{aligned}\big\langle \Delta_q(t)\omega, \omega\big\rangle \geq &(1-2C_4\alpha^2)\langle \Delta_q\omega_2, \omega_2\rangle + (1-\delta-2C_4\alpha^2)\langle \Delta_q\omega_1, \omega_1\rangle \\ +\big(C_1t^2 - C_2t - 2C_4(1+\alpha^{-2})\big)\|\omega_2\|^2 &+ \big(t(\delta C_3 - (1-\delta)C_2) - 2C_4(1+\alpha^{-2})\big)\|\omega_1\|^2 \ .\end{aligned}$$

First choose $0 < \delta < 1$ sufficiently close to 1 so that $C_6 := \delta C_3 - (1-\delta)C_2 > 0$. Then choose $\alpha > 0$ sufficiently small so that $1 - \delta - 2C_4\alpha^2 > 0$. With these choices we obtain

$\langle\Delta_q(t)\omega,\omega\rangle \geq (C_1t^2-C_2t-4C_4(1+\alpha^{-2}))\|\omega_2\|^2+(tC_6-4C_4(1+\alpha^{-2}))\|\omega_1\|^2$.

Together with $2(\|\omega_1\|^2+\|\omega_2\|^2)\geq\|\omega\|^2$ this establishes property (4).

To prove (5.11) choose $C_1 := \inf_{z\in M\setminus\cup_{x\in\mathrm{Cr}(h)}U_x}|\nabla h(z)|^2$ and $C_2 = \sup_{x\in M}\|L_q(x)\|$. The estimate (5.11) then follows from (5.1).

To prove (5.12) it suffices to notice that the support of ω_1 is contained in $\cup_{x\in\mathrm{Cr}(h)}U_x$ and ω_1 is orthogonal to $\psi_{x,i}(x\in\mathrm{Cr}_q(h), 1\leq i\leq l)$. Thus (5.12) follows from (5.7) with $C_3 := C_0(\epsilon)$.

Formula (5.13) is a direct consequence of (5.1).

To find the lower bound (5.14) note that $|\operatorname{Re}\langle L_q\omega_1,\omega_2\rangle|\leq C_2|\operatorname{Re}\langle\omega_1,\omega_2\rangle| = C_2|\langle\omega_1,\omega_2\rangle|$ and, using that $\mathrm{supp}(\omega_2)$ does not intersect any of the U_x's, $\langle|\nabla h|^2\omega_1,\omega_2\rangle\geq C_1\langle\chi(1-\chi)\omega,\omega\rangle\geq 0$. Combining with (5.1) one concludes that $\langle\Delta_q(t)\omega_1,\omega_2\rangle = \langle\Delta_q\omega_1,\omega_2\rangle + t^2\langle|\nabla h|^2\omega_1,\omega_2\rangle + t\langle L_q\omega_1,\omega_2\rangle$ can be estimated

$$\operatorname{Re}\langle\Delta_q(t)\omega_1,\omega_2\rangle \geq \operatorname{Re}\langle\Delta_q\omega_1,\omega_2\rangle + (C_1t^2-C_2t)\langle\omega_1,\omega_2\rangle\ .$$

As $\langle\omega_1,\omega_2\rangle = \langle\chi\omega,(1-\chi)\omega\rangle$ is real and nonnegative, one thus obtains for $t > C_2/C_1$

$$\operatorname{Re}\langle\Delta_q(t)\omega_1,\omega_2\rangle \geq \operatorname{Re}\langle\Delta_q\omega_1,\omega_2\rangle\ . \tag{5.16}$$

Therefore, the lower bound (5.14) follows from Lemma 5.3 below together with $2(\|\omega_1\|^2+\|\omega_2\|^2)\geq\|\omega\|^2$. □

LEMMA 5.3. *Let the q-forms ω, ω_1 and ω_2 be defined as above. Then there exists a constant $C_4>0$ so that, for any $\alpha>0$,*

$$\operatorname{Re}\langle\Delta_q\omega_1,\omega_2\rangle \geq -C_4(1+\alpha^{-2})\|\omega\|^2 - C_4\alpha^2\langle\Delta_q\omega_2,\omega_2\rangle - C_4\alpha^2\langle\Delta_q\omega_1,\omega_1\rangle\ . \tag{5.17}$$

Proof. Write $\Delta_q = d_{q-1}d^*_{q-1}+d^*_qd_q$ where $d^*_{q-1} = -(-1)^{dq+d+1}R_{d-q+1}d_{d-q}R_q$, and $* = R_q$ denotes the Hodge $*$ operator. Using that $\omega_1=\chi\omega$ and $\omega_2=(1-\chi)\omega$ one obtains

$$\begin{aligned}\langle\Delta_q\omega_1,\omega_2\rangle = \langle d\omega_1,d\omega_2\rangle + \langle d*\omega_1,d*\omega_2\rangle &\geq A+B-\|d\chi\wedge\omega\|^2-\|d\chi\wedge *\omega\|^2\\ &+\langle\chi d\omega,(1-\chi)d\omega\rangle + \langle\chi d*\omega,(1-\chi)d*\omega\rangle\ ,\end{aligned}$$

where

$$\begin{aligned}A &:= \langle d\chi\wedge\omega, u(1-\chi)d\omega\rangle + \langle d\chi\wedge *\omega, u(1-\chi)d*\omega\rangle\ , \qquad (5.18)\\ B &:= -\langle\chi d\omega, ud\chi\wedge\omega\rangle - \langle\chi d*\omega, ud\chi\wedge *\omega\rangle\ ,\end{aligned}$$

where u is the characteristic function of $M\setminus\operatorname{supp}\chi$. Notice that $\langle\chi d\omega,(1-\chi)d*\omega\rangle$ are real and nonnegative and therefore

$$\operatorname{Re}\langle\Delta_q\omega_1,\omega_2\rangle \geq \operatorname{Re}A + \operatorname{Re}B - \|d\chi\wedge\omega\|^2 - \|d\chi\wedge *\omega\|^2\ .$$

In order to estimate the expressions A and B we introduce the constant $C_5 := \sup_{1\leq k\leq d} \|K_k\|$, where $K_k : L_2(\Lambda^k(M;\mathcal{E})) \to L_2(\Lambda^{k+1}(M;\mathcal{E}))$ is the exterior multiplication by $d\chi$. Note that $\|K_k\| = \|K_k^*\|$ where K_k^* denotes the adjoint of K_k and $\|\omega\| = \| * \omega\|$. A straightforward calculation yields

$$\begin{aligned}|A| &\leq C_5\|\omega\|\big(\|(1-\chi)d\omega\| + \|(1-\chi)d * \omega\|\big)\\ &\leq C_5\|\omega\|\big(\|d\omega_2\| + \|d\chi \wedge \omega\| + \|d * \omega_2\| + \|d\chi \wedge *\omega\|\big)\\ &\leq C_5\|\omega\|\big(\|d\omega_2\| + \|d * \omega_2\| + 2C_5\|\omega\|\big)\\ &\leq \sqrt{2}C_5\|\omega\|\langle\Delta_q\omega_2, \omega_2\rangle^{1/2} + 2C_5^2\|\omega\|^2 \ .\end{aligned}$$

Thus for any $\alpha > 0$, and in view of the inequality $bc \leq (b/\alpha)^2 + (\alpha c)^2$ one obtains

$$|A| \leq (2C_5^2 + C_5\alpha^{-2})\|\omega\|^2 + \alpha^2\langle\Delta_q\omega_2, \omega_2\rangle \ . \tag{5.19}$$

A similar computation leads to

$$|B| \leq (2C_5^2 + C_5\alpha^{-2})\|\omega\|^2 + \alpha^2\langle\Delta_q\omega_1, \omega_1\rangle \ . \tag{5.20}$$

Choosing C_4 appropriately leads to the claimed statement. □

5.2 Asymptotic properties of the small subcomplex. In this subsection we require that $\mathcal{W}$ is a free $\mathcal{A}$-Hilbert module. Recall that Proposition 5.2 yields, for t sufficiently large, a decomposition of $(\Lambda^q(M;\mathcal{E}), d_q(t))$

$$\big(\Lambda^q(M;\mathcal{E}), d_q(t)\big) = \big(\Lambda^q(M;\mathcal{E})_{\rm sm}, d_q(t)\big) \oplus \big(\Lambda^q(M;\mathcal{E})_{\rm la}, d_q(t)\big) \ , \tag{5.21}$$

where $\Lambda^q(M;\mathcal{E})_{\rm sm} \equiv \Lambda^q(M;\mathcal{E})_{\rm sm}(t)$ is the image of $Q(1,t)$, the spectral projection of $\Delta_q(t)$ corresponding to the interval $(-\infty, 1]$, and $\Lambda^q(M;\mathcal{E})_{\rm la}$ denotes the orthogonal complement of $\Lambda^q(M;\mathcal{E})_{\rm sm}$. Accordingly, one can decompose $\Delta_q(t) = \Delta_q(t)_{\rm sm} + \Delta_q(t)_{\rm la}$ where $\Delta_q(t)_{\rm sm}$, resp. $\Delta_q(t)_{\rm la}$, denotes the restriction of $\Delta_q(t)$ to $\Lambda^q(M;\mathcal{E})_{\rm sm}$, resp. the restriction to $\Lambda^q(M;\mathcal{E})_{\rm la}$.

Using the forms $\psi_{x,i}(t)$, introduced in subsection 5.1, we will construct, following Helffer and Sjöstrand (cf. [HSj1] or [BZ1]), an orthonormal base (in the sense of Definition 1.3), $\varphi_{q;j,i}(t)$ $(1 \leq j \leq m_q, 1 \leq i \leq l)$ of the $\mathcal{A}$-Hilbert module $\Lambda^q(M;\mathcal{E})_{\rm sm}(t)$.

Since the $\psi_{x,i}(t)$ are regular elements of the $\mathcal{A}$-Hilbert module $L_2(\Lambda^q(M;\mathcal{E}))$, and satisfy (5.4), the map

$$J_q(t)\Bigg(\sum_{x\in Cr_q(x),i} a_{x,i}e_{x,i}\Bigg) := \sum_{x,i} a_{x,i}\psi_{x,i}(t) \tag{5.22}$$

$(e_{x_{q;j},i} \equiv e_{q;j,i})$ extends to a bounded $\mathcal{A}$-linear isometric embedding

$$J_q(t) : \mathcal{C}^q = \bigoplus_{x\in {\rm Cr}_q(h)} \mathcal{E}_x \to L_2\big(\Lambda^q(M;\mathcal{E})\big) \ . \tag{5.23}$$

The following proposition is a generalization of [BZ1, Theorem 8.8, p. 128].

PROPOSITION 5.4. *For $\epsilon > 0$ (in (5.3′)) sufficiently small, there exists a constant $c > 0$, so that*

$$\left\|(Q_q(1,t)J_q(t)v - J_q(t)v)(y)\right\| = O(e^{-ct})\|v\| \tag{5.24}$$

uniformly for $y \in M$ and $v \in \mathcal{C}^q$. Similar estimates hold for the derivatives.

Proof. We proceed as in [BZ1, p. 128]. In view of Proposition 5.2, for $t \geq t_0$, $Q_q(1,t)$ is given by the Riesz projector

$$Q_q(1,t) = \frac{1}{2\pi i}\int_{S^1} \left(\lambda - \Delta_q(t)\right)^{-1} d\lambda \tag{5.25}$$

where S^1 is the unit circle in $\mathbb{C}$, centered at the origin and where $(\lambda - \Delta_q(t))^{-1}$ is the resolvent of $\Delta_q(t)$. The operator $Q_q(1,t)J_q(t) - J_q(t)$ can therefore be represented by a Cauchy integral whose integrand is given by

$$\left(\lambda - \Delta_q(t)\right)^{-1} J_q(t) - \lambda^{-1} J_q(t) = \lambda^{-1}\left(\lambda - \Delta_q(t)\right)^{-1} \Delta_q(t) J_q(t)\ . \tag{5.26}$$

By (5.5), for any Sobolev norm $\|\cdot\|_{2r}$, with r a nonnegative integer, there exists a constant $c_{2r} > 0$ such that

$$\left\|\Delta_q(t)J_q(t)(v)\right\|_{2r} = O(e^{-c_{2r}t})\|v\|\ , \tag{5.27}$$

uniformly in $v \in \mathcal{C}^q$. We write $\|\cdot\|$ instead of $\|\cdot\|_0$. To estimate $(\lambda - \Delta_q(t))^{-1}$ first notice that, by the ellipticity of $\Delta_q = \Delta_q(0)$, there exists c'_{2r} so that

$$\|u\|_{2r} \leq c'_{2r}\left(\|\Delta_q u\|_{2r-2} + \|u\|\right) \tag{5.28}$$

for $u \in \Lambda^q(M;\mathcal{E})$. By (5.1) there exists c''_{2r} so that for $\lambda \in S^1$ and $t \geq 1$

$$\left\|(\lambda - \Delta_q(t) + \Delta_q)u\right\|_{2r} \leq c''_{2r} t^2 \|u\|_{2r}\ . \tag{5.29}$$

Combining (5.28) and (5.29) and using that $\|u\|_k \leq \|u\|_{k+1}$, one concludes that there exist C'_{2r} and C_{2r} so that for $u \in \Lambda^q(M;\mathcal{E})$

$$\|u\|_{2r} \leq C'_{2r}\left(\|(\lambda - \Delta_q(t))u\|_{2r-2} + t^2\|u\|_{2r-2}\right)\ ,$$

and, by iterating the estimate and using that $\|u\|_k \leq \|u\|_{k+1}$,

$$\|u\|_{2r} \leq C_{2r} t^{2r}\left(\|(\lambda - \Delta_q(t))u\|_{2r-2} + \|u\|\right)\ . \tag{5.30}$$

We want to apply the estimate (5.30) for $u = (\lambda - \Delta_q(t))^{-1}\tilde{u}$ with $\tilde{u} \in \Lambda^q(M;\mathcal{E})$. To this end we observe that, by Proposition 5.2, there exists $t_0 > 0$ so that $(\lambda - \Delta_q(t))^{-1}$ is a L_2-bounded operator for $t \geq t_0$ uniformly in $\lambda \in S^1$, i.e. there exists $C''' > 0$ so that for $\lambda \in S^1$, $t \geq t_0$

$$\left\|(\lambda - \Delta_q(t))^{-1}\tilde{u}\right\| \leq C'''\|\tilde{u}\|\ . \tag{5.31}$$

As $(\lambda - \Delta_q(t))^{-1}\tilde{u} \in \Lambda^q(M;\mathcal{E})$, for $\tilde{u} \in \Lambda^q(M;\mathcal{E})$ we can apply (5.30) to find $C_r > 0$ so that for t sufficiently large and $\lambda \in S^1$

$$\left\|(\lambda - \Delta_q(t))^{-1}\tilde{u}\right\|_{2r} \leq C_r t^{2r}\|\tilde{u}\|_{2r-2}\ . \tag{5.32}$$

Combining (5.32) and (5.27), one sees that for any $0 < c' < c_{2r}$ there exists $t_0 > 0$ so that for $t \geq t_0$ and $v \in \mathcal{C}^q$,

$$\left\|(\lambda - \Delta_q(t))^{-1}\Delta_q(t)J_q(t)v\right\|_{2r} = O(e^{-c't})\|v\| . \tag{5.33}$$

Choose an integer $r > d/2$ and use the Sobolev embedding theorem for $\mathcal{E}$-valued q forms to obtain (5.24) from (5.26) and (5.33). By choosing r even larger one obtains similar estimates for the derivatives. □

Proposition 5.4 insures that, for sufficiently small $\epsilon > 0$, there exists t_0 so that for $t \geq t_0$, $Q_q(1,t)J_q(t)$ is an isomorphism of $\mathcal{C}^q$ onto $Y(t) := Q_q(1,t)J_q(t)(\mathcal{C}^q)$ with $Y(t)$ a closed subspace and $\mathcal{A}$-submodule of $\Lambda^q(M;\mathcal{E})_{\rm sm}$. We claim that, for t large enough, $Y(t) = \Lambda^q(M;\mathcal{E})_{\rm sm}$. To verify this we will show that $u \in \Lambda^q(M;\mathcal{E})_{\rm sm}$ and u orthogonal to $Y(t)$ imply $u = 0$. Indeed, since $Q_q(1,t)$ is a selfadjoint projector and $Q_q(1,t)(u) = u$, we have ($x \in Cr_q(h)$, $1 \leq i \leq l$)

$$\langle \psi_{x,i}(t), u\rangle = \langle Q_q(1,t)(\psi_{x,i}(t)), u\rangle = 0 .$$

Then, by (5.7), $\langle\Delta_q(t)u, u\rangle \geq C(\epsilon)t\|u\|^2$. On the other hand, $\langle\Delta_q(t)u,u\rangle \leq \|u\|^2$ because $u \in \Lambda^q(M;\mathcal{E})_{\rm sm}$ and hence $u = 0$. Let $i_q(t) := Q_q(1,t)J_q(t)$. Then, for t sufficiently large, $Q_q(1,t)(\psi_{x,i}) = i_q(t)e_{q;j,i}$ ($x = x_{q;j} \in Cr_q(h)$, $1 \leq i \leq l$) is a basis of $\Lambda^q(M;\mathcal{E})_{\rm sm}$. By the remark following Definition 1.3, this basis can be used to obtain an orthonormal basis

$$\varphi_{q;j,i}(t) := i_q(t)\big(i_q(t)^* i_q(t)\big)^{-1/2} i_q(t)(e_{q;j,i}) .$$

The main properties of the base $\varphi_{q;j,i}(t)$ are stated in Theorem 5.5 below. To formulate Theorem 5.5 we need some additional definitions.

Consider the cochain complex $\mathcal{C}(M,\tau,O_h,\mathcal{W})$, which has been introduced in section 4, and define the orthonormal base of $\mathcal{C}^q$, $E_{q;j,i}$ with $1 \leq j \leq m_q$ and $1 \leq i \leq l$, by

$$E_{q;j,i}(x_{q;j'}) = \begin{cases} e_{q;j,i} & \text{if } j' = j \\ 0 & \text{if } j' \neq j . \end{cases}$$

With respect to this basis the differential δ_q can be written as

$$\delta_q(E_{q;j,i}) = \sum_{1 \leq j' \leq m_{q+1}, 1 \leq i' \leq l} \gamma_{q;ji,j'i'} E_{q+1;j',i'}$$

where $\gamma_{q;ji,j'i'} \in \mathcal{A}^{\rm op}$. If one identifies $\mathcal{C}^q$ to $\sum_{x \in Cr_q(h)} \mathcal{E}_x$ then the elements $E_{q;j,i}$ correspond to $e_{q;j,i}$.

Introduce the $\mathcal{A}$-linear maps $f_k(t) : \Lambda^k(M;\mathcal{E})_{\rm sm} \to \mathcal{C}^k$ defined by

$$f_k(t) = \left(\left(\frac{\pi}{t}\right)^{\frac{d-2k}{4}} e^{-tk}\right) \operatorname{Int}^{(k)} e^{th} , \tag{5.34}$$

where $\operatorname{Int}^{(k)} : \Lambda^k(M;\mathcal{E}) \to \mathcal{C}^k$ is the integration (considered in section 4) on the k-cells of the generalized triangulation τ. The k-cells of this generalized triangulation are the unstable manifolds $W^-_{k;j}$. Recall that the closure of $W^-_{k;j}$ is a compact smooth manifold with conical singularities (cf. Appendix by F. Laudenbach in [BZ1]) and therefore the integration makes sense. The maps $\mathrm{Int}^{(k)}$ are continuous, $\mathcal{A}$-linear and satisfy $\mathrm{Int}^{(k+1)}d_k = \delta_k \mathrm{Int}^{(k)}$. Consequently

$$\{f_k(t)\} : (\Lambda^k(M;\mathcal{E})_{\mathrm{sm}}, \tilde{d}_k(t)) \to (\mathcal{C}^k, \delta_k) ,$$

where

$$\tilde{d}_k(t) := e^t \left(\frac{t}{\pi}\right)^{-1/2} d_k(t) ,$$

and $d_k(t) = e^{-th} d_k e^{th}$, is a morphism of $\mathcal{A}$-cochain complexes.

Let

$$\alpha_q(t) := \sup \{0, \mathrm{spec}(Q_q(1,t)\Delta_q(t))\} \tag{5.35}$$

$$\beta_q(t) := \inf \mathrm{spec}\left((\mathrm{Id} - Q_q(1,t))\Delta_q(t)\right) . \tag{5.36}$$

Theorem 5.5 ([HSj1],[BZ1]). (1) *For $t \to \infty$, $\alpha_q(t) \to 0$ and $\beta_q(t) \to \infty$.*

(2) *There exists a constant t_1 so that for $t > t_1$ the elements $\varphi_{q;j,i}(t) \in \Lambda^q(M;\mathcal{E})_{\mathrm{sm}}$ ($1 \le j \le m_q$, $1 \le i \le l$) constructed above provide an orthonormal basis for $\Lambda^q(M;\mathcal{E})_{\mathrm{sm}}$. Hence $\Lambda^q(M;\mathcal{E})_{\mathrm{sm}}$ is a free $\mathcal{A}$-Hilbert module of rank $l \times \#\,\mathrm{Cr}_q(h)$.*

(3) *There exist $\eta > 0$, $t_0 > 0$ and $C > 0$ such that for $t \ge t_0$ and $1 \le r \le l$, $1 \le j \le m_q$,*

$$\sup_{y \in M\setminus U_{qj}} \|\varphi_{q;j,r}(y,t)\| \le C e^{-\eta t} . \tag{5.37}$$

(4) *When expressed in H-coordinates on $W^-_{q,j} \cap \phi(B_r)$, the q-forms $\varphi_{q;j,i}(y,t)$ satisfy the estimate*

$$\varphi_{q;j,i}(y,t) = \left(\frac{t}{\pi}\right)^{d/4} e^{-t|y|^2/2}\left(dy_1 \wedge dy_2 \cdots \wedge dy_q \otimes e_{q;j,i} + O(t^{-1})\right) . \tag{5.38}$$

where $0 < r := \min\{\alpha, \epsilon/2, \sqrt{2c(\epsilon/2)}\}$, with α as in Definition 5.1 and $c(\epsilon/2)$ given by Proposition 5.4.

(5) *The bounded $\mathcal{A}$-linear maps $f_k(t) : \Lambda^k(M;\mathcal{E})_{\mathrm{sm}} \to \mathcal{C}^k$, defined above, satisfy the estimate*

$$f_k(t)\left(\varphi_{k;j,r}(t)\right) = E_{k;j,r} + O(t^{-1}) \tag{5.39}$$

and therefore, for t sufficiently large, define an isomorphism of cochain complexes of $\mathcal{A}$-Hilbert modules of finite type.

(6) *Representing $d_q(t)$ with respect to the bases given in* (2)

$$d_q(t)\varphi_{q;j,i}(t) = \sum_{1\leq j'\leq m_{q+1}, 1\leq i'\leq l} \eta_{q;ji,j'i'}(t)\varphi_{q+1;j',i'}(t) ,$$

with $\eta_{q;ji,j'i'} \in \mathcal{A}^{\mathrm{op}}$, the coefficients $\eta_{q;ji,j'i'}$ satisfy

$$\eta_{q;ji,j'i'}(t) = e^{-t}\left(\frac{t}{\pi}\right)^{1/2}\left(\gamma_{q;ji,j'i'} + O(t^{-1/2})\right)$$

with $\gamma_{q;ji,j'i'}$ as above.

Proof. (1) follows from Proposition 5.2, (2) was proven above and (3) follows from Proposition 5.4.

(4) By Proposition 5.4, $(i_q(t))^* i_q(t) = \mathrm{Id} + O(e^{-ct})$, thus it suffices to verify (5.38) with $Q_q(1,t)\psi_{q;j,i}(t)$ instead of $\varphi_{q;j,i}(t)$. By Proposition 5.4,

$$Q_q(1,t)\psi_{q;j,i}(y,t) = \psi_{q;j,i}(y,t) + O(e^{-ct}) ,$$

uniformly in y, and (4) follows.

To prove (5.39) we have to show that for any cell $W^-_{q;j'}$

$$\int_{W^-_{q;j'}} \varphi_{q;j,r}(y,t)e^{th(y)} = \left(\frac{t}{\pi}\right)^{(d-2q)/4} e^{tq}\left(\delta_{jj'}e_{q;j,r} + O(t^{-1})\right) .$$

Note that, due to Theorem 5.5 (3) and (4), it suffices to consider the case where $j = j'$. By Theorem 5.5 (4) and Proposition 5.4 we conclude that (with B_r as in (4))

$$\begin{aligned}\int_{W^-_{q;j}} &\varphi_{q;j,r}(y,t)e^{th(y)} \\ &= \left(\frac{t}{\pi}\right)^{d/4} e^{qt}\int_{W^-_{q;j}\cap\phi(B_r)} e^{-t\sum_1^q y_k^2}\left(dy_1\wedge ..\wedge dy_q \otimes e_{q;j,r} + O(t^{-1})\right) \\ &\quad + e^{qt}\int_{W^-_{q;j}\setminus\phi(B_r)} e^{(h(y)-q)t}\varphi_{q;j,r}(y,t) \\ &= e^{qt}\left(\frac{t}{\pi}\right)^{d/4}\left(\frac{t}{\pi}\right)^{-q/2}\left(e_{q;j,r} + O(t^{-1})\right) .\end{aligned}$$

The second integral on the right-hand side of the above equation decays exponentially when $t \to \infty$ because

$$\left\|\phi_{q;j,r}(y,t)\right\| = O(e^{-ct})$$

for $y \in W^-_{q;j}\setminus\phi(B_r)$ (Proposition 5.4) and $h(y) - q \leq 0$ on $W^-_{q;j}$. The second part of the statement follows from (2). (6) follows from (5). □

5.3 Applications. We present two applications of Theorem 5.5. We point out that in this subsection we do not assume that $\mathcal{W}$ is a free $\mathcal{A}$-Hilbert module. First we state and prove Proposition 5.6, a generalized version of a result of Gromov-Shubin (cf. [GrSh], also [E1,2]) which we stated in Proposition 1 in the introduction.

PROPOSITION 5.6 ([GrSh]). *Let M be a closed manifold and $\mathcal{W}$ be an $(\mathcal{A}, \Gamma^{\mathrm{op}})$-Hilbert module of finite type (not necessarily $\mathcal{A}$-free). Then the following statements are true:*

(1) *Suppose g is a Riemannian metric and $\tau = (h, g')$ is a generalized triangulation of M. Then the system $(M, g, \mathcal{W})$ is of a-determinant class iff $(M, \tau, \mathcal{W})$ is of c-determinant class.*

(2) *If M_1 and M_2 are two homotopy equivalent connected manifolds and τ_1 and τ_2 are generalized triangulations of M_1, respectively M_2, then $(M_1, \tau_1, \mathcal{W})$ is of c-determinant class iff $(M_2, \tau_2, \mathcal{W})$ is of c-determinant class.*

Proof. (A) First we prove the two results in the case where $\mathcal{W}$ is $\mathcal{A}$-free. Notice that statement (2) follows directly from Proposition 1.18. To prove (1) consider the cochain complexes

$$\left(\Lambda^k(M;\mathcal{E}), \tilde{d}_k(t)\right) ; \quad \tilde{d}_k(t) = e^t \left(\frac{t}{\pi}\right)^{-1/2} d_k(t) \tag{5.40}$$

$$\left(\Lambda^k(M;\mathcal{E}), d_k(t)\right) ; \quad d_k(t) = e^{-th} d_k e^{th} \tag{5.41}$$

$$\left(\Lambda^k(M;\mathcal{E}), d_k\right) \tag{5.42}$$

and let $\tilde{\Delta}_k(t), \Delta_k(t), \Delta_k$ be the Laplacians of (5.40), (5.41) and (5.42) with respect to the Riemannian metric g'. By Theorem 5.5 (5) there exists $t_0 > 0$ so that for $t \geq t_0$ the maps $f_k(t)$ introduced in (5.34) are isomorphisms between the cochain complexes $(\Lambda^k(M;\mathcal{E})_{\mathrm{sm}}, \tilde{d}_k(t))$ and $(\mathcal{C}^k, \delta_k)$. Therefore, by Proposition 1.18, $(M, \tau, \mathcal{W})$ is of *c-determinant* class iff there exists $t > t_0$ with

$$\int_{0+}^{1} \log \lambda dN_{\tilde{\Delta}_k(t)}(\lambda) > -\infty\ . \tag{5.43}$$

As $\tilde{\Delta}_k(t) = e^{2t}\left(\frac{t}{\pi}\right)^{-1} \Delta_k(t)$, (5.43) is equivalent to

$$\int_{0+}^{1} \log \lambda dN_{\Delta_k(t)}(\lambda) > -\infty\ . \tag{5.44}$$

Since multiplication by e^{th} defines a L_2-bounded isomorphism between the L_2-completion of cochain complexes (5.41) and (5.42) we conclude from

[GrSh, Proposition 4.1] that (5.44) holds iff

$$\int_{0+}^{1} \log \lambda dN_{\Delta_k}(\lambda) > -\infty \,. \tag{5.45}$$

As the L_2-completions of (5.42) with respect the Riemannian metric g' and g are isomorphic by a bounded isomorphism we apply once more [GrSh, Proposition 4.1] to conclude that (5.45) is equivalent to

$$\int_{0+}^{1} \log \lambda dN_{\Delta'_k}(\lambda) > -\infty \tag{5.46}$$

where Δ'_k denotes the Laplacian with respect to the Riemannian metric g. The inequality (5.46) says that $(M, g, \mathcal{W})$ is of $a-$determinant class and thus (1) follows.

(B) To prove (1) and (2) in general let us make the following observations:

(O1) If $\mathcal{W}$ is Γ^{op}-trivial then any system $(M, g, \mathcal{W})$ resp. $(M, \tau, \mathcal{W})$ is of *a-determinant* resp. *c-determinant* class. Indeed in this case the spectral distribution functions of the corresponding (analytic resp. combinatorial) Laplacians are step functions like in the case $\mathcal{A} = \mathbb{C}$.

(O2) Let $\mathcal{W}_i$, $i = 1, 2$, be two $(\mathcal{A}, \Gamma^{\text{op}})$-Hilbert modules of finite type. The system $(M, g, \mathcal{W}_1 \oplus \mathcal{W}_2)$ is of *a-determinant* class resp. $(M, \tau, \mathcal{W}_1 \oplus \mathcal{W}_2)$ is of *c-determinant* class, iff the systems $(M, g, \mathcal{W}_i)$, $i = 1, 2$, are of *a-determinant* class resp. the systems $(M, \tau, \mathcal{W}_i)$, $i = 1, 2$, are of *c-determinant* class. This equivalence follows by comparing the spectral distribution functions of the Laplacians (analytic and combinatorial) in the complexes associated to $\mathcal{W}_i$, $i = 1, 2$, and to $\mathcal{W}_1 \oplus \mathcal{W}_2$. An inequality between these functions in the combinatorial case follows from Proposition 1.10 (2). The analogous inequality holds in the analytic case as well. (As in the combinatorial case the inequality can be deduced easily from the definition of the spectral distribution function.)

(O3) Given an $(\mathcal{A}, \Gamma^{\text{op}})$-Hilbert module $\mathcal{W}$ of finite type there exists an $(\mathcal{A}, \Gamma^{\text{op}})$-Hilbert module $\mathcal{W}'$ of finite type, which is Γ^{op} trivial so that $\mathcal{W} \oplus \mathcal{W}'$ is $\mathcal{A}$-free. Indeed it suffices to take an $\mathcal{A}$-Hilbert module $\mathcal{W}'$ of finite type so that $\mathcal{W} \oplus \mathcal{W}'$ is $\mathcal{A}$-free and equip it with the trivial Γ^{op} action.

Combine (O1), (O2), (O3) to conclude that $(M, g, \mathcal{W} \oplus \mathcal{W}')$ resp. $(M, \tau, \mathcal{W} \oplus \mathcal{W}')$ is of determinant class iff $(M, g, \mathcal{W})$ resp. $(M, \tau, \mathcal{W})$ is. In view of (A), we therefore have proved (1) and (2) as stated. □

Proposition 5.6 suggests the following definition. (cf. Definition 4.1)

DEFINITION 5.7. *The pair* $(M, \mathcal{W})$ *with* M *a closed manifold and* $\mathcal{W}$ *an* $(\mathcal{A}, \Gamma^{\text{op}})$*-Hilbert module of finite type is of determinant class iff for some (and then for any) Riemannian metric* g, *the system* $(M, g, \mathcal{W})$ *is of a-determinant class.*

Proposition 5.6 is used to reduce the proof of Theorem 2 to the case where $\mathcal{W}$ is a free $\mathcal{A}$-Hilbert module of finite type.

PROPOSITION 5.8. *Let* M *be a closed manifold,* g *a Riemannian metric,* τ *a generalized triangulation and* $\mathcal{A}$ *a finite von Neumann algebra. The following two statements are equivalent:*

(1) *For any free* $(\mathcal{A}, \Gamma^{\text{op}})$*-Hilbert module* $\mathcal{W}$ *of finite type, the property* $(M,\mathcal{W})$ *being of determinant class implies* $T_{\text{an}}(M,g,\mathcal{W})=T_{\text{Re}}(M,g,\tau,\mathcal{W})$.
(2) *For any* $(\mathcal{A}, \Gamma^{\text{op}})$*-Hilbert module* $\mathcal{W}$ *of finite type, the property* $(M, \mathcal{W})$ *being of determinant class implies* $T_{\text{an}}(M, g, \mathcal{W}) = T_{\text{Re}}(M, g, \tau, \mathcal{W})$.

Proof. We have to prove that (1) implies (2). We will do this in four steps.

(S1) When $\mathcal{A} = \mathbb{C}$, $(M, \mathcal{W})$ is of determinant class and the equality of the two torsion holds by (1).

(S2) If $\mathcal{W}$ is Γ^{op}- trivial then by observation (O1) above, $(M, \mathcal{W})$ is of determinant class, and

$$\log T_{\text{an}}(M, g, \mathcal{W}) = \dim_{\mathcal{N}}(\mathcal{W}) \log T_{\text{an}}(M, g, 1_{\mathbb{C}}) \ ,$$

where $1_{\mathbb{C}}$ denotes the complex line $\mathbb{C}$ with the trivial Γ^{op} action. A similar formula holds for T_{Re}. Hence the equality of the two torsions follows from (S1).

(S3) Let $\mathcal{W}_1$ and $\mathcal{W}_2$ be two $(\mathcal{A}, \Gamma^{\text{op}})$-Hilbert modules of finite type and set $\mathcal{W} = \mathcal{W}_1 \oplus \mathcal{W}_2$. By observation (O2) above, $(M, g, \mathcal{W})$ is of determinant class iff $(M, g, \mathcal{W}_i)$, $i = 1, 2$, are of determinant class, and by the sum formula (Proposition 4.3), the analytic torsion of $(M, g, \mathcal{W})$ resp. the Reidemeister torsion of $(M, g, \tau, \mathcal{W})$ is the product of the analytic torsions of $(M, g, \mathcal{W}_i)$, $i = 1, 2$, resp. the Reidemeister torsions of $(M, g, \tau, \mathcal{W}_i)$, $i = 1, 2$.

(S4) Let $\mathcal{W}'$ be as in observation (O3) above. By (O1), $(M, \mathcal{W}')$ is of determinant class and by (O2), $(M, \mathcal{W})$ is of *determinant* class iff $(M, \mathcal{W} \oplus \mathcal{W}')$ is of *determinant* class. Combine statement (1), (S1) and (S3) to conclude the equality of the two torsions for $\mathcal{W}$. □

The second application of Theorem 5.5 concerns the proof of the identity (4.9) which leads to the formula (4.8‴),

$$\log T_{\text{met}}(M, g, \tau_D, \mathcal{W}) = (-1)^{-d+1} \log T_{\text{met}}(M, g, \tau, \mathcal{W}) \ .$$

Let (M, g) be a closed Riemannian manifold and $h : M \to \mathbb{R}$ be a Morse function so that $\tau = (h, g)$ is a generalized triangulation. Denote by

$\tau_D = (d-h, g)$ its dual triangulation. Choose orientations O_h (cf. section 4) and let $\mathcal{W}$ be a $(\mathcal{A}, \Gamma^{\mathrm{op}})$-Hilbert module of finite type with $\Gamma = \pi_1(M)$. As in section 4 denote by $\overline{\mathrm{Int}}_\tau^{(q)}$ the restriction of $\pi_{q,\tau}\mathrm{Int}_\tau^{(q)}$ to $\mathcal{H}_q$ where $\pi_{q,\tau}$ is the orthogonal projection on $\mathrm{Null}(\Delta_{q,\tau}^{\mathrm{comb}})$ and by $(\overline{\mathrm{Int}}_\tau^{(q)})^*$ the adjoint of $\overline{\mathrm{Int}}_\tau^{(q)}$. With respect to the Hodge decomposition of $\Lambda^q(M;\mathcal{E})$ and $\mathcal{C}_\tau^q$ we can write $F^q \equiv F_{\tau,q} := \mathrm{Int}_\tau^{(q)}$ as a 3×3 matrix

$$\begin{pmatrix} F_{11}^q & 0 & F_{13}^q \\ F_{21}^q & F_{22}^q & F_{23}^q \\ 0 & 0 & F_{33}^q \end{pmatrix}$$

and obtain $\overline{\mathrm{Int}}_\tau^{(q)} = F_{11}^q \equiv F_{11}^{\tau,q}$. The critical points of h of index q and their corresponding unstable manifolds of $-\mathrm{grad}_g h$ identify to the critical points of $d-h$ of index $d-q$ and their corresponding stable manifolds (i.e. with respect to $-\mathrm{grad}_g(d-h)$). The orientations O_h induce the orientations O_{d-h}.

Let $\mathcal{C}_\tau$ resp. $\mathcal{C}_{\tau_D}$ be the cochain complex associated to the generalized triangulation τ and the orientations O_h resp. the generalized triangulation τ_D and the orientations O_{d-h}. The above identification of the critical points of h and $d-h$ provides isometries $PD_q : \mathcal{C}_\tau^q \to \mathcal{C}_{\tau_D}^{d-q}$ which intertwine the coboundary operator δ_q^τ in $\mathcal{C}_\tau$ with the adjoint $(\delta_{d-q-1}^{\tau_D})^*$ of the coboundary operator in $\mathcal{C}_{\tau_D}$. Similarly the Hodge $*$ operator $R_q : \Lambda^q(M;\mathcal{E}) \to \Lambda^{d-q}(M;\mathcal{E})$ intertwines d_q resp. $\tilde{d}_q(t)$ with d_{d-q-1}^* resp. $\tilde{d}(t)_{d-q-1}^*$ where $\tilde{d}_q(t)$ is given in (5.40).

PROPOSITION 5.9. *With the above notation we have*

$$R_q|_{\mathcal{H}_q} = (\overline{\mathrm{Int}}_{\tau_D}^{(d-q)})^* \cdot PD_q \cdot \overline{\mathrm{Int}}_\tau^{(q)}\ . \tag{5.47}$$

Proof. Let

$$I_q(t) : \big(\Lambda^q(M;\mathcal{E}), \tilde{d}_q(t)\big) \to \big(\Lambda^q(M;\mathcal{E}), d_q\big)$$

denote the multiplication with $\left(\frac{\pi}{t}\right)^{\frac{d-2q}{4}} e^{-tq}$ and

$$J_q(t) : \big(\Lambda^q(M;\mathcal{E})_{\mathrm{sm}}, \tilde{d}_q(t)\big) \to \big(\Lambda^q(M;\mathcal{E}), \tilde{d}_q(t)\big)$$

the canonical inclusion. The $I_q(t)$'s define a morphism of cochain complexes and $F_q \cdot I_q(t) = f_q(t)$ with $f_q(t)$ defined by (5.34). Notice that the matrix representations of PD_q and R_q with respect to the Hodge decompositions of $\mathcal{C}_i^*$ with $i = \tau, \tau_D$ and $(\Lambda^q(M,\mathcal{E}), d_q)$ are of the form

$$\begin{pmatrix} PD_{11}^q & 0 & 0 \\ 0 & 0 & PD_{23}^q \\ 0 & PD_{32}^q & 0 \end{pmatrix} ;$$

$$\begin{pmatrix} R^q_{11} & 0 & 0 \\ 0 & 0 & R^q_{23} \\ 0 & R^q_{32} & 0 \end{pmatrix} .$$

Equation (5.47) therefore becomes

$$R^q_{11} = (\overline{\mathrm{Int}}^{(d-q)}_{\tau_D})^* \cdot PD^q_{11} \cdot \overline{\mathrm{Int}}^{(q)}_{\tau} . \tag{5.48}$$

Consider the (typically not commutative) diagram

$$\begin{array}{ccc} (\Lambda^q(M;\mathcal{E})_{\mathrm{sm}}, \tilde{d}_q(t)) & \xrightarrow{f_q(t)} & (\mathcal{C}^q_\tau, \delta_q) \\ {\scriptstyle R_q}\downarrow & & \downarrow{\scriptstyle PD_q} \\ (\Lambda^{d-q}(M;\mathcal{E})_{\mathrm{sm}}, \tilde{d}^*_{d-q}(t)) & \xleftarrow[f^*_{d-q}(t)]{} & (\mathcal{C}^{d-q}_{\tau_D}, \delta^*_{d-q}) \end{array}$$

Notice that with respect to the bases considered in Theorem 5.5 the map $R_q|_{\Lambda(M,\mathcal{E})_{\mathrm{sm}}}$ (by Theorem 5.5(3) and (4)),the maps $f_q(t)$ and $f_{d-q}(t)^*$ (by Theorem 5.5(5)) are of the form $Id + O(\frac{1}{t})$ and PD_q is equal to the identity. Therefore

$$R_q|_{\Lambda^q(M;\mathcal{E})_{\mathrm{sm}}} = \left(F_{\tau_D,d-q} \cdot I_{d-q}(t) \cdot J_{d-q}(t)\right)^* \cdot PD_q \cdot F_{\tau,q} \cdot I_q(t) \cdot J_q(t) + O(1/t) . \tag{5.49}$$

The representation of R_q with respect to the Hodge decomposition of $(\Lambda^q(M,\mathcal{E}), \tilde{d}_q(t))$ is again of the form

$$\begin{pmatrix} R^q_{11}(t) & 0 & 0 \\ 0 & 0 & R^q_{23}(t) \\ 0 & R^q_{32}(t) & 0 \end{pmatrix} .$$

and

$$J^q_{11}(t) = Id , \tag{5.50}$$

Then (5.49) and (5.50) imply

$$R^q_{11}(t) = \left(I^{d-q}_{11}(t)\right)^* \cdot (F^{\tau_D,d-q}_{11})^* \cdot PD^q_{11} \cdot F^{\tau,q}_{11} \cdot I^q_{11}(t) + O(1/t) , \tag{5.51}$$

Notice

$$R^q_{11}(t) = \left(I^{d-q}_{11}(t)\right)^* \cdot R^q_{11} \cdot I^q_{11}(t) . \tag{5.52}$$

and therefore (5.51) and (5.52) imply

$$R^q_{11} - (\overline{\mathrm{Int}}^{(d-q)}_{\tau_D})^* \cdot PD^q_{11} \cdot \overline{\mathrm{Int}}^{(d)}_{\tau} = O(1/t) . \tag{5.53}$$

Since the left side of the equality (5.53) is independent from t, (5.48) follows. □

6. The Main Results

6.1 Asymptotic expansion of Witten's deformation of the analytic torsion. Let (M,g) be a closed Riemannian manifold with fundamental group $\Gamma = \pi_1(M)$ and $h : M \to \mathbb{R}$ a Morse function so that $\tau = (h,g)$ is a generalized triangulation. Let $\mathcal{A}$ be a finite von Neumann algebra and $\mathcal{W}$ an $(\mathcal{A}, \Gamma^{\text{op}})$-Hilbert module of finite type. The bundle $p : \mathcal{E} \to M$ associated to $\mathcal{W}$ (cf. section 1.4) is equipped with a canonical flat connection and a Hermitian structure μ on $\mathcal{E} \to M$. Throughout this subsection we assume that $(M, \mathcal{W})$ is of determinant class.

DEFINITION. *A function $a : \mathbb{R} \to \mathbb{R}$ is said to have an asymptotic expansion for $t \to \infty$ if there exists a sequence $i_1 > i_2 > \ldots > i_N = 0$ and constants $(a_k)_{1 \le k \le N}, (b_k)_{1 \le k \le N}$ such that*

$$a(t) = \sum_1^N a_k t^{i_k} + \sum_1^N b_k t^{i_k} \log t + o(1) \ . \tag{6.1}$$

For convenience we denote by $\mathrm{FT}(a(t))$ the coefficient a_N in the asymptotic expansion of $a(t)$ corresponding to t^0.

Recall that in section 4 we introduced $T_{\text{an}}(h,t) \equiv T_{\text{an}}(g,h,t)$, $T_{\text{Re}}(\tau) \equiv T_{\text{Re}}(g,\tau)$, $T_{\text{comb}}(\tau)$ and $T_{\text{met}}(\tau) \equiv T_{\text{met}}(g,\tau)$, and in section 5 we introduced $T_{\text{sm}}(h,t) \equiv T_{\text{sm}}(g,h,t)$ and $T_{\text{la}}(h,t) \equiv T_{\text{la}}(g,h,t)$. In this subsection we prove the following:

Theorem A. *Let (M,g) be a closed Riemannian manifold of odd dimension, $\mathcal{W}$ an $(\mathcal{A}, \Gamma^{\text{op}})$-Hilbert module of finite type with $l = \dim_{\mathrm{N}} \mathcal{W}$ and $h : M \to \mathbb{R}$ a Morse function. Assume that $(M, \mathcal{W})$ is of determinant class and that $\tau = (h,g)$ is a generalized triangulation. Denote by m_q the number of critical points of index q of h and let $\beta_q := \dim_N H^q(M;\mathcal{W})$. Then the following statements hold:*

(1) *The functions $\log T_{\text{an}}(h,t), \log T_{\text{sm}}(h,t)$ and $\log T_{\text{la}}(h,t)$ admit asymptotic expansions for $t \to \infty$.*

(2) *The asymptotic expansion of $\log T_{\text{an}}(h,t)$ is of the form*

$$\log T_{\text{an}}(h,t) = \log T_{\text{an}}(h,0) - \log T_{\text{met}}(\tau) + \tfrac{1}{2}\left(\sum_{q=0}^{d} (-1)^{q+1} q\beta_q\right)(2t - \log t + \log \pi) + O(t^{-1}) \ . \tag{6.2}$$

(3) *The asymptotic expansion of $\log T_{\text{sm}}(h,t)$ is of the form*

$$\log T_{\text{comb}}(\tau) + \tfrac{1}{2}\left(\sum_{q=0}^{d} (-1)^{q+1} (q\beta_q - q m_q l)\right)(2t - \log t + \log \pi) + o(1) \ . \tag{6.3}$$

Using the same arguments as in the proof of Proposition 5.8 one can show that it suffices to prove the statements for $\mathcal{W}$ a free $\mathcal{A}$-module.

We begin by deriving an alternative formula for the analytic torsion (cf. [RSin] ,[Ch] and [BuFrKa1]). The space of q-forms can be decomposed into orthogonal subspaces:

$$\Lambda^q(M;\mathcal{E}) = \Lambda_t^{+,q}(M;\mathcal{E}) \oplus \Lambda_t^{-,q}(M;\mathcal{E}) \oplus \mathcal{H}_t^q \tag{6.4}$$

where

$$\begin{aligned}\Lambda_t^{+,q}(M;\mathcal{E}) :&= closure\big(d_{q-1}(t)\Lambda^{q-1}(M;\mathcal{E})\big)\\ &= closure\big(e^{-th}d_{q-1}\Lambda^{q-1}(M;\mathcal{E})\big)\ ;\end{aligned} \tag{6.5}$$

$$\begin{aligned}\Lambda_t^{-,q}(M;\mathcal{E}) :&= closure\big(d_q(t)^*\Lambda^{q+1}(M;\mathcal{E})\big)\\ &= closure\big(e^{th}d_q^*\Lambda^{q+1}(M;\mathcal{E})\big)\ ;\end{aligned} \tag{6.6}$$

$$\mathcal{H}_t^q := \big\{\omega \in \Lambda^q(M;\mathcal{E}); \Delta_q(t)\omega = 0\big\}, \tag{6.7}$$

where the word closure refers to the closure with respect to the C^∞ topology. Note that the spaces $\Lambda_t^{\pm,q}(M;\mathcal{E})$ are invariant with respect to the Laplacian $\Delta_q(t)$. Denote by $\Delta_q^{\pm}(t)$ the restriction of $\Delta_q(t)$ to $\Lambda_t^{\pm,q}(M;\mathcal{E})$ given by $\Delta_q^+(t) = d_{q-1}(t)d_{q-1}(t)^*$ and $\Delta_q^-(t) = d_q(t)^*d_q(t)$. The operator $d_q(t)$ maps the space $\Lambda_t^{-,q}(M;\mathcal{E})$ injectively onto a dense subspace of $\Lambda_t^{+,q+1}(M;\mathcal{E})$ and it intertwines $\Delta_q^-(t)$ and $\Delta_{q+1}^+(t)$. By the same arguments as for (4.18),

$$N_q^-(\lambda,t) = N_{q+1}^+(\lambda,t)\ . \tag{6.8}$$

Note that both $\Delta_q^+(t)$ and $\Delta_q^-(t)$ are of determinant class, more precisely, for a given compact interval $I \subset [0,\infty)$ there exists $B > -\infty$ so that for $t \in I$

$$\int_{0+}^1 \log\lambda dN_q^{\pm}(\lambda,t) > B\ . \tag{6.9}$$

Using formula (6.8) one obtains

$$\int_1^\infty \frac{dx}{x}\mathrm{tr}_{\mathrm{N}}(e^{-x\Delta_q^-(t)}) = \int_1^\infty \frac{dx}{x}\mathrm{tr}_{\mathrm{N}}(e^{-x\Delta_{q+1}^+(t)}) \tag{6.10}$$

and, for $\Re s$ sufficiently large,

$$\int_0^1 dx x^{s-1}\mathrm{tr}_{\mathrm{N}}(e^{-x\Delta_q^-(t)}) = \int_0^1 dx x^{s-1}\mathrm{tr}_{\mathrm{N}}(e^{-x\Delta_{q+1}^+(t)})\ . \tag{6.11}$$

The integrals in formula (6.10) converge because the operators $\Delta_q^{\pm}(t)$ are of determinant class (Proposition 2.12 (3), cf. also [Lo]), and the integrals in (6.11) converge because $\int_0^1 dx x^{s-1}\mathrm{tr}_{\mathrm{N}}(e^{-x\Delta(t)}) < \infty$, as can be seen from the heat trace expansion of $\mathrm{tr}_{\mathrm{N}}(e^{-x\Delta_q(t)})$ at $x=0$ (cf. (6.22) below).

Introduce

$$\xi_{t,q}(s) := \frac{1}{\Gamma(s)} \int_0^1 x^{s-1} \mathrm{tr}_{\mathrm{N}} \big(e^{-x\Delta_q(t)}(Id - Q_q(0,t))\big) dx \qquad (6.12')$$
$$+ s \int_1^\infty \frac{1}{x} \mathrm{tr}_{\mathrm{N}} \big(e^{-x\Delta_q(t)}(Id - Q_q(0,t))\big) dx$$

$$\xi^{\pm}_{t,q}(s) := \frac{1}{\Gamma(s)} \int_0^1 x^{s-1} \mathrm{tr}_{\mathrm{N}} (e^{-x\Delta^{\pm}_q(t)}) dx$$
$$+ s \int_1^\infty \frac{1}{x} \mathrm{tr}_{\mathrm{N}} (e^{-x\Delta^{\pm}_q(t)}) dx \ . \qquad (6.12'')$$

The functions $\xi_{t,q}(s)$ and $\xi^{\pm}_{t,q}(s)$ are defined for $t > 0$ and $\Re s > \frac{d}{2}$, smooth in t, s and holomorphic in s. Further, as $\Delta_q(t)$ is a differential operator the function $\xi_{t,q}(s)$ is meromorphic and has $s = 0$ as a regular value. From (2.17),

$$\log \det{}_N \Delta_q(t) = -\frac{d}{ds}\Big|_{s=0} \xi_{t,q}(s) \ . \qquad (6.13)$$

Denote by $\log T_{\mathrm{an}}(h,t,s)$ and $\log T^{\pm}_{\mathrm{an}}(h,t)$ the functions defined by $\big(\Re s > \frac{d}{2}\big)$

$$\log T_{\mathrm{an}}(h,t,s) := \frac{1}{2} \sum_{q=0}^{d} (-1)^{q+1} q \left(-\frac{\partial}{\partial s} \xi_{t,q}(s)\right) \qquad (6.14')$$

and

$$\log T^{\pm}_{\mathrm{an}}(h,t,s) := \frac{1}{2} \sum_{q=0}^{d} (-1)^{q} \left(-\frac{\partial}{\partial s} \xi^{\pm}_{t,q}(s)\right) \ . \qquad (6.14'')$$

In view of (6.10) and (6.11) we have $\xi^{-}_{t,q}(s) = \xi^{+}_{t,q+1}(s)$. As $\xi_{t,q}(s) = \xi^{+}_{t,q}(s) + \xi^{-}_{t,q}(s)$ this leads to $\big(\Re s > \frac{d}{2}\big)$

$$\log T_{\mathrm{an}}(h,t,s) = -\log T^{+}_{\mathrm{an}}(h,t,s) = \log T^{-}_{\mathrm{an}}(h,t,s) \ . \qquad (6.15)$$

Although we do not know if $\xi^{\pm}_{t,q}(s)$ have an analytic continuation at $s = 0$ it follows from (6.15) that $\log T^{\pm}_{\mathrm{an}}(h,t,s)$ have. By (6.13), $\log T_{\mathrm{an}}(h,t) = \log T_{\mathrm{an}}(h,t,0)$ and therefore

$$\log T_{\mathrm{an}}(h,t) = \mp \log T^{\pm}_{\mathrm{an}}(h,t,0) \ .$$

Our first goal is to compute $\frac{d}{dt} \log T_{\mathrm{an}}(h,t,s)$. To analyze the t-dependence of $\xi^{\pm}_{t,q}(s)$ we treat the two terms on the right-hand side of (6.12″) separately. To illustrate the new difficulties which arise (as compared with the classical situation) we point out that the differentiability of $\int_1^\infty x^{-1} \mathrm{tr}_{\mathrm{N}} e^{-x\Delta^{+}_q(t)} dx$ with respect to t is far from being obvious.

We begin by computing $\frac{d}{dt}(\mathrm{tr}_{\mathrm{N}} e^{-x\Delta_q^+(t)})$. Note that $\Delta_q^+(t) : \Lambda_t^{+,q}(M;\mathcal{E}) \to \Lambda_t^{+,q}(M;\mathcal{E})$ where the space $\Lambda_t^{+,q}(M;\mathcal{E}) = e^{-th}\Lambda^{+,q}(M;\mathcal{E})$ depends on t. It is therefore convenient to introduce $\widetilde{\Delta_q^+}(t) = e^{th}\Delta_q^+(t)e^{-th} : \Lambda^{+,q}(M;\mathcal{E}) \to \Lambda^{+,q}(M;\mathcal{E})$ which is isospectral with $\Delta_q^+(t)$. Hence, $\mathrm{tr}_{\mathrm{N}} e^{-x\Delta_q^+(t)} = \mathrm{tr}_{\mathrm{N}} e^{-x\widetilde{\Delta_q^+}(t)}$. Now one computes $\frac{d}{dt}\mathrm{tr}_{\mathrm{N}} e^{-x\widetilde{\Delta_q^+}(t)}$ using Duhamel's formula and the identity

$$\widetilde{\Delta_q^+}(t) = e^{2th}(d_{q-1}d_{q-1}^* + 2tdh \wedge d_{q-1}^*)e^{-2th} ,$$

one obtains

$$\begin{aligned}\frac{d}{dt}(\mathrm{tr}_{\mathrm{N}} e^{-x\widetilde{\Delta_q^+}(t)}) &= -x\mathrm{tr}_{\mathrm{N}}\left(\frac{d}{dt}(\widetilde{\Delta_q^+}(t))e^{-x\widetilde{\Delta_q^+}(t)}\right) \\ &= \mathrm{tr}_{\mathrm{N}}\big(2[h, -x\widetilde{\Delta_q^+}(t)]e^{-x\widetilde{\Delta_q^+}(t)}\big) \\ &\quad - 2x\mathrm{tr}_{\mathrm{N}}(e^{2th}dh \wedge d_{q-1}^* e^{-2th}e^{-x\widetilde{\Delta_q^+}(t)})\end{aligned}$$

where $[A, B]$ denotes the commutator of the two operators A and B. Using

$$\mathrm{tr}_{\mathrm{N}}\big(2[h, -x\widetilde{\Delta_q^+}(t)]e^{-x\widetilde{\Delta_q^+}(t)}\big) = 0 ,$$

$e^{th}d_{q-1}^*e^{-th} = d_{q-1}(t)^*$ and that $e^{-th}e^{-x\widetilde{\Delta_q^+}(t)}e^{th} = e^{-x\Delta_q^+(t)}$ we obtain

$$\frac{d}{dt}\big(\mathrm{tr}_{\mathrm{N}}(e^{-x\Delta_q^+(t)})\big) = -2x\mathrm{tr}_{\mathrm{N}}\big(dh \wedge d_{q-1}(t)^* e^{-x\Delta_q^+(t)}\big) .$$

Further observe that, despite of the fact that $d_{q-1}(t) : \Lambda_t^{-,q-1}(M;\mathcal{E}) \to \Lambda_t^{+,q}(M;\mathcal{E})$ is not invertible (it might not be onto), we can form

$$d_{q-1}(t)^* = d_{q-1}(t)^{-1}d_{q-1}(t)d_{q-1}(t)^* = d_{q-1}(t)^{-1}\Delta_q^+(t)$$

where the domain of definition of $d_{q-1}(t)^{-1}$ is the range of $d_{q-1}(t)$. We note that

$$dh \wedge d_{q-1}(t)^{-1}\Delta_q^+(t) = \big(d_{q-1}(t)hd_{q-1}(t)^{-1} - h\big)\Delta_q^+(t) .$$

This leads to the following formula

$$\begin{aligned}\frac{d}{dt}\big(\mathrm{tr}_{\mathrm{N}}(e^{-x\Delta_q^+(t)})\big) &= -2x\mathrm{tr}_{\mathrm{N}}\big(d_{q-1}(t)hd_{q-1}(t)^{-1}\Delta_q^+(t)e^{-x\Delta_q^+(t)}\big) \\ &\quad + 2x\mathrm{tr}_{\mathrm{N}}\big(h\Delta_q^+(t)e^{-x\Delta_q^+(t)}\big) .\end{aligned}$$

Next we observe that

$$\mathrm{tr}_{\mathrm{N}}\big(h\Delta_q^+(t)e^{-x\Delta_q^+(t)}\big) = -\frac{d}{dx}\big(\mathrm{tr}_{\mathrm{N}}(he^{-x\Delta_q^+(t)})\big)$$

and that

$$\begin{aligned}
&\mathrm{tr}_{\mathrm{N}}\left(d_{q-1}(t)hd_{q-1}(t)^{-1}\Delta_q^+(t)e^{-x\Delta_q^+(t)}\right)\\
&\qquad = \mathrm{tr}_{\mathrm{N}}\left(hd_{q-1}(t)^{-1}\Delta_q^+(t)e^{-x\Delta_q^+(t)}d_{q-1}(t)\right)\\
&\qquad = \mathrm{tr}_{\mathrm{N}}\left(h\Delta_{q-1}^-(t)e^{-x\Delta_{q-1}^-(t)}\right)\\
&\qquad = -\frac{d}{dx}\left(\mathrm{tr}_{\mathrm{N}}(he^{-x\Delta_{q-1}^-(t)})\right) .
\end{aligned}$$

We have therefore proved that

$$\frac{d}{dt}\left(\mathrm{tr}_{\mathrm{N}}(e^{-x\Delta_q^+(t)})\right) = 2x\frac{d}{dx}\left(\mathrm{tr}_{\mathrm{N}}(he^{-x\Delta_{q-1}^-(t)})\right) - 2x\frac{d}{dx}\left(\mathrm{tr}_{\mathrm{N}}(he^{-x\Delta_q^+(t)})\right) .$$

This leads to

$$\begin{aligned}
&\frac{1}{2}\sum_{q=0}^{d}(-1)^{q+1}\frac{d}{dt}\left(\mathrm{tr}_{\mathrm{N}}(e^{-x\Delta_q^+(t)})\right)\\
&\qquad = x\sum_{q=0}^{d}(-1)^{q+1}\frac{d}{dx}\left(\mathrm{tr}_{\mathrm{N}}(he^{-x\Delta_{q-1}^-(t)})\right)\\
&\qquad\quad - x\sum_{q=0}^{d}(-1)^{q+1}\frac{d}{dx}\left(\mathrm{tr}_{\mathrm{N}}(he^{-x\Delta_q^+(t)})\right)\\
&\qquad = x\frac{d}{dx}\sum_{q=0}^{d}(-1)^{q}\left(\mathrm{tr}_{\mathrm{N}}(he^{-x\Delta_q^+(t)}t) + \mathrm{tr}_{\mathrm{N}}(he^{-x\Delta_q^-(t)})\right)\\
&\qquad = x\frac{d}{dx}\left(\sum_{q=0}^{d}(-1)^{q}\mathrm{tr}_{\mathrm{N}}\left(he^{-x\Delta_q(t)}(\mathrm{Id} - Q_q(0,t))\right)\right) . \qquad (6.16)
\end{aligned}$$

We use the above formula to prove that $-\int_1^\infty \frac{1}{x}\frac{1}{2}\sum_q(-1)^{q+1}\mathrm{tr}_{\mathrm{N}}(e^{-x\Delta_q^+(t)})dx$ has a continuous derivative with respect to t as follows. By the Leibniz rule for improper integrals, it suffices to verify that $f(x,t) := -\frac{1}{x}\frac{1}{2}\sum_q(-1)^q\mathrm{tr}_{\mathrm{N}}(e^{-x\Delta_q^+(t)})$ and $\frac{\partial f}{\partial t}(x,t)$ are both continuous and the integrals $\int_1^\infty f(x,t)dx$ and $\int_1^\infty \frac{\partial f}{\partial t}(x,t)dx$ are both convergent uniformly with respect to t (t varying in a compact interval). Using Duhamel's formula for $e^{-\Delta_q^+(t)}$ one sees that $f(x,t)$ is continuous and, by the above formula,

$$\frac{\partial f}{\partial t}(x,t) = \frac{d}{dx}\sum_{q=0}^{d}(-1)^q\mathrm{tr}_{\mathrm{N}}\left(he^{-x\Delta_q(t)}(\mathrm{Id} - Q_q(0,t))\right)$$

is continuous too. The uniform convergence of the integrals $\int_1^\infty f(x,t)dx$ and $\int_1^\infty \frac{\partial f}{\partial t}(x,t)dx$ follows from

LEMMA 6.1. *Let I be an arbitrary compact interval contained in $[0,\infty)$. Then*

(1) $\lim_{u\to\infty}\int_1^u \frac{1}{x}\mathrm{tr_N}(e^{-x\Delta_q^+(t)})dx$ *exists uniformly in t for $t\in I$.*
(2) $\lim_{u\to\infty}\int_1^u \frac{\partial f}{\partial t}(x,t)dx$ *exists uniformly in t for $t\in I$.*

Proof. (1) Note that the integrand $\frac{1}{x}\mathrm{tr_N}(e^{-x\Delta_q^+(t)})$ is positive. Thus, for $u\geq 1$,

$$\begin{aligned}
0 &\leq \int_u^\infty \frac{1}{x}\mathrm{tr_N}(e^{-x\Delta_q^+(t)})dx = \int_{0+}^\infty dN_{\Delta_q^+(t)}(\mu)\int_{\mu u}^\infty \frac{1}{s}e^{-s}ds \qquad (6.17)\\
&\leq \int_{u^{-\frac12}}^\infty dN_{\Delta_q^+(t)}(\mu)\frac{e^{-\mu u}}{\mu u}\\
&\quad + \int_{0+}^{u^{-\frac12}} dN_{\Delta_q^+(t)}(\mu)\Big(\int_{\min(1,\mu u)}^1 \frac{1}{s}ds + \int_1^\infty e^{-s}ds\Big)\\
&\leq u^{-\frac12}\int_{u^{-\frac12}}^\infty dN_{\Delta_q^+(t)}(\mu)e^{-\mu u}\\
&\quad + \int_{0+}^{u^{-\frac12}} dN_{\Delta_q^+(t)}(\mu)\big(-\log(\min(1,\mu u)) + e^{-1}\big)\\
&\leq A(u,t) + B(u,t)
\end{aligned}$$

where

$$A \equiv A(u,t) := u^{-1/2}\int_{0+}^\infty dN_{\Delta_q^+(t)}(\mu)e^{-\mu}\ ,$$

$$B \equiv B(u,t) := \int_{0+}^{u^{-\frac12}} dN_{\Delta_q^+(t)}(\mu)\big(-\log(\mu) + e^{-1}\big)\ .$$

To obtain the inequalities in (6.17) we used that for $0<\mu\leq u^{-1/2}$ we have $|\log(\mu u)|\leq|\log\mu|$. In fact, $1\leq u\leq\frac{1}{\mu^2}$ and thus $\mu\leq\mu u\leq\frac{1}{\mu}$.

The terms A and B are estimated separately. Integrate by parts and use the asymptotics (2.15) to conclude that there exists $C>0$ so that for $t\in I$

$$\int_{0+}^\infty dN_{\Delta_q^+(t)}(\mu)e^{-\mu}d\mu = \int_{0+}^\infty N_{\Delta_q^+(t)}(\mu)e^{-\mu}d\mu \leq C\int_{0+}^\infty \mu^{d/2}e^{-\mu}d\mu < \infty\ .$$

This shows that $\lim_{u\to\infty}A(u,t)=0$ uniformly in $t\in I$. Concerning $B(u,t)$ note that by Proposition 1.18, there exists $C>0$ so that for $t\in I$, $N_{\Delta_q^+(t)}(\mu)\leq N_{\Delta_q^+(0)}(C\mu)$. By Lemma 1.20 it then follows that

$$B(u,t)\leq\int_{0+}^{u^{-1/2}} dN_{\Delta_q^+(0)}(C\mu)(-\log\mu + e^{-1})\ .$$

As $(M,\mathcal{W})$ is of determinant class one concludes that $\lim_{u\to\infty}B(u,t)=0$ uniformly for t in I.

(2) From (6.16), $\int_1^u \frac{\partial f}{\partial t}(x,t)dx = -\sum_{q=0}^d (-1)^q \mathrm{tr_N}(he^{-x\Delta_q^+(t)}(\mathrm{Id} - Q_q(0,t)))|_1^u$. Therefore it suffices to prove that, for $0 \le q \le d$ and uniformly for t in I,

$$\lim_{x\to\infty} \left(he^{-x\Delta_q(t)}(\mathrm{Id} - Q_q(0,t))\right) = 0 \; . \tag{6.18}$$

To prove (6.18) note that, according to (2.15), there exists $C > 0$, so that for $t \in I$ we obtain, integrating by parts,

$$\begin{aligned}\left|\mathrm{tr_N}(he^{-x\Delta_q(t)}(\mathrm{Id} - Q_q(0,t)))\right| &\le \|h\|_{L^\infty} \int_{0+}^{\infty} e^{-x\lambda} dN_q(\lambda,t) \\ &\le \|h\|_{L^\infty} \int_{0+}^{\infty} \frac{e^{-x\lambda}}{x} N_q(\lambda,t)d\lambda \\ &\le \|h\|_{L^\infty} \int_{1}^{\infty} \frac{e^{-x\lambda}}{x} C\lambda^{d/2} d\lambda + \|h\|_{L^\infty} \frac{1}{x} N_q(1,t)\end{aligned}$$

and (6.18) follows. □

By Leibniz rule and Lemma 6.1 we have thus shown that $\int_1^\infty \frac{1}{x}\frac{1}{2}\sum_q (-1)^q \mathrm{tr_N}(e^{-x\Delta_q^+(t)})dx$ has a continuous derivative with respect to t given by

$$\begin{aligned}&\frac{d}{dt}\left(\int_1^\infty \frac{1}{x}\frac{1}{2}\sum_q (-1)^q \mathrm{tr_N}(e^{-x\Delta_q^+(t)})dx\right) \\ &\quad = \sum_{q=0}^d (-1)^q \mathrm{tr_N}\left(he^{-\Delta_q(t)}(\mathrm{Id} - Q_q(0,t))\right) .\end{aligned} \tag{6.19}$$

Next we analyze the t-derivative of $\frac{\partial}{\partial s}\frac{1}{\Gamma(s)}\int_0^1 x^{s-1}\frac{1}{2}\sum_q (-1)^q \mathrm{tr_N}(e^{-x\Delta_q^+(t)})dx$.

We first apply formula (6.16) and then integrate by parts to obtain for $\Re s > d/2$

$$\begin{aligned}&-\frac{d}{dt}\left(\frac{1}{\Gamma(s)}\int_0^1 x^{s-1}\frac{1}{2}\sum_q (-1)^q \mathrm{tr_N}(e^{-x\Delta_q^+(t)})dx\right) \\ &= \frac{1}{\Gamma(s)}\int_0^1 x^s \frac{d}{dx}\left(\sum_{q=0}^d (-1)^q \mathrm{tr_N}\left(he^{-x\Delta_q(t)}(\mathrm{Id} - Q_q(0,t))\right)dx\right) \\ &= \frac{1}{\Gamma(s)}\sum_{q=0}^d (-1)^q \mathrm{tr_N}\left(he^{-\Delta_q(t)}(\mathrm{Id} - Q_q(0,t))\right) \\ &\quad - \frac{s}{\Gamma(s)}\int_0^1 x^{s-1}\sum_{q=0}^d (-1)^q \mathrm{tr_N}\left(he^{-x\Delta_q(t)}(\mathrm{Id} - Q_q(0,t))\right)dx \; .\end{aligned}$$

Both terms on the right-hand side of this last equation are smooth functions in s and t, meromorphic in s and holomorphic in a neighborhood of $s = 0$ independent of t, and therefore so is the left-hand side. These properties are obvious for the first term. For the second one they follow from Theorem 2.9 (2) and formula (2.9). In view of $\frac{1}{\Gamma(s)} = \frac{s}{\Gamma(s+1)}$ and $\Gamma(1) = 1$, we obtain

$$-\frac{d}{dt}\left(\frac{\partial}{\partial s}\Big|_{s=0}\frac{1}{\Gamma(s)}\int_0^1 x^{s-1}\frac{1}{2}\sum_q(-1)^q\mathrm{tr}_{\mathrm{N}}(e^{-x\Delta_q^+(t)})dx\right) \tag{6.20}$$

$$= \sum_{q=0}^{d}(-1)^q\mathrm{tr}_{\mathrm{N}}\left(he^{-\Delta_q(t)}(\mathrm{Id} - Q_q(0,t))\right)$$

$$- \mathrm{F.p.}_{s=0}\frac{1}{\Gamma(s)}\int_0^1 x^{s-1}\sum_{q=0}^{d}(-1)^q\mathrm{tr}_{\mathrm{N}}\left(he^{-x\Delta_q(t)}(\mathrm{Id} - Q_q(0,t))\right)dx\ ,$$

where $\mathrm{F.p.}_{s=0}f$ denotes the constant term of the Laurent series expansion of a meromorphic function f at 0. Combining (6.15), (6.19) and (6.20) we conclude that $\log T_{\mathrm{an}}(h,t) = \log T_{\mathrm{an}}(h,t,0)$ is continuous, has a continuous derivative with respect to t and

$$\frac{d}{dt}\log T_{\mathrm{an}}(h,t)$$

$$= \mathrm{F.p.}_{s=0}\frac{1}{\Gamma(s)}\int_0^1 x^{s-1}\sum_{q=0}^{d}(-1)^q\mathrm{tr}_{\mathrm{N}}\left(he^{-x\Delta_q(t)}(\mathrm{Id} - Q_q(0,t))\right)dx\ . \tag{6.21}$$

Next, $\mathrm{tr}_{\mathrm{N}}(he^{-x\Delta_q(t)}(\mathrm{Id} - Q_q(0,t))) = \mathrm{tr}_{\mathrm{N}}(he^{-x\Delta_q(t)}) - \mathrm{tr}_{\mathrm{N}}(hQ_q(0,t))$. We want to verify that

$$\mathrm{F.p.}_{s=0}\frac{1}{\Gamma(s)}\int_0^1 x^{s-1}\sum_{q=0}^{d}(-1)^q\mathrm{tr}_{\mathrm{N}}(he^{-x\Delta_q(t)})dx = 0\ . \tag{6.21'}$$

To verify this, note that

$$\mathrm{F.p.}_{s=0}\frac{1}{\Gamma(s)}\int_0^1 x^{s-1}dx = \mathrm{F.p.}_{s=0}\frac{s}{\Gamma(s+1)}\frac{1}{s} = 1$$

and the heat kernel expansion for the Schwartz kernel $K_q(y,y',x,t)$ of $e^{-x\Delta_q(t)}$ on the diagonal $y = y'$ is of the form

$$K_q(y,y,x,t) = \sum_{j=0}^{d} x^{\frac{j-d}{2}}l_{q,j}(y,t) + O_t(x^{\frac{1}{2}}) \tag{6.22}$$

where $l_{q,j}(y,t)$ are densities defined on M with values in $\mathcal{B}$ and the error term $O_t(x^{\frac{1}{2}})$ is a density which can be bounded by $Cx^{1/2}$, where C is independent of $t \in I$. Thus

$$\mathrm{F.p.}_{s=0}\frac{1}{\Gamma(s)}\int_0^1 x^{s-1}\sum_{q=0}^{d}(-1)^q\mathrm{tr_N}(he^{-x\Delta_q(t)})dx = \sum_{q=0}^{d}(-1)^q\mathrm{tr_N}\left(hl_{q,d}(\cdot,t)\right) .$$

which by Theorem 2.9 (2) and (2.16) is zero. This proves (6.21′). As $Q_q(0,t)$ is a projection $\mathrm{tr_N}(hQ_q(0,t)) = \mathrm{tr_N}(Q_q(0,t)hQ_q(0,t))$ and thus (6.21) and (6.21′) lead to the following:

PROPOSITION 6.2. $\frac{d}{dt}\log T_{\mathrm{an}}(h,t) = \sum_{q=0}^{d}(-1)^{q+1}\mathrm{tr_N}(Q_q(0,t)hQ_q(0,t))$.

Next, we express the terms $\mathrm{tr_N}(Q_q(0,t)hQ_q(0,t))$ in a more explicit way. It is convenient to introduce $P_q(t) = Q_q(0,t)$. Consider $K_q(t) : \mathcal{H}_t^q(M;\mathcal{E}) \to \mathcal{H}_q(M;\mathcal{E})$ defined by

$$K_q(t)(\omega) := P_q(0)e^{th}\omega .$$

Using the decomposition ($\omega \in \mathcal{H}_q(M;\mathcal{E})$) $e^{-th}\omega = e^{-th}\omega_+(t) + \omega_0(t) \in \Lambda_t^{+,q}(M;\mathcal{E}) \oplus \mathcal{H}_t^q(M;\mathcal{E})$ where $\omega_+(t) \in \Lambda^{+,q}(M;\mathcal{E})$ and $\omega_0(t) \in \mathcal{H}_t^q(M;\mathcal{E})$, one verifies that $P_q(t)e^{-th}$ is the right inverse of $K_q(t)$. Therefore, $K_q(t)$ is an isomorphism. This implies that $(K_q(t)K_q(t)^*)^{\frac{1}{2}}$ is a selfadjoint, positive, $\mathcal{A}$-linear operator on $\mathcal{H}_q(M;\mathcal{E})$ and thus admits a determinant with $\det_N(K_q(t)K_q(t)^*)^{1/2} > 0$. Note that $K_q(t)^*$ is given by $P_q(t)e^{th}$ and thus $K_q(t)K_q(t)^*$ can be written as

$$K_q(t)K_q(t)^* = P_q(0)e^{th}P_q(t)e^{th}P_q(0) . \tag{6.23}$$

LEMMA 6.3. $\mathrm{tr_N}(P_q(t)hP_q(t)) = \frac{d}{dt}\log\det_N(K_q(t)K_q(t)^*)^{\frac{1}{2}}$.

Proof. Using Proposition 1.9 we note that

$$\begin{aligned}\frac{d}{dt}\log\det_N\left(K_q(t)K_q(t)^*\right)^{\frac{1}{2}} &= \frac{1}{2}\frac{d}{dt}\log\det_N\left(K_q(t)K_q(t)^*\right) \qquad (6.24)\\ &= \frac{1}{2}\mathrm{tr_N}\left(\frac{d}{dt}\left(K_q(t)K_q(t)^*\right)\left(K_q(t)K_q(t)^*\right)^{-1}\right) .\end{aligned}$$

Notice that $P_q(t) = P_q^+(t)P_q^-(t)$ where $P_q^+(t)$, respectively $P_q^-(t)$, denote the orthogonal projection onto $\mathrm{Null}(d_q(t))$, respectively $\mathrm{Null}(d_{q-1}(t)^*)$. Therefore, $P_q(t)$ is smooth in t as both $P_q^{\pm}(t)$ are smooth as can be deduced from the representation

$$P_q^{\pm}(t) = \frac{1}{2\pi i}\int_C \left(\lambda - e^{\mp th}P_q^{\pm}(0)e^{\mp th}\right)^{-1}d\lambda ,$$

where C is a circle in $\mathbb{C}$ centered at the origin with radius $r > e^{t\|h\|_{L^\infty}}$.

Using (6.23) and writing $\dot{P}_q(t) = \frac{d}{dt}P_q(t)$, we obtain

$$\begin{aligned}\frac{d}{dt}\left(K_q(t)K_q(t)^*\right) &= P_q(0)he^{th}P_q(t)e^{th}P_q(0) + P_q(0)e^{th}\dot{P}_q(t)e^{th}P_q(0)\\ &\quad + P_q(0)e^{th}P_q(t)he^{th}P_q(0) . \qquad (6.25)\end{aligned}$$

To compute $\dot{P}_q(t) = \frac{d}{dt}(P_q(t)^2) = \dot{P}_q(t)P_q(t) + P_q(t)\dot{P}_q(t)$ we consider the orthogonal decomposition $\Lambda^q(M;\mathcal{E}) = \mathcal{H}_t^q(M;\mathcal{E}) \oplus \Lambda_t^{+,q}(M;\mathcal{E}) \oplus \Lambda_t^{-,q}(M;\mathcal{E})$. An element $\omega \in \Lambda^q(M;\mathcal{E})$ can be uniquely written as (cf. (6.4)-(6.7))

$$\omega = \omega_0(t) + e^{-th}\omega_+(t) + e^{th}\omega_-(t)$$

where $\omega_\pm(t) \in \Lambda^{\pm,q}(M;\mathcal{E})$ and $\omega_0(t) = P_q(t)\omega$. We conclude that

$$0 = \frac{d}{dt}\omega = \dot{\omega}_0(t) + e^{-th}\dot{\omega}_+(t) + e^{th}\dot{\omega}_-(t) - he^{-th}\omega_+(t) + he^{th}\omega_-(t) \ . \quad (6.26)$$

Note that $\dot{\omega}_\pm(t) \in \Lambda^{\pm,q}(M;\mathcal{E})$ and therefore $e^{-th}\dot{\omega}_+(t) \in \Lambda_t^{+,q}(M;\mathcal{E})$ and $e^{th}\dot{\omega}_-(t) \in \Lambda_t^{-,q}(M;\mathcal{E})$. Applying $P_q(t)$ to (6.26) leads to

$$0 = P_q(t)\dot{\omega}_0(t) - P_q(t)he^{-th}\omega_+(t) + P_q(t)he^{th}\omega_-(t) \ .$$

In terms of $P_q^\pm(t)$, the orthogonal projectors $\Lambda^q(M;\mathcal{E}) \to \Lambda_t^{\pm,q}(M;\mathcal{E})$, the above equality becomes

$$P_q(t)\dot{P}_q(t) = P_q(t)hP_q^+(t) - P_q(t)hP_q^-(t) \ . \quad (6.27)$$

Observe that the projectors $P_q(t)$ and therefore $\dot{P}_q(t)$ are selfadjoint to conclude that

$$\dot{P}_q(t)P_q(t) = P_q^+(t)hP_q(t) - P_q^-(t)hP_q(t) \ . \quad (6.28)$$

Combining (6.27) and (6.28) we obtain

$$\begin{aligned}\dot{P}_q(t) &= \dot{P}_q(t)P_q(t) + P_q(t)\dot{P}_q(t)\\ &= P_q(t)hP_q^+(t) + P_q^+(t)hP_q(t) - P_q(t)hP_q^-(t) - P_q^-(t)hP_q(t) \ . \quad (6.29)\end{aligned}$$

Compose (6.29) to the left with $P_q(0)e^{th}$ and to the right with $e^{th}P_q(0)$ to get

$$\begin{aligned}&P_q(0)e^{th}\dot{P}_q(t)e^{th}P_q(0)\\ &\quad = P_q(0)e^{th}P_q(t)hP_q^+(t)e^{th}P_q(0) + P_q(0)e^{th}P_q^+(t)hP_q(t)e^{th}P_q(0)\\ &\qquad - P_q(0)e^{th}P_q(t)hP_q^-(t)e^{th}P_q(0)\\ &\qquad\quad - P_q(0)e^{th}P_q^-(t)hP_q(t)e^{th}P_q(0) \ . \quad (6.30)\end{aligned}$$

To simplify (6.30), notice that $d_q(t)^* = e^{th}d_q^*e^{-th}$ and therefore $e^{th}\mathcal{H}_q(M;\mathcal{E}) \subset \mathcal{H}_t^q(M;\mathcal{E}) \oplus \Lambda_t^{-,q}(M;\mathcal{E})$ which implies that $P_q^+(t)e^{th}P_q(0) = 0$. Taking the adjoint, we conclude that $P_q(0)e^{th}P_q^+(t) = 0$. Thus, the first two terms on the right-hand side of (6.30) are zero. Note also that $P_q^-(t)e^{th}P_q(0) = (\mathrm{Id} - P_q(t))e^{th}P_q(0)$ and $P_q(0)e^{th}P_q^-(t) = P_q(0)e^{th}(\mathrm{Id} - P_q(t))$. Applying these three observations to (6.25) and taking into consideration the definition of

$K_q(t)$ one obtains

$$\begin{aligned}
&\frac{d}{dt}\big(K_q(t)K_q(t)^*\big)\\
&\quad = P_q(0)he^{th}P_q(t)e^{th}P_q(0) - P_q(0)e^{th}P_q(t)h\big(\mathrm{Id} - P_q(t)\big)e^{th}P_q(0)\\
&\qquad + P_q(0)e^{th}P_q(t)he^{th}P_q(0) - P_q(0)e^{th}\big(\mathrm{Id} - P_q(t)\big)hP_q(t)e^{th}P_q(0)\\
&\quad = 2P_q(0)e^{th}P_q(t)hP_q(t)e^{th}P_q(0)\\
&\quad = 2K_q(t)P_q(t)hP_q(t)K_q(t)^* \ . \qquad (6.31)
\end{aligned}$$

Substituting (6.31) into (6.24) one obtains

$$\begin{aligned}
\frac{d}{dt}\log\det{}_N\big(K_q(t)K_q(t)^*\big)^{\frac{1}{2}} &= \mathrm{tr}_{\mathrm{N}}\big(K_q(t)P_q(t)hP_q(t)K_q(t)^*\big)\big(K_q(t)K_q(t)^*\big)^{-1}\\
&= \mathrm{tr}_{\mathrm{N}}\big(K_q(t)P_q(t)hP_q(t)K_q(t)^{-1}\big)\\
&= \mathrm{tr}_{\mathrm{N}}\big(P_q(t)hP_q(t)\big)
\end{aligned}$$

which concludes the proof of the lemma. □

Using that $K_q(0) = \mathrm{Id}$ and therefore that $\det_N(K_q(0)K_q(0)^*)^{\frac{1}{2}} = 1$, Proposition 6.2 together with Lemma 6.3 lead to

$$\begin{aligned}
\log T_{\mathrm{an}}(h,t) &= \log T_{\mathrm{an}}(h,0) + \sum_{q=0}^{d}(-1)^{q+1}\int_0^t \frac{d}{dt}\log\det{}_N\big(K_q(t)K_q(t)^*\big)^{\frac{1}{2}}dt\\
&= \log T_{\mathrm{an}}(h,0) + \sum_{q=0}^{d}(-1)^{q+1}\log\det{}_N\big(K_q(t)K_q(t)^*\big)^{\frac{1}{2}} \ . \qquad (6.32)
\end{aligned}$$

In section 4.1 we introduced the $\mathcal{A}$-linear isomorphisms

$$\theta_q : \mathrm{Null}\,\Delta_q^{\mathrm{comb}} \longrightarrow \mathcal{H}_q(M;\mathcal{E}) \ ,$$

the inverse of $\overline{\mathrm{Int}}^{(q)}$ and the metric part of the Reidemeister torsion $T_{\mathrm{met}}(\tau) \equiv T_{\mathrm{met}}(M,g,\tau,\mathcal{W})$, cf. (4.4), defined by

$$\log T_{\mathrm{met}}(\tau) = \frac{1}{2}\sum_{q=0}^{d}(-1)^q\log\det{}_N(\theta_q^*\theta_q) \ .$$

By applying Theorem 5.5 (5) we show that

LEMMA 6.4. *For t sufficiently large, the following statements hold:*

$$\begin{aligned}
&\log\det{}_N\big(K_q(t)K_q(t)^*\big)^{\frac{1}{2}}\\
&\quad = \log\det{}_N(\theta_q^*\theta_q)^{\frac{1}{2}} + q\beta_q t + \beta_q\left(\frac{d-2q}{4}\right)\log\left(\frac{t}{\pi}\right) + O(t^{-1}) \qquad (6.33)
\end{aligned}$$

and

$$\sum_{q=0}^{d}(-1)^{q+1}\log\det_N\left(K_q(t)K_q(t)^*\right)^{\frac{1}{2}}$$
$$= -\log T_{\mathrm{met}}(M,g,\tau,\mathcal{W}) + \sum_{q=0}^{d}(-1)^{q+1}q\beta_q t$$
$$+\sum_{q=0}^{d}(-1)^{q+1}\beta_q\frac{d-2q}{4}\log\left(\frac{t}{\pi}\right) + O(t^{-1})\ . \qquad (6.34)$$

Proof. Summing with respect to q, statement (6.34) follows directly from statement (6.33) and the definition $\log T_{\mathrm{met}}(\tau)=\frac{1}{2}\sum_{q=0}^{d}(-1)^q\log\det_N(\theta_q^*\theta_q)$. To prove (6.33) observe that if one represents $F_q = \mathrm{Int}^{(q)} : \Lambda^q(M,\mathcal{E}) \to \mathcal{C}^q$, $M_q : \Lambda^q(M,\mathcal{E}) \to \Lambda^q(M,\mathcal{E})$, the multiplication by e^{th}, and $f_q(t) : \Lambda^q(M,\mathcal{E})_{\mathrm{sm}} \to \mathcal{C}^q$ (cf. Theorem 5.5 (5)) as 3×3 matrices, with respect to the Hodge decompositions, cf. (1.20), then $(\theta_q)^{-1} = F_{11}^q$, $K_q = M_{11}^q$ and $f_{11}^q(t) = (\frac{\pi}{t})^{\frac{d-2q}{4}}e^{-tq}F_{11}^qM_{11}^q$. Theorem 5.5 (5) implies that

$$\log\det_N\left(f_{11}^q(t)f_{11}^q(t)^*\right) = O(t^{-1})\ , \qquad (6.35)$$

which combined with Proposition 1.9 implies that

$$\log\det_N\left(K_q(t)K_q(t)^*\right) - \beta_q\frac{(d-2q)}{2}\log\left(\frac{t}{\pi}\right)$$
$$- \beta_q 2qt - \log\det_N\left(\theta_q(t)\theta_q(t)^*\right) = O(t^{-1})\ . \ (6.36)$$

hence (6.33). □

LEMMA 6.5. *For $t\to\infty$,*

$$\log T_{\mathrm{sm}}(h,t) = \log T_{\mathrm{comb}}(\tau)$$
$$+\tfrac{1}{2}\left(\sum_{q=0}^{d}(-1)^{q+1}q(\beta_q - m_q l)\right)(2t - \log t + \log\pi) + o(1)\ . \qquad (6.37)$$

Proof. Recall that $\log T_{\mathrm{sm}}(h,t)$ is a real number defined by

$$\log T_{\mathrm{sm}}(h,t) = \tfrac{1}{2}\left(\sum_{q=0}^{d}(-1)^{q+1}q\log\det_{\mathrm{N}}\Delta_q(t)_{\mathrm{sm}}\right) \qquad (6.38)$$

and that for any $0 < C < \infty$ (cf. (6.13))

$$\log \det_N \Delta_q(t)_{sm} = -\frac{\partial}{\partial s}\Big|_{s=0} \frac{1}{\Gamma(s)} \int_0^C \frac{dx}{x} x^s \big(\mathrm{tr}_N(e^{-x\Delta_q(t)_{sm}}) - \beta_q\big) - \int_C^\infty \frac{dx}{x} \big(\mathrm{tr}_N(e^{-x\Delta_q(t)_{sm}}) - \beta_q\big) \tag{6.39}$$

where $\beta_q = \dim_N \mathrm{Null}\, \Delta_q^{\mathrm{comb}} = \dim_N \mathrm{Null}\, \Delta_q(t)$.

Consider $\widetilde{\Delta_q(t)}_{\mathrm{sm}}$ the Laplacian of the cochain complex $(\Lambda^q(M,\mathcal{E})_{\mathrm{sm}}(t), \tilde{d}_q(t))$ and note that

$$\widetilde{\Delta(t)}_{q,\mathrm{sm}} = \frac{\pi}{t} e^{2t} \Delta_q(t)_{\mathrm{sm}} \ . \tag{6.40}$$

Note also that

$$\int_C^\infty \frac{dx}{x} \big(\mathrm{tr}_N(e^{-xA}) - \beta_q\big) = \int_{0+}^\infty \left(\int_C^\infty \frac{1}{x} e^{-\mu x} dx \right) dF_A(\mu) = \int_{0+}^\infty \left(\int_C^\infty e^{-\mu y} dy \right) F_A(\mu) \tag{6.41}$$

where $F_A(\mu)$ has been introduced in (1.9) and A is $\widetilde{\Delta(t)}_{q,\mathrm{sm}}, \Delta(t)_{q,\mathrm{sm}}$, or Δ_q^{comb}. The last equality follows from integration by parts.

By a change of variable of integration, $y = \frac{t}{\pi} e^{-2t} x$, and in view of (6.41) we obtain

$$\int_{C\frac{\pi}{t}e^{2t}}^\infty \frac{dx}{x} \big(\mathrm{tr}_N(e^{-x\Delta_q(t)_{sm}}) - \beta_q\big) = \int_{0+}^\infty \left(\int_C^\infty e^{-\mu y} dy \right) F_{\widetilde{\Delta_q(t)}_{sm}}(\mu) d\mu \ . \tag{6.42}$$

From Theorem 5.5 (5) and Proposition 1.18 we conclude, arguing as in Proposition 5.6, that there exists $t_0 > 0$ such that, for $t \geq t_0$ and $0 \leq q \leq d$, $F_{\widetilde{\Delta_q(t)}_{sm}}(\mu) \leq F_{\Delta_q^{\mathrm{comb}}}(10\mu)$. For $t \geq t_0$ and $0 \leq q \leq d$, the above computations lead to

$$\int_{C\frac{\pi}{t}e^{2t}}^\infty \frac{dx}{x} \big(\mathrm{tr}_N(e^{-x\Delta_q(t)_{sm}}) - \beta_q\big) \leq \int_{\frac{C}{10}}^\infty \frac{dx}{x} \big(\mathrm{tr}_N(e^{-x\Delta_q^{\mathrm{comb}}}) - \beta_q\big) \ . \tag{6.43}$$

Taking into account that $(M, \mathcal{W})$ is of determinant class, one can choose $C > 0$ such that

$$\int_{\frac{C}{10}}^\infty \frac{dx}{x} \big(\mathrm{tr}_N(e^{-x\Delta_q^{\mathrm{comb}}}) - \beta_q\big) \leq \epsilon \ . \tag{6.44}$$

Therefore, for all $t \geq t_0$, $0 \leq q \leq d$, it suffices to consider

$$-\frac{\partial}{\partial s}\Big|_{s=0} \frac{1}{\Gamma(s)} \int_0^{C\frac{\pi}{t}e^{2t}} \frac{dx}{x} x^s \big(\mathrm{tr}_N(e^{-x\Delta_q(t)_{sm}}) - \beta_q\big) \ . \tag{6.45}$$

When expressed with respect to the basis given in Theorem 5.5, $\Delta_q(t)_{\text{sm}}$ is of the form $\frac{t}{\pi}e^{-2t}(\Delta_q^{\text{comb}} + O(t^{-\frac{1}{2}}))$. By a change of variable of integration, $y = \frac{t}{\pi}e^{-2t}x$, the expression (6.45) can be written as a sum of two terms, $I_q + II_q$, where

$$\begin{aligned} I_q &= -(\log\pi - \log t + 2t)III_q(s)_{|_{s=0}} \\ II_q &= -\frac{d}{ds}\bigg|_{s=0} III_q(s) \end{aligned} \tag{6.47}$$

and

$$III_q(s) = \frac{1}{\Gamma(s)}\int_0^C \frac{dy}{y} y^s \left(\mathrm{tr}_{\mathrm{N}}(e^{-y(\Delta_q^{\text{comb}} + O(t^{-\frac{1}{2}}))}) - \beta_q\right). \tag{6.48}$$

We first evaluate $III_q(s)$ at $s = 0$. Use $\frac{1}{\Gamma(s)} = \frac{s}{\Gamma(s+1)}$ and integrate by parts to obtain, for arbitrary $\delta > 0$,

$$\begin{aligned} III_q(s)|_{s=0} &= \frac{1}{\Gamma(s+1)}\int_0^\delta dy\frac{d}{dy}(y^s)\left(\mathrm{tr}_{\mathrm{N}}(e^{-y(\Delta_q^{\text{comb}} + O(t^{-\frac{1}{2}}))}) - \beta_q\right)\Big|_{s=0} \\ &= \mathrm{tr}_{\mathrm{N}}(e^{-\delta(\Delta_q^{\text{comb}} + O(t^{-\frac{1}{2}}))}) - \beta_q \\ &\quad + \int_0^\delta dy\,\mathrm{tr}_{\mathrm{N}}\left((\Delta_q^{\text{comb}} + O(t^{-\frac{1}{2}}))e^{-y(\Delta_q^{\text{comb}} + O(t^{-\frac{1}{2}}))}\right). \end{aligned} \tag{6.49}$$

Taking the limit as $\delta \to 0$ in (6.49) leads to

$$III_q(s)\big|_{s=0} = m_q l - \beta_q\,. \tag{6.50}$$

To compute $II_q = -\frac{d}{ds}\big|_{s=0} III_q(s)$ recall that

$$\begin{aligned} \log\det{}_{\mathrm{N}}\,\Delta_q^{\text{comb}} &= -\frac{\partial}{\partial s}\bigg|_{s=0}\frac{1}{\Gamma(s)}\int_0^C \frac{dy}{y}y^s\left(\mathrm{tr}_{\mathrm{N}}(e^{-y(\Delta_q^{\text{comb}})}) - \beta_q\right) \\ &\quad - \int_C^\infty \frac{dy}{y}\left(\mathrm{tr}_{\mathrm{N}}(e^{-y(\Delta_q^{\text{comb}})}) - \beta_q\right) \end{aligned} \tag{6.51}$$

and use the estimate $(0 \le y \le C)$

$$\left|\mathrm{tr}_{\mathrm{N}}(e^{-y(\Delta_q^{\text{comb}})}) - \mathrm{tr}_{\mathrm{N}}(e^{-y(\Delta_q^{\text{comb}} + O(t^{-\frac{1}{2}}))})\right| \le yO(t^{-\frac{1}{2}}) \tag{6.52}$$

to conclude, together with (6.49), that

$$|II_q - \log\det{}_{\mathrm{N}}\,\Delta_q^{\text{comb}}| \le \epsilon + O(t^{-\frac{1}{2}})\,. \tag{6.53}$$

Combining (6.40)-(6.43) and (6.50)-(6.53) we conclude that for given $\epsilon > 0$, there exists $t_\epsilon > 0$ so that for all $t > t_\epsilon$,

$$\left| \log T_{\mathrm{sm}}(h,t) - \log T_{\mathrm{comb}}(\tau) - \frac{1}{2}\sum_{q=0}^{d}(-1)^{q+1}q(\beta_q - m_q l)\left(2t - \log\frac{t}{\pi}\right)\right| \leq 3\epsilon \,. \tag{6.54}$$

□

Proof of Theorem A. First note that $\log T_{\mathrm{la}}(h,t) = \log T_{\mathrm{an}}(h,t) - \log T_{\mathrm{sm}}(h,t)$. Therefore the asymptotic expansion of $\log T_{\mathrm{la}}(h,t)$ is obtained from the expansions of $\log T_{\mathrm{an}}(h,t)$ and $\log T_{\mathrm{sm}}(h,t)$. The asymptotic expansion for $\log T_{\mathrm{an}}(h,t)$ follows from (6.32) and Lemma 6.4 together with the fact that, since d is odd, $\chi(M,\tau) = \sum_{q=0}^{d}(-1)^q\beta_q = 0$. The asymptotic expansion for $\log T_{\mathrm{sm}}(h,t)$ is contained in Lemma 6.5. □

6.2 Comparison theorem for Witten's deformation of the analytic torsion. The family of operators $\Delta_q(t)$ is a family with parameter of order 2 and weight 1 (cf. [Sh1] and section 3). It fails to be elliptic with parameter, but only at the critical points of the Morse function h. We can therefore use the Mayer-Vietoris type formula for determinants (cf. section 3) to localize the failure of the family $\Delta_q(t)$ to be elliptic with parameter and thus obtain a relative result which compares the asymptotic expansions of Witten's deformation of the analytic torsion corresponding to two different systems $(M^d, h, g, \mathcal{W})$ and $(M'^d, h', g', \mathcal{W})$ where the manifolds M and M' have the same fundamental group Γ and (h,g) and (h',g') are generalized triangulations for M respectively M'.

Theorem B. *Let d be odd. Suppose that $\tau = (h,g)$ and $\tau' = (h',g')$ are generalized triangulations with $\#\mathrm{Cr}_q(h) = \#\mathrm{Cr}_q(h')$ $(0 \leq q \leq d)$, and that $(M,\mathcal{W})$ and $(M',\mathcal{W})$ are of determinant class. Then the following statements hold:*

(1) *The free term $\mathrm{FT}(\log T_{\mathrm{la}}(h,t) - \log T_{\mathrm{la}}(h',t))$ of the asymptotic expansion of $\log T_{\mathrm{la}}(h,t) - \log T_{\mathrm{la}}(h',t)$ is given by*

$$\mathrm{FT}(\log T_{\mathrm{la}}(h,t) - \log T_{\mathrm{la}}(h',t)) = \int_{M\setminus \mathrm{Cr}(h)} a_0(h,\epsilon=0) - \int_{M'\setminus \mathrm{Cr}(h')} a_0(h',\epsilon=0) \tag{6.55}$$

where the densities $a_0(h,\epsilon=0)$ and $a_0(h',\epsilon=0)$ are smooth forms of degree d and are given by explicit local formulas and the difference appearing in the right-hand side of (6.55) is taken in the sense (6.56) explained in the remark below.

(2) *Due to the assumption that d is odd,*

$$a_0(h,\epsilon=0,x) + a_0(d-h,\epsilon=0,x) = 0 \,.$$

REMARK: The integral $\int_{M\setminus\mathrm{Cr}(h)} a_0(h,\epsilon=0)$ need not be convergent and the difference on the right-hand side of (6.55) should be understood in the following sense: In view of the definition of a generalized triangulation, there exist neighborhoods V of $\mathrm{Cr}(h)$ and V' of $\mathrm{Cr}(h')$, a diffeomorphism $f: V \to V'$ and a smooth bundle isomorphism $F: \mathcal{E}_{|V} \to \mathcal{E}'_{|V'}$ so that f and F intertwine the functions h and h', the metrics g and g' and the Laplace operators Δ_q and Δ'_q. Define

$$\int\limits_{M\setminus\mathrm{Cr}(h)} a_0(h,\varepsilon=0) - \int\limits_{M'\setminus\mathrm{Cr}(h')} a_0(h',\varepsilon=0) = \int\limits_{M\setminus V} a_0(h,\varepsilon=0) - \int\limits_{M'\setminus V'} a_0(h',\varepsilon=0)\,. \tag{6.56}$$

Clearly, the definition is independent of the choice of V and V'.

As an application of Theorem A and Theorem B we obtain the following result:

COROLLARY C. *Let M and M' be two closed manifolds with the same fundamental group Γ and the same dimension d and let $\mathcal{W}$ be an $(\mathcal{A},\Gamma^{\mathrm{op}})$-Hilbert module of finite type. Suppose that $\tau=(h,g)$ and $\tau'=(h',g')$ are generalized triangulations with $\#\mathrm{Cr}_q(h)=\#\mathrm{Cr}_q(h')$ $(0\le q\le d)$, and that $(M,\mathcal{W})$ and $(M',\mathcal{W})$ are of determinant class. Let $T'_{\mathrm{an}}=T_{\mathrm{an}}(M',g',\mathcal{W})$ and $T'_{\mathrm{Re}}=T_{\mathrm{Re}}(M',\tau',\mathcal{W})$. Then*

$$\log T_{\mathrm{an}} - \log T'_{\mathrm{an}} = \log T_{\mathrm{Re}} - \log T'_{\mathrm{Re}}\ .$$

In the remainder of this subsection we prove Theorem B and Corollary C.

Let M be a manifold equipped with a generalized triangulation $\tau=(h,g)$. Let $x_{q;j}\in\mathrm{Cr}_q(h)$ be a critical point of h of index q and U_{qj} a system of H-neighborhoods of $x_{q;j}$ (cf. Definition 5.1). Introduce the manifolds

$$M_I := M\setminus \cup_{q,j} U'_{qj}\ ;\quad M_{II} := \cup_{q,j}\overline{U'_{qj}}\ ,$$

where U'_{qj} is defined as in Definition 5.1. Both manifolds M_I and M_{II} have the same boundary, given by a disjoint union of spheres of dimension $d-1$.

To make the notation more pleasant, we will write, as in section 5, ∇h for $\mathrm{grad}_g h$. Fix $\varepsilon>0$ and consider the operator $\Delta_q(t)+\varepsilon$. Its symbol with respect to arbitrary coordinates (φ,Φ) of $(M,\mathcal{E}\to M)$ is of the form

$$a_2(x,\xi)+t^2\|\nabla h\|^2 + a_1(x,\xi)+tL_q(x)+\epsilon \tag{6.57}$$

where $a_i:\mathbb{R}^d\times\mathbb{R}^d\to End(\Lambda^q(\mathbb{R}^d)\otimes\mathcal{W})$ $(i=1,2)$ are homogeneous of degree i in ξ, where $\|\nabla h\|^2:\mathbb{R}^d\to\mathbb{R}$ is given by

$$\|\nabla h\|^2=\sum_{1\le i,j\le d} g^{ij}\frac{\partial h}{\partial x_i}\frac{\partial h}{\partial x_j}$$

and where $L_q:\mathbb{R}^d\to End(\Lambda^q(\mathbb{R}^d))$ is the operator $L_q=\mathcal{L}_{\nabla h}+\mathcal{L}^*_{\nabla h}$ of order

0 with $\mathcal{L}_{\nabla h}$ denoting the Lie-derivative of q-forms along the vector field

$$\nabla h = \sum_{i,j} g^{ij} \frac{\partial h}{\partial x_i} \frac{\partial}{\partial x_j} \, .$$

The operator $\mathcal{L}^*_{\nabla h}$ is the adjoint of $\mathcal{L}_{\nabla h}$ with respect to the metric g and is given by (cf. [HSj1, Appendix formulas A.1.9, A.1.6])

$$\mathcal{L}^*_{\nabla h} = -(-1)^{q(d+q)} R_{d-q} \mathcal{L}_{\nabla h} R_q \tag{6.58}$$

where $R_q : \Lambda^q(\mathbb{R}^d) \to \Lambda^{d-q}(\mathbb{R}^d)$ is the Hodge operator associated to the metric $\varphi^* g$. Recall that we have denoted by $\mathrm{Cr}(h)$ the set of all critical points of h. Set $M^* := M \setminus \mathrm{Cr}(h)$. For an arbitrary chart (φ, Φ) of $(M^*, \mathcal{E}|_{M^*} \to M^*)$, define, as discussed in section 3.2, the symbol expansion $\sum_{j \geq 0} r_{-2-j}(h, \varepsilon, x, \xi, t, \mu)$ of the resolvent $(\mu - \Delta_q(t) - \varepsilon)^{-1}$ inductively:

$$r_{-2}(h, \varepsilon, x, \xi, t, \mu) = \left(\mu - a_2(x, \xi) - t^2 \|\nabla h\|^2\right)^{-1}$$

and, for $j \geq 1$,

$$\begin{aligned} r_{-2-j} = {} & \left(\mu - a_2 - t^2 \|\nabla h\|^2\right)^{-1} \sum_{\substack{\leq |\alpha| \leq 2 \\ l + |\alpha| = j}} \frac{1}{\alpha!} \partial_\xi^\alpha a_2 (D_x)^\alpha r_{-2-l} \\ & + \left(\mu - a_2 - t^2 \|\nabla h\|^2\right)^{-1} \sum_{\substack{0 \leq |\alpha| \leq 1 \\ l + |\alpha| = j}} \partial_\xi^\alpha (a_1 + t L_q)(D_x)^\alpha r_{-2-l} \\ & + \left(\mu - a_2 - t^2 \|\nabla h\|^2\right)^{-1} \varepsilon r_{-j} \, . \end{aligned} \tag{6.59}$$

Note that r_{-2-j} has the following homogeneity property: for $\lambda \in \mathbb{R} \setminus \{0\}$

$$r_{-2-j}(h, \varepsilon, x, \lambda\xi, \lambda t, \lambda^2 \mu) = \lambda^{-2-j} r_{-2-j}(h, \varepsilon, x, \xi, t, \mu) \, . \tag{6.60}$$

For later use, we introduce the densities $a_0(h, \varepsilon, x)$ on M^* with values in $\mathbb{R}$ (cf. (3.4)), defined with respect to the chart (φ, Φ) and arbitrary ε as

$$\begin{aligned} a_0(h, \varepsilon, x) = {} & \frac{\partial}{\partial s}\bigg|_{s=0} \left(\frac{1}{2\pi}\right)^d \frac{1}{2\pi i} \int_{\mathbb{R}^d} d\xi \\ & \cdot \int_\Gamma d\mu \mu^{-s} \operatorname{tr}_N r_{-2-d}(h, \varepsilon, x, \xi, t = 1, \mu) \\ = {} & \frac{-1}{(2\pi)^d} \int_{\mathbb{R}^d} d\xi \int_0^\infty d\mu \operatorname{tr}_N \left(r_{-2-d}(h, \varepsilon, x, \xi, t = 1, -\mu)\right) \, . \end{aligned} \tag{6.61}$$

Since we will be obliged to work with two functions h and h' at the same time we will write $\Delta_q(H, t)$ instead of (the abbreviated form) $\Delta_q(t)$.

PROPOSITION 6.6. *Assume that $(M^d, \tau = (h,g), \mathcal{W})$ and $(M'^d, \tau' = (h',g'), \mathcal{W})$ satisfy the hypothesis of Theorem B except with d not necessarily odd. Then for any $\varepsilon > 0$*

(i) $\log\det_N(\Delta_q(h,t)+\varepsilon) - \log\det_N(\Delta_q(h',t)+\varepsilon)$ *has a asymptotic expansion for* $t \to \infty$ *whose free term is denoted by* $\overline{a}_0 := \overline{a}_0(h,h',\varepsilon)$.

(ii) *The coefficient* $\overline{a}_0$ *can be represented in the form*

$$\overline{a}_0 = \int_{M_I} a_0(h,\varepsilon,x) - \int_{M'_I} a_0(h',\varepsilon,x') \tag{6.62}$$

where $a_0(h,\varepsilon,x)$ *and* $a_0(h',\varepsilon,x')$ *are the densities introduced in* (6.61) *for arbitrary* ε.

(iii) *If* $\dim M = d$ *is odd then for arbitrary* $\varepsilon > 0$, $x \in M$

$$a_0(h,\varepsilon,x) + a_0(d-h,\varepsilon,x) = 0\ . \tag{6.63}$$

With the same identity for h' *one then obtains*

$$\overline{a}_0(h,h',\varepsilon) + \overline{a}_0(d-h,d-h',\varepsilon) = 0\ .$$

Proof. The proof is based on Theorem 3.6 (Mayer-Vietoris type formula). Note that $\Delta_q(h,t)+\varepsilon$ is a family of invertible, selfadjoint, elliptic operators with parameter t of order 2 and weight 1 for any $\varepsilon > 0$. The same is true for the operators $(\Delta_q^I(h,t)+\varepsilon)_D$ and $(\Delta_q^{II}(h,t)+\varepsilon)_D$ obtained by restricting $\Delta_q(h,t)+\varepsilon$ to M_I and M_{II}, respectively, and imposing Dirichlet boundary conditions. We can therefore apply Theorem 3.6. Denote by $R_{DN}(h,t,\varepsilon)$ the Dirichlet to Neumann operator defined in section 3.3. We conclude from Theorem 3.6 (4) that $R_{DN}(h,t,\varepsilon)$ is an invertible pseudodifferential operator with parameter t of order 1 and weight 2 and from Theorem 3.6 (2) we conclude that $R_{DN}(h,t,\varepsilon)$ is elliptic with parameter t. It follows from Theorem 3.2 (2), in view of the fact that $R_{DN}(h,t,\varepsilon)$ is elliptic with parameter, self adjoint, positive and invertible that π is an Agmon angle uniformly in t for a fixed ϵ. According to Theorem 3.4, $\log\det_N R_{DN}(h,t,\varepsilon)$ has an asymptotic expansion for $t \to \infty$. Inspecting the principal symbol of $(\Delta_q^I(h,t)+\varepsilon)_D$ one observes that $(\Delta_q^I(h,t)+\varepsilon)_D$ is a family of invertible, selfadjoint differential operators with parameter of order 2 and weight 1 which is elliptic with parameter. From Theorem 3.5 we therefore conclude that $\log\det_N(\Delta_q^I(h,t)+\varepsilon)_D$ admits an asymptotic expansion as $t \to \infty$. Finally $(\Delta_q^{II}(h,t)+\varepsilon)_D$ is a family of invertible selfadjoint operators with parameter of order 2 and weight 1, which is, however, not elliptic with parameter.

Of course the same considerations can be made for the system (M',h',g') to conclude that $\log\det_N R_{DN}(h',t,\varepsilon)$ and $\log\det_N(\Delta_q^I(h',t)+\varepsilon)_D$ have both asymptotic expansions for $t \to \infty$. Applying the Mayer-Vietoris type formula (Theorem 3.6 (3)) for $\log\det_N(\Delta_q(h,t)+\varepsilon)$ and $\log\det_N(\Delta_q(h',t)+\varepsilon)$

we obtain for the difference

$$\begin{aligned}
&\log\det{}_N\big(\Delta_q(h,t)+\varepsilon\big)-\log\det{}_N\big(\Delta_q(h',t)+\varepsilon\big) \qquad (6.64)\\
&\quad=\log\det{}_N\big(\Delta_q^I(h,t)+\varepsilon\big)_D-\log\det{}_N\big(\Delta_q^I(h',t)+\varepsilon\big)_D\\
&\qquad+\log\det{}_N\big(\Delta_q^{II}(h,t)+\varepsilon\big)_D-\log\det{}_N\big(\Delta_q^{II}(h',t)+\varepsilon\big)_D\\
&\qquad+\log\det{}_N R_{DN}(h,t,\varepsilon)-\log\det{}_N R_{DN}(h',t,\varepsilon)\\
&\qquad+\log\bar c(h,t,\varepsilon)-\log\bar c(h',t,\varepsilon)\ .
\end{aligned}$$

Note that M_{II} and M'_{II} are isometric and $\mathcal{E}|_{M_{II}}$ as well as $\mathcal{E}'|_{M'_{II}}$ are trivial. Consequently

$$\log\det{}_N\big(\Delta_q^{II}(h,t)+\varepsilon\big)_D=\log\det{}_N\big(\Delta_q^{II}(h',t)+\varepsilon\big)_D\ .$$

Due to our definition of H-coordinates the isometry between M_{II} and M'_{II} extends to neighborhoods of M_{II} and M'_{II}. As a consequence we conclude from Theorem 3.6(3)and Theorem 3.4 that $\bar c(h,t,\varepsilon)=\bar c(h',t,\varepsilon)$ and that $\log\det_N R_{DN}(h,t,\varepsilon)$ and $\log\det_N R_{DN}(h',t,\varepsilon)$ have identical asymptotic expansions. We have therefore proved that

$$\log\det{}_N\big(\Delta_q(h,t)+\varepsilon\big)-\log\det{}_N\big(\Delta_q(h',t)+\varepsilon\big)$$

has an asymptotic expansion as $t\to\infty$ which is identical to the asymptotic expansion for $\log\det{}_N(\Delta_q^I(h,t)+\varepsilon)_D-\log\det{}_N(\Delta_q^I(h',t)+\varepsilon)_D$. According to Theorem 3.5 the free term in the asymptotic expansions of both $\log\det_N(\Delta_q^I(h,t)+\varepsilon)_D$ and $\log\det_N(\Delta_q^I(h',t)+\varepsilon)_D$ consists of a boundary contribution and a contribution from the interior. Recall that ∂M_I and $\partial M'_I$ are isometric and that in collar neighborhoods of ∂M_I and of $\partial M'_I$ the symbols of $(\Delta_q^I(h,t)+\varepsilon)_D$ and $(\Delta_q^I(h',t)+\varepsilon)_D$ are identical when expressed in (H)-coordinates. Therefore the boundary contributions are the same and the free term in the asymptotic expansion of $\log\det_N(\Delta_q^I(h,t)+\varepsilon)_D-\log\det_N(\Delta_q^I(h',t)+\varepsilon)_D$ is given by (cf. formula (3.4))

$$\bar a_0=\int_{M_I}a_0(h,\varepsilon,x)-\int_{M'_I}a_0(h',\varepsilon,x') \qquad (6.65)$$

where the densities $a_0(h,\varepsilon,x)$ and $a_0(h',\varepsilon,x')$ are given by (6.61).

Noting that $a_0(h,\varepsilon,x)$ and $a_0(h',\varepsilon,x')$ are identical on $M_{II}\backslash\mathrm{Cr}(h)\cong M'_{II}\backslash\mathrm{Cr}(h')$ statement (ii) follows. Towards (iii), observe that as M is of odd dimension, the quantity $r_{-d-2}(h,\varepsilon,x,\xi,t,\mu)$ defining $a_0(h,\varepsilon,x)$ satisfies, according to (6.57) and (6.59),

$$r_{-d-2}(d-h,\varepsilon,x,\xi,t,\mu)=r_{-d-2}(h,\varepsilon,x,\xi,-t,\mu) \qquad (6.66)$$

and, according to (6.60)

$$r_{-d-2}(h,\varepsilon,x,-\xi,-t,\mu)=-r_{-d-2}(h,\varepsilon,x,\xi,t,\mu)\ . \qquad (6.67)$$

Therefore, $r_{-d-2}(h,\varepsilon,x,\xi,t,\mu)+r_{-d-2}(d-h,\varepsilon,x,\xi,t,\mu)$ is an odd function of ξ. Integrating over $|\xi|=1$ we conclude that $a_0(h,\varepsilon,x)+a_0(d-h,\varepsilon,x)=0$. □

For any $\varepsilon>0$ introduce the following perturbed version of $\log T(h,t)$

$$A(h,t,\varepsilon):=\frac{1}{2}\sum_{q=0}^{d}(-1)^{q+1}q\log\det{}_{\mathcal{N}}\left(\Delta_q(h,t)+\varepsilon\right). \tag{6.68}$$

Note that $A(h,t,\varepsilon)$ can be written as a sum

$$A(h,t,\varepsilon)=A_{\rm sm}(h,t,\varepsilon)+A_{\rm la}(h,t,\varepsilon) \tag{6.69}$$

where $A_{\rm sm}$ is defined in a fashion analogous to $\log T_{\rm sm}(h,t)$,

$$A_{\rm sm}(h,t,\varepsilon):=\frac{1}{2}\sum_{q=0}^{d}(-1)^{q+1}q\log\det{}_{\mathcal{N}}\left(\Delta_q(h,t)_{\rm sm}+\varepsilon\right)$$

with

$$\Delta_q(h,t)_{\rm sm}:=\Delta_q(h,t)|_{\Lambda^q(M;\mathcal{E})_{\rm sm}}$$

and $A_{\rm la}(t,h,\varepsilon)$ is given by $A(h,t,\varepsilon)-A_{\rm sm}(h,t,\varepsilon)$. Observe that the spectrum of the operator $\Delta_{q,sm}(h,t)$ tends to 0 as $t\to\infty$ and therefore, by Theorem 5.5 (6)

$$\log\det{}_{\mathcal{N}}\left(\Delta_q(h,t)_{\rm sm}+\varepsilon\right)=m_q l\log\varepsilon+O\left(\frac{1}{\varepsilon}te^{-2t}\right)$$

for $t\to\infty$ where $l=\dim_{\mathcal{N}}\mathcal{W}$. This shows that $A_{\rm sm}(h,t,\varepsilon)-A_{\rm sm}(h',t,\varepsilon)$ is exponentially small as $t\to\infty$ and hence, for any fixed $\varepsilon>0$, it has a trivial asymptotic expansion for $t\to\infty$. In view of (6.69) and Proposition 6.6 we conclude that for any $\varepsilon>0$, $A(h,t,\varepsilon)-A(h',t,\varepsilon)$ and $A_{\rm la}(h,t,\varepsilon)-A_{\rm la}(h',t,\varepsilon)$ have asymptotic expansions for $t\to\infty$ and, moreover, these expansions are identical. In particular we conclude that the free terms of the two expansions are identical

$$FT\left(A_{\rm la}(h,t,\varepsilon)-A_{\rm la}(h',t,\varepsilon)\right)=FT\left(A(h,t,\varepsilon)-A(h',t,\varepsilon)\right).$$

Using Proposition 6.6 (ii) and the fact that the densities $a_0(h,\varepsilon,x)$ and $a_0(h',\varepsilon,x)$, defined in (6.61), are continuous in ε we obtain

LEMMA 6.7. (i) *For any $\varepsilon>0$, $A_{\rm la}(h,t,\varepsilon)-A_{\rm la}(h',t,\varepsilon)$ has a asymptotic expansion for $t\to\infty$ which is identical to the asymptotic expansion for $A(h,t,\varepsilon)-A(h',t,\varepsilon)$.*

(ii) *The limit*

$$\lim_{\varepsilon\to 0}FT\left(A_{\rm la}(h,t,\varepsilon)-A_{\rm la}(h',t,\varepsilon)\right)$$

exists and is given by

$$\lim_{\varepsilon\to 0} FT\big(A_{\mathrm{la}}(h,t,\varepsilon) - A_{\mathrm{la}}(h',t,\varepsilon)\big) \qquad (6.70)$$
$$= \int_{M_I} a_0(h,\varepsilon=0,x) - \int_{M'_I} a_0(h',\varepsilon=0,x') \ .$$

We proceed to investigate the left-hand side of (6.70). For this we need the following estimate for the spectral distribution function $N_q(t,\lambda)$ of $\Delta_q(t) \equiv \Delta_q(h,t)$.

LEMMA 6.8. *There exist a constant $C > 0$ and $t_0 > 0$ so that for $t \geq t_0$ and $\lambda \geq 1$*

$$N_q(t,\lambda) \leq C\lambda^d \ .$$

Proof. Denote by $\Delta^I_{q,N}(t)$, resp. $\Delta^{II}_{q,N}(t)$, the operator $\Delta_q(t)$ restricted to M^I, resp. M^{II}, with Neumann boundary conditions, i.e. restricted to $C^\infty_N(\mathcal{E}|_{M^I}) := \{u \in C^\infty(\mathcal{E}|_{M^I}) \mid Nu = 0\}$ resp. $C^\infty_N(\mathcal{E}|_{M^{II}}) := \{u \in C^\infty(\mathcal{E}|_{M^{II}}) \mid Nu = 0\}$ where Nu denotes the covariant derivative of u with respect to the unit (inward) vector field normal to the boundary. $\Delta^I_{q,N}(t)$ and $\Delta^{II}_{q,N}(t)$ have well defined spectral distribution functions denoted by $N^I_q(t,\lambda)$ and $N^{II}_q(t,\lambda)$, and in view of their variational characterization one has

$$N_q(t,\lambda) \leq N^I_q(t,\lambda) + N^{II}_q(t,\lambda) \ . \qquad (6.71)$$

We estimate $N^I_q(t,\lambda)$ and $N^{II}_q(t,\lambda)$ separately. For t sufficiently large, $\Delta_q(t) \geq \Delta_q(0) = \Delta_q$ on M_I and thus $N^I_q(t,\lambda) \leq N^I_q(0,\lambda)$. The asymptotic estimate (2.15) (by the same arguments used to verify (2.15)) is also valid for $N^I_q(0,\lambda)$, $N^I_q(0,\lambda) \leq C\lambda^{d/2}$. Therefore there exist $t_0 > 0$ and $C_1 > 0$ such that for $t \geq t_0$, $\lambda \geq 1$

$$N^I_q(t,\lambda) \leq C_1\lambda^{d/2} \ . \qquad (6.72)$$

Now let us estimate $N^{II}_q(t,\lambda)$. Recall that $M_{II} = \cup_{k,j} U_{kj}$. On each of the discs U_{kj}, when expressed in (H)-coordinates (ϕ,Φ), $\Delta_{q;k}(t) = \tilde{\Delta}_{q;k}(t) \otimes \mathrm{Id}$ with $\tilde{\Delta}_{q;k} : \Lambda^q(\mathbb{R}^d;\mathbb{R}) \to \Lambda^q(\mathbb{R}^d;\mathbb{R})$ given by (cf. (5.2))

$$\tilde{\Delta}_{q;k}(t) = \tilde{\Delta}_q + t^2|x|^2 - t(d-2k) + 2t(N^+_{q;k} - N^-_{q;k}) \ .$$

Note that the spectral distribution function of $\Delta_{q;k}(t)$ on B_α (α as in Definition 5.1) with Neumann boundary conditions is equal to the product of $\dim_N \mathcal{W}$ and the spectral distribution function of $\tilde{\Delta}_{q;k}(t)$ on B_α with Neumann boundary conditions. Introduce the scaling operator S_t defined by

$$S_t f(x) := t^{1/2} f(tx) \ .$$

Then $S_{t^{1/2}} \cdot tK \cdot S_{t^{1/2}}^{-1} = \tilde{\Delta}_{q;k}(t)$ where

$$K \equiv K_{q;k} := \tilde{\Delta}_q + |x|^2 - (d-2k) + 2(N_{q;k}^+ - N_{q;k}^-) \ .$$

Therefore the Neumann spectrum of $\tilde{\Delta}_{q;k}(t)$ on B_α is the same as the Neumann spectrum of tK when considered on $B_{\sqrt{t}\alpha}$. Denote by $N_{tK;\sqrt{t}}(\lambda)$ the spectral distribution function of the operator tK on $B_{\sqrt{t}\alpha}$ with Neumann boundary conditions. Using the variational characterization of a spectral distribution function one can compare $N_{tK;\sqrt{t}}(\lambda)(= N_{K;\sqrt{t}}(\frac{\lambda}{t}))$ with the spectral distribution function of K on the whole space $\mathbb{R}^d$ to conclude that there exists $C_2 > 0$ so that for $t \geq 1$, $\lambda \geq 1$,

$$N_{tK;\sqrt{t}}(\lambda) \leq C_2 \left(\frac{\lambda}{t}\right)^d \leq C_2 \lambda^d \ .$$

Hence we have shown that for the spectral distribution function $N_q^{II}(t,\lambda)$ of the operator $\Delta_q(t)$ on M_{II} with Neumann boundary conditions there exists $C_3 > 0$ so that for $\lambda \geq 1$, $t \geq 1$,

$$N_q^{II}(t,\lambda) \leq C_3 \lambda^d \ . \tag{6.73}$$

The claimed result now follows from (6.71), (6.72) and (6.73). □

For t sufficiently large we introduce the trace of the heat kernel of $\Delta_{q;\mathrm{la}}(t)$,

$$\theta_q(t,\mu) = \int_1^\infty e^{-\mu\lambda} dN_q^{\mathrm{la}}(t,\lambda)$$

where $N_q^{\mathrm{la}}(t,\lambda)$ is the spectral distribution function of $\Delta_{q;\mathrm{la}}(t)$.

COROLLARY 6.9. (i) *There exist $t_0 > 0$ and a constant $C > 0$ such that, for $t \geq t_0$ and $\mu > 0$,*

$$\theta_q(t,\mu) \leq C\mu^{-d} \ . \tag{6.74}$$

(ii) *There exist constants $t_0 > 0$, $C > 0$ and $\beta > 0$ such that, for $t \geq t_0$ and $\mu \geq 1/\sqrt{t}$,*

$$\theta_q(t,\mu) \leq Ce^{-\beta t\mu} \ . \tag{6.75}$$

Proof. (i) By Proposition 5.2 there exist $t_0 > 0$ and a constant $C_1 > 0$ such that for $t \geq t_0$ $\mathrm{spec}(\Delta_{q;la}(t)) \subset [C_1 t, \infty)$ and therefore

$$\theta_q(t,\mu) = \int_{C_1 t}^\infty e^{-\mu\lambda} dN_q^{\mathrm{la}}(t,\lambda) \ .$$

Integrating by parts we obtain

$$\theta_q(t,\mu) \leq \mu \int_{C_1 t}^\infty e^{-\mu\lambda} N_q^{\mathrm{la}}(t,\lambda) d\lambda \ . \tag{6.76}$$

Notice that $N_q^{\mathrm{la}}(t,\lambda) \leq N_q(t,\lambda)$ and therefore, by Lemma 6.8, one concludes that

$$\theta_q(t,\mu) \le \frac{C}{\mu^d}\int_{C_1 t\mu}^{\infty} e^{-\lambda}\lambda^d d\lambda \le \tilde{C}/\mu^d \, .$$

(ii) From (6.76) and Lemma 6.8 we obtain

$$\theta_q(t,\mu) \le C\mu e^{-C_1 t\mu/2}\int_{C_1 t}^{\infty} e^{-\mu\lambda/2}\lambda^d d\lambda \le \frac{\tilde{C}}{\mu^d} e^{-C_1 t\mu/2} \, .$$

By choosing $\beta < C_1/2$ and $C > 0$ sufficiently large we obtain (ii). □

Recall from Theorem A that $\log T_{\rm la}(h,t)$ has an asymptotic expansion for $t \to \infty$.

PROPOSITION 6.10.

$$\lim_{\varepsilon\to 0} FT\big(A_{\rm la}(h,t,\varepsilon) - A_{\rm la}(h',t,\varepsilon)\big) = FT\big(\log T_{\rm la}(h,t)\big) - FT\big(\log T_{\rm la}(h',t)\big) \, . \tag{6.77}$$

Proof. We verify below that the function, defined for $\varepsilon > 0$ and t sufficiently large by

$$H(t,\varepsilon) := A_{\rm la}(h,t,\varepsilon) - A_{\rm la}(h',t,\varepsilon) - \log T_{\rm la}(h,t) + \log T_{\rm la}(h',t)$$

is of the form

$$H(t,\varepsilon) = \sum_{k=1}^{d} \varepsilon^k f_k(t) + g(t,\varepsilon) \tag{6.78}$$

where $g(t,\varepsilon) = o(1)$ uniformly in ε. The statement of the proposition can be deduced from (6.78) as follows: Recall that for $\varepsilon > 0$, $H(t,\varepsilon)$ has an asymptotic expansion for $t \to \infty$. As $g(t,\varepsilon) = o(1)$ uniformly in ε we conclude that for any $\varepsilon > 0$, $\sum_{k=1}^{d} \varepsilon^k f_k(t)$ has an asymptotic expansion for $t \to \infty$. By taking d different values $0 < \varepsilon_1 < \ldots < \varepsilon_d$ for ε and using that the Vandermonde determinant is nonzero

$$\det\begin{pmatrix} \varepsilon_1 & \ldots & \varepsilon_1^d \\ \vdots & & \vdots \\ \varepsilon_d & \ldots & \varepsilon_d^d \end{pmatrix} \neq 0$$

we conclude that for any $1 \le k \le d$, $f_k(t)$ has an asymptotic expansion for $t \to \infty$ and that for any $\varepsilon > 0$

$$FT\big(H(t,\varepsilon)\big) = \sum_{k=1}^{d} \varepsilon^k FT\big(f_k(t)\big) \, .$$

Hence $\lim_{\varepsilon\to 0} FT(H(t,\varepsilon))$ exists and $\lim_{\varepsilon\to 0} FT(H(t,\varepsilon)) = 0$. To prove (6.78) we introduce the zeta function $\zeta_{q,{\rm la}}$ of $\Delta_q(t)_{\rm la} + \varepsilon$,

$$\zeta_{q,{\rm la}}(t,\varepsilon,s) = \frac{1}{\Gamma(s)}\int_0^{\infty} \mu^{s-1}\theta_q(t,\mu)e^{-\varepsilon\mu} d\mu \tag{6.79}$$

with $\theta_q(t,\mu)$ given as above. The integral in (6.79) can be split into two parts

$$\zeta^{I}_{q,\mathrm{la}}(t,\varepsilon,s) = \frac{1}{\Gamma(s)}\int_{1/\sqrt{t}}^{\infty} \mu^{s-1}\theta_q(t,\mu)e^{-\varepsilon\mu}d\mu \tag{6.80}$$

and

$$\zeta^{II}_{q,\mathrm{la}}(t,\varepsilon,s) = \frac{1}{\Gamma(s)}\int_{0}^{1/\sqrt{t}} \mu^{s-1}\theta_q(t,\mu)e^{-\varepsilon\mu}d\mu\ . \tag{6.81}$$

First let us consider

$$\zeta^{I}_{q,\mathrm{la}}(t,\varepsilon,s) - \zeta^{I}_{q,\mathrm{la}}(t,\varepsilon=0,s) = \frac{1}{\Gamma(s)}\int_{1/\sqrt{t}}^{\infty} \mu^{s}\theta_q(t,\mu)\frac{e^{-\varepsilon\mu}-1}{\mu}d\mu\ . \tag{6.82}$$

Note that

$$\zeta^{I}_{q,\mathrm{la}}(t,\varepsilon,s) - \zeta^{I}_{q,\mathrm{la}}(t,\varepsilon=0,s)$$

is, by Corollary 6.9 (ii), an entire function of s. Recall that

$$\frac{d}{ds}\left(\frac{1}{\Gamma(s)}\right)\bigg|_{s=0} = 1$$

and that $1-e^{-\varepsilon\mu} \le \varepsilon\mu$, we obtain

$$\begin{aligned}
&\left|\frac{d}{ds}\bigg|_{s=0}\left(\zeta^{I}_{q,\mathrm{la}}(t,\varepsilon,s) - \zeta^{I}_{q,\mathrm{la}}(t,\varepsilon=0,s)\right)\right| \\
&\quad= \left|\int_{1/\sqrt{t}}^{\infty}\theta_q(t,\mu)\frac{e^{-\varepsilon\mu}-1}{\mu}d\mu\right| \\
&\quad\le \varepsilon C\int_{1/\sqrt{t}}^{\infty} e^{-\beta t\mu}d\mu = \frac{\varepsilon C}{\beta t}e^{-\beta\sqrt{t}}
\end{aligned}$$

where we have used Corollary 6.9. To analyze the term

$$\frac{d}{ds}\bigg|_{s=0}\left(\zeta^{II}_{q,\mathrm{la}}(t,\varepsilon,s) - \zeta^{II}_{q,\mathrm{la}}(t,\varepsilon=0,s)\right)\ ,$$

first expand $(e^{-\varepsilon\mu}-1)/\mu$

$$(e^{-\varepsilon\mu}-1)/\mu = \sum_{k=1}^{d}\frac{(-1)^k}{k!}\varepsilon^k\mu^{k-1} + \varepsilon^{d+1}\mu^d e(\varepsilon,\mu)$$

where the error term is given by

$$e(\varepsilon,\mu) = \left(\sum_{k=d+1}^{\infty}\frac{(-1)^k}{k!}\varepsilon^k\mu^{k-1}\right)\Big/\varepsilon^{d+1}\mu^d\ .$$

Note that, by Corollary 6.9, $\mu^d\theta_q(t,\mu) \le C$ and therefore

$$\int_0^{1/\sqrt{t}}\mu^s\theta_q(t,\mu)\varepsilon^{d+1}\mu^d e(\varepsilon,\mu)d\mu$$

is a meromorphic function of s, with $s = 0$ a regular point and, for t sufficiently large

$$\left| \frac{d}{ds}\Big|_{s=0} \left(\frac{1}{\Gamma(s)} \int_0^{1/\sqrt{t}} \mu^s \theta_q(t,\mu) \varepsilon^{d+1} \mu^d e(\varepsilon,\mu) d\mu \right) \right| \leq \varepsilon^{d+1} C/\sqrt{t}$$

where C is independent of t and ε, $0 \leq \varepsilon \leq 1$. $\theta_q(t,\mu)$ admits an expansion for $\mu \to 0+$ of the form

$$\theta_q(t,\mu) = \sum_{j=0}^{d} C_j(t) \mu^{(j-d)/2} + \theta'_q(t,\mu)$$

where $\theta'_q(t,\mu)$ is continuous in $\mu \geq 0$, because so does $\Delta_q(t)$. Therefore, for $1 \leq k \leq d$,

$$\frac{1}{\Gamma(s)} \int_0^{1/\sqrt{t}} \mu^s \theta_q(t,\mu) \frac{(-1)^k}{k!} \varepsilon^k \mu^{k-1} d\mu$$

is analytic in s at $s = 0$ and

$$\sum_{k=1}^{d} \frac{d}{ds}\Big|_{s=0} \left(\frac{1}{\Gamma(s)} \int_0^{1/\sqrt{t}} \mu^s \theta_q(t,\mu) \frac{(-1)^k}{k!} \varepsilon^k \mu^{k-1} d\mu \right)$$

is of the form $\sum_{k=1}^{d} \varepsilon^k f_k(t)$. This establishes (6.78). □

Proof of Theorem B. From Theorem A we know that $\log T_{\mathrm{la}}(h,t) - \log T_{\mathrm{la}}(h',t)$ has an asymptotic expansion for $t \to \infty$. By Proposition 6.10, the free term $\overline{a}_0$ of the asymptotic expansion is given by

$$\overline{a}_0 = \lim_{\varepsilon \to 0} FT\big(A_{\mathrm{la}}(h,t,\varepsilon) - A_{\mathrm{la}}(h',t,\varepsilon)\big) \ .$$

By Lemma 6.7 (ii) we conclude that

$$\overline{a}_0 = \int_{M_I} a_0(h, \varepsilon = 0, x) - \int_{M'_I} a_0(h', \varepsilon = 0, x') \ .$$

which proves part 1 of Theorem B. Part (2) follows from (6.63). □

Proof of Corollary C. Choose a bijection $\Theta : \mathrm{Cr}(h) \to \mathrm{Cr}(h')$ so that $\Theta(x_{q;j})$ is a critical point $x'_{q;j}$ of h' of index q. By assumption Θ extends to an isometry $\Theta : \cup_{q,j} U_{qj} \to \cup_{q,j} U'_{qj}$ where (U_{qj}) and (U'_{qj}) are systems of H-neighborhoods for h, respectively h'. Denote by τ, respectively, τ' the triangulation induced by (h,g), respectively (h',g'), and by $\tau_{\mathcal{D}}$, respectively $\tau'_{\mathcal{D}}$ the triangulations $\tau_{\mathcal{D}} = (d-h, g)$, respectively $\tau'_{\mathcal{D}} = (d-h', g')$. It follows from (4.8′)-(4.8‴) and d odd that $\log T_{\mathrm{met}}(\tau) = \log T_{\mathrm{met}}(\tau_{\mathcal{D}})$ and $\log T_{\mathrm{comb}}(\tau) = \log T_{\mathrm{comb}}(\tau_{\mathcal{D}})$.

Using Theorem A for both h and $d-h$, we obtain

$$\begin{aligned}2\log T_{\mathrm{an}} - 2\log T'_{\mathrm{an}} = & FT\big(\log T_{\mathrm{an}}(h,t) - \log T_{\mathrm{an}}(h',t)\big) \\ & + FT\big(\log T_{\mathrm{an}}(d-h,t) - \log T_{\mathrm{an}}(d-h',t)\big) \\ & + 2\log T_{\mathrm{met}}(\tau) - 2\log T_{\mathrm{met}}(\tau') \ .\end{aligned}$$

Decomposing $\log T_{\mathrm{an}}(h,t) = \log T_{\mathrm{la}}(h,t) + \log T_{\mathrm{sm}}(h,t)$ and taking into account the definition of the Reidemeister torsion (4.5) and the asymptotics of $\log T_{\mathrm{sm}}(h,t)$ (cf. Theorem A) we conclude that

$$\begin{aligned}2\log T_{\mathrm{an}} - 2\log T'_{\mathrm{an}} = & 2\log T_{\mathrm{Re}}(\tau) - 2\log T_{\mathrm{Re}}(\tau') \\ & + FT\big(\log T_{\mathrm{la}}(h,t) - \log T_{\mathrm{la}}(h',t)\big) \\ & + FT\big(\log T_{\mathrm{la}}(d-h,t) - \log T_{\mathrm{la}}(d-h',t)\big)\end{aligned}$$

from which Corollary C follows by (6.55) and statement 2 of Theorem B. □

6.3 Proof of Theorem 2. In this subsection we provide the proof of Theorem 2 using Corollary C of subsection 6.2 together with the product formulas for the Reidemeister torsion and the analytic torsion established in Proposition 4.1.

First we need the following result concerning the metric anomaly of the analytic torsion, which is a generalization of a classical result due to Ray-Singer (cf. [RSin, Theorem 2.1]), and can be proved by the same arguments used to verify Proposition 6.2. (For the convenience of the reader, a proof is included in Appendix 3.)

LEMMA 6.11. *Let M^d be a closed manifold of odd dimension d such that $(M,\mathcal{W})$ is of determinant class. Assume that $g(u)$ is a smooth one-parameter family of Riemannian metrics on M. Then $\log T_{\mathrm{an}}(M,g(u),\mathcal{W})$ is a smooth function of u whose derivative is given by*

$$\frac{d}{du}\log T_{\mathrm{an}}\left(M,g(u),\mathcal{W}\right) = \frac{1}{2}\frac{d}{du}\sum_{q=0}^{d}(-1)^q \log\det{}_{\mathrm{N}}\left(\sigma_q(u)^*\sigma_q(u)\right) \qquad (6.83)$$

where $\sigma_q(u)$ is the $\mathcal{A}$-linear, bounded isomorphism $\sigma_q(u) : \mathrm{Null}\,\Delta_q(u_0) \to \mathrm{Null}\,\Delta_q(u)$ (the projection on $\mathrm{Null}\,\Delta_q(u)$), provided by Hodge theory and u_0 is arbitrary but fixed.

Given generalized triangulations $\tau = (h,g')$ and $\tau' = (h',g'')$ of M, τ' is called a subdivision of τ if

(1) $\mathrm{Cr}_q(h) \subset \mathrm{Cr}_q(h')$ $(0 \le q \le d)$
(2) $W_x^{\pm}(h',g'') \subset W_x^{\pm}(h,g')$ for any $x \in \mathrm{Cr}_q(h)$.

The following result can be found in [Mi2].

LEMMA 6.12. *Let $\tau = (h, g')$ be a generalized triangulation, q an integer $0 \leq q \leq d-1$ and x, y two distinct points in $M \setminus \mathrm{Cr}(h)$. Then there exists a generalized triangulation $\tau' = (h', g'')$ with the following properties*

(1) $\mathrm{Cr}_k(h') = \mathrm{Cr}_k(h)$ *for* $k \neq q, q+1$;
(2) $\mathrm{Cr}_q(h') = \mathrm{Cr}_q(h) \cup \{x\}$; $\mathrm{Cr}_{q+1}(h') = \mathrm{Cr}_{q+1}(h) \cup \{y\}$;
(3) τ' *is a subdivision of* τ;
(4) $W_y^- \cap W_x^+$ *is connected.*

Since the Reidemeister torsion does not change under subdivision (cf. [Mil]), one obtains

$$T_{\mathrm{Re}}(M, g, \mathcal{W}, \tau) = T_{\mathrm{Re}}(M, g, \mathcal{W}, \tau') .$$

Proof of Theorem 2. By Proposition 5.6 it suffices to consider the case where $\mathcal{W}$ is free. Further, by Lemma 6.11 and in view of the definition (4.4) of T_{met}, it suffices to prove Theorem 2 in the case where $g = g'$, $\tau = (h, g')$.

Consider the sphere $S^6 = \{x = (x_1, \ldots, x_7) \in \mathbb{R}^7; \sum x_j^2 = 1\}$ with an arbitrary generalized triangulation $\tau_1 = (h_1, g_1)$. Let $\tau = (h, g)$ be a generalized triangulation for M and consider $M \times S^6$, endowed with the Riemannian metric $g \times g_1$ and the triangulation $\tau \times \tau_1 = (h + h_1, g \times g_1)$. Note that $\Gamma = \pi_1(M) = \pi_1(M \times S^6)$. By assumption, $(M, \mathcal{W})$ is of determinant class and thus $(M \times S^6, \mathcal{W})$ is of determinant class as well. Moreover, by the product formulas of Proposition 4.2,

$$\log T_{\mathrm{an}}(M \times S^6, g \times g_1, \mathcal{W}) = 2 \log T_{\mathrm{an}}(M, g, \mathcal{W}) \tag{6.84}$$

and

$$\log T_{\mathrm{Re}}(M \times S^6, g \times g_1, \mathcal{W}, \tau \times \tau_1) = 2 \log T_{\mathrm{Re}}(M, g, \mathcal{W}, \tau) \tag{6.85}$$

where we used that $\chi(S^6) = 2$ and that $\chi(M, \mathcal{W}) = 0$ (as M is of odd dimension).

Next, consider the product $S^3 \times S^3$ of the 3-spheres, $S^3 = \{x = (x_1, \ldots, x_4) \in \mathbb{R}^4; \sum x_j^2 = 1\}$, with an arbitrary generalized triangulation $\tau_2 = (h_2, g_2)$. Arguing as above, we conclude that $(M \times S^3 \times S^3, \mathcal{W})$ is of determinant class and that, by the product formulas of Proposition 4.1,

$$\log T_{\mathrm{an}}(M \times S^3 \times S^3, g \times g_2, \mathcal{W}) = 0$$

and

$$\log T_{\mathrm{Re}}(M \times S^3 \times S^3, g \times g_2, \mathcal{W}, \tau \times \tau_2) = 0$$

where we used that $\chi(S^3 \times S^3) = 0$ and that $\chi(M, \mathcal{W}) = 0$.

Choose a subdivision $\tau' = (h', g'')$ of the generalized triangulation $\tau \times \tau_1$ in $M \times S^6$ and a subdivision $\tau'' = (h'', g'')$ of the generalized triangulation $\tau \times \tau_2$ in $M \times S^3 \times S^3$ so that, for $0 \leq q \leq d+6$, $\# \mathrm{Cr}_q(h') = \# \mathrm{Cr}_q(h'')$. In view of Lemma 6.12 this is possible because $M \times S^6$ and $M \times S^3 \times S^3$ are both of odd dimension and therefore $\chi(M \times S^3 \times S^3, \mathcal{W}) = \chi(M \times S^6, \mathcal{W}) = 0$. We conclude from the above, Corollary C, Lemma 6.11 and Lemma 6.12 that

$$\begin{aligned}
&2\log T_{\mathrm{an}}(M, g, \mathcal{W}) - 2\log T_{\mathrm{Re}}(M, g, \mathcal{W}, \tau) \\
&\quad = \log T_{\mathrm{an}}(M \times S^6, g \times g_1, \mathcal{W}) - \log T_{\mathrm{Re}}(M \times S^6, g \times g_1, \mathcal{W}, \tau \times \tau_1) \\
&\quad = \log T_{\mathrm{an}}(M \times S^6, g', \mathcal{W}) - \log T_{\mathrm{Re}}(M \times S^6, g', \mathcal{W}, \tau') \\
&\quad = \log T_{\mathrm{an}}(M \times S^3 \times S^3, g'', \mathcal{W}) - \log T_{\mathrm{Re}}(M \times S^3 \times S^3, g'', \mathcal{W}, \tau'') \\
&\quad = \log T_{\mathrm{an}}(M \times S^3 \times S^3, g \times g_2, \mathcal{W}) \\
&\qquad - \log T_{\mathrm{Re}}(M \times S^3 \times S^3, g \times g_2, \mathcal{W}, \tau \times \tau_2) = 0 \ .
\end{aligned}$$

This proves Theorem 2.

A. Appendices

A.1 Appendix 1. In this appendix we prove Proposition 1.9, Proposition 1.10 and formula (1.32) stated in section 1.

Proof of Proposition 1.9. Proof of (1): First we show that for any $s \in \mathbb{C}$

$$\frac{d}{dt} \operatorname{tr}_N (f_t)^s = s \operatorname{tr}_N \left((f_t)^{s-1} \frac{df_t}{dt} \right) \ . \tag{A1.1}$$

As the interval I can be assumed to be compact one can write

$$(f_t)^s = \frac{1}{2\pi i} \int_C \lambda^s (\lambda - f_t)^{-1} d\lambda$$

where C, a circle centered at zero, contains $\operatorname{spec} f_t$ for $t \in I$. Then

$$\begin{aligned}
\frac{d}{dt} \operatorname{tr}_N (f_t)^s &= \frac{1}{2\pi i} \int_C \lambda^s \operatorname{tr}_N \left((\lambda - f_t)^{-1} \frac{df_t}{dt} (\lambda - f_t)^{-1} \right) d\lambda \\
&= \operatorname{tr}_N \left(\frac{df_t}{dt} \frac{1}{2\pi i} \int_C \lambda^s (\lambda - f_t)^{-2} d\lambda \right) \ .
\end{aligned}$$

Integrating by parts one obtains

$$\frac{1}{2\pi i} \int_C \lambda^s (\lambda - f_t)^{-2} d\lambda = \frac{1}{2\pi i} \int_C s\lambda^{s-1} (\lambda - f_t)^{-1} d\lambda = s(f_t)^{s-1}$$

and (A1.1) follows. Using (A1.1) one obtains

$$\begin{aligned}
\frac{d}{dt} \log \det_N (f_t) &= \frac{d}{dt} \frac{d}{ds}\Big|_{s=0} (\operatorname{tr}_N (f_t)^s) \\
&= \frac{d}{ds}\Big|_{s=0} \left(s \operatorname{tr}_N (f_t)^{s-1} \frac{df_t}{dt} \right) = \operatorname{tr}_N \left(f_t^{-1} \frac{df_t}{dt} \right) \ .
\end{aligned}$$

Proof of (2). Since $f_1 = \alpha^{-1} f_2 \alpha$ one has $\operatorname{spec} f_1 = \operatorname{spec} f_2$ and, for $\lambda \notin \operatorname{spec} f_2$, $(\lambda - f_1)^{-1} = \alpha^{-1}(\lambda - f_2)^{-1}\alpha$. If $0 \notin \operatorname{spec} f_2$

$$\begin{aligned}\log\det{}_N(f_1) &= \frac{d}{ds}\Big|_{s=0} \frac{1}{2\pi i}\int_C \lambda^s \operatorname{tr}_N(\lambda - f_1)^{-1} d\lambda \\ &= \frac{d}{ds}\Big|_{s=0} \frac{1}{2\pi i}\int_C \lambda^s \operatorname{tr}_N \left(\alpha^{-1}(\lambda - f_2)^{-1}\alpha\right) d\lambda \\ &= \frac{d}{ds}|_{s=0} \operatorname{tr}_N(f_2)^s = \log\det{}_N(f_2)\ .\end{aligned}$$

Proof of (3) (a). For $\lambda \notin \operatorname{spec} f_1 \cup \operatorname{spec} f_2$, both $(\lambda - f_1)^{-1}$ and $(\lambda - f_2)^{-1}$ exist and $(\lambda - f)$ has an inverse given by

$$(\lambda - f)^{-1} = \begin{pmatrix} (\lambda - f_1)^{-1} & 0 \\ -(\lambda - f_2)^{-1} g (\lambda - f_1)^{-1} & (\lambda - f_2)^{-1} \end{pmatrix} .$$

This shows that $\operatorname{spec} f \subseteq \operatorname{spec} f_1 \cup \operatorname{spec} f_2$. Reversing the argument one can see that $\operatorname{spec} f_1 \cup \operatorname{spec} f_2 \subseteq \operatorname{spec} f$. Further, with C a circle which contains $\operatorname{spec} f$

$$\begin{aligned}\log\det{}_N(f) &= \frac{d}{ds}\Big|_{s=0} \frac{1}{2\pi i}\int_C \lambda^s \operatorname{tr}_N(\lambda - f)^{-1} d\lambda \\ &= \frac{d}{ds}\Big|_{s=0} \frac{1}{2\pi i}\int_C \lambda^s \operatorname{tr}_N(\lambda - f_1)^{-1} d\lambda \\ &\quad + \frac{d}{ds}|_{s=0} \frac{1}{2\pi i}\int_C \lambda^s \operatorname{tr}_N(\lambda - f_2)^{-1} d\lambda \\ &= \log\det{}_N(f_1) + \log\det{}_N(f_2)\ .\end{aligned}$$

Proof of (3) (b). Notice that f is an isomorphism if $0 \notin spec(f)$. Therefore, in view of (a), it remains to check (1.5B). Consider a one parameter family

$$h_t = \begin{pmatrix} f_1^* & g^* \\ 0 & f_2^* \end{pmatrix} \cdot \begin{pmatrix} f_1 & 0 \\ tg & f_2 \end{pmatrix} .$$

Then

$$2 \log \operatorname{Vol}_N(f) = \log\det{}_N(h_1) \tag{A1.2}$$

and

$$\log\det{}_N(h_0) = \log\det{}_N \begin{pmatrix} f_1^* f_1 & g^* f_2 \\ 0 & f_2^* f_2 \end{pmatrix} = \log\det{}_N(f_1^* f_1) + \log\det{}_N(f_2^* f_2) \tag{A1.3}$$

where for the last inequality we used (3) (a) . According to (1.4) we have

$$\frac{d}{dt}\log \mathrm{Vol}_N(h_t) = \mathrm{tr}_N\left(h_t^{-1}\frac{d}{dt}h_t\right) = \mathrm{tr}_N\left(\begin{pmatrix} f_1 & 0 \\ tg & f_2\end{pmatrix}^{-1}\frac{d}{dt}\begin{pmatrix} f_1 & 0 \\ tg & f_2\end{pmatrix}\right)$$
$$= \mathrm{tr}_N\left(\begin{pmatrix} f_1^{-1} & 0 \\ -tg & f_2^{-1}\end{pmatrix}\begin{pmatrix} 0 & 0 \\ g & 0\end{pmatrix}\right) = 0$$

and (1.5B) follows from (A1.1)-(A1.3).

Proof of (4). We have to prove

$$\log\det_N f_1^* f_2^* f_2 f_1 - \log\det_N f_1^* f_1 = \log\det_N f_2^* f_2 \ .$$

Consider the 1-parameter family $C(t)$ of positive, selfadjoint operators, $C(t) := f_1^*(f_2^* f_2)^t f_1$, defined on $\mathcal{W}_1$. Using formula (1.4) one verifies that

$$\frac{d}{dt}\log\det_N f_1^*(f_2^* f_2)^t f_1 = \frac{d}{dt}\log\det_N (f_2^* f_2^*)^t \ .$$

This leads to the claimed formula,

$$\log\det_N f_1^* f_2^* f_2 f_1 - \log\det_N f_1^* f_1 = \int_0^1 \frac{d}{dt}\log\det_N \left(f_1^*(f_2^* f_2)^t f_1\right) dt$$
$$= \int_0^1 \frac{d}{dt}\log\det_N (f_2^* f_2)^t dt = \log\det_N (f_2^* f_2) \ .$$

Proof of (5). Note that (5) follows from (4) by observing that for an isometry α_1, $\mathrm{spec}(\alpha_1^*\alpha_1) = \{1\}$ and therefore $\log \mathrm{Vol}_N(\alpha_1) = 0$. □

Proof of Proposition 1.10. (1) is obvious from the definition. In order to check (2), first note that f is a weak isomorphism iff both f_1 and f_2 are weak isomorphisms. Using special elements of the form $u+0$ and $0+v$ of $\mathcal{W}_1 \oplus \mathcal{W}_2$, one concludes that $F_{f_1^*}(\lambda) \le F_{f^*}(\lambda)$ and $F_{f_2}(\lambda) \le F_f(\lambda)$ which, by (1), implies $\max\{F_{f_1}(\lambda), F_{f_2}(\lambda)\} \le F_f(\lambda)$. For any $w = u+v \in \mathcal{W}_1 \oplus \mathcal{W}_2$, one has the inequality

$$\langle f^* f w, w\rangle \ge \|f_1 u\|^2 + \|gu + f_2 v\|^2 \ge \|f_1 u\|^2 \ . \tag{A1.4}$$

Let $\mathcal{L}$ be a $\mathcal{A}$-Hilbert submodule of $\mathcal{W}_1 \oplus \mathcal{W}_2$ with $\|f(w)\| \le \lambda\|w\|$, $w \in \mathcal{L}$. Let $\mathcal{L}_1 := \overline{\pi_1(\mathcal{L})}$ where $\pi_1 : \mathcal{W}_1 \oplus \mathcal{W}_2 \to \mathcal{W}_1$ is the canonical projection, and let $\mathcal{L}_2 := \mathrm{Null}(\pi_1|_{\mathcal{L}})$. Then $\dim_N \mathcal{L} = \dim_N \mathcal{L}_1 + \dim_N \mathcal{L}_2$. By (A1.4), $\|f_1(u)\| \le \lambda\|u\|$ for $u \in \mathcal{L}_1$ and, by the definition of $\mathcal{L}_2$, $\|f_2(w)\| \le \lambda\|w\|$ for $w \in \mathcal{L}_2$. Therefore $(j = 1, 2)$

$$\dim_N \mathcal{L}_j \le F_{f_j}(\lambda) \ .$$

As $\mathcal{L}$ is arbitrary one then concludes that $F_f(\lambda) \le F_{f_1}(\lambda) + F_{f_2}(\lambda)$. □

Finally we prove formula (1.32)

$$\zeta_C(\lambda, s) = \frac{1}{2}\sum_i (-1)^i i \frac{1}{\Gamma(s)}\int_0^\infty \eta^{s-1}\,\mathrm{tr}_N\, e^{-\eta(\Delta_i+\lambda)} d\eta \ . \tag{1.32}$$

Proof of (1.32). It suffices to show that for $\lambda > 0$ and $\Re s > 0$

$$\operatorname{tr}_N(\Delta_i + \lambda)^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{(-t\lambda)} \operatorname{tr}_N e^{-t\Delta_i} dt \ . \tag{A1.5}$$

By the spectral theorem

$$\operatorname{tr}_N e^{-\eta(\Delta_i+\lambda)} = \int_0^\infty e^{(-\eta\mu)} dN_{\Delta_i+\lambda}(\mu) \tag{A1.6}$$

and thus

$$\operatorname{tr}_N(\Delta_i + \lambda)^{-s} = \int_0^\infty \mu^{-s} dN_{\Delta_i+\lambda}(\mu) \ . \tag{A1.7}$$

Use that

$$\frac{1}{\Gamma(s)} \int_0^\infty \eta^{s-1} e^{-\eta\mu} d\eta = \mu^{-s} \tag{A1.8}$$

to conclude from (A1.7) and (A1.6) that

$$\operatorname{tr}_N(\Delta_i + \lambda)^{-s} = \int_0^\infty \left(\frac{1}{\Gamma(s)} \int_0^\infty \eta^{s-1} e^{-\eta\mu} d\eta \right) dN_{\Delta_i+\lambda}(\mu) =$$

$$\frac{1}{\Gamma(s)} \int_0^\infty \eta^{s-1} \left(\int_0^\infty e^{-\eta\mu} dN_{\Delta_i+\lambda}(\mu) \right) d\eta = \frac{1}{\Gamma(s)} \int_0^\infty \eta^{s-1} \operatorname{tr}_N e^{-\eta(\Delta_i+\lambda)} d\eta \ .$$

□

A.2 Appendix 2. In this appendix we prove formulas (5.5)–(5.7). Recall that (cf. section 5 for complete statements)

$$\left|\Delta_{q;q}(t)\psi_{q,i}(x,t)\right| \le C_0(\epsilon) e^{-C(\epsilon)t} \ ; \tag{5.5}$$

$$\left\langle \Delta_{q;k}(t)\psi_{q,i}(t), \psi_{q,i}(t) \right\rangle \ge 2t|q-k| \ ; \tag{5.6}$$

$$\left\langle \Delta_{q;q}(t)\omega, \omega \right\rangle \ge C(\epsilon) t \|\omega\|^2 \ . \tag{5.7}$$

To prove these formulas first notice that

$$(N^+_{q;k} - N^-_{q;k})(dx_1 \wedge \ldots \wedge dx_q) = n_{q;k} dx_1 \wedge \ldots \wedge dx_q$$

where

$$n_{q;k} = -q \ \text{ if } k \ge q \ \text{ and } \ n_{q;k} = q - 2k \ \text{ if } k < q \ .$$

Thus, with $\tilde\Delta := -\sum_1^d \partial^2_{x_j}$,

$$(\tilde\Delta + t^2|x|^2 - t(d-2k) + 2tn_{q;k}) e^{-t|x|^2/2} = 2t|q-k| e^{-t|x|^2/2} \ . \tag{A2.1}$$

Write

$$\tilde\Delta\big(\nu_\epsilon(|x|) e^{-t|x|^2/2}\big) = e^{-t|x|^2/2} \tilde\Delta\big(\nu_\epsilon(|x|)\big) - 2\sum_1^d \partial_{x_j}\nu_\epsilon\big(|x|\big)\partial_{x_j} e^{-t|x|^2/2}$$

$$+ \nu_\epsilon\big(|x|\big)\tilde\Delta\big(e^{-t|x|^2/2}\big) \ . \tag{A2.2}$$

Proof of (5.5). Use $k = q$ to conclude from (A2.2) and (A2.1) that

$$\begin{aligned}&\left|(\tilde{\Delta} + t^2|x|^2 - t(d-2k) + 2tn_{q;q})(\nu_\epsilon(|x|)e^{-t|x|^2/2})\right| \\ &\leq \left| -2\sum_1^d \partial_{x_j}(\nu_\epsilon(|x|))\partial_{x_j} e^{-t|x|^2/2}\right| + \left| -\sum_1^d \partial^2_{x_j}(\nu_\epsilon(|x|))e^{-t|x|^2/2}\right| \\ &\leq |\dot{\nu}_\epsilon(|x|)|2t|x|e^{-t|x|^2/2} + |\dot{\nu}_\epsilon(|x|)|\frac{d-1}{|x|}e^{-t|x|^2/2} + |\ddot{\nu}_\epsilon(|x|)|e^{-t|x|^2/2}\end{aligned}$$

where $\dot{\nu}_\epsilon(t) = \frac{d}{dt}\nu_\epsilon(t)$ and $\ddot{\nu}_\epsilon(t) = \frac{d^2}{dt^2}\nu_\epsilon(t)$. Use that $\operatorname{supp}(\dot{\nu}_\epsilon)$ and $\operatorname{supp}(\ddot{\nu}_\epsilon)$ are contained in $[\epsilon/2, \epsilon]$ to conclude that

$$\begin{aligned}&\left|(\tilde{\Delta} + t^2|x|^2 - t(d-2k) + 2tn_{q;q})(\nu_\epsilon(|x|)e^{-t|x|^2/2})\right| \\ &\quad\leq \left(\|\dot{\nu}_\epsilon\|_{L^\infty}\left(2\epsilon t e^{-t\epsilon^2/16} + 2\frac{d-1}{\epsilon}e^{-t\epsilon^2/16}\right) + \|\ddot{\nu}_\epsilon\|_{L^\infty}e^{t\epsilon^2/16}\right)e^{-t\epsilon^2/16} .\end{aligned} \tag{A2.3}$$

To estimate the normalizing constant $\beta(t)$ (cf. (5.3′)), notice that with $t_0 := (2/\epsilon)^2$, one obtains for $t \geq t_0$,

$$\int_{\mathbb{R}^d} \nu_\epsilon(|x|)^2 e^{-t|x|^2}dx \geq \int_{|x|<\epsilon/2} e^{-t|x|^2}dx \geq C' t^{-\frac{d}{2}} \int_0^1 e^{-s^2}s^{d-1}ds \ . \tag{A2.4}$$

Combining (A2.3), (A2.4) one concludes that there exists $C > 0$ so that for $t \geq t_0(\epsilon)$

$$|\Delta_{q;q}(t)\psi_{q,i}(x,t)| \leq C\left\{\|\dot{\nu}_\epsilon\|_{L^\infty}\left(2\epsilon t e^{-t\epsilon^2/16} + \frac{d-1}{2}\right) + \|\ddot{\nu}_\epsilon\|_{L^\infty}\right\} t^{-\frac{d}{2}}e^{-t\epsilon^2/16}. \tag{A2.5}$$

This leads to the estimate (5.5).

Proof of (5.6). Integrating by parts, we obtain

$$\begin{aligned}&\int_{\mathbb{R}^d} e^{-t|x|^2/2}(\tilde{\Delta}\nu_\epsilon)e^{-t|x|^2/2}\nu_\epsilon dx = -\sum_1^d \int_{\mathbb{R}^d} e^{-t|x|^2}\nu_\epsilon \partial^2_{x_j}\nu_\epsilon dx \qquad \text{(A2.6)} \\ &= \sum_1^d \int_{\mathbb{R}^d} (\partial_{x_j}\nu_\epsilon)^2 e^{-t|x|^2}dx + \sum_1^d \int_{\mathbb{R}^d} \nu_\epsilon \frac{\partial \nu_\epsilon}{\partial x_j} 2\left(\frac{\partial}{\partial x_j}e^{-t|x|^2/2}\right)e^{-t|x|^2/2}dx .\end{aligned}$$

Combining (A2.2) and (A2.6)one obtains

$$\begin{aligned}&\int_{\mathbb{R}^d} \tilde{\Delta}(\nu_\epsilon e^{-t|x|^2/2})\nu_\epsilon e^{-t|x|^2/2}dx \\ &\qquad = \int_{\mathbb{R}^d}\left(\sum_1^d (\partial_{x_j}\nu_\epsilon)^2 e^{-t|x|^2} + \nu_\epsilon^2\tilde{\Delta}(e^{-t|x|^2/2})e^{-t|x|^2/2}\right)dx .\end{aligned} \tag{A2.7}$$

Combining (A2.1) and (A2.7) one obtains

$$\int_{\mathbb{R}^d} (\tilde{\Delta} + t^2|x|^2 - t(d-2k) + 2tn_{q;k})(\nu_\epsilon e^{-t|x|^2/2}) \cdot (\nu_\epsilon e^{-t|x|^2/2})dx \tag{A2.8}$$

$$= \int_{\mathbb{R}^d} \left(2t|q-k|\nu_\epsilon^2 e^{-t|x|^2} + \sum_1^d (\partial_{x_j}\nu_\epsilon)^2 e^{-t|x|^2} \right) dx$$

$$\geq 2t|q-k| \int_{\mathbb{R}^d} (\nu_\epsilon)^2 e^{-t|x|^2} dx \ .$$

Taking into account the normalization factor $\beta(t)$ we obtain

$$\langle \Delta_{q;k}(t)\psi_{q,i}(t), \psi_{q,i}(t) \rangle =$$

$$\frac{\int_{\mathbb{R}^d} (\tilde{\Delta} + t^2|x|^2 - t(d-2k) + 2tn_{q;q})(\nu_\epsilon e^{-t|x|^2/2}) \cdot (\nu_\epsilon e^{-t|x|^2/2})dx}{\int_{\mathbb{R}^q} (\nu_\epsilon)^2 e^{-t|x|^2} dx} \geq 2t|q-k|.$$

This proves (5.6).

Proof of (5.7). It suffices to consider $\omega \in \Lambda^q(\mathbb{R}^d; \mathcal{W})$ of the form $\omega = \varphi dx_{i_1} \wedge \cdots \wedge dx_{i_q} \otimes v$ with $v \in \mathcal{W}$ and $\varphi \in C^\infty(\mathbb{R}^d; \mathbb{C})$ with compact support. Thus it remains to show that there exists $t_0 = t_0(\epsilon)$ and $C_0(\epsilon)$ so that for any $\varphi \in C^\infty(\mathbb{R}^d; \mathbb{C})$ with compact support, satisfying

$$\int_{\mathbb{R}^d} \varphi(x)\nu_\epsilon(|x|)e^{-t|x|^2/2}dx = 0 \tag{A2.9}$$

the following estimate holds

$$\int_{\mathbb{R}^d} (\tilde{\Delta} + t^2|x|^2 - td)\varphi(x) \cdot \overline{\varphi(x)} dx \geq C_0(\epsilon)t \int_{\mathbb{R}^d} |\varphi(x)|^2 dx \ . \tag{A2.10}$$

To prove (A2.10) introduce $\varphi_2 := \varphi - \varphi_1$ with

$$\varphi_1(x) := \frac{\int_{\mathbb{R}^d} \varphi(x)e^{-t|x|^2/2}dx}{\int_{\mathbb{R}^d} e^{-t|x|^2}dx} e^{-t|x|^2/2}$$

being the orthogonal projection of φ onto the function $e^{-t|x|^2/2}$. Notice that $(\tilde{\Delta} + t^2|x|^2 - td)\varphi_1 = 0$ and, due to (HO1), $\mathrm{spec}(\tilde{\Delta} + t^2|x|^2 - td) \subseteq t\mathbb{Z}_{\geq 0}$. Therefore

$$\left| \int_{\mathbb{R}^d} (\tilde{\Delta} + t^2|x|^2 - td)\varphi(x) \cdot \overline{\varphi(x)} dx \right| \geq \left| \int_{\mathbb{R}^d} (\tilde{\Delta} + t^2|x|^2 - td)\varphi_2(x)\overline{\varphi_2(x)} dx \right| \tag{A2.11}$$

$$\geq t \int_{\mathbb{R}^d} |\varphi_2|^2 dx = t\left(\int_{\mathbb{R}^d} |\varphi|^2 dx - \int_{\mathbb{R}^d} |\varphi_1|^2 dx \right) \ .$$

It remains to estimate $\int_{\mathbb{R}^d} |\varphi_1|^2 dx = \left| \int_{\mathbb{R}^d} \varphi e^{-t|x|^2/2} dx \right|^2$. Use (A2.9) to conclude, by the Cauchy-Schwartz inequality,

$$\left|\int_{\mathbb{R}^d}\varphi(x)e^{-t|x|^2/2}dx\right|^2=\left|\int_{\mathbb{R}^d}\varphi(x)\big(1-\nu_\epsilon(|x|)\big)e^{-t|x|^2/2}dx\right|^2 \quad (A2.12)$$
$$\le\int_{\mathbb{R}^d}|\varphi|^2dx\int_{\mathbb{R}^d}\big(1-\nu_\epsilon(|x|)\big)^2e^{-t|x|^2}dx\ .$$

Further, with $t_0\equiv t_0(\epsilon)\ge(\frac{2}{\epsilon})^2$ and $t\ge t_0$

$$\int_{\mathbb{R}^d}\big(1-\nu_\epsilon(|x|)\big)^2e^{-t|x|^2}dx\le\int_{|x|>\frac{\epsilon}{2}}e^{-t_0|x|^2}\le Ct_0^{-d/2}\int_1^\infty e^{-s^2}s^{d-1}ds\ . \quad (A2.13)$$

By (A2.13) we can choose $t_0\ge(\frac{2}{\epsilon})^2$ so large that

$$\int_{\mathbb{R}^d}\big(1-\nu_\epsilon(|x|)\big)^2e^{-t|x|^2}dx\le 1/2\ . \quad (A2.14)$$

Combine (A2.11), (A2.12) and (A2.14) to see that for $t\ge t_0$

$$\int_{\mathbb{R}^d}(\tilde\Delta+t^2|x|^2-td)\varphi(x)\cdot\overline{\varphi(x)}dx\ge\frac{t}{2}\int_{\mathbb{R}^d}|\varphi|^2dx\ . \qquad \square$$

A.3 Appendix 3.

Proof of Proposition 1.6. We follow the line of arguments given in [RSin]. Let $T_{\rm an}(u):=T_{\rm an}(M,g(u),\mathcal{W})$ and denote by $\Delta_q^\pm(u)$ the restrictions of the Laplacians to $\Lambda_u^{\pm,q}(M;\mathcal{E})$ (defined in (4.17)) with respect to the Riemannian metric $g(u)$. By (6.15) (and the argument which follows it), $\log T_{\rm an}(u)=-\log T_{\rm an}^+(0,u)$ where $T_{\rm an}^+(s,u)$ is given by

$$\log T_{\rm an}^\pm(s,u):=\frac{1}{2}\sum_{q=0}^d(-1)^q\left(-\frac{\partial}{\partial s}\xi_{q,u}^\pm(s)\right) \quad (A3.1)$$

with $\xi_{q,u}^\pm(s)$ given by (6.12″) (for $t=0, g=g(u)$)

$$\xi_{q,u}^\pm=\frac{1}{\Gamma(s)}\int_0^1x^{s-1}\,{\rm tr}_N(e^{-x\Delta^\pm(u)})dx+s\int_1^\infty\frac{1}{x}{\rm tr}_N(e^{-x\Delta^\pm(u)})dx\ . \quad (A3.2)$$

We want to compute $\frac{d}{du}\log T_{\rm an}^+(s,u)$. Notice that $\Lambda_u^{+,q}(M;\mathcal{E})=d_{q-1}(\Lambda^{q-1}(M;\mathcal{E}))$ does not depend on the metric and therefore is independent of u. By Duhamel's formula

$$\frac{d}{du}{\rm tr}_N(e^{-x\Delta_q^+(u)})=-x\,{\rm tr}_N\left(e^{-x\Delta_q^+(u)}\frac{d}{du}\Delta_q^+(u)\right)\ . \quad (A3.3)$$

To compute $\frac{d}{du}\Delta_q^+(u)=d_{q-1}\frac{d}{du}(d_{q-1}^*(u))$ write for $\omega\in\Lambda^q(M;\mathcal{E})$ and $\eta\in\Lambda^{q-1}(M;\mathcal{E})$

$$\left\langle d_{q-1}^*(u)\omega,\eta\right\rangle_u=\langle\omega,d_{q-1}\eta\rangle_u \quad (A3.4)$$

and define $R_q(u) : \Lambda^q(M;\mathcal{E}) \to \Lambda^q(M;\mathcal{E})$ by $(\omega, \omega' \in \Lambda^q(M;\mathcal{E}))$

$$\frac{d}{du}\langle\omega,\omega'\rangle_u = \left\langle R_q(u)\omega,\omega'\right\rangle_u . \tag{A3.5}$$

Taking the derivative of (A3.4) and using (A3.5) one obtains $(\omega \in \Lambda^q(M;\mathcal{E}))$

$$\frac{d}{du}d^*_{q-1}(u)\omega + R_{q-1}(u)d^*_{q-1}(u)\omega = d^*_{q-1}(u)R_q(u)\omega . \tag{A3.6}$$

Recall that $\Delta^+_q d_{q-1} = d_{q-1}\Delta^-_{q-1}$ and thus, by functional calculus,

$$e^{-x\Delta^+_q(u)}d_{q-1} = d_{q-1}e^{-x\Delta^-_{q-1}(u)}.$$

Substituting (A3.6) into (A3.3) and using that $\mathrm{tr}_N(AB) = \mathrm{tr}_N(BA)$ then leads to

$$\frac{d}{du}\mathrm{tr}_N\left(e^{-x\Delta^+_q}\right) = x\,\mathrm{tr}_N\left(R_{q-1}(u)\Delta^-_{q-1}(u)e^{-x\Delta^-_q(u)}\right) - x\,\mathrm{tr}_N\left(R_q(u)\Delta^+_q(u)e^{-x\Delta^+_q(u)}\right)$$

and, summing up,

$$\begin{aligned}
&\frac{d}{du}\left(\sum_{q=0}^{d}(-1)^q\,\mathrm{tr}_N(e^{-x\Delta^+_q(u)})\right)\\
&= -x\sum_0^d(-1)^q\,\mathrm{tr}_N\left(R_q(u)(e^{-x\Delta^+_q(u)}\Delta^+_q(u) + e^{-x\Delta^-_q(u)})\Delta^-_q(u)\right)\\
&= -x\sum_0^d(-1)^q\,\mathrm{tr}_N\left(R_q(u)(e^{-x\Delta_q(u)}\Delta_q(u))\right) \qquad \text{(A3.7)}\\
&= x\frac{d}{dx}\sum_0^d(-1)^q\,\mathrm{tr}_N\left(R_q(u)(e^{-x\Delta_q(u)}(Id - Q_q(0,u)))\right)
\end{aligned}$$

where $Q_q(0,u)$ denotes the orthogonal projection onto the space of harmonic q-forms $\mathrm{Null}(\Delta_q(u))$. By assumption, $(M,\mathcal{W})$ is of determinant class and therefore

$$-\infty < \int_{0+}^{1}\log\mu dN_{\Delta_q(u)}(\mu) .$$

We now compute

$$\frac{d}{du}\log T^+_{\mathrm{an}}(s,u) = \frac{d}{du}\frac{\partial}{\partial s}I(s,u) + \frac{d}{du}II(u)$$

for $s = 0$, where

$$I(s,u) := -\frac{1}{2}\sum_{q=0}^{d}(-1)^q\frac{1}{\Gamma(s)}\int_0^1 x^{s-1}\,\mathrm{tr}_N(e^{-x\Delta^+_q(u)})dx$$

and

$$II(u) := -\frac{1}{2}\sum_{q=0}^{d}(-1)^q \int_1^\infty \frac{1}{x}\,\mathrm{tr}_N(e^{-x\Delta_q^+(u)})dx \;.$$

In view of (A3.7) we get

$$\frac{d}{du}II(u) = -\frac{1}{2}\sum_{q=0}^{d}(-1)^q \int_1^\infty \frac{d}{dx}\,\mathrm{tr}_N\left(R_q(u)e^{-x\Delta_q(u)}(Id - Q_q(0,u))\right)dx$$

$$= \frac{1}{2}\sum_{q=0}^{d}(-1)^q\,\mathrm{tr}_N\left(R_q(u)e^{-\Delta_q(u)}(Id - Q_q(0,u))\right)dx \qquad \text{(A3.8)}$$

where we used that

$$\lim_{x\to\infty}\mathrm{tr}_N\left(R_q(u)e^{-x\Delta_q(u)}(Id - Q_q(0,u))\right) = 0 \;.$$

Again applying (A3.7), and integrating by parts

$$\frac{d}{du}\frac{\partial}{\partial s}\Big|_{s=0} I(s,u) = -\frac{1}{2}\frac{\partial}{\partial s}\Big|_{s=0}\frac{1}{\Gamma(s)}\int_0^1 x^{s-1}\sum_{q=0}^{d}(-1)^q\frac{d}{du}\,\mathrm{tr}_N(e^{-x\Delta_q^+(u)})dx$$

$$= -\frac{1}{2}\sum_{q=0}^{d}(-1)^q\,\mathrm{tr}_N\left(R_q(u)e^{-\Delta_q(u)}(Id - Q_q(0,u))\right) + \qquad \text{(A3.9)}$$

$$+\mathrm{F.p.}_{s=0}\frac{1}{\Gamma(s)}\int_0^1 x^{s-1}\frac{1}{2}\sum_{q=0}^{d}(-1)^q\,\mathrm{tr}_N\left(R_q(u)e^{-\Delta_q(u)}(Id - Q_q(0,u))\right)dx \;.$$

Combining (A3.8) and (A3.9) one obtains

$$\frac{d}{du}\log T_{\mathrm{an}}(u) = -\frac{d}{du}\log T_{\mathrm{an}}^+(0,u) = -\frac{\partial}{\partial s}\Big|_{s=0} I(s,u) - II(u) = \qquad \text{(A3.10)}$$

$$-\mathrm{F.p.}_{s=0}\frac{1}{\Gamma(s)}\int_0^1 x^{s-1}\frac{1}{2}\sum_{q=0}^{d}(-1)^q\,\mathrm{tr}_N\left(R_q(u)e^{-\Delta_q(u)}(Id - Q_q(0,u))\right)dx \;.$$

The same arguments which were used to verify (6.21′) can be applied in this situation as well: Note that the heat kernel expansion for the Schwartz kernel $K_q(y,y',x,u)$ of $e^{-x\Delta_q(u)}$ on the diagonal $y = y'$ is of the form

$$K_q(y,y,x,u) = \sum_{j=0}^{d} x^{(j-d)/2} l_{q,j}(y,u) + O_u(x^{1/2})$$

where $l_{q,j}(y,u)$ are densities defined on M with values in $\mathcal{B}$ and the error term $O_u(x^{1/2})$ is a density which can be bounded by $Cx^{1/2}$ with C

independent of u in a compact subset of the parameter domain. Thus

$$\text{F.p.}_{s=0}\frac{1}{\Gamma(s)}\int_0^1 x^{s-1}\sum_{q=0}^{d}(-1)^q \operatorname{tr}_N\left(R_q(u)e^{-\Delta_q(u)}\right)dx$$
$$=\sum_{q=0}^{d}(-1)^q \operatorname{tr}_N\left(R_q(u)l_{q,d}(\cdot,u)\right) .$$

Taking into account that M is of odd dimension and the operator $\Delta_q(u)$ is of even order one sees by a parity argument that $\operatorname{tr}_N(R_q(u)l_{q,d}(\cdot,u))=0$. Therefore

$$\text{F.p.}_{s=0}\frac{1}{\Gamma(s)}\int_0^1 x^{s-1}\operatorname{tr}_N\left(R_q(u)e^{-x\Delta_q(u)}\right)dx=0$$

and (A3.10) leads to

$$\frac{d}{du}\log T_{\text{an}}(u)=\frac{1}{2}\sum_{q=0}^{d}(-1)^q \operatorname{tr}_N\left(Q_q(0,u)R_q(u)Q_q(0,u)\right) . \tag{A3.11}$$

Let $P_q(u):=Q_q(0,u)$ and define (for $u_0\in\mathbb{R}$ arbitrary, but fixed)

$$\sigma_q(u):\mathcal{H}^q_{u_0}(M;\mathcal{E})\to\mathcal{H}^q_u(M;\mathcal{E}) \tag{A3.12}$$

where $\mathcal{H}^q_u(M;\mathcal{E})$ denotes the nullspace $\operatorname{Null}(\Delta_q(u))$ and $\sigma_q(u)(\omega)$ is the unique element in $\mathcal{H}^q_u(M;\mathcal{E})$ satisfying $[\omega]=[\sigma_q(u)(\omega)]$. Thus $\frac{d}{du}\sigma_q(u)(\omega)$ is cohomologous to 0, i.e. there exists a sequence $(\eta_j(u))_{j\geq1}$ in $\Lambda^{q-1}(M;\mathcal{E})$ such that

$$\frac{d}{du}\sigma_q(u)(\omega)=\lim_{j\to\infty}d_{q-1}\eta_j(u) .$$

To prove the lemma it remains to show that

$$\operatorname{tr}_N\left(R_q(u)P_q(u)\right)=\frac{d}{du}\log\det{}_N\left(\sigma_q(u)^*\sigma_q(u)\right) . \tag{A3.13}$$

To verify this identity notice that $\sigma_q(u)$ is an isomorphism. Therefore

$$\log\det{}_N\left(\sigma_q(u)^*\sigma_q(u)\right)>-\infty$$

and thus

$$\frac{d}{du}\log\det_N\left(\sigma_q(u)^*\sigma_q(u)\right)=\operatorname{tr}_N\left(\left(\sigma_q(u)^*\sigma_q(u)\right)^{-1}\frac{d}{du}\left(\sigma_q(u)^*\sigma_q(u)\right)\right) . \tag{A3.14}$$

To compute $\frac{d}{du}(\sigma_q(u)^*\sigma_q(u))$, consider ($\omega,\omega'\in\mathcal{H}^q_{u_0}(M;\mathcal{E})$)

$$\frac{d}{du}\left\langle\sigma_q(u)^*\sigma_q(u)\omega,\omega'\right\rangle_{u_0}=\frac{d}{du}\left\langle\sigma_q(u)\omega,\sigma_q(u)\omega'\right\rangle_u=I(u)+II(u)+III(u)$$

where

$$I(u) := \left\langle \frac{d}{du}\sigma_q(u)\omega, \sigma_q(u)\omega' \right\rangle_u ,$$

$$II(u) := \left\langle \sigma_q(u)\omega, \frac{d}{du}\sigma_q(u)\omega' \right\rangle_u ,$$

and

$$III(u) := \langle R_q(u)\sigma_q(u)\omega, \sigma_q(u)\omega' \rangle_u .$$

In view of (A3.12), $I(u) = 0$ and $II(u) = 0$ and thus

$$\frac{d}{du}\big(\sigma_q(u)^*\sigma_q(u)\big) = \sigma_q(u)^* R_q(u)\sigma_q(u) .$$

Substituting this into (A3.14) leads to

$$\begin{aligned}\frac{d}{du}\log\det{}_N \big(\sigma_q(u)^*\sigma_q(u)\big) &= \mathrm{tr}_N \big(\sigma(u)^{-1}(\sigma_q(u)^*)^{-1}\sigma_q(u)^* R_q(u)\sigma_q(u)\big) \\ &= \mathrm{tr}_N \big(R_q(u)P_q(u)\big)\end{aligned}$$

which establishes (A3.13). □

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