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Linear Interval Equations

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Abstract — Zusammenfassung

Linear Interval Equations. Necessary and sufficient criteria are given for the existence and uniqueness of solutions of linear interval equations. Explicit formulas are given for the solution set when the solution set is convex. Necessary and sufficient conditions are given for the convexity of the solution set.

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Lineare Intervallgleichungen. Es werden notwendige und hinreichende Kriterien für die Existenz und Eindeutigkeit von Lösungen linearer Intervallgleichungen vorgestellt. Ist die Lösungsgesamtheit konvex, wofür auch notwendige und hinreichende Bedingungen angegeben werden, ist sie durch geschlossene Formeln charakterisierbar.

1. Introduction

Let $\mathbb{I}$ be the set of real compact intervals and let $A * B = \{a * b : a \in A,\ b \in B\}$ for A, $B \in \mathbb{I}$ and $* \in \{+, -, \cdot, :\}$, where it is understood that $A : B$ is not defined when $0 \in B$. This interval arithmetic is of great interest both for the automatic roundoff analysis on computers and for general error analysis, see for example Moore [5]. Even though interval arithmetic may be considered to be a real arithmetic where errors are taken into account, it differs from real arithmetic in some essential points. The distributive law is for example only valid in exceptional cases. The equations $A + X = B$ and $AY = B$ over $\mathbb{I}$ do not have $X = B - A$ and $Y = B : A$ as solutions. (A survey of the algebraic properties of $\mathbb{I}$ may be found in [9].)

Since the distributive law is not valid it follows that the map $X \mapsto AX$ with $A, X \in \mathbb{I}$ as well as the "linear" interval equation systems

$$\sum_{i=1}^{n} A_{ji} X_i = B_j \quad \text{with} \quad A_{ji}, B_j \in \mathbb{I}\ (j = 1, \ldots, m) \tag{ES}$$

are not linear. They are at most "formally linear". Instead of (ES) we write the system abbreviated as $\mathfrak{A}\mathfrak{x} = \mathfrak{b}$ with an $m \times n$ interval matrix $\mathfrak{A} = (A_{ji})$ and interval vectors $\mathfrak{b} = (B_1 \ldots B_m)^T$ and $\mathfrak{x} = (X_1 \ldots X_n)^T$.

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Systems of equations formulated as interval systems of equations are of interest numerically when the coefficients of a system over $\mathbb{R}$, the reals, are only known to a certain tolerance. Coefficients of this type occur in systems of equations arising from measurements, observations, numerical calculations etc.

In this sense we define $\mathbb{L}_{\text{num}}=\{\underset{\cdot}{\mathfrak{x}}:\underset{\cdot}{\mathfrak{A}}\,\underset{\cdot}{\mathfrak{x}}=\underset{\cdot}{\mathfrak{b}},\ \underset{\cdot}{\mathfrak{A}}\in\mathfrak{A},\ \underset{\cdot}{\mathfrak{b}}\in\mathfrak{b}\}\subset\mathbb{R}^n$ as the *numerical solution* of (ES). The notation $\underset{\cdot}{\mathfrak{A}},\underset{\cdot}{\mathfrak{x}},\underset{\cdot}{\mathfrak{b}}$ means matrices and vectors over $\mathbb{R}$, that is point matrices [vectors] as opposed to interval matrices [vectors]. By $\underset{\cdot}{\mathfrak{A}}\in\mathfrak{A}$ it is implied that $a_{ji}\in A_{ji}$ for all i,j when $\underset{\cdot}{\mathfrak{A}}=(a_{ji})$ etc. The set $\mathbb{L}_{\text{num}}$ is generally not representable as an arithmetic expression over $\mathbb{I}^n$. The *interval arithmetic solution* of (ES) is therefore defined as the interval hull $\mathbb{L}_{\text{int}}$ of $\mathbb{L}_{\text{num}}$ which is the smallest interval vector $\mathbb{L}_{\text{int}}$ for which

$$\underset{\cdot}{\mathfrak{x}}\in\mathbb{L}_{\text{num}}\Rightarrow\underset{\cdot}{\mathfrak{x}}\in\mathbb{L}_{\text{int}}. \tag{1}$$

We also allow the interval vector $\mathbb{L}_{\text{int}}$ to contain unbounded interval components. The converse of (1) is generally not valid.

The interpretation of this concept is based upon the fact that $\mathfrak{A}$ is the matrix of tolerances of an (not exactly known) matrix $\underset{\cdot}{\mathfrak{A}}$ and $\mathfrak{b}$ the column vector of tolerances of an (not exactly known) column vector $\underset{\cdot}{\mathfrak{b}}$. Relation (1) implies that each solution $\underset{\cdot}{\mathfrak{x}}$ of the system $\underset{\cdot}{\mathfrak{A}}\,\underset{\cdot}{\mathfrak{x}}=\underset{\cdot}{\mathfrak{b}}$ lies in $\mathbb{L}_{\text{num}}$, that means, if there exist solutions then they are included both in $\mathbb{L}_{\text{num}}$ and $\mathbb{L}_{\text{int}}$. (A survey of the kind of problems described above is found for example in [1], [2], [4].)

The numerical solution $\mathbb{L}_{\text{num}}$ is contrasted with the (*algebraic*) *solutions* of (ES). These algebraic solutions are defined as interval vectors $\mathfrak{x}$ satisfying $\mathfrak{A}\,\mathfrak{x}=\mathfrak{b}$ (corresponding to the usual concept of a solution). Let $\mathbb{L}=\{\mathfrak{x}:\mathfrak{A}\,\mathfrak{x}=\mathfrak{b}\}$ be the *totality of solutions*. While $\mathbb{L}_{int}$ gives an inclusion of $\mathbb{L}_{num}$ according to (1) it follows that $\mathbb{L}$ does guarantee the *existence* of vectors from $\mathbb{L}_{\text{num}}$, that is,

$$\underset{\cdot}{\mathfrak{x}}\in\mathfrak{x},\ \mathfrak{x}\in\mathbb{L}\Rightarrow\underset{\cdot}{\mathfrak{x}}\in\mathbb{L}_{\text{num}}.$$

It is known that the converse of this implication is not valid. The equation

$$[1,2]\,X=[1,4]$$

has for example the solution set $\mathbb{L}=\{[1,2]\}$. Furthermore

$$\mathbb{L}_{\text{num}}=[1,4]/[1,2]=[1/2,4]=\mathbb{L}_{\text{int}}.$$

For $x=4$ we have $x\in\mathbb{L}_{\text{num}}$. There is, however, no $X\in\mathbb{L}$ for which $x\in X$.

It is the aim of this paper to investigate the solution set $\mathbb{L}$ of (ES) for the case $m=1$. This case is of particular theoretical interest since it is possible to give criteria for the existence, uniqueness and convexity of $\mathbb{L}$ through explicit formulas (see also [10]). The question of convexity of $\mathbb{L}$ is of importance since it is possible to represent the solution set in an elegant manner in this case.

It is not possible to solve (ES) using the methods for linear systems of equations over $\mathbb{R}$ since $\mathbb{I}$ does not have the same algebraic structure as $\mathbb{R}$.

2. Interval Analytic Tools

The definition of the arithmetic operations on $\mathbb{I}$ leads to the following formulas for addition and multiplication:

$$[a,b]+[c,d]=[a+c,\ b+d],$$

$$[a,b]\,[c,d]=[\min\{a\,c, a\,d, b\,d, b\,c\},\ \max\{a\,c, a\,d, b\,d, b\,c\}].$$

Intervals $[a,a]$ are called *point intervals*. They are written $\underset{\bullet}{a}$ and also a so that $a+[c,d]$, $a\,[c,d]$ etc. are defined. The set $\mathbb{I}'$ denotes the set $\mathbb{I}$ without the point interval 0. The *subdistributive law* is fundamental, that is,

$$A\,(B+C)\subset AB+AC \quad \text{for} \quad A, B, C\in\mathbb{I}.$$

Some interval functionals are useful for describing certain classes of intervals: For $A=[a,b]$ and $B=[c,d]\neq 0$ let

$$\varphi\,A=(a+b)/2\ (\textit{midpoint}),\ \lambda\,A=(b-a)\ (\textit{width}),$$

$$\psi\,A=\max\{|a|,|b|\}\ (\textit{modulus}),\ \sigma\,A=\operatorname{sgn}(a+b)\,(\textit{sign}),$$

$$\sigma'\,A=\begin{cases} 1, & \text{if } \sigma\,A\geq 0,\\ -1, & \text{otherwise,}\end{cases} \qquad \chi\,B=\begin{cases} c/d, & \text{if } |c|\leq|d|,\\ d/c, & \text{otherwise.}\end{cases}$$

The functionals $\varphi:(\mathbb{I},+)\to(\mathbb{R},+)$ and $\lambda:(\mathbb{I},+)\to(\mathbb{R},+)$ are homomorphisms as well as $\sigma:(\mathbb{I},\cdot)\to(\mathbb{R},\cdot)$, $\psi:(\mathbb{I},\cdot)\to(\mathbb{R},\cdot)$ and $\chi:(\mathbb{I}',\cdot)\to([-1,1],\theta)$ with $x\,\theta\,y=x\,y$, if $x, y\geq 0$, respectively $\min\{x,y\}$ otherwise.

An interval A is uniquely determined by the values $\varphi\,A$ and $\lambda\,A$ as well as by the values $\sigma\,A$, $\psi\,A$, $\chi\,A$ if $A\neq 0$, and it follows that (details are found in [8]):

$$A=\varphi\,A+[-1;1]\,(\lambda\,A)/2 \quad\text{and}\quad A=(\sigma'\,A)\,(\psi\,A)\,[\chi\,A, 1]. \tag{2}$$

Two intervals are called *unidirectional* if $(\sigma\,A)(\sigma\,B)\geq 0$. A and B are called *proportional* if $A=t\,B$ for some $t>0$. An interval $A\in\mathbb{I}'$ is a point interval iff $\chi A=1$. Let A, $B\in\mathbb{I}'$. It then follows that

a) $\chi(A\,B)\leq\min\{\chi A, \chi B\}$; (3)

b) $\chi A=\chi B$ iff $A=t\,B$ for some $t\in\mathbb{R}$, $t\neq 0$;

c) $\chi(A+B)\leq\max\{\chi A, \chi B\}$, if $A+B\neq 0$;
equality holds iff A and B are either both proportional or both point intervals;

d) $\chi(A+B)\geq\min\{\chi A, \chi B\}$, if A and B are unidirectional;
equality holds iff A and B are proportional.

The components of interval vectors are always denoted by the corresponding Latin letters without explicit mention. That is

$$\mathfrak{x}=(X_1,\dots X_n),\ \mathfrak{y}=(Y_1,\dots Y_n),\ \mathfrak{z}=(Z_1,\dots Z_n)\ \text{for}\ \mathfrak{x},\mathfrak{y},\mathfrak{z}\in\mathbb{I}^n.$$

A finite sum $\mathfrak{y}=\sum a_i\,\mathfrak{x}_i$, with $a_i\in\mathbb{R}$, $\mathfrak{x}_i\in\mathbb{I}^n$ is called a *linear combination of the* $\mathfrak{x}_i$. If $\sum a_i=1$ and $a_i\geq 0$ then $\mathfrak{y}$ is called a *convex combination* of the $\mathfrak{x}_i$. The *linear hull* $[\mathbb{M}]$

and the *convex hull* con $\mathbb{M}$ of a set $\mathbb{M} \subset \mathbb{I}^n$ is defined as the set of linear respectively convex combinations of vectors from $\mathbb{M}$.

If con $\{x, y\} \subset \mathbb{M}$ for all $\mathfrak{x}, \mathfrak{y} \in \mathbb{M}$ then we call $\mathbb{M}$ *convex*. If $\mathfrak{x} \in \mathbb{I}^n$ then we write

$$\psi(\mathfrak{x}) = \max\{\psi X_i : i = 1, \ldots, n\}$$

and we take ψ to be a "modulus" on $\mathbb{I}^n$. In this manner we may use the concepts of *bounded* and *unbounded* subsets of $\mathbb{I}^n$.

3. A Criterion for a Solution, the Structure of the Solution Set

We investigate the interval arithmetic equation

$$A_1 X_1 + \ldots + A_n X_n = B \text{ for } n \geq 2, A_1, \ldots, A_n \in \mathbb{I}', B \in \mathbb{I}, \tag{E}$$

where we have assumed $\chi A_1 = \max\{\chi A_1, \ldots, \chi A_n\}$ without loss of generality. The functional χ answers the question (as in [8] for $n = 1$) of the solvability of (E), even though χ only is a homomorphism with respect to the multiplication (see Section 2).

Theorem 1: *The equation* (E) *is always solvable if* $B = 0$. *If* $B \neq 0$ *then* (E) *has a solution iff* $\chi A_1 \geq \chi B$.

Proof: Let $B \neq 0$. Suppose $\mathfrak{y}$ is a solution and suppose that $J = \{i : Y_i \neq 0\}$. From (3) it follows that

$$\chi B = \chi\left(\sum_{i \in J} A_i Y_i\right) \leq \max\{\chi(A_i Y_i) : i \in J\} \leq \max\{\chi A_i : i \in J\} \leq \chi A_1.$$

Conversely suppose $\chi A_1 \geq \chi B$. From [8] it follows that there is a $Y_1 \in \mathbb{I}'$ with $A_1 Y_1 = B$, and $(Y_1, 0, \ldots, 0)$ therefore solves (E).

The following equation is dependent on fewer parameters than (E) and therefore easier to solve than (E),

$$[\chi A_1, 1] X_1 + \ldots + [\chi A_n, 1] X_n = [\chi B, 1], \text{ if } B \neq 0. \tag{NE}$$

This equation is called the *normalized equation* (NE) of (E) and Theorem 2 and Remark 1 ensure that (E) and (NE) are in some kind equivalent.

Theorem 2: *Suppose* $B \neq 0$. *Then* (E) *is solvable iff* (NE) *is solvable.*

Proof: Since $A_i \neq 0$, $\chi A_i = \chi[\chi A_i, 1]$, and $\chi B = \chi[B, 1]$ it follows that the conditions for solvability of (E) and (NE) are both either satisfied or not satisfied.

Remark 1: Suppose $B \neq 0$. Then both (E) and (NE) have the same solution set: Suppose $\mathfrak{x}$ is a solution of (E). Then it follows that $\mathfrak{y}$ where

$$Y_i = \frac{(\sigma' A_i)\psi A_i}{(\sigma' B)\psi B} X_i$$

is a solution of (NE). If $\mathfrak{y}$ is a solution of (NE) then it follows that $\mathfrak{x}$ where

$$X_i = \frac{(\sigma' B)\psi B}{(\sigma' A_i)\psi A_i} Y_i$$

is a solution of (E). The verification of these statements is done by simple calculations.

For describing the solution set $\mathbb{L}$ of (E), it is of great importance to point out the connections between the convexity properties of subsets $\mathbb{M} \subset \mathbb{L}$ and the distributivity relation

$$\mathbb{D} = \{(A, B, C) \in \mathbb{I}^3 : A(B+C) = AB + AC\},$$

introduced in [7].

Lemma 1: *Let $\mathbb{L}$ be the solution set of* (E) *and let $\mathbb{M} \subset \mathbb{L}$. The set $\mathbb{M}$ is convex iff $(A_i, Y_i, Z_i) \in \mathbb{D}$ for all $\mathfrak{y}, \mathfrak{z} \in \mathbb{M}$.*

Proof: Let $\mathbb{M}$ be convex. It then follows that

$$\sum A_i(Y_i + Z_i) = 2 \sum A_i(Y_i/2 + Z_i/2) = 2B = \sum (A_i Y_i + A_i Z_i).$$

From

$$A_i(Y_i + Z_i) \subset A_i Y_i + A_i Z_i$$

(subdistributivity) it follows that

$$(A_i, Y_i, Z_i) \in \mathbb{D} \text{ for } i = 1, \ldots, n.$$

It then follows that

$$(A_i, \alpha Y_i, \beta Z_i) \in \mathbb{D}$$

according to [7].

Furthermore

$$\sum A_i(\alpha Y_i + \beta Z_i) = \alpha \sum A_i Y_i + \beta \sum A_i Z_i = (\alpha + \beta) B = B$$

according to [7].

The following notations simplify the description of the connections between distributivity and convexity: Let $t \in [-1, 0]$ and define

$$\begin{aligned} \mathbb{U}_t &= \{X \in \mathbb{I}' : \sigma X \geq 0, \chi X \geq t\} \cup \{0\} \\ \mathbb{V}_t &= \{X \in \mathbb{I}' : \chi X \leq t\} \cup \{0\}, \\ \mathbb{W}_t &= \{X \in \mathbb{I}' : \sigma X \leq 0, \chi X \geq t\} \cup \{0\}. \end{aligned}$$

If one defines $\tau A = \min\{\chi A, 0\}$ for $A \in \mathbb{I}'$, then one obtains the following lemma directly from [7]:

Lemma 2: *Let $A \in \mathbb{I}'$ with $\chi A < 1$ and $B, C \in \mathbb{I}$. Then it follows that $(A, B, C) \in \mathbb{D}$ iff $B, C \in \mathbb{U}_{\tau A}$ or $B, C \in \mathbb{V}_{\tau A}$ or $B, C \in \mathbb{W}_{\tau A}$.*

The structure of the solution set is described by:

Theorem 3: *The solution set $\mathbb{L}$ of* (E) *is a union of at most 3^n convex sets.*

Proof: Let $i = 1, \ldots, n$ and $j = 0, \ldots, 3^{n-1}$ and let

$$j = \sum_{v=0}^{n-1} j_{n-v} 3^v$$

be the triadical expansion of j. Then we define

$$\mathbb{M}_{ji}=\left.\begin{array}{ll}\mathbb{U}_{\tau A_i}, & \text{if } j_i=0,\\ \mathbb{V}_{\tau A_i}, & \text{if } j_i=1,\\ \mathbb{W}_{\tau A_i}, & \text{if } j_i=2,\end{array}\right\} \text{ and } \mathbb{M}_j=\mathbb{M}_{j1}\times\ldots\times\mathbb{M}_{jn}.$$

It follows that $\mathbb{I}^n=\bigcup\{\mathbb{M}_j: j=0,\ldots,3^{n-1}\}$. From Lemma 2 and Lemma 1 it follows that the sets $\mathbb{M}_j$ and $\mathbb{K}_j=\mathbb{L}\cap\mathbb{M}_j$ are convex. From this it follows that $\mathbb{L}=\bigcup\{K_j: j=0,\ldots,2^{n-1}\}$.

4. Uniquely Solvable Equations

In contrast to the equation $a_1 x_1+\ldots+a_n x_n=b$ with $a_i, b\in\mathbb{R}$, $n>1$ it is possible that (E) is uniquely solvable. The following lemma gives criteria for this.

Lemma 3: *Let $\chi A_1=\chi B>\chi A_i$ for $i=2,\ldots,n$. It then follows that $Y_2=Y_3=\ldots=Y_n=0$ for every solution $\mathfrak{y}$ of* (E).

Proof: Let $\mathfrak{y}$ be a solution of (E). Writing $C=A_1 Y_1$ and $D=A_2 Y_2+\ldots+A_n Y_n$ and assuming $D\neq 0$ we prove the lemma by contradiction. Using formula (3) and the assumptions it follows that

$$\chi D\leq\chi(A_i Y_i)\leq\chi A_i<\chi A_1=\chi B=\chi(C+D)\leq\max\{\chi C,\chi D\}$$

for $i=2,\ldots,n$.

Because of the proper inequality, the chain may be continued by

$$\max\{\chi C,\chi D\}=\chi C=\chi(A_1 Y_1)\leq\chi A_1=\chi B.$$

This means that $\chi(C+D)=\max\{\chi C,\chi D\}$ and C and D are either proportional or point intervals by (3). Therefore it follows that $\chi C=\chi D$ from (3 b) which gives the required contradiction.

Theorem 4: *Let $B\neq 0$. Equation* (E) *is uniquely solvable iff $0<\chi B=\chi A_1>\chi A_i$ for $i=2,\ldots,n$. In this case the solution is given by*

$$\left(\frac{(\sigma' B)\psi B}{(\sigma' A_1)\psi A_1}, 0,\ldots,0\right).$$

Proof: Let (E) be uniquely solvable. From Theorem 1 it follows that $\chi A_1\geq\chi B$. Furthermore we have $\chi A_i<\chi B$ for $i=2,\ldots,n$. Contrary, if $\chi A_2\geq\chi B$ (without loss of generality) then there would exist intervals $Z_1, Z_2\in\mathbb{I}$ for which $A_1 Z_1=B$ and $A_2 Z_2=B$ according to [8]. The interval vectors $(Z_1, 0,\ldots,0)$ and $(0, Z_2, 0,\ldots,0)$ would then be different solutions of (E). Further we get $\chi A_1=\chi B$. If contrary $\chi A_1>\chi B$ then it would follow that

$$\left(\frac{\chi B-\chi A_2}{\chi A_1-\chi A_2}, \frac{\chi A_1-\chi B}{\chi A_1-\chi A_2}, 0,\ldots,0\right) \text{ and } ([\alpha,1], 0,\ldots,0)$$

were different solutions of (NE) when $\alpha=\chi A_1/\chi B$ for $\chi B>0$ and $\alpha=\chi B$ for $\chi B\leq 0$. From Remark 1 it would follow that (E) had several solutions which is a contradiction.

Let now $0<\chi B=\chi A_1>\chi A_i$ for $i=2,\ldots,n$. From Theorem 1 it follows that there exists a solution of (E). This solution has the property that $Z_2=\ldots=Z_n=0$ from Lemma 3 and we therefore have $A_1 Z_1=B$. From [8] we have that there exists only one such Z_1 since $0<\chi B$.

If the solution of (E) is unique then it is easy to see that the given interval vector is a solution.

Example: The equation $[2,3]X_1+[1,2]X_2+[-4,5]X_3=[-6,-4]$ has the unique solution $(-2,0,0)$.

Theorem 5: *Let $B=0$. The equation* (E) *is uniquely solvable iff $\chi A_i<1$ for $i=2,\ldots,n$. The solution is in this case* $(0,\ldots,0)$.

Proof: Let (E) be uniquely solvable. Assume without loss of generality that $\chi A_2=1$. This means that A_1 and A_2 are point intervals, that is $A_1, A_2 \in \mathbb{R}$. The equation $A_1 X_1+A_2 X_2=0$ has several solutions in $\mathbb{R}$ and therefore in (E). This gives the required contradiction. Conversely let $\chi A_i<1$ for $i=2,\ldots,n$. Clearly $(0,\ldots,0)$ is a solution of (E). Each further solution $\mathfrak{y}$ satisfies

$$0=\lambda B=\sum_{i=1}^{n} \lambda(A_i Y_i)$$

from the additivity of λ (cf. Sec. 2) from which $Y_i=0$ since $A_i \neq 0$.

Remark 2: Contrary to the case of linear equations over $\mathbb{R}$ the solution set of (E) may be bounded if (E) is not uniquely solvable, cf. Example in Sec. 6.

5. Convexity of the Solution Set

If the solution set $\mathbb{L}$ is convex then it is possible to give an explicit representation for the set (see Sec. 6). In the present section we give criteria for the convexity of $\mathbb{L}$. From Remark 1 it follows that the solution set of (E) is convex iff the solution set of (NE) is convex. It is not necessary to consider the case of $B=b\in\mathbb{R}$ and therefore also the case $B=0$ since $\mathbb{L}$ may then be determined as the solution set of an equation over $\mathbb{R}$ which implies that $\mathbb{L}$ is convex. We therefore restrict our attention to the case $\chi B<1$ here and in the sequel. As before we assume $\chi A_1=\max\{\chi A_1,\ldots,\chi A_n\}$.

Lemma 4: *Let $\chi A_1=\chi B$ and let $\mathfrak{y}$ be a solution of* (E). *From $\chi A_i<\chi B$ it follows that $Y_i=0$ $(i=2,\ldots,n)$.*

Proof: Let $\chi A_1=\ldots=\chi A_r=\chi B>\chi A_i$ for $i=r+1,\ldots,n$ without loss of generality. Let $J=\{j: 1\leq j\leq r,\ Y_j\neq 0\}$. We have that $J\neq\emptyset$ since we would otherwise get a contradiction by setting

$$\chi B=\chi\left(\sum_{i=r+1}^{n} A_i Y_i\right)\leq\max\{\chi A_i: i=r+1,\ldots,n\}<\chi B.$$

Similarly one shows that

$$A=\sum_{j\in J} A_j Y_j \neq 0.$$

Since $\chi A \leq \chi B$ there exists a $Z_1 \in \mathbb{I}'$ from [8] for which $BZ_1 = A$. We then get

$$BZ_1 + \sum_{i=r+1}^{n} A_i Y_i = B,$$

from which $Y_i = 0$ for $i = r+1, \ldots, n$ according to Lemma 3.

Remark 3: From the assumption $\chi A_1 = \chi B$ and Lemma 4 one may ignore coefficients A_i for which $\chi A_i < \chi B$. This means that (NE) may be written in the form

$$[\chi B, 1] X_1 + \ldots + [\chi B, 1] X_n = [\chi B, 1]. \tag{NE*}$$

Lemma 5: *Let* $-1 < \chi A_i = \chi B$ *for* $i = 1, \ldots, n$ *and let* $\mathfrak{y}$ *be a solution of* (NE). *It then follows that*:

a) $\sum_{i=1}^{n} \psi Y_i \leq 1$,

b) $\sigma Y_i = 1$, *if* $Y_i \neq 0$ *and*

c) $\chi B \leq \chi Y_i$ *for* $\chi B \leq 0$ *respectively* $\chi Y_i = 1$ *for* $\chi B > 0$, *provided* $Y_i \neq 0$.

Proof: The assertion is proven using Lemma 4 and some simple calculations where the relation

$$\lambda(CD) = (\psi C)(\psi D)[1 - (\chi C \,\theta\, \chi D)] \quad \text{for} \quad C, D \in \mathbb{I}'$$

is used.

Theorem 6: *Let* $\chi B < 1$ *and assume that* (E) *is solvable. The solution set* $\mathbb{L}$ *is convex iff*

a) $\chi A_i = 1$ *for* $i = 1, \ldots, n$ *or*
b) $\chi A_1 = \chi B > -1$.

Proof: If both a) and b) do not hold we show that $\mathbb{L}$ is not convex. We therefore assume without loss of generality that

$$\chi A_2 < 1 \quad \text{and} \quad \alpha)\ \chi A_1 > \chi B \quad \text{or} \quad \beta)\ \chi B = -1.$$

For the case α) we consider the following two solutions of (NE),

$$\mathfrak{y} = \left(\frac{1 - (\chi A_2)\chi B}{1 - (\chi A_1)\chi A_2}, \frac{\chi B - \chi A_1}{1 - (\chi A_1)\chi A_2}, 0, \ldots, 0 \right)$$

and

$$\mathfrak{z} = \begin{cases} \left(\dfrac{\chi B - \chi A_2}{\chi A_1 - \chi A_2}, \dfrac{\chi A_1 - \chi B}{\chi A_1 - \chi A_2}, 0, \ldots, 0 \right), & \text{if } \chi B \geq \chi A_2 \\[2ex] \left(\dfrac{\chi B - \chi A_2}{1 - (\chi A_1)\chi A_2}, \dfrac{1 - (\chi A_1)\chi B}{1 - (\chi A_1)\chi A_2}, 0, \ldots, 0 \right), & \text{if } \chi B < \chi A_2. \end{cases}$$

Since $\chi A_1 < 1$ it follows that $(A_2, Y_2, Z_2) \notin \mathbb{D}$ by [7], that is, $\mathbb{L}$ is not convex (Lemma 1). For the case β) we assume $\chi A_1 = \chi B$ since $\chi A_1 > \chi B$ was already considered in α). The vectors $(1, 0, \ldots, 0)$ and $(-1, 0, \ldots, 0)$ are solutions of (NE). The convex combination $(0, \ldots, 0)$ of these two vectors is, however, not a solution.

Conversely we assume that a) is valid. The distributivity relations required in Lemma 1 follow immediately from [7]. If b) holds then, using Remark 3, we consider (NE*). If $\mathfrak{y}$ and $\mathfrak{z}$ are solutions of (NE*) then one obtains $([\chi B, 1], Y_i, Z_i) \in \mathbb{D}$ from Lemma 5 and [7] and from this the convexity of $\mathbb{L}$ from Lemma 1.

6. The Representation of Convex Solution Sets

It is sufficient to represent the solution sets of (NE) by Remark 1. Let $\mathbb{M}, \mathbb{M}_1 \subset \mathbb{I}^n$ and define

$$\mathbb{M} + \mathbb{M}_1 = \{\mathfrak{x} + \mathfrak{y} : \mathfrak{x} \in \mathbb{M}, \mathfrak{y} \in \mathbb{M}_1\}.$$

Let the vectors

$$\mathfrak{e}_i = (E_{i1}, \ldots, E_{in}), \;\; \mathfrak{f}_i = (F_{i1}, \ldots, F_{in}) \in \mathbb{I}^n$$

be defined by $E_{ij} = \delta_{ij}$ (the Kronecker symbol) and $F_{ij} = \delta_{ij}[\chi B, 1]$. The remaining notation was introduced in Sec. 2.

Theorem 7: *The solution set* $\mathbb{L}$ *of* (NE) *with* $\chi B < 1$ *is*

a) $\mathbb{L} = [\mathfrak{e}_1 - \mathfrak{e}_2, \ldots, \mathfrak{e}_1 - \mathfrak{e}_n] + \operatorname{con}\{\mathfrak{f}_1, \ldots, \mathfrak{f}_n\}$, *if* $\chi A_i = 1$ for $i = 1, \ldots, n$,

b) $\mathbb{L} = \operatorname{con}\{\mathfrak{e}_1, \ldots, \mathfrak{e}_n\}$, *if* $0 < \chi A_i = \chi B$ *for* $i = 1, \ldots, n$,

c) $\mathbb{L} = \operatorname{con}\{\mathfrak{e}_1, \ldots, \mathfrak{e}_n, \mathfrak{f}_1, \ldots, \mathfrak{f}_n\}$, *if* $-1 < \chi A_i = \chi B \leq 0$ *for* $i = 1, \ldots, n$.

Proof: It is easy to check that the above sets are subsets of the solution sets of (NE). We only show that $\mathbb{L}$ is contained in the above sets:

Case a): Let $\mathfrak{y}$ be a solution of (NE) with $Y_i = [y_{i1}, y_{i2}]$. It then follows that

$$\lambda Y_i \leq \lambda[\chi B, 1] = 1 - \chi B.$$

Setting

$$\eta_i = \frac{\lambda \mathrm{Y}_i}{1 - \chi \mathrm{B}_i} \quad \text{and} \quad \xi_i = y_{i1} - \eta_i \chi B,$$

one obtains

$$Y_i = \xi_i + \eta_i [\chi B, 1].$$

where

$$\sum_{i=1}^{n} \eta_i = 1 \quad \text{and} \quad \eta_i \geq 0.$$

From

$$\sum_{i=1}^{n} \xi_i = 0$$

it follows that

$$(\xi_1, \ldots, \xi_n) \in [\mathfrak{e}_1 - \mathfrak{e}_2, \ldots, \mathfrak{e}_1 - \mathfrak{e}_n],$$

that is there exists numbers $\beta_2, \ldots, \beta_n \in \mathbb{R}$, so that

$$\mathfrak{y} = \sum_{i=2}^{n} \beta_i (\mathfrak{e}_1 - \mathfrak{e}_i) + \sum_{j=1}^{n} \eta_j \mathfrak{f}_j$$

is on the desired form.

Case b): Let $\mathfrak{y}$ be a solution of (NE*). We then have $\sigma Y_i = 1 = \chi Y_i$ from Lemma 5, that is $Y_i = y_i \in \mathbb{R}$, $y_i > 0$. From [7] it follows that

$$[\chi B, 1] = [\chi B, 1] \sum_{i=1}^{n} Y_i.$$

Since the equation $[\chi B, 1] = [\chi B, 1]\, X$ has a unique solution 1 because of $0 < \chi B$, cf. [8], it follows that

$$\sum_{i=1}^{n} y_i = 1$$

and $\mathfrak{y}$ is on the required form.

Case c): Let $\mathfrak{y}$ be a solution of (NE*). We obtain first $Y_i = \alpha_i + \beta_i [\chi B, 1]$ with

$$\alpha_i = \varphi Y_i \frac{\chi Y_i - \chi B}{1 - \chi B} \geq 0, \quad \beta_i = \varphi Y_i \frac{1 - \chi Y_i}{1 - \chi B} \geq 0$$

and $\alpha_i + \beta_i \leq 1$. It is to show that

$$\sum_{i=1}^{n} (\alpha_i + \beta_i) = 1,$$

that is $\mathfrak{y}$ is on the desired form.

Since $\alpha_i, \beta_i [\chi B, 1] \in \mathbb{U}_{\chi B}$ it follows that

$$[\chi B, 1]\, Y_i = \alpha_i [\chi B, 1] + \beta_i [\chi B, 1][\chi B, 1] = (\alpha_i + \beta_i)[\chi B, 1]$$

from Lemma 2 and the fact that $\chi B \leq 0$. This means that (NE*) takes on the form

$$[\chi B, 1] = [\chi B, 1] \sum_{i=1}^{n} (\alpha_i + \beta_i),$$

from which

$$\sum_{i=1}^{n} (\alpha_i + \beta_i) = 1$$

follows according to [8].

Example: The solution set of the equation

$$[-1, 2]\, X_1 + [-3, 3/2]\, X_2 = [-2, 4]$$

is

$$\mathbb{L} = \operatorname{con}\{(2, 0), (0, -4/3), ([-1, 2], 0), (0, [-4, 2]/3)\}.$$

7. Concluding Remarks

If the solution set $\mathbb{L}$ of (NE) is not convex then Theorem 3 implies that $\mathbb{L}$ is the union of finitely many convex sets $\mathbb{K}_j$. Using the algorithm of Motzkin-Burger [3] it is possible to connect each $\mathbb{K}_j$ with certain fragments of (NE), which are interval equations in two variables,

$$[\chi A_i, 1]\, X_i + [\chi A_j, 1]\, X_j = [\chi B, 1].$$

The solution sets $\mathbb{L}_{ij}$ are explicitly representable and the sets $\mathbb{K}_j$ may be derived from the sets $\mathbb{L}_{ij}$ in a simple manner, cf. [10].

Considering the solution of a linear interval equation system (ES) it is possible to form the solution set as the intersection of the solution sets of the individual equations. It is, however, almost impossible to give explicit formulas. The details may be found in [10].

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