Computing 26, 73—77 (1981)

**Computing**



# On the Natural Interpolation Formula for Cauchy Type Singular Integral Equations of the First Kind

**N. I. Ioakimidis,** Patras

Received April 8, 1980

**Abstract — Zusammenfassung**

**On the Natural Interpolation Formula for Cauchy Type Singular Integral Equations of the First Kind.** A Cauchy type singular integral equation of the first kind can be numerically solved either directly, through the use of a Gaussian numerical integration rule, or by reduction to an equivalent Fredholm integral equation of the second kind, where the Nyström method is applicable. In this note it is proved that under appropriate but reasonable conditions the expressions of the unknown function of the integral equation, resulting from the natural interpolation formulae of the direct method, as well as of the Nyström method, are identical along the whole integration interval.

*Key words and phrases:* Cauchy type singular integral equations, natural interpolation formulae, Gauss-Chebyshev quadrature rule, Nyström (quadrature) method for integral equations, direct numerical solution of Cauchy type singular integral equations.

*1980 Mathematics Subject Classification numbers:* Primary: 65R20, Secondary: 45E05, 45L10, 65D05.

**Über die natürliche Interpolationsformel für singuläre Integralgleichungen vom Cauchy-Typ erster Art.** Eine singuläre Integralgleichung erster Art vom Cauchy-Typ kann entweder direkt, mittels einer Gaußschen numerischen Integrationsformel, oder durch Reduktion auf eine äquivalente Fredholmsche Integralgleichung zweiter Art, wo die Nyström-Methode anwendbar ist, gelöst werden. In dieser Arbeit wird bewiesen, daß unter geeigneten und sinnvollen Bedingungen die Ausdrücke der unbekannten Funktion der Integralgleichung, die einerseits bei den natürlichen Integrationsformeln der direkten Methode und anderseits bei der Nyström-Methode entstehen, im ganzen Integrationsintervall gleich sind.

## 1. Introduction

For the numerical solution of the Cauchy type singular integral equation of the first kind

$$\int_{-1}^{1} (1-t^2)^{-\frac{1}{2}} \left[\frac{1}{\pi(t-x)}+k(t,x)\right] g(t)\,dt=f(x), \quad -1<x<1, \tag{1}$$

accompanied by the condition

0010-485X/81/0026/0073/$ 01.00

$$\int_{-1}^{1} (1-t^2)^{-\frac{1}{2}} g(t)\, dt = C, \tag{2}$$

where $g(t)$ is the unknown function, $k(t, x)$ a regular (Fredholm) kernel, $f(x)$ the known right-hand side function and $C$ a known constant, one can use the direct quadrature method [2, 6] to reduce it to the following approximate system of linear algebraic equations

$$\sum_{i=1}^{n} A_i \left[\frac{1}{\pi(t_i - x_k)} + k(t_i, x_k)\right] g_1(t_i) = f(x_k),\ k = 1\,(1)\,(n-1), \tag{3 a}$$

$$\sum_{i=1}^{n} A_i\, g_1(t_i) = C, \tag{3 b}$$

where $g_1(t_i)$ are approximations to $g(t_i)$ and the nodes $t_i$, the weights $A_i$ and the collocation points $x_k$ are defined by

$$T_n(t_i) = 0,\ A_i = \pi/n,\ i = 1\,(1)\,n,\ U_{n-1}(x_k) = 0,\ k = 1\,(1)\,(n-1), \tag{4}$$

with $T_n(x)$ and $U_n(x)$ denoting the Chebyshev polynomials of degree $n$ of the first and the second kind, respectively.

Alternatively, one can use the regularization procedure [3] to reduce (1, 2) to the following system of linear algebraic equations [5]

$$g_2(t_j) + \frac{1}{\pi} \sum_{i=1}^{n} A_i \left[\sum_{k=1}^{n-1} B_k \frac{k(t_i, x_k)}{t_j - x_k}\right] g_2(t_i) = \frac{1}{\pi} \sum_{k=1}^{n-1} B_k \frac{f(x_k)}{t_j - x_k} + \frac{C}{\pi},\ j = 1\,(1)\,n, \tag{5}$$

where the weights $B_k$ are now determined by [5]

$$B_k = \pi(1 - x_k^2)/n,\ k = 1\,(1)\,(n-1), \tag{6}$$

and $g_2(t_i)$ are also approximations to $g(t_i)$.

As regards the approximation $g_1(x)$ to $g(x)$ by the first method along the whole interval $[-1, 1]$ (with the exception of the nodes $t_i$, where $g_1(t_i)$ are determined from (3), and the collocation points $x_k$), this can be based on the error term of the Gauss-Chebyshev quadrature rule used in (3) and (5) [7], which, on the basis of the results of [4], leads to the natural interpolation formula

$$g_1(x) = \frac{T_n(x)}{U_{n-1}(x)} \left[f^*(x) - \frac{1}{\pi} \sum_{i=1}^{n} A_i \frac{g_1(t_i)}{t_i - x}\right],\ x \neq t_i,\ i = 1\,(1)\,n, \quad x \neq x_k,\ k = 1\,(1)\,(n-1), \tag{7}$$

with

$$f^*(x) = f(x) - \sum_{i=1}^{n} A_i\, k(t_i, x)\, g_1(t_i), \tag{8}$$

based on the values of $g_1(t_i)$. Similarly, the approximation $g_2(x)$ to $g(x)$ by the second method along the whole interval $[-1, 1]$ can be based on the Nyström natural interpolation formula for Fredholm integral equations of the second kind [1], which, by using the numerical technique of [5] for the reduction of (1, 2) to a Fredholm integral equation of the second kind and the results of [4], can be seen to have the form

$$g_2(x)=\frac{T_n(x)}{U_{n-1}(x)}f^*(x)-\frac{1}{\pi}\sum_{k=1}^{n-1}B_k\frac{f^*(x_k)}{x_k-x}+\frac{C}{\pi},\ x\neq x_k,\ k=1\,(1)\,(n-1), \tag{9}$$

with $g_2(t_i)$ used instead of $g_1(t_i)$ in (8).

In [5] it was shown that

$$g_1(t_i)=g_2(t_i),\ i=1\,(1)\,n, \tag{10}$$

that is the systems of linear algebraic equations (3) and (5) are equivalent. Here we will show that

$$g_1(x)\equiv g_2(x) \tag{11}$$

along the whole integration interval $[-1,1]$.

## 2. Proof of the Equivalence

By comparing the natural interpolation formulas (7) and (9), we see that in order to prove (11) (taking also into account (10)), we have to show that

$$\frac{T_n(x)}{U_{n-1}(x)}\sum_{i=1}^{n}A_i\frac{g_1(t_i)}{t_i-x}=\sum_{k=1}^{n-1}B_k\frac{f^*(x_k)}{x_k-x}-C,\ x\neq t_i,\ i=1\,(1)\,n,$$
$$x\neq x_k,\ k=1\,(1)\,(n-1). \tag{12}$$

Now, because of (3 a) and (8), we have

$$f^*(x_k)=\frac{1}{\pi}\sum_{i=1}^{n}A_i\frac{g_1(t_i)}{t_i-x_k},\ k=1\,(1)\,(n-1). \tag{13}$$

By taking into account (3 b), (10) and (13), we can write (12) as

$$\frac{T_n(x)}{U_{n-1}(x)}\sum_{i=1}^{n}A_i\frac{g_1(t_i)}{t_i-x}=\sum_{i=1}^{n}A_i\,g_1(t_i)\left[\frac{1}{\pi}\sum_{k=1}^{n-1}\frac{B_k}{(x_k-x)(t_i-x_k)}-1\right],$$
$$x\neq t_i,\ i=1\,(1)\,n,\ x\neq x_k,\ k=1\,(1)\,(n-1). \tag{14}$$

To show the validity of (14), it is sufficient to show that

$$\frac{T_n(x)}{U_{n-1}(x)}=\frac{1}{\pi}\sum_{k=1}^{n-1}\frac{B_k(t_i-x)}{(x_k-x)(t_i-x_k)}-t_i+x,\ i=1\,(1)\,n,$$
$$x\neq x_k,\ k=1\,(1)\,(n-1). \tag{15}$$

But since

$$\frac{t_i-x}{(x_k-x)(t_i-x_k)}=\frac{1}{x_k-x}+\frac{1}{t_i-x_k} \tag{16}$$

and [5]

$$\sum_{k=1}^{n-1}\frac{B_k}{x_k-x}-\frac{\pi\,T_n(x)}{U_{n-1}(x)}=-\pi\,x,\ x\neq x_k,\ k=1\,(1)\,(n-1), \tag{17 a}$$

$$\sum_{k=1}^{n-1}\frac{B_k}{x_k-t_i}=-\pi\,t_i,\ i=1\,(1)\,n, \tag{17 b}$$

(15) is valid. This completes the proof of (11).

## 3. Discussion

It was shown above that the natural interpolation formula for the approximation to the solution of a Cauchy type singular integral equation of the first kind [7] is equivalent to the Nyström natural interpolation formula [1] for the equivalent Fredholm integral equation of the second kind [3] if the Gauss-Chebyshev quadrature rule is used [5]. A similar result can also be proved in the case when the Lobatto-Chebyshev quadrature rule is used instead of the Gauss-Chebyshev quadrature rule [6]. Moreover, the above result remains valid if the index of (1) [3] is equal to 0 and not equal to 1 as assumed previously. (In this case, (2) is no longer necessary.)

Furthermore, by comparing the natural interpolation formulae (7) and (9) (based on the solutions $g_1(t_i)$ and $g_2(t_i)$ of (3) and (5), respectively), we observe that the evaluation of $g_1(x)$ requires less computational effort than the evaluation of $g_2(x)$ because of the appearance of the quantities $f^*(x_k)$ in the right-hand side of (9). In another wording and because of (11), (7) is a more convenient writing of (9).

As was stated in [7] and verified in numerical examples, the use of a natural interpolation formula during the numerical solution of a Cauchy type singular integral equation significantly improves the accuracy of the numerical results in comparison with the use of Lagrangian interpolation methods based on the values of $g_1(t_i)$. This is a quite analogous result to the use of the Nyström natural interpolation formula for Fredholm integral equations of the second kind. Finally, (11) may prove useful for an alternative proof of the convergence of the direct quadrature method of numerical solution of (1) on the basis of well-known results for Fredholm integral equations of the second kind [1]. Yet, this is not trivial since quadrature rules are used in the Fredholm integral equation equivalent to (1, 2) not only for the approximation to the integral term, but also to the kernel itself [5]. In spite of this difficulty, it is believed that (11) may prove of fundamental importance for theoretical investigations of the direct quadrature methods for the numerical solution of Cauchy type singular integral equations of the first kind.

**Acknowledgement**

This research was sponsored by a grant of the National Hellenic Research Foundation. The support of this Foundation is gratefully acknowledged.

**References**

[1] Baker, C. T. H.: The numerical treatment of integral equations, 1st ed., pp. 356—375. Oxford: Clarendon Press 1977.

[2] Erdogan, F., Gupta, G. D.: On the numerical solution of singular integral equations. Quart. Appl. Math. *29*, 525—534 (1972).

[3] Gakhov, F. D.: Boundary value problems, 1st English ed., pp. 472—479. Oxford: Pergamon Press and Addison-Wesley 1966. [Translation of the 2nd Russian ed., Moscow: Fizmatgiz 1963.]

[4] Ioakimidis, N. I., Theocaris, P. S.: On the numerical evaluation of Cauchy principal value integrals. Rev. Roumaine Sci. Tech. Sér. Méc. Appl. *22*, 803—818 (1977).

[5] Ioakimidis, N. I., Theocaris, P. S.: A comparison between the direct and the classical numerical methods for the solution of Cauchy type singular integral equations. SIAM J. Numer. Anal. *17*, 115—118 (1980).
[6] Theocaris, P. S., Ioakimidis, N. I.: Numerical integration methods for the solution of singular integral equations. Quart. Appl. Math. *35*, 173—183 (1977).
[7] Theocaris, P. S., Ioakimidis, N. I.: A remark on the numerical solution of singular integral equations and the determination of stress-intensity factors. J. Engrg. Math. *13*, 213—222 (1979).

Prof. N. I. Ioakimidis
Chair of Mathematics B′
School of Engineering
University of Patras
P.O. Box 25B
Patras
Greece