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Computing the Range of Values of Real Functions with Accuracy Higher than Second Order

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Abstract — Zusammenfassung

Computing the Range of Values of Real Functions with Accuracy Higher than Second Order. Given a continuous function $f: D \to \mathbb{R}$ on a compact interval $D \subseteq \mathbb{R}$ we consider the problem of finding an interval $V(f, X)$ that contains the range of the values of f, $W(f, X) := \{f(x) \mid x \in X\}$, on a subinterval $X \subseteq D$. To reach this goal we use methods from interval-arithmetic. When $V(f, X)$ is computed by one of the well-known methods from literature for a sequence $\{X_n\}$ of intervals with decreasing diameters $d(X_n) \to 0$, then generally the overestimation of $W(f, X_n)$ by $V(f, X_n)$ will decrease at most quadratically with $d(X_n)$. The method presented in this paper, however, allows the computation of $V(f, X_n)$ such that this overestimation decreases with an arbitrary power $s > 0$ of $d(X_n)$. Theoretically any power $s \in \mathbb{N}$ is possible, in practice, however, $1 \leq s \leq 4$ can be reached with little or moderate amount of work and $s = 5$ or $s = 6$ with some more work. A generalization to functions $f: \mathbb{R}^n \to \mathbb{R}$ is given at the end of the paper.

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Die Berechnung des Wertebereiches reeller Funktionen mit Verfahren von höherer als zweiter Ordnung. Für eine stetige Funktion $f: D \to \mathbb{R}$ auf einem kompakten Intervall $D \subseteq \mathbb{R}$ betrachten wir das Problem, ein Intervall $V(f, X)$ zu finden, das den Wertebereich $W(f, X) := \{f(x) \mid x \in X\}$ von f auf einem Teilintervall $X \subseteq D$ enthält. Um dies zu erreichen, verwenden wir Methoden der Intervallrechnung. Wird $V(f, X)$ mit einer aus der Literatur bekannten Methode für eine Folge $\{X_n\}$ von Intervallen mit abnehmendem Durchmesser $d(X_n) \to 0$ berechnet, dann wird i.a. die Überschätzung von $W(f, X_n)$ durch $V(f, X_n)$ höchstens quadratisch mit $d(X_n)$ abnehmen. Das in dieser Arbeit vorgestellte Verfahren erlaubt es, $V(f, X_n)$ so zu berechnen, daß diese Überschätzung mit einer beliebigen Potenz $s > 0$ von $d(X_n)$ abnimmt. Theoretisch ist jede Potenz $s \in \mathbb{N}$ erreichbar, in der Praxis jedoch kann $1 \leq s \leq 4$ mit wenig oder mäßigem Aufwand und $s = 5$ oder $s = 6$ mit etwas größerem Aufwand erreicht werden. Eine Verallgemeinerung auf Funktionen $f: \mathbb{R}^n \to \mathbb{R}$ wird zum Schluß der Arbeit angegeben.

1. Introduction

Let $f: D \to \mathbb{R}$ be a continuous function defined on a compact interval $D \subseteq \mathbb{R}$ and let $X := [\underline{x}, \bar{x}] \subseteq D$ be any subinterval of D. Then the range of values of f on X

$$W(f, X) := \{f(x) \mid x \in X\} \tag{1}$$

is obviously again a compact interval in $\mathbb{R}$. In general the numerical computation of this interval $W(f, X)$ is not possible. Thus it is desirable to have computational procedures for the construction of an interval $V(f,X)$ that approximates $W(f,X)$ with a high degree of accuracy such that $W(f,X)$ is contained in $V(f,X)$:

$$W(f,X) \subseteq V(f,X). \tag{2}$$

Sometimes also an inner approximation $\underline{V}(f,X)$ of $W(f,X)$ is useful:

$$W(f,X) \supseteq \underline{V}(f,X). \tag{3}$$

In order to be able to get a high accuracy of the approximations the function f will have to satisfy additional assumptions:

$$\left\{\begin{array}{l}\text{let } f \text{ be } m \text{ times differentiable and let each} \\ \text{derivative } f^{(k)},\ k=1\,(1)\,m, \\ \text{have an interval evaluation for any interval} \\ X \subseteq D \text{ (see Section 2).}\end{array}\right. \tag{4}$$

If we introduce a metric $q(\cdot,\cdot)$ for (nonempty) real intervals, then $q(W(f,X), V(f,X))$ is a measure for the amount by which $V(f,X)$ overestimates $W(f,X)$. If there holds an estimation of the form

$$q(W(f,X), V(f,X)) \leq c \cdot d(X)^n \tag{5}$$

with fixed $n \in \mathbb{N}$ and $c = c(D) \geq 0$, where $d(X) = \bar{x} - \underline{x}$ is the diameter of $X \subseteq D$, then $V(f,X)$ is called an n-th order approximation of $W(f,X)$. It is obvious that for small $d(X)$ it is desirable to have approximations of as high order as possible. To the knowledge of the authors there are at the present time only methods available where $n=1$ or $n=2$ in (5) can be reached. Some of these methods will be described shortly in Section 2 where we also introduce our notations and basic definitions. Herzberger [8] mentions a special case where a higher than second order can be reached in (5) for a certain class of intervals, however, in this case f has to satisfy a very strong condition which is almost never satisfied in practice.

After the discussion of known results in Section 2 we present a basic theorem in Section 3 which allows a great variety of procedures for the construction of approximations $V(f,X)$ and $\underline{V}(f,X)$. In Section 4 several realizations of such possible procedures are presented and discussed which use interpolation and Taylor-expansions. This section contains the most important results for practical applications: among others we present explicit expressions (42), (43) and (44) for third-order approximations which can be evaluated directly using interval-arithmetic. In Section 5 we discuss some connections between our results and the mean value form. Section 6 illustrates the methods with numerical examples and finally in Section 7 we discuss a generalization of our method to functions $f: \mathbb{R}^n \to \mathbb{R}$.

2. Notations, Definitions and Known Results

In this paper we will use capital letters $A, B, C, \ldots$ for real compact (nonempty) intervals, and the set of all these intervals is denoted by $I(\mathbb{R})$:

$$I(\mathbb{R}) := \{X = [\underline{x}, \bar{x}] \mid \underline{x} \leq \bar{x},\ \underline{x}, \bar{x} \in \mathbb{R}\};$$

analogously the set of all intervals contained in D is called $I(D)$:

$$I(D):=\{X=[\underline{x},\bar{x}] \mid \underline{x}\le\bar{x},\ \underline{x},\bar{x}\in D\},$$

where $\underline{x}$ and $\bar{x}$ are the lower and the upper bound of X. Arithmetic operations are defined for intervals according to Moore [14], [15] or Alefeld/Herzberger [2] by

$$X+Y:=[\underline{x}+\underline{y},\bar{x}+\bar{y}],\ X-Y:=[\underline{x}-\bar{y},\bar{x}-\underline{y}],$$
$$X\cdot Y:=[\min\{\underline{x}\underline{y},\underline{x}\bar{y},\bar{x}\underline{y},\bar{x}\bar{y}\},\ \max\{\underline{x}\underline{y},\underline{x}\bar{y},\bar{x}\underline{y},\bar{x}\bar{y}\}],$$
$$X\ /\ Y:=X\cdot\left[\frac{1}{\bar{y}},\frac{1}{\underline{y}}\right],\ \text{if } 0\notin Y$$

where there hold

$$X*Y=\{x*y \mid x\in X, y\in Y\},\ *\in\{+,-,\cdot,/\},$$

and the inclusion monotonicity

$$A\subseteq X,\ B\subseteq Y\Rightarrow A*B\subseteq X*Y,\ *\in\{+,-,\cdot,/\}.$$

For $x\in X$ and $y\in Y$ this is the inclusion property

$$x*y\in X*Y,\ *\in\{+,-,\cdot,/\}.$$

Using the notations from Alefeld/Herzberger [2] we define

$d(X):=\bar{x}-\underline{x}$, the diameter of X,

$|X|:=\max_{x\in X}|x|=\max\{|\underline{x}|,|\bar{x}|\}$, the absolute value of X,

$q(X,Y):=\max\{|\underline{x}-\underline{y}|,|\bar{x}-\bar{y}|\}$, the distance between X and Y.

The distance-function $q(\cdot,\cdot)$ is the Hausdorff-distance which makes $I(\mathbb{R})$ to a complete metric space.

We will need the following properties which hold for all $X, Y, Z\in I(\mathbb{R})$:

$$0\in X\Rightarrow |X|\le d(X), \tag{6}$$

$$a\in\mathbb{R}\Rightarrow d(a\pm X)=d(X), \tag{7}$$

$$d(X)\le 2|X|, \tag{8}$$

$$0\in X,\ 0\in Y\Rightarrow d(X\cdot Y)\le d(X)\cdot d(Y), \tag{9}$$

$$X\subseteq Y\subseteq Z\Rightarrow q(X,Y)\le q(X,Z). \tag{10}$$

With the exception of (9) the proofs can be found in Alefeld/Herzberger [2] or Moore [14], [15]. Property (9) can be shown as follows:

Since

$$0\le-\underline{x},\bar{x}\le d(X)\quad\text{and}\quad 0\le-\underline{y},\bar{y}\le d(Y)$$

we have:

$$\begin{aligned}d(X\cdot Y)&=\max\{\underline{x}\underline{y},\bar{x}\bar{y}\}-\min\{\underline{x}\bar{y},\bar{x}\underline{y}\}=\\&=\max\{\underline{x}\underline{y},\bar{x}\bar{y}\}+\max\{-\underline{x}\bar{y},-\bar{x}\underline{y}\}=\\&=\max\{\underline{x}\underline{y}-\underline{x}\bar{y},\underline{x}\underline{y}-\bar{x}\underline{y},\bar{x}\bar{y}-\underline{x}\bar{y},\bar{x}\bar{y}-\bar{x}\underline{y}\}=\\&=\max\{(-\underline{x})d(Y),(-\underline{y})d(X),\bar{y}d(X),\bar{x}d(Y)\}\le d(X)\cdot d(Y).\end{aligned}$$ □

The following theorems are well known in literature. The interval evaluations used there are defined as in Alefeld/Herzberger [2] or in Moore [14], [15].

Theorem 1:
Let the function $f: D \to \mathbb{R}$ be represented by an expression $f(x)$ in which the independent variable x occurs exactly one time. Then for the interval evaluation $f(X)$, $X \in I(D)$, there holds:

$$f(X) = W(f, X). \tag{11}$$

Theorem 2:
If $f: D \to \mathbb{R}$ satisfies certain Lipschitz-conditions (for details see Alefeld/Herzberger [2] or Moore [14], [15]) then the interval evaluation of f satisfies for all $X \in I(D)$:

$$\begin{cases} W(f, X) \subseteq f(X), \\ q(W(f, X), f(X)) \leq c_1 \cdot d(X) \text{ with } c_1 = \text{const.} \geq 0, \\ d(f(X)) \leq c_2 \cdot d(X) \text{ with } c_2 = \text{const.} \geq 0. \end{cases} \tag{12}$$

Thus, if $d(X_n) \to 0$ for a sequence $\{X_n\}$ of intervals, then the overestimation of $W(f, X_n)$ by $f(X_n)$ converges to zero at least linearly with $d(X)$.

Theorem 3:
Let $f: D \to \mathbb{R}$ be differentiable and let f' have an interval evaluation that satisfies Theorem 2. Then for the mean value form

$$F_y(f, X) := f(y) + f'(X) \cdot (X - y), \; y \in X \text{ fixed}, \tag{13}$$

there holds for all $X \in I(D)$:

$$\begin{cases} W(f, X) \subseteq F_y(f, X), \\ q(W(f, X), F_y(f, X)) \leq c_3 \cdot d(X)^2, \; c_3 = \text{const.} \geq 0. \end{cases} \tag{14}$$

Thus, if $d(X_n) \to 0$ for a sequence $\{X_n\}$ of intervals, then the overestimation of $W(f, X_n)$ by $F_y(f, X_n)$ converges to zero at least quadratically with $d(X)$.

Many investigations have been done during the last fifteen years concerning this mean value form and related forms. One of the related forms is the centered form

$$f(X) := f(y) + h(X - y) \cdot (X - y)$$

for polynomials and rational functions which has also the properties (14) and thus quadratic convergence. Krawczyk and Nickel [10] defined a more general centered form

$$F(X) := f(y) + H(X, y) \cdot (X - y)$$

and showed quadratic convergence in the case that H satisfies a Lipschitz-condition in the first argument. They also showed that the mean value form (13) is a special case of this centered form. Ratschek [16] introduced centered forms of "higher order" for rational functions followed by further investigations by Ratschek and Rokne [18], Ratschek [17], Ratschek and Schröder [19] and Alefeld and Rokne [3]. Although these centered forms use higher derivatives of f and give in general better approximations than a centered form of "lower order", their order of convergence is still quadratic. Adams and Lohner [1] use Taylor-approximations with remainder terms of higher order, however, the overestimation is still of second order. The only case with higher than quadratic convergence that seems to be mentioned in literature is in the paper by Herzberger [8] cited already in the introduction and this is a very unrealistic case. A construction of an outer *and* an inner approximation of $W(f, X)$ is presented by Krawczyk [9]. He also shows quadratic convergence of these approximations to $W(f, X)$.

3. The Basic Theorem

Let $f: D \to \mathbb{R}$ have a representation of the form

$$f(x) = g(x) + r(x) \quad \text{for all} \quad x \in D \tag{15}$$

with continuous functions g and r. Furthermore, let $R(X)$, $R(D) \in I(\mathbb{R})$ be intervals such that

$$r(x) \in R(X) \subseteq R(D) \quad \text{for all} \quad x \in X, \ X \in I(D). \tag{16}$$

The function g can be interpreted as an approximation of f and r is the corresponding remainder term. The intervals $R(X)$ resp. $R(D)$ are estimations of this remainder term for all $x \in X$ resp. $x \in D$.

Now we define the basic evaluation procedure for (15) on an interval $X \in I(D)$: the *remainder form* of the representation (15) of f is defined as

$$V(f, X) := W(g, X) + R(X). \tag{17}$$

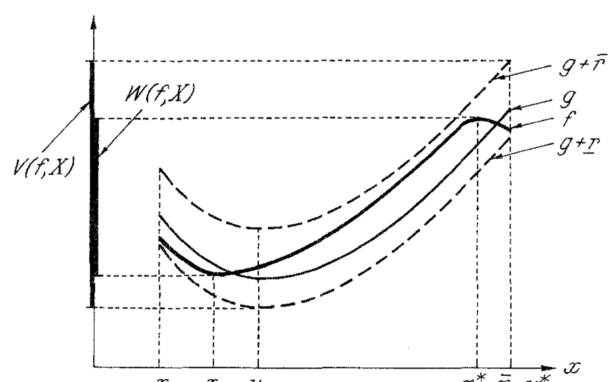

Fig. 1

Clearly $V(f, X)$, $W(g, X) \in I(\mathbb{R})$. Note that we do *not* use an interval evaluation of $g(x)$ but rather we use the *exact range of* g on X, $W(g, X)$, to compute $V(f, X)$. However, $R(X)$ can be an interval evaluation of $r(x)$ or any other estimation of $r(x)$ on X. The use of $W(g, X)$ implies that, in practical applications, we can choose only very simple functions g, e.g. polynomials of degree at most 5, or monotone functions. Fig. 1 indicates that $W(f, X)$ is contained in $V(f, X)$ and that the maximum distance between both intervals is not greater than $d(R(X))$. This is formally proved in the following

Theorem 4:

Let the continuous function $f: D \to \mathbb{R}$ have the representation (15) *and let $V(f, X)$ be the remainder form* (17). *Then for all $X \in I(D)$ there holds*

$$\begin{cases} (a) \quad W(f,X) \subseteq V(f,X), \\ (b) \quad q\big(W(f,X), V(f,X)\big) \le d\big(R(X)\big) \le 2\,|R(X)|. \end{cases} \tag{18}$$

Proof:

Because of (15) and (16) there holds for any $x \in X$:

$$f(x) = g(x) + r(x) \in g(x) + R(X) \subseteq W(g,X) + R(X) = V(f,X). \tag{19}$$

Since this is true for all $x \in X$, we have shown (a).

Now we prove part (b). Since f is continuous on the compact interval X, there exist two points $x_*, x^* \in X$ where f takes its minimum and its maximum:

$$\begin{cases} \quad f(x_*) \le f(x) \le f(x^*) \ \text{ for all } \ x \in X, \\ \Rightarrow W(f,X) = [f(x_*), f(x^*)]. \end{cases} \tag{20}$$

Since g is also continuous there exist two points $y_*, y^* \in X$ with

$$\begin{cases} \quad g(y_*) \le g(x) \le g(y^*) \ \text{ for all } \ x \in X, \\ \Rightarrow W(g,X) = [g(y_*), g(y^*)]. \end{cases} \tag{21}$$

With $[\underline{r}, \bar{r}] := R(X)$ we then have

$$\begin{aligned} q\big(W(f,X), V(f,X)\big) &= q\big([f(x_*), f(x^*)], [g(y_*), g(y^*)] + [\underline{r}, \bar{r}]\big) = \\ &= \max\{|f(x_*) - g(y_*) - \underline{r}|, |f(x^*) - g(y^*) - \bar{r}|\}. \end{aligned} \tag{22}$$

Estimating the arguments separately yields:

$$\begin{aligned} |f(x_*) - g(y_*) - \underline{r}| &\underset{(a)}{=} f(x_*) - g(y_*) - \underline{r} \underset{(20)}{\le} f(y_*) - g(y_*) - \underline{r} \le \\ &\underset{(19)}{\le} \big(g(y_*) + \bar{r}\big) - g(y_*) - \underline{r} = \bar{r} - \underline{r} = d\big(R(X)\big) \end{aligned}$$

and

$$\begin{aligned} |f(x^*) - g(y^*) - \bar{r}| &\underset{(a)}{=} g(y^*) + \bar{r} - f(x^*) \underset{(20)}{\le} g(y^*) + \bar{r} - f(y^*) \le \\ &\underset{(19)}{\le} g(y^*) + \bar{r} - \big(g(y^*) + \underline{r}\big) = \bar{r} - \underline{r} = d\big(R(X)\big). \end{aligned}$$

Together with (22) and (8) finally:

$$q\big(W(f,X), V(f,X)\big) \le d\big(R(X)\big) \le 2\,|R(X)|$$

which completes the proof. □

The estimation $(18\,(b))$ of this theorem shows that $d\big(R(X)\big)$ is a *direct* measure for the amount of the overestimation of $W(f,X)$ by $V(f,X)$. This fact can be used to construct inner approximations $\underline{V}(f,X)$ of $W(f,X)$: if there holds the relation

$$g(y_*) + \underline{r} + d\big(R(X)\big) = g(y_*) + \bar{r} \le g(y^*) + \underline{r} = g(y^*) + \bar{r} - d\big(R(X)\big) \tag{23}$$

then

$$[g(y_*)+\bar{r}, g(y^*)+\underline{r}] \subseteq W(f,X)$$

is an inner approximation of $W(f,X)$ since for any

$$z \in [g(y_*)+\bar{r}, g(y^*)+\underline{r}]$$

there holds

$$f(y_*) \underset{(19)}{\leq} g(y_*)+\bar{r} \leq z \leq g(y^*)+\underline{r} \underset{(19)}{\leq} f(y^*).$$

Because of the continuity of f there is a $y \in X$ (between y_* and y^*) with $z=f(y) \in W(f,X)$. We can even remove condition (23) in the case when $f(x_0)$ has been computed for at least one value $x_0 \in X$. This is in general no restriction since all the methods discussed in the next section which have at least quadratic convergence require the computation of f for one or more arguments. Thus, if the computation of $V(f,X)$ requires $m \geq 1$ evaluations of f at $x_1, \ldots, x_m \in X$, then we define

$$\begin{aligned} \underline{V}(f,X) := [&\min \{g(y_*)+\bar{r}, f(x_1), \ldots, f(x_m)\}, \\ &\max \{g(y^*)+\underline{r}, f(x_1), \ldots, f(x_m)\}] \end{aligned} \tag{24}$$

and it is very easy to see that $\underline{V}(f,X)$ is always defined and nonempty whenever $V(f,X)$ is defined and that $\underline{V}(f,X)$ is an inner approximation of $W(f,X)$, i.e. $\underline{V}(f,X) \subseteq W(f,X)$. For $\underline{V}(f,X)$ we also have the same estimation as for $V(f,X)$:

$$q(W(f,X), \underline{V}(f,X)) \leq d(R(X)) \leq 2\,|R(X)|$$

thus, whenever $V(f,X)$ is an outer approximation of n-th order then $\underline{V}(f,X)$ is an inner approximation of n-th order. We omit the simple proof of these facts. In the following sections we will only discuss outer approximations, the construction of a corresponding inner approximation is then obvious.

4. Realizations of High Order Approximations

For numerical applications of Theorem 4 it is important to use simple functions g in order to be able to compute $W(g,X)$. On the other hand the remainder r should be small enough to get a small value of $d(R(X))$ and thus a good approximation $V(f,X)$ of $W(f,X)$. Interpolation- and Taylorpolynomials of moderate degree (≤ 5, say) seem to be well suited for this purpose. Both are special cases of the Hermite interpolation problem.

Let $p_s(x)$ be the uniquely defined interpolation polynomial of degree $s \geq 0$ solving the Hermite interpolation problem

$$\left\{ \begin{array}{l} p_s^{(j)}(x_i) = f^{(j)}(x_i), \quad j=0(1)m_i-1, m_i \in \mathbb{N}, \\ \qquad\qquad\qquad i=0(1)n, n \geq 0, \\ \text{where } x_0, \ldots, x_n \in X \text{ are } n+1 \text{ distinct points in } X \\ \text{and } m_0, \ldots, m_n > 0 \text{ are such that } s+1 = \sum_{i=0}^{n} m_i. \end{array} \right. \tag{25}$$

If f is $s+1$ times continuously differentiable, then for all $x \in X$

$$f(x) = p_s(x) + \frac{1}{(s+1)!} f^{(s+1)}(\xi(x)) \prod_{i=0}^{n} (x-x_i)^{m_i}, \xi(x) \in X. \tag{26}$$

Let $L_{s+1}(X)$ and $L_{s+1}(D)$ be intervals with

$$f^{(s+1)}(x) \in L_{s+1}(X) \subseteq L_{s+1}(D) \text{ for all } x \in X, X \in I(D). \tag{27}$$

If we take (26) as a representation of f of the form (15), then we can define the *interpolation form* $V_s(f, X)$ according to (17) by:

$$V_s(f, X) := W(p_s, X) + \frac{1}{(s+1)!} L_{s+1}(X) \prod_{i=0}^{n} (X - x_i)^{m_i} \tag{28}$$

where the powers $(X - x_i)^{m_i}$ are to be evaluated as products of the same interval:

$$(X - x_i)^{m_i} := \prod_{j=1}^{m_i} (X - x_i).$$

If $f^{(s+1)}$ has an interval evaluation $f^{(s+1)}(X)$ over X satisfying Theorem 2, and if $m^{(s+1)}$ is an arbitrary but fixed point out of $f^{(s+1)}(X)$,

$$m^{(s+1)} \in f^{(s+1)}(X) \tag{29}$$

then we use

$$f(x) = q_{s+1}(x) + \frac{1}{(s+1)!} \left(f^{(s+1)}(\xi(x)) - m^{(s+1)}\right) \prod_{i=0}^{n} (x - x_i)^{m_i} \tag{30}$$

where

$$q_{s+1}(x) := p_s(x) + \frac{m^{(s+1)}}{(s+1)!} \prod_{i=0}^{n} (x - x_i)^{m_i}, \tag{31}$$

as a representation of the form (15) and define the *interpolation form* $U_s(f, X)$ by

$$U_s(f, X) := W(q_{s+1}, X) + \frac{1}{(s+1)!} \left(f^{(s+1)}(X) - m^{(s+1)}\right) \prod_{i=0}^{n} (X - x_i)^{m_i}. \tag{32}$$

Clearly $V_s(f, X)$ and $U_s(f, X)$ are intervals. The following theorem shows that both, V_s and U_s can be approximations of high order:

Theorem 5:

Let $V_s(f, X)$ be defined as in (28) *such that* (27) *holds, and let $U_s(f, X)$ be defined as in* (32) *such that Theorem* 2 *holds for $f^{(s+1)}$. Then for all $X \in I(D)$ there hold*

$$\begin{cases} (a) & W(f, X) \subseteq V_s(f, X), \\ (b) & q\left(W(f, X), V_s(f, X)\right) \leq \alpha_{s+1}\, d(X)^{s+1} \text{ with } \alpha_{s+1} = \alpha_{s+1}(D) \geq 0, \end{cases} \tag{33}$$

and

$$\begin{cases} (a) & W(f, X) \subseteq U_s(f, X), \\ (b) & q\left(W(f, X), U_s(f, X)\right) \leq \beta_{s+2}\, d(X)^{s+2} \text{ with } \beta_{s+2} = \beta_{s+2}(D) \geq 0. \end{cases} \tag{34}$$

Proof:

Since both $V_s(f, X)$ and $U_s(f, X)$ are special cases of the remainder form (17) it follows immediately from Theorem 4 that $(33(a))$ and $(34(a))$ must hold.

In the case of $V_s(f, X)$ we have (see (28)):

$$R(X) = \frac{1}{(s+1)!} L_{s+1}(X) \prod_{i=0}^{n} (X - x_i)^{m_i}$$

$$\underset{(18(b))}{\Rightarrow} q(W(f, X), V_s(f, X)) \leq 2\,|R(X)| = \frac{2}{(s+1)!} |L_{s+1}(X)| \prod_{i=0}^{n} |X - x_i|^{m_i}$$

$$\underset{(6),(27)}{\leq} \underbrace{\frac{2}{(s+1)!} |L_{s+1}(D)|}_{=:\alpha_{s+1} \geq 0} \prod_{i=0}^{n} d(X)^{m_i} \underset{(25)}{=} \alpha_{s+1}\, d(X)^{s+1}$$

where $\alpha_{s+1} = \alpha_{s+1}(D)$. Thus $(33(b))$ is true.

In the case of $U_s(f, X)$ we have (see (32)):

$$R(X) = \frac{1}{(s+1)!} \left(f^{(s+1)}(X) - m^{(s+1)}\right) \prod_{i=0}^{n} (X - x_i)^{m_i}$$

$$\underset{(18(b))}{\Rightarrow} q(W(f, X), U_s(f, X)) \leq 2\,|R(X)| =$$

$$= \frac{2}{(s+1)!} \left|f^{(s+1)}(X) - m^{(s+1)}\right| \prod_{i=0}^{n} |X - x_i|^{m_i}$$

$$\underset{(6),(7),(29)}{\leq} \frac{2}{(s+1)!} d\left(f^{(s+1)}(X)\right) \prod_{i=0}^{n} d(X)^{m_i}$$

$$\underset{(12),(25)}{\leq} \underbrace{\frac{2}{(s+1)!} c_2}_{=:\beta_{s+2} \geq 0} \cdot d(X) \cdot d(X)^{s+1} = \beta_{s+2}\, d(X)^{s+2}$$

where $\beta_{s+2} = \beta_{s+2}(D)$. Thus $(34(b))$ is also true. This completes the proof of Theorem 5. □

Now we know that $V_s(f, X)$ is an $(s+1)$-st order outer approximation of $W(f, X)$ and $U_s(f, X)$ is an $(s+2)$-nd order outer approximation. In order to get higher than quadratic convergence it is sufficient to use $V_s(f, X)$ with quadratic interpolation or with a quadratic Taylor-polynomial $(s=2)$ and for $U_s(f, X)$ it is even sufficient to use linear interpolation or a linear Taylorapproximation $(s=1)$! Obviously in these cases the computation of $W(g, X)$ is almost trivial (see $(42)-(44)$) since g is a quadratic polynomial.

Remarks:

1. Comparing the definitions of V_s, (28), and U_s, (32), we see that the highest derivative of f used in both forms is $f^{(s+1)}$ and that for both forms $s+2$ evaluations of f and/or derivatives of f are necessary. Nevertheless with U_s we get an order of $s+2$ and with V_s an order of only $s+1$. The reason for this is that in U_s

we use the stronger assumption on $f^{(s+1)}$ to improve the approximation p_s of f and to estimate $d(f^{(s+1)}(X)) \leq c_2 \cdot d(X)$ by Theorem 2. The improvement of p_s can also be done in the case of V_s, however, the resulting term $d(L_{s+1}(X))$ in the estimation of $|R(X)|$ cannot be estimated further. Thus, if we want an approximation $V(f, X)$ of a given, fixed order $\tilde{s}$, it is cheaper to use $U_{\tilde{s}-2}$, where only $f^{(\tilde{s}-1)}$ and $\tilde{s}$ function-/derivative-evaluations are needed, as compared to $V_{\tilde{s}-1}$, where $f^{(\tilde{s})}$ and $\tilde{s}+1$ evaluations are needed. In both cases the exact range of a polynomial of degree $\tilde{s}-1$ must be computed.

2. The forms $V_s(f, X)$ resp. $U_s(f, X)$ reduce the problem of computing the range of f to the simpler problem of computing the range of a polynomial of degree s resp. $s+1$. The reduced problem can be easily solved for $0 \leq s \leq 3$ resp. $0 \leq s \leq 2$ thus obtaining a maximum order of 4, and it can be solved with some more work for $4 \leq s \leq 5$ resp. $3 \leq s \leq 4$ thus obtaining a maximum order of 6.

3. A further advantage of U_s as compared to V_s is, that when using rounded interval arithmetic (which must be used on a computer) the highest coefficient of q_{s+1} can be chosen as a machine representable number, whereas all other coefficients of q_{s+1} and all coefficients of p_s will be non-degenerate intervals in general, because of rounding errors.

4. If $0 \in f^{(s+1)}(X)$ then $m^{(s+1)} := 0$ can be chosen in U_s; instead of $W(q_{s+1}, X)$ only $W(p_s, X)$ has to be computed then!

5. If $f^{(s+1)}$ has an interval evaluation for all $X \in I(D)$, then of course $L_{s+1}(X) := f^{(s+1)}(X)$ can be chosen in (27) and there holds $f^{(s+1)}(X) \subseteq f^{(s+1)}(D)$ for all $X \in I(D)$.

We can still further improve the approximations V_s and U_s by refining the method of evaluating the products in the remainder terms. Since

$$\prod_{i=0}^{n} (x-x_i)^{m_i} \in W\left(\prod_{i=0}^{n} (x-x_i)^{m_i}, X\right) \subseteq \prod_{i=0}^{n} W((x-x_i)^{m_i}, X) \subseteq$$
$$\subseteq \prod_{i=0}^{n} (X-x_i)^{m_i}, \; x \in X, \; X \in I(D), \tag{35}$$

we can improve (28) by defining

$$\hat{V}_s(f, X) := W(p_s, X) + \frac{1}{(s+1)!} L_{s+1}(X) \cdot W\left(\prod_{i=0}^{n} (x-x_i)^{m_i}, X\right), \tag{36}$$

and

$$\tilde{V}_s(f, X) := W(p_s, X) + \frac{1}{(s+1)!} L_{s+1}(X) \prod_{i=0}^{n} W((x-x_i)^{m_i}, X). \tag{37}$$

Also (32) can be improved analogously by defining:

$$\hat{U}_s(f, X) := W(q_{s+1}, X) + \frac{1}{(s+1)!} \left(f^{(s+1)}(X) - m^{(s+1)}\right) \cdot$$
$$\cdot W\left(\prod_{i=0}^{n} (x-x_i)^{m_i}, X\right), \tag{38}$$

and

$$\tilde{U}_s(f,X) := W(q_{s+1},X) + \frac{1}{(s+1)!}\left(f^{(s+1)}(X) - m^{(s+1)}\right) \cdot \prod_{i=0}^{n} W\left((x-x_i)^{m_i}, X\right). \tag{39}$$

Because of (35) it is obvious that

$$W(f,X) \subseteq \hat{V}_s(f,X) \subseteq \tilde{V}_s(f,X) \subseteq V_s(f,X) \tag{40}$$

and with (10) it follows that

$$\begin{aligned} q\left(W(f,X), \hat{V}_s(f,X)\right) &\leq q\left(W(f,X), \tilde{V}_s(f,X)\right) \leq \\ &\leq q\left(W(f,X), V_s(f,X)\right) \leq \alpha_{s+1} \cdot d(X)^{s+1}. \end{aligned} \tag{41}$$

For $\hat{U}_s$ and $\tilde{U}_s$ (40) and (41) hold analogously. Thus Theorem 5 also holds for these modified forms and $\hat{V}_s$ resp. $\hat{U}_s$ are in general the best ones as compared to $\tilde{V}_s$ and V_s resp. $\tilde{U}_s$ and U_s. At the end of this section we give the explicit expressions for some forms with cubic convergence which can be evaluated directly by using interval arithmetic.

(I) $U_1(f,X)$ and $\hat{U}_1(f,X)$ using Taylorexpansion at $y \in X$:

Using

$$\begin{aligned} f(x) &= f(y) + f'(y)(x-y) + \frac{1}{2} f''(\xi)(x-y)^2 = \\ &= f(y) + f'(y)(x-y) + \frac{m}{2}(x-y)^2 + \frac{1}{2}\left(f''(\xi) - m\right)(x-y)^2 = \\ &= f(y) + \frac{m}{2}\left[\left(x - y + \frac{f'(y)}{m}\right)^2 - \left(\frac{f'(y)}{m}\right)^2\right] + \frac{1}{2}\left(f''(\xi) - m\right)(x-y)^2 \end{aligned}$$

we get (with Theorem 1):

$$\begin{aligned} U_1(f,X) = f(y) + \frac{m}{2}\left[\left(X - y + \frac{f'(y)}{m}\right)^2 - \left(\frac{f'(y)}{m}\right)^2\right] + \\ + \frac{1}{2}\left(f''(X) - m\right)(X-y)^2 \end{aligned} \tag{42}$$

where $0 \neq m \in f''(X)$. The first expression of the difference in the square brackets must be evaluated exactly whereas $(X-y)^2 = (X-y) \cdot (X-y)$. If this latter expression is also evaluated exactly, then we obtain the form $\hat{U}_1(f,X)$ (which is identical with $\tilde{U}_1$ in this case).

(II) $U_1(f, X)$ *and* $\hat{U}_1(f, X)$ *using linear interpolation of* $f(\underline{x})$ *and* $f(\bar{x})$:

Using

$$f(x)=f(\underline{x})+\frac{f(\bar{x})-f(\underline{x})}{\bar{x}-\underline{x}}(x-\underline{x})+\frac{1}{2}f''(\xi)(x-\bar{x})(x-\underline{x})=$$

$$=f(\underline{x})+\frac{f(\bar{x})-f(\underline{x})}{\bar{x}-\underline{x}}(x-\underline{x})+\frac{m}{2}(x-\underline{x})(x-\bar{x})+$$

$$+\frac{1}{2}(f''(\xi)-m)(x-\underline{x})(x-\bar{x})=$$

$$=\frac{f(\underline{x})+f(\bar{x})}{2}+\frac{m}{2}\left[\left(x-\frac{\underline{x}+\bar{x}}{2}+\frac{\Delta}{m}\right)^2-\left(\frac{\Delta}{m}\right)^2-\frac{(\bar{x}-\underline{x})^2}{4}\right]+$$

$$+\frac{1}{2}(f''(\xi)-m)(x-\underline{x})(x-\bar{x})$$

where

$$\Delta:=\frac{f(\bar{x})-f(\underline{x})}{\bar{x}-\underline{x}} \quad \text{and} \quad 0\neq m\in f''(X),$$

we get (again with Theorem 1):

$$U_1(f, X)=\frac{f(\underline{x})+f(\bar{x})}{2}+\frac{m}{2}\left[\left(X-\frac{\underline{x}+\bar{x}}{2}+\frac{\Delta}{m}\right)^2-\left(\frac{\Delta}{m}\right)^2-\frac{(\bar{x}-\underline{x})^2}{4}\right]+$$
$$+\frac{1}{2}(f''(X)-m)(X-\underline{x})(X-\bar{x}). \tag{43}$$

Since

$$W((x-\underline{x})(x-\bar{x}), X)=\left[-\frac{1}{4}d(X)^2, 0\right]$$

we get for $\hat{U}_1(f, X)$:

$$\hat{U}_1(f, X)=\frac{f(\underline{x})+f(\bar{x})}{2}+\frac{m}{2}\left[\left(X-\frac{\underline{x}+\bar{x}}{2}+\frac{\Delta}{m}\right)^2-\left(\frac{\Delta}{m}\right)^2-\frac{(\bar{x}-\underline{x})^2}{4}\right]+$$
$$+(f''(X)-m)\left[-\frac{d(X)^2}{8}, 0\right]. \tag{44}$$

In (43) and (44) the first expression in the square brackets must also be evaluated exactly, like in (42).

5. Connections with the Mean Value Form

One of the most commonly used forms for the approximation of $W(f, X)$ is the mean value form $F_y(f, X)$, see (13), whose convergence is quadratic when f' satisfies Theorem 2. Thus, if we want to compare it with one of our forms then we have to do this with $U_0(f, X)$ which is also quadratically convergent under the same hypotheses. In general there does not hold equality because of subdistributivity of interval arithmetic (see e.g. Alefeld/Herzberger [2] or Moore [14], [15]). If $m \in f'(X)$ is arbitrary then:

$$\begin{cases} F_y(f,X) = f(y) + f'(X)(X-y) = f(y) + (f'(X) - m + m)(X-y) \subseteq \\ \qquad \subseteq f(y) + m(X-y) + (f'(X) - m)(X-y) = \\ \qquad = W(f(y) + m(x-y), X) + (f'(X) - m)(X-y) = U_0(f,X). \end{cases} \tag{45}$$

($U_0(f,X)$ is obtained from interpolation at $y \in X$ with the constant $p_0(x) \equiv f(y)$).

However, equality can always be forced in (45) when m is chosen in a suitable way. This is the first statement of the following Theorem 6. The second statement shows another case where both forms are equal.

Theorem 6:
Let $F_y(f,X)$ and $U_0(f,X)$ exist and let f' satisfy Theorem 2.

(*a*) *If $f'(X) =: [\underline{k}, \bar{k}]$ and m in $U_0(f,X)$ is chosen as*

$$m := \begin{cases} \bar{k}, & \text{if } \bar{k} < 0, \\ 0, & \text{if } \underline{k} \le 0 \le \bar{k}, \\ \underline{k}, & \text{if } \underline{k} > 0, \end{cases}$$

then $F_y(f,X) = U_0(f,X)$.

(*b*) *Let* $\text{mid}(A) := \dfrac{\underline{a} + \bar{a}}{2}$ *denote the midpoint of an interval* $A = [\underline{a}, \bar{a}] \in I(\mathbb{R})$.

If $y = \text{mid}(X)$ and $m = \text{mid}(f'(X))$ then $F_y(f,X) = U_0(f,X)$.

We omit the proof which makes use of the distributive law

$$(a+b)A = aA + bA \quad \text{for } a, b \in \mathbb{R},\ A \in I(\mathbb{R}),\ \text{if } a \cdot b \ge 0,$$

and elementary interval operations.

Theorem 6 and (45) show that in the quadratically convergent case the mean value form $F_y(f,X)$ should be preferred to $U_0(f,X)$ since it requires less arithmetic operations.

6. Numerical Results

We have compared the intervals $U_1(f,X)$ from (43), $\hat{U}_1(f,X)$ from (44) and $\hat{U}_1(f,X)$ from (42) – denoted here as $U_1^T(f,X)$ – with the mean value form $F_y(f,X)$ and the interval arithmetic evaluation $f(X)$ of f. The computations were done in the programming language PASCAL-SC, which is an extension of PASCAL with a maximum accuracy arithmetic. Furthermore the type "INTERVAL" is a standard type in PASCAL-SC, so that one can use all operations on intervals just as for floating point numbers. The numbers are represented in the decimal system with a 12 digit mantissa. See for details Kulisch/Ullrich [11].

The assertions of Theorem 5 are only interesting for a small diameter $d(X)$ of X. Therefore all the examples are as follows: given a midpoint $m \in \mathbb{R}$ we consider the intervals

$$X_i := m + 10^{-i} \cdot [-1, 1],\ i = 0(1)7,$$

for which the range of values $W_i := W(f, X_i)$ is to be enclosed. The following tables contain the rounded distances $q(\cdot,\cdot)$ from $W(f, X_i) = W_i$ to the individual intervals mentioned above.

Table 1

Example 1: $f(x) = \dfrac{x^2 - 5x + 9}{x - 5}$, midpoint $m = 2$, intervals $X_i = 2 + 10^{-i}[-1, 1]$, $i = 0(1)7$.

(X^2 was computed as $X^2 = W(x^2, X)$)

i	$d(X_i)$	$q(W_i, F(X_i))$	$q(W_i, F_m(f, X_i))$	$q(W_i, U_1(f, X_i))$	$q(W_i, \hat{U}_1(f, X_i))$	$q(W_i, U_1^T(f, X_i))$
0	$2 \cdot 10^{0}$	$5 \cdot 10^{0}$	$4 \cdot 10^{0}$	$2 \cdot 10^{0}$	$8 \cdot 10^{-1}$	$6 \cdot 10^{-1}$
1	$2 \cdot 10^{-1}$	$3 \cdot 10^{-1}$	$2 \cdot 10^{-2}$	$1 \cdot 10^{-3}$	$4 \cdot 10^{-4}$	$3 \cdot 10^{-4}$
2	$2 \cdot 10^{-2}$	$3 \cdot 10^{-2}$	$2 \cdot 10^{-4}$	$1 \cdot 10^{-6}$	$3 \cdot 10^{-7}$	$3 \cdot 10^{-7}$
3	$2 \cdot 10^{-3}$	$3 \cdot 10^{-3}$	$2 \cdot 10^{-6}$	$1 \cdot 10^{-9}$	$4 \cdot 10^{-10}$	$3 \cdot 10^{-10}$
4	$2 \cdot 10^{-4}$	$3 \cdot 10^{-4}$	$2 \cdot 10^{-8}$	$2 \cdot 10^{-11}$	$2 \cdot 10^{-11}$	$1 \cdot 10^{-11}$
5	$2 \cdot 10^{-5}$	$3 \cdot 10^{-5}$	$2 \cdot 10^{-10}$	$2 \cdot 10^{-11}$	$2 \cdot 10^{-11}$	$1 \cdot 10^{-11}$
6	$2 \cdot 10^{-6}$	$3 \cdot 10^{-6}$	$2 \cdot 10^{-12}$	$2 \cdot 10^{-11}$	$2 \cdot 10^{-11}$	$1 \cdot 10^{-11}$
7	$2 \cdot 10^{-7}$	$3 \cdot 10^{-7}$	$1 \cdot 10^{-12}$	$2 \cdot 10^{-11}$	$2 \cdot 10^{-11}$	$1 \cdot 10^{-11}$

Table 2

Example 2: $f(x) = \dfrac{x + 2}{\sqrt{x}}$, midpoint $m = 2$, intervals $X_i = 2 + 10^{-i}[-1, 1]$, $i = 0(1)7$.

i	$d(X_i)$	$q(W_i, F(X_i))$	$q(W_i, F_m(f, X_i))$	$q(W_i, U_1(f, X_i))$	$q(W_i, \hat{U}_1(f, X_i))$	$q(W_i, U_1^T(f, X_i))$
0	$2 \cdot 10^{0}$	$2 \cdot 10^{0}$	$5 \cdot 10^{-1}$	$3 \cdot 10^{0}$	$1 \cdot 10^{0}$	$9 \cdot 10^{-1}$
1	$2 \cdot 10^{-1}$	$2 \cdot 10^{-1}$	$2 \cdot 10^{-3}$	$7 \cdot 10^{-4}$	$2 \cdot 10^{-4}$	$2 \cdot 10^{-4}$
2	$2 \cdot 10^{-2}$	$1 \cdot 10^{-2}$	$2 \cdot 10^{-5}$	$7 \cdot 10^{-7}$	$2 \cdot 10^{-7}$	$2 \cdot 10^{-7}$
3	$2 \cdot 10^{-3}$	$1 \cdot 10^{-3}$	$2 \cdot 10^{-7}$	$7 \cdot 10^{-10}$	$2 \cdot 10^{-10}$	$2 \cdot 10^{-10}$
4	$2 \cdot 10^{-4}$	$1 \cdot 10^{-4}$	$2 \cdot 10^{-9}$	$5 \cdot 10^{-11}$	$5 \cdot 10^{-11}$	$2 \cdot 10^{-11}$
5	$2 \cdot 10^{-5}$	$1 \cdot 10^{-5}$	$2 \cdot 10^{-11}$	$6 \cdot 10^{-11}$	$6 \cdot 10^{-11}$	$1 \cdot 10^{-11}$
6	$2 \cdot 10^{-6}$	$1 \cdot 10^{-6}$	$1 \cdot 10^{-11}$	$5 \cdot 10^{-11}$	$5 \cdot 10^{-11}$	$1 \cdot 10^{-11}$
7	$2 \cdot 10^{-7}$	$1 \cdot 10^{-7}$	$1 \cdot 10^{-11}$	$4 \cdot 10^{-11}$	$4 \cdot 10^{-11}$	$1 \cdot 10^{-11}$

Table 3

Example 3: $f(x)=\frac{\ln x}{x}$, midpoint $m=1.5$,
intervals $X_i=1.5+10^{-i}[-1,1]$ $i=0(1)7$.

i	$d(X_i)$	$q(W_i,F(X_i))$	$q(W_i,F_m(f,X_i))$	$q(W_i,U_1(f,X_i))$	$q(W_i,\hat{U}_1(f,X_i))$	$q(W_i,U_1^T(f,X_i))$
0	$2\cdot 10^{0}$	$2\cdot 10^{0}$	$7\cdot 10^{0}$	$4\cdot 10^{1}$	$2\cdot 10^{1}$	$2\cdot 10^{1}$
1	$2\cdot 10^{-1}$	$4\cdot 10^{-2}$	$1\cdot 10^{-2}$	$4\cdot 10^{-3}$	$9\cdot 10^{-4}$	$7\cdot 10^{-4}$
2	$2\cdot 10^{-2}$	$4\cdot 10^{-3}$	$1\cdot 10^{-4}$	$3\cdot 10^{-6}$	$9\cdot 10^{-7}$	$6\cdot 10^{-7}$
3	$2\cdot 10^{-3}$	$4\cdot 10^{-4}$	$1\cdot 10^{-6}$	$3\cdot 10^{-9}$	$9\cdot 10^{-10}$	$6\cdot 10^{-10}$
4	$2\cdot 10^{-4}$	$4\cdot 10^{-5}$	$1\cdot 10^{-8}$	$4\cdot 10^{-12}$	$1\cdot 10^{-12}$	$1\cdot 10^{-12}$
5	$2\cdot 10^{-5}$	$4\cdot 10^{-6}$	$1\cdot 10^{-10}$	$1\cdot 10^{-12}$	$1\cdot 10^{-12}$	$1\cdot 10^{-12}$
6	$2\cdot 10^{-6}$	$4\cdot 10^{-7}$	$2\cdot 10^{-12}$	$1\cdot 10^{-12}$	$1\cdot 10^{-12}$	$2\cdot 10^{-12}$
7	$2\cdot 10^{-7}$	$4\cdot 10^{-8}$	$1\cdot 10^{-12}$	$1\cdot 10^{-12}$	$1\cdot 10^{-12}$	$2\cdot 10^{-12}$

Table 4

Example 4: $f(x)=e^{x-\sin x}-1$, midpoint $m=-1.5$,
intervals $X_i=-1.5+10^{-i}[-1,1]$, $i=0(1)7$.

i	$d(X_i)$	$q(W_i,F(X_i))$	$q(W_i,F_m(f,X_i))$	$q(W_i,U_1(f,X_i))$	$q(W_i,\hat{U}_1(f,X_i))$	$q(W_i,U_1^T(f,X_i))$
0	$2\cdot 10^{0}$	$7\cdot 10^{-1}$	$3\cdot 10^{0}$	$9\cdot 10^{0}$	$2\cdot 10^{0}$	$3\cdot 10^{0}$
1	$2\cdot 10^{-1}$	$1\cdot 10^{-2}$	$1\cdot 10^{-2}$	$4\cdot 10^{-3}$	$1\cdot 10^{-3}$	$8\cdot 10^{-4}$
2	$2\cdot 10^{-2}$	$9\cdot 10^{-4}$	$1\cdot 10^{-4}$	$3\cdot 10^{-6}$	$6\cdot 10^{-7}$	$8\cdot 10^{-7}$
3	$2\cdot 10^{-3}$	$9\cdot 10^{-5}$	$1\cdot 10^{-6}$	$3\cdot 10^{-9}$	$6\cdot 10^{-10}$	$8\cdot 10^{-10}$
4	$2\cdot 10^{-4}$	$9\cdot 10^{-6}$	$1\cdot 10^{-8}$	$3\cdot 10^{-12}$	$1\cdot 10^{-12}$	$2\cdot 10^{-12}$
5	$2\cdot 10^{-5}$	$9\cdot 10^{-7}$	$1\cdot 10^{-10}$	$1\cdot 10^{-12}$	$1\cdot 10^{-12}$	$2\cdot 10^{-12}$
6	$2\cdot 10^{-6}$	$9\cdot 10^{-8}$	$2\cdot 10^{-12}$	$1\cdot 10^{-12}$	$1\cdot 10^{-12}$	$2\cdot 10^{-12}$
7	$2\cdot 10^{-7}$	$9\cdot 10^{-9}$	$1\cdot 10^{-12}$	$1\cdot 10^{-12}$	$1\cdot 10^{-12}$	$2\cdot 10^{-12}$

Table 5

Example 5: $f(x)=(16x^2-24x+5)e^{-x}$, midpoint $m=2.9$,
intervals $X_i=2.9+10^{-i}[-1,1]$, $i=0(1)7$.
(X^2 computed as $X^2=W(x^2,X)$).

i	$d(X_i)$	$q(W_i,F(X_i))$	$q(W_i,F_m(f,X_i))$	$q(W_i,U_1(f,X_i))$	$q(W_i,\hat{U}_1(f,X_i))$	$q(W_i,U_1^T(f,X_i))$
0	$2\cdot 10^{0}$	$3\cdot 10^{1}$	$3\cdot 10^{1}$	$6\cdot 10^{1}$	$1\cdot 10^{1}$	$1\cdot 10^{1}$
1	$2\cdot 10^{-1}$	$1\cdot 10^{0}$	$1\cdot 10^{-1}$	$2\cdot 10^{-2}$	$6\cdot 10^{-3}$	$7\cdot 10^{-3}$
2	$2\cdot 10^{-2}$	$1\cdot 10^{-1}$	$9\cdot 10^{-4}$	$2\cdot 10^{-5}$	$6\cdot 10^{-6}$	$6\cdot 10^{-6}$
3	$2\cdot 10^{-3}$	$1\cdot 10^{-2}$	$9\cdot 10^{-6}$	$2\cdot 10^{-8}$	$6\cdot 10^{-9}$	$6\cdot 10^{-9}$
4	$2\cdot 10^{-4}$	$1\cdot 10^{-3}$	$9\cdot 10^{-8}$	$3\cdot 10^{-11}$	$1\cdot 10^{-11}$	$2\cdot 10^{-11}$
5	$2\cdot 10^{-5}$	$1\cdot 10^{-4}$	$9\cdot 10^{-10}$	$1\cdot 10^{-11}$	$1\cdot 10^{-11}$	$2\cdot 10^{-11}$
6	$2\cdot 10^{-6}$	$1\cdot 10^{-5}$	$3\cdot 10^{-11}$	$1\cdot 10^{-11}$	$1\cdot 10^{-11}$	$3\cdot 10^{-11}$
7	$2\cdot 10^{-7}$	$1\cdot 10^{-6}$	$4\cdot 10^{-11}$	$1\cdot 10^{-11}$	$1\cdot 10^{-11}$	$3\cdot 10^{-11}$

7. The n-Dimensional Case

If $f: D \to \mathbb{R}$ where $D \subseteq \mathbb{R}^n$, $n \in \mathbb{N}$, is an n-dimensional interval, i.e.

$$D := D_1 \times \ldots \times D_n,\ D_i \in I(\mathbb{R}),\ i = 1(1)n,$$

and f is continuous on D, then Section 3, and thus Theorem 4, can be applied literally in the same way as in the one-dimensional case. The outer approximation $\bar{V}(f, X)$ and the corresponding inner approximation $\underline{V}(f, X)$ of $W(f, X)$ on a subinterval X of D can be defined as in (17) and (24) whenever a representation like (15) holds. Theoretically it is also possible to obtain approximations of arbitrary order. However, in general it is very hard to find functions g such that $W(g, X)$ can be computed exactly. E.g. if we want to construct a third-order approximation by using Taylorexpansion at $y \in X$ with a second-order remainder term like in (42), then we have to compute the maximum and the minimum of a quadratic form in n variables over the interval X. This can be done in principle, it is, however, in most cases a very lengthy computation.

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