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Iterative Improvement of Componentwise Errorbounds for Invariant Subspaces Belonging to a Double or Nearly Double Eigenvalue

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Abstract — Zusammenfassung

Iterative Improvement of Componentwise Errorbounds for Invariant Subspaces Belonging to a Double or Nearly Double Eigenvalue. In this paper we present a systematic method which computes bounds for invariant subspaces belonging to a double or nearly double eigenvalue. Furthermore an algorithm based on interval arithmetic tools is introduced which improves these bounds systematically.

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Iterative Verbesserung von komponentenweisen Fehlerschranken für invariante Teilräume, die zu einem doppelten oder fast doppelten Eigenwert gehören. In dieser Arbeit bringen wir eine systematische Methode zur Berechnung von Schranken für invariante Teilräume, die zu einem doppelten oder fast doppelten Eigenwert gehören. Außerdem wird ein auf intervallarithmetischen Hilfsmitteln aufgebauter Algorithmus angegeben, der diese Schranken systematisch verbessert.

0. Introduction

In this paper we consider the eigenvalue problem for a real (n, n) matrix which has a real double eigenvalue or a real pair of nearly equal eigenvalues.

Starting with sufficiently good approximations we construct bounds for the elements of a two by two matrix whose eigenvalues are eigenvalues of the given (n, n) matrix. Furthermore using the Jordan normal form of this $(2, 2)$ matrix the generators of the invariant subspace belonging to these eigenvalues can be enclosed.

The first section contains a careful description of the problem and a reformulation which was already discussed in [3]. See also [6]. In Section 2 we construct bounds for the unknown terms. Furthermore we introduce an iterative method which improves these bounds and which has the property that these bounds are converging to the exact values.

The computation of these bounds and the iterative improvement is based on the contents of Theorem 1 from Section 1. This theorem was implicitly already used in [3].

In Section 3 we list some numerical examples. These examples were computed on an APPLE IIe using the programming language PASCAL SC which has available the exact scalar product. See [4], [5].

In concluding we note that in principle it is not difficult to discuss the more general case in which the matrix and the eigenpairs are complex. Furthermore we also could consider the case of an eigenvalue which has multiplicity larger than two or a set of more than two eigenvalues which are nearly equal. These problems will be discussed in a future paper.

1. Formulation of the Problem

Assume that $A=(a_{ij})$ is a real (n, n)-matrix which has the real eigenvalues γ_1 and γ_2. We assume *either* that

a) $\gamma_1 \neq \gamma_2$ and γ_1, γ_2 are both simple eigenvalues *or* that

b) $\gamma_1=\gamma_2=\gamma$ holds and that γ is double eigenvalue. Then γ either has two linear elementary divisors or one quadratic elementary divisor.

From a numerical point of view we are in case a) especially interested in the situation that γ_1 is close to γ_2.

Under our assumptions there exist two linearly independent real vectors $\bar{u}$ and $\bar{v}$ from $\mathbb{R}^n$ such that

$$A(\bar{u}, \bar{v})=(\bar{u}, \bar{v})\bar{A} \tag{1.1}$$

where

$$\bar{A}=\begin{pmatrix} \gamma_1 & \bar{b} \\ 0 & \gamma_2 \end{pmatrix}, \quad \bar{b} \in \mathbb{R} \tag{1.2}$$

and where $(\bar{u}, \bar{v})$ is an $(n, 2)$-matrix which has $\bar{u}$ and $\bar{v}$ as its columns.

We now assume that approximations $x^1=(x_i^1)$ and $x^2=(x_i^2)$ to $\bar{u}$ and $\bar{v}$ are given. Furthermore m_{11}, m_{12} and m_{22} are given real numbers which are considered as approximations to γ_1, $\bar{b}$ and γ_2. In [3] it is described in detail how to find such approximations using the QR-algorithm. See also [6].

We are now looking for vectors $\tilde{y}^1=(\tilde{y}_i^1)$ and $\tilde{y}^2=(\tilde{y}_i^2)$ and furthermore for real numbers μ_{11}, μ_{12}, μ_{21} and μ_{22} such that

$$A(x^1+\tilde{y}^1, x^2+\tilde{y}^2)=(x^1+\tilde{y}^1, x^2+\tilde{y}^2)\begin{pmatrix} m_{11}+\mu_{11} & m_{12}+\mu_{12} \\ \mu_{21} & m_{22}+\mu_{22} \end{pmatrix} \tag{1.3}$$

holds.

Defining the $(n, 2)$ matrix X by

$$X=(x^1+\tilde{y}^1, x^2+\tilde{y}^2) \tag{1.4}$$

and the (2, 2) matrix D by

$$D=\begin{pmatrix} m_{11}+\mu_{11} & m_{12}+\mu_{12} \\ \mu_{21} & m_{22}+\mu_{22} \end{pmatrix} \tag{1.5}$$

then (1.3) can be written as

$$AX=XD. \tag{1.6}$$

If the (2, 2)-matrix $Y=(y_{ij})$ transforms D by a similarity transformation to upper Jordan normal form then

$$AXY=XYY^{-1}DY,$$

or

$$A(u,v)=(u,v)\Lambda$$

where

$$\Lambda=Y^{-1}DY=\begin{pmatrix} \gamma_1 & b \\ 0 & \gamma_2 \end{pmatrix} \tag{1.7}$$

and

$$(u,v)=XY. \tag{1.8}$$

If $\gamma_1 \neq \gamma_2$ or if $\gamma_1=\gamma_2=\gamma$ and if there are linear elementary divisors belonging to γ then $b=0$. The vectors u and v are then eigenvectors of A corresponding to the eigenvalues γ_1 and γ_2.

If, however, $\gamma_1=\gamma_2=\gamma$ is a double eigenvalue with a quadratic elementary divisor then $b=1$. The vector u then is an eigenvector of A belonging to the eigenvalue γ and v is a principal vector of grade two belonging to the eigenvalue γ.

In every case the linearly independent vectors u and v span a linear space called the invariant subspace belonging to the eigenvalues γ_1 and γ_2 or γ, respectively, u and v are called generators of the subspace.

After having solved (1.3) the computation of γ_1 and γ_2 and of the generators u and v is, because of (1.7) and (1.8), essentially reduced to the eigenvalue problem for the (2, 2)-matrix D, defined by (1.5).

The system (1.3) is a nonlinear system of $2n$ equations and $2n+4$ unknowns. In order to compute a unique solution we perform the same normalization as in [3]. (See also [6]):

If $x^1=(x_i^1)$ and $x^2=(x_i^2)$ are sufficiently good approximations for the (linearly independent) vectors $\bar{u}$ and $\bar{v}$ then they are also linearly independent. Therefore, if we define the integers p and q such that

$$|x_p^1| = \max_{1\le i\le n} |x_i^1| \tag{1.9}$$

and

$$|x_p^1 x_q^2 - x_q^1 x_p^2| = \max_{1\le i\le n} |x_p^1 x_i^2 - x_i^1 x_p^2| \tag{1.10}$$

then $p \neq q$.

The vectors $\tilde{y}^1=(\tilde{y}_i^1)$ and $\tilde{y}^2=(\tilde{y}_i^2)$ are now determined in such a manner that

$$\tilde{y}_p^1=\tilde{y}_p^2=\tilde{y}_q^1=\tilde{y}_q^2=0. \tag{1.11}$$

If we also define vectors $y^1=(y_i^1)$ and $y^2=(y_i^2)$ by

$$y_i^1=\begin{cases}\tilde{y}_i^1, & i\neq p,q\\ \mu_{11}, & i=p\\ \mu_{21}, & i=q\end{cases} \tag{1.12}$$

and

$$y_i^2=\begin{cases}\tilde{y}_i^2, & i\neq p,q\\ \mu_{12}, & i=p\\ \mu_{22}, & i=q,\end{cases} \tag{1.13}$$

respectively, then (1.3) can be written as

$$\begin{cases}B_1\,y^1=r^1+y_p^1\cdot\tilde{y}^1+y_q^1\cdot\tilde{y}^2\\ B_2\,y^2=r^2+m_{12}\cdot\tilde{y}^1+y_p^2\cdot\tilde{y}^1+y_q^2\cdot\tilde{y}^2\end{cases} \tag{1.14}$$

where

$$\begin{cases}r^1=m_{11}\,x^1-A\,x^1\\ r^2=m_{12}\,x^1+m_{22}\,x^2-A\,x^2\end{cases} \tag{1.15}$$

and where B_i, $i=1,2$, is $A-m_{ii}\,I$ with columns p and q replaced by $-x^1$ and $-x^2$ respectively. The rewriting of the original system (1.3) into (1.14) was already performed in [3]. See also [6].

This rewriting has *two important advantages*:

1. The right-hand sides of (1.14) can be computed by computing $2n$ scalar products. This has a tremendous impact on the numerical precision if one is computing the unknowns $\tilde{y}^1$, $\tilde{y}^2$, μ_{11}, μ_{12}, μ_{21} and μ_{22}.
2. For sufficiently good approximations x^1, x^2, m_{11}, m_{12} and m_{21} the matrices B_1 and B_2 are nonsingular. By continuity arguments this follows from the next theorem which was implicitly already used in [3].

Theorem 1.1: *Assume that the real (n,n) matrix $A=(a_{ij})$ has the eigenvalues γ_1 and γ_2 and that u and v are the generators of the invariant subspace belonging to γ_1 and γ_2. Let the integers p and q be defined analogously to* (1.9) *and* (1.10).

If $\gamma_1\neq\gamma_2$ then assume that γ_1 and γ_2 are both simple eigenvalues. If $\gamma_1=\gamma_2=\gamma$ then assume that γ is a double eigenvalue with two linear elementary divisors or one quadratic elementary divisor. Then the matrices B_i, $i=1,2$, where B_i is $A-\gamma_i I$ with columns p and q replaced by $-u$ and $-v$, respectively, are nonsingular.

The preceding Theorem 1 is the basis for a method which computes bounds for $\tilde{y}^1$, $\tilde{y}^2$, μ_{11}, μ_{12}, μ_{21} and μ_{22} and for a method which improves these bounds. This is the topic of the next section.

2. Computing Bounds and Their Iterative Improvement

Assume that L_1 and L_2 are approximations to the inverses of B_1 and B_2 (or the inverses themselves). Then the equations (1.14) can be written as

$$\begin{cases} y^1 = g_1(y^1, y^2) \\ y^2 = g_2(y^1, y^2) \end{cases} \tag{2.1}$$

where

$$g_1(y^1, y^2) = L_1 r^1 + K_1 y^1 + L_1(y_p^1 \cdot \tilde{y}^1) + L_1(y_q^1 \cdot \tilde{y}^2),$$

$$g_2(y^1, y^2) = L_2 r^2 + K_2 y^2 + L_2(m_{12} \cdot \tilde{g}_1(y^1, y^2)) + L_2(y_p^2 \cdot \tilde{y}^1) + L_2(y_q^2 \cdot \tilde{y}^2)$$

and

$$K_1 = I - L_1 B_1, \quad K_2 = I - L_2 B_2. \tag{2.2}$$

Furthermore $\tilde{g}_1(y^1, y^2)$ is $g_1(y^1, y^2)$ with components p and q set equal to zero. By

$$g(y^1, y^2) = \begin{pmatrix} g_1(y^1, y^2) \\ g_2(y^1, y^2) \end{pmatrix} \tag{2.3}$$

a mapping of the $\mathbb{R}^{2n}$ into itself is defined and a fixed point $\begin{pmatrix} y^{1,*} \\ y^{2,*} \end{pmatrix} \in \mathbb{R}^{2n}$ is a solution of (1.14) (and therefore also of (1.3)) if L_1 and L_2 are nonsingular.

Starting with x^1, x^2, m_{11}, m_{12} and m_{22} we are now constructing interval vectors $[y]^1$ and $[y]^2$ such that $\begin{pmatrix} [y]^1 \\ [y]^2 \end{pmatrix}$ contains a fixed point of g.

For interval vectors $[y]^1 = ([y]_i^1)$ and $[y]^2 = ([y]_i^2)$ we define interval vectors $[\tilde{y}]^1 = ([\tilde{y}]_i^1)$ and $[\tilde{y}]^2 = ([\tilde{y}]_i^2)$ by

$$[\tilde{y}]_i^1 = \begin{cases} [y]_i^1, & i \neq p, q \\ 0, & i = p, q \end{cases}$$

and

$$[\tilde{y}]_i^2 = \begin{cases} [y]_i^2, & i \neq p, q \\ 0, & i = p, q, \end{cases}$$

respectively. We now determine $[y]^1$ and $[y]^2$ in such a manner that

$$g([y]^1, [y]^2) = \begin{pmatrix} g_1([y]^1, [y]^2) \\ g_2([y]^1, [y]^2) \end{pmatrix} \subseteq \begin{pmatrix} [y]^1 \\ [y]^2 \end{pmatrix} \tag{2.4}$$

holds. Because of the inclusion monotonicity of interval arithmetic (see [1], p. 6)

$$g(y^1, y^2) \in \begin{pmatrix} [y]^1 \\ [y]^2 \end{pmatrix}$$

holds for all

$$\begin{pmatrix} y^1 \\ y^2 \end{pmatrix} \in \begin{pmatrix} [y]^1 \\ [y]^2 \end{pmatrix}.$$

Since g is continuous and since $\begin{pmatrix}[y]^1\\ [y]^2\end{pmatrix}$ is a compact and convex set in $\mathbb{R}^{2n}$ the fixed point theorem of Brouwer implies that there is at least one fixed point of g in $\begin{pmatrix}[y]^1\\ [y]^2\end{pmatrix}$.

In order to determine $[y]^1$ and $[y]^2$ such that (4) holds we set

$$[y]^1=[y]^2=[-\beta,\beta]\,e \tag{2.5}$$

where

$$\beta\in\mathbb{R},\ \beta\geq 0 \text{ and } e=\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}\in\mathbb{R}^n.$$

Defining

$$e_{pq}=((e_{pq})_i)\in\mathbb{R}^n$$

by

$$(e_{pq})_i=\begin{cases}1, & i\neq p,q\\ 0, & i=p,q,\end{cases}$$

then

$$\begin{aligned}[y]_p^1\cdot[\tilde{y}]^1=[y]_q^1\cdot[\tilde{y}]^2=[y]_p^2\cdot[\tilde{y}]^1=\\ =[y]_q^2\cdot[\tilde{y}]^2=[-\beta^2,\beta^2]\cdot e_{pq}.\end{aligned}$$

Denoting by $|K_1|$, $|K_2|$, $|L_1|$ and $|L_2|$ the real matrices which one gets from K_1, K_2, L_1 and L_2, respectively, by forming elementwise the absolute value then we obtain for the *diameters* (see [1], p. 14 and p. 125) $d\,g_1([y]^1,[y]^2)$ and $d\,g_2([y]^1,[y]^2)$

$$d\,g_1([y]^1,[y]^2)=2\beta\,|K_1|\,e+4\beta^2\,|L_1|\,e_{pq} \tag{2.6}$$

and

$$\begin{aligned}d\,g_2([y]^1,[y]^2)&=|m_{12}|\,|L_2|\cdot d\tilde{g}_1([y]^1,[y]^2)+\\ &\quad+2\beta\,|K_2|\,e+4\beta^2\,|L_2|\,e_{pq}\\ &\leq|m_{12}|\,|L_2|\,\{2\beta\,|K_1|\,e+4\beta^2\,|L_1|\,e_{pq}\}+\\ &\quad+2\beta\,|K_2|\,e+4\beta^2\,|L_2|\,e_{pq}\\ &=4\beta^2\,|L_2|\,\{I+|m_{12}|\,|L_1|\}\,e_{pq}+\\ &\quad+2\beta\,\{|m_{12}|\,|L_2|\,|K_1|+|K_2|\}\,e.\end{aligned} \tag{2.7}$$

For the *centers* (*midpoints*) of $[y]^1$, $[y]^2$, $g_1([y]^1,[y]^2)$ and $g_2([y]^1,[y]^2)$ we have $m[y]^1=m[y]^2=0$ and

$$m\,g_1([y]^1,[y]^2)=L_1\,r^1 \tag{2.8}$$

$$m\,g_2([y]^1,[y]^2)=L_2\,(r^2+m_{12}\,\widetilde{L_1\,r^1}). \tag{2.9}$$

We have

$$g([y]^1,[y]^2)=\begin{pmatrix}g_1([y]^1,[y]^2)\\ g_2([y]^1,[y]^2)\end{pmatrix}\subseteq\begin{pmatrix}[y]^1\\ [y]^2\end{pmatrix}$$

if and only if

$$|m[y]^i - m g_i([y]^1,[y]^2)| + \frac{1}{2} d g_i([y]^1,[y]^2) \le \frac{1}{2} d[y]^i, \quad i=1,2. \tag{2.10}$$

Because of (2.6) – (2.9) this is the case if and only if both

$$|L_1 r^1| + \beta |K_1| e + 2\beta^2 |L_1| e_{pq} \le \beta e$$

and

$$|L_2(r^2 + m_{12} \widetilde{L_1 r^1})| + \beta(|m_{12}||L_2||K_1| + |K_2|) e + + 2\beta^2 |L_2| (|m_{12}||L_1| + I) e_{pq} \le \beta e$$

hold.

Defining real numbers

$$\rho_1 = \max_{1\le i\le n} \{|L_1 r^1|_i\}, \quad \rho_2 = \max_{1\le i\le n} \{|L_2(r^2 + m_{12} \widetilde{L_1 r^1})|\},$$

$$\rho = \max\{\rho_1, \rho_2\}, \tag{2.11}$$

$$\kappa_1 = \max_{1\le i\le n} \left\{\sum_{j=1}^n |(\mathrm{K}_1)_{ij}|\right\}, \quad \kappa_2 = \max_{1\le i\le n} \left\{\sum_{j=1}^n (|m_{12}||L_2||K_1| + |K_2|)_{ij}\right\},$$

$$\kappa = \max\{\kappa_1, \kappa_2\}, \tag{2.12}$$

$$l_1 = \max_{1\le i\le n} \left\{\sum_{j=1}^n |(L_1)_{ij}|\right\}, \quad l_2 = \max_{1\le i\le n} \left\{\sum_{j=1}^n (|L_2|(|m_{12}||L_1| + I))_{ij}\right\},$$

$$l = \max\{l_1, l_2\}, \tag{2.13}$$

then (2.10) surely holds if the inequality

$$\rho + (\kappa - 1)\beta + 2 l \beta^2 \le 0$$

holds. Assuming $\kappa < 1$ and $(\kappa-1)^2 - 8\rho l \ge 0$ then this inequality holds for all $\beta \in [\beta_1, \beta_2]$ where

$$\beta_{1/2} = \frac{1 - \kappa \mp \sqrt{(1-\kappa)^2 - 8\rho l}}{4 l}. \tag{2.14}$$

Hence we have the proof for the following

Theorem 2.1: *Let ρ, κ and l be defined via* (2.11), (2.12) *and* (2.13). *If $\kappa < 1$ and $(1-\kappa)^2 - 8\rho l \ge 0$ and if $\beta \in [\beta_1, \beta_2]$ where β_1 and β_2 are defined by* (2.14) *then the mapping g defined by* (2.3) *has at least one fixed point $\begin{pmatrix} y^{1,*} \\ y^{2,*} \end{pmatrix}$ in $\begin{pmatrix} [y]^1 \\ [y]^2 \end{pmatrix}$ where*

$$[y]^1 = [y]^2 = [y]^0 := [-\beta, \beta] e.$$

□

A fixed point $\begin{pmatrix} y^{1,*} \\ y^{2,*} \end{pmatrix}$ of g is surely a solution of (1.14) (and hence of (1.3)) if L_1 and L_2 are nonsingular. Under the assumption $\kappa < 1$ of the preceding Theorem 1 this is always the case. This can be seen as follows: From the definition of κ we have that

$$\| K_1 \|_\infty = \| I - L_1 B_1 \|_\infty < 1 .$$

Therefore the inverse of $(I-(I-L_1 B_1)) = L_1 B_1$ exists which implies that L_1^{-1} exists. Similarly the existence of L_2^{-1} follows.

We now consider the *iteration method*

$$\begin{cases} [y]^{1,k+1} = g_1([y]^{1,k}, [y]^{2,k}) \\ [y]^{2,k+1} = g_2([y]^{1,k}, [y]^{2,k}) \end{cases} \tag{V}$$

$$k = 0, 1, 2, \ldots,$$

where

$$[y]^{1,0} = [y]^{2,0} = [y]^0 = [-\beta, \beta]\, e .$$

Theorem 2.2: *Let $\kappa < 1$ and $\beta_1 \neq \beta_2$ where β_1 and β_2 are defined by* (2.14). *If then*

$$\beta_1 \le \beta < \frac{\beta_1 + \beta_2}{2}$$

and $[y]^0 = [-\beta, \beta]\, e$ then (V) *is well-defined.*
(V) *delivers two sequences $\{[y]^{1,k}\}_{k=0}^{\infty}$ and $\{[y]^{2,k}\}_{k=0}^{\infty}$ of interval vectors for which $y^{1,*} \in [y]^{1,k}$, $y^{2,*} \in [y]^{2,k}$ and*

$$\lim_{k\to\infty} [y]^{1,k} = y^{1,*}; \quad \lim_{k\to\infty} [y]^{2,k} = y^{2,*}$$

hold. $\begin{pmatrix} y^{1,*} \\ y^{2,*} \end{pmatrix}$ *is the unique fixed point of g in* $\begin{pmatrix} [y]^{1,0} \\ [y]^{2,0} \end{pmatrix}$.

Proof: By the proof of Theorem 1 we have

$$\begin{pmatrix} [y]^{1,1} \\ [y]^{2,1} \end{pmatrix} = g([y]^{1,0}, [y]^{2,0}) \subseteq \begin{pmatrix} [y]^{1,0} \\ [y]^{2,0} \end{pmatrix}.$$

By complete induction it follows that

$$\begin{pmatrix} [y]^{1,k+1} \\ [y]^{2,k+1} \end{pmatrix} \subseteq \begin{pmatrix} [y]^{1,k} \\ [y]^{2,k} \end{pmatrix}, \quad k = 0, 1, 2, \ldots .$$

Hence the convergence of the sequences $\{[y]^{1,k}\}_{k=0}^{\infty}$ and $\{[y]^{2,k}\}_{k=0}^{\infty}$ follows, that is we have $\lim_{k\to\infty} [y]^{1,k} = [y]^{1,*}$ and $\lim_{k\to\infty} [y]^{2,k} = [y]^{2,*}$ where $[y]^{1,*}$ and $[y]^{2,*}$ are interval vectors. By Theorem 1 there exists at least one fixed point $\begin{pmatrix} y^{1,*} \\ y^{2,*} \end{pmatrix}$ of g in $\begin{pmatrix} [y]^{1,0} \\ [y]^{2,0} \end{pmatrix}$.

Using the inclusion monotonicity (see [1], p. 6) it follows by complete induction and passing to the limit afterwards that

$$\begin{pmatrix} y^{1,*} \\ y^{2,*} \end{pmatrix} \in \begin{pmatrix} [y]^{1,*} \\ [y]^{2,*} \end{pmatrix}.$$

Therefore, if we are able to prove that

$$\lim_{k\to\infty} d[y]^{1,k} = \lim_{k\to\infty} d[y]^{2,k} = 0$$

holds then (2.15) and the uniqueness of $\begin{pmatrix} y^{1,*} \\ y^{2,*} \end{pmatrix}$ in $\begin{pmatrix} [y]^{1,0} \\ [y]^{2,0} \end{pmatrix}$ follow.

We define d_k to be

$$d_k = \max\left\{\max_{1\le i\le n} \{d[y]_i^{1,k}\}, \max\{d[y]_i^{2,k}\}\right\}$$

and take into account that for two real intervals $[a]$ and $[b]$ (see [1], p. 15)

$$d([a]\cdot[b]) \le |[a]|\, d[b] + d[a]\, |[b]|.$$

The absolute value of interval terms is defined in [1], Chapter 2.

Because of

$$[y]^{1,k},\ [y]^{2,k} \subseteq [y]^0 = [-\beta, \beta]\, e$$

we then get

$$\begin{aligned} d[y]^{1,k+1} &= |K_1|\, d[y]^{1,k} + |L_1|\, d([y]_p^{1,k}\cdot[\tilde{y}]^{1,k}) + \\ &\quad + |L_1|\, d([y]_q^{1,k}\cdot[\tilde{y}]^{2,k}) \\ &\le d_k\cdot(|K_1| + 4\beta|L_1|)\, e \end{aligned} \tag{2.16}$$

and

$$\begin{aligned} d[y]^{2,k+1} &= |m_{12}|\,|L_2|\, d\widetilde{[y]}^{1,k+1} + |K_2|\, d[y]^{2,k} + \\ &\quad + |L_2|\, d([y]_p^{2,k}\cdot[\tilde{y}]^{1,k}) + |L_2|\, d([y]_q^{2,k}\cdot\widetilde{[y]}^{2,k}) \\ &\le d_k |m_{12}|\,|L_2|(|K_1| + 4\beta|L_1|)\, e + d_k |K_2|\, e + 4 d_k \beta |L_2|\, e \\ &= d_k\big(|K_2| + 4\beta|L_2| + |m_{12}|\,|L_2|(|K_1| + 4\beta|L_1|)\big)\, e \\ &= d_k\big((|m_{12}|\,|L_2|\,|K_1| + |K_2|) + 4\beta|L_2|(|m_{12}|\,|L_1| + I)\big)\, e. \end{aligned} \tag{2.17}$$

Using the definition of κ and l it follows from the last two inequalities that

$$d_{k+1} \le (\kappa + 4\beta l)\, d_k. \tag{2.18}$$

Because of

$$\beta < \frac{\beta_1 + \beta_2}{2} = \frac{1-\kappa}{4l}$$

we have

$$\kappa + 4\beta l < 1.$$

Therefore from (2.18) it follows that $\lim_{k\to\infty} d_k = 0$ and therefore (2.15) holds. □

3. Numerical Examples

1. We consider the (7, 7)-matrix

$$A=\begin{pmatrix} -6 & 0 & 0 & -1 & -4 & -4 & 0\\ 0 & 4 & 1 & 0 & 0 & 0 & 2\\ 0 & 1 & 4 & 0 & 0 & 0 & 0\\ -1 & 0 & 0 & -6 & -4 & -4 & 0\\ -4 & 0 & 0 & -4 & -6 & -1 & 0\\ -4 & 0 & 0 & -4 & -1 & -6 & 0\\ 0 & 1 & 1 & 0 & 0 & 0 & 4 \end{pmatrix}$$

which has the eigenvalues

$$\lambda_1=6,\ \lambda_2=\lambda_3=3,\ \lambda_4=1,\ \lambda_5=\lambda_6=-5,\ \lambda_7=-15.$$

To the double eigenvalue $\lambda_2=\lambda_3=3$ belongs a quadratic elementary divisor, to $\lambda_5=\lambda_6=-5$ belong two linear elementary divisors. The eigenvectors and the principal vector belonging to the eigenvalue 3 are as follows:

$$u^1=\frac{1}{4}\begin{pmatrix}0\\4\\2\\0\\0\\0\\3\end{pmatrix};\quad \underbrace{u^2=\begin{pmatrix}0\\1\\-1\\0\\0\\0\\0\end{pmatrix}}_{\text{eigenvector}};\quad \underbrace{u^3=\begin{pmatrix}0\\0\\-1\\0\\0\\0\\1\end{pmatrix}}_{\text{principalvector}};\quad u^4=\begin{pmatrix}1\\0\\0\\1\\-1\\-1\\0\end{pmatrix}$$

(eigenvector principalvector belonging to the eigenvalue 3)

$$u^5=\begin{pmatrix}1\\0\\0\\-1\\-1\\1\\0\end{pmatrix};\quad u^6=\begin{pmatrix}1\\0\\0\\-1\\1\\-1\\0\end{pmatrix};\quad u^7=\begin{pmatrix}1\\0\\0\\1\\1\\1\\0\end{pmatrix}$$

u^5 and u^6 are two linearly independent eigenvectors belonging to the double eigenvalue -5.

As approximations to the real numbers m_{11}, m_{12} and m_{22} in (1.3) we choose

$$\begin{pmatrix}m_{11} & m_{12}\\ & m_{22}\end{pmatrix}=\begin{pmatrix}-4.999\,999\,99 & 1\,E-8\\ & -5.000\,000\,01\end{pmatrix}.$$

Correspondingly we consider

$$x^1 = \begin{pmatrix} 9.999\,999\,9\,E-1 \\ 1\,E-8 \\ -1\,E-8 \\ -1 \\ -9.999\,999\,9\,E-1 \\ 9.999\,999\,9\,E-1 \\ 1\,E-8 \end{pmatrix} \quad \text{and} \quad x^2 = \begin{pmatrix} 1 \\ 1\,E-8 \\ -1\,E-8 \\ -9.999\,999\,9\,E-1 \\ 9.999\,999\,9\,E-1 \\ -9.999\,999\,9\,E-1 \\ 1\,E-8 \end{pmatrix}$$

as approximations to the eigenvectors u^5 and u^6. Using the iteration method (V) from Section 2 we get the following inclusions for the solution of (1.3):

$$\begin{pmatrix} m_{11}+\mu_{11} & m_{12}+\mu_{12} \\ \mu_{21} & m_{22}+\mu_{22} \end{pmatrix} \in$$

$$\begin{pmatrix} [-5.000\,000\,000\,01; \; -4.999\,999\,999\,99] & [-1\,E-19; \;\; 3\,E-20] \\ [-2.6\,E-20; \; 2.6\,E-20] & [-5.000\,000\,000\,01; \; -4.999\,999\,999\,99] \end{pmatrix}$$

$$x^1+\tilde{y}1 \in \begin{pmatrix} [9.999\,999\,999\,99\,E-1; \; 1.000\,000\,000\,01] \\ [-1\,E-19; \;\; 2\,E-20] \\ [-2\,E-20; \;\; 1\,E-19] \\ [-1; -1] \\ [-9.999\,999\,9\,E-1; \; -9.999\,999\,9\,E-1] \\ [9.999\,999\,899\,99\,E-1; \; 9.999\,999\,999\,01\,E-1] \\ [-1\,E-19; \;\; 2\,E-20] \end{pmatrix} \qquad x^2+\tilde{y}^2 \in \begin{pmatrix} [9.999\,999\,899\,99\,E-1; \; 9.999\,999\,900\,01\,E-1] \\ [-2\,E-19; \;\; 1.2\,E-19] \\ [-3\,E-20; \;\; 1.0\,E-19] \\ [-9.999\,999\,9\,E-1; \; -9.999\,999\,9\,E-1] \\ [9.999\,999\,9\,E-1; \; 9.999\,999\,9\,E-1] \\ [-9.999\,999\,900\,01\,E-1; \; -9.999\,999\,899\,99\,E-1] \\ [-1\,E-19; \;\; 3\,E-20] \end{pmatrix}$$

2. In this example we replace the zero elements of the preceding matrix by numbers ε_{ij} for which $|\varepsilon_{ij}|=1\,E-8$. We denote this matrix by A_1. We now choose

$$\begin{pmatrix} m_{11} & m_{12} \\ & m_{22} \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ & 3 \end{pmatrix},$$

$$x^1 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad x^2 = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

In this case we get the following inclusions:

$$\begin{pmatrix} m_{11}+\mu_{11} & m_{12}+\mu_{12} \\ \mu_{21} & m_{22}+\mu_{22} \end{pmatrix} \in$$

$$\begin{pmatrix} [2.999\,999\,999\,99; & [1.000\,000\,006\,66; \\ \;3.000\,000\,000\,00] & \;1.000\,000\,006\,67] \\ [-5.925\,925\,936\,25\,E-17; & [3.000\,000\,003\,33; \\ \;-5.925\,925\,936\,21\,E-17] & \;3.000\,000\,003\,34] \end{pmatrix}$$

$$x^1+\tilde{y}^1 \in \qquad\qquad x^2+\tilde{y}^2 \in$$

$$\begin{pmatrix} [4.027\,777\,777\,77\,E-9; & [4.828\,317\,894\,28\,E-9; \\ \;4.027\,777\,777\,80\,E-9] & \;4.828\,317\,894\,33\,E-9] \\ [1;\;\;1] & [0;\;\;0] \\ [-1;-1] & [-1;-1] \\ [1.527\,777\,777\,77\,E-9; & [2.640\,817\,897\,29\,E-9; \\ \;1.527\,777\,777\,79\,E-9] & \;2.640\,817\,897\,33\,E-9] \\ [-2.222\,222\,222\,24\,E-9; & [-2.515\,432\,088\,06\,E-9; \\ \;-2.222\,222\,222\,22\,E-9] & \;-2.515\,432\,088\,02\,E-9] \\ [-2.222\,222\,222\,24\,E-9 & [-5.015\,432\,095\,36\,E-9; \\ \;-2.222\,222\,222\,22\,E-9] & \;-5.015\,432\,095\,31\,E-9] \\ [-7.037\,037\,067\,12\,E-17; & [1.000\,000\,003\,33; \\ \;-7.037\,037\,067\,07\,E-17] & \;1.000\,000\,003\,34] \end{pmatrix}$$

3. As a final example we consider the (7, 7) matrix

$$A=\begin{pmatrix} -6 & 0 & -4 & -1 & 0 & 0 & -4 \\ 0 & b & 0 & 0 & 0 & 0 & 0 \\ -4 & 0 & -6 & -4 & 0 & 0 & -1 \\ -1 & 0 & -4 & -6 & 0 & 0 & -4 \\ 0 & 0 & 0 & 0 & c & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & a & 0 \\ -4 & 0 & -1 & -4 & 0 & 0 & -6 \end{pmatrix} \qquad a \neq b$$

The eigenvalues and the corresponding eigenvectors are as follows:

Eigenvalues:

$$\lambda_1=a; \quad \lambda_2=b; \quad \lambda_3=c; \quad \lambda_4=-15; \quad \lambda_5=\lambda_6=-5; \quad \lambda_7=1;$$

Eigenvectors:

$$u^1=\begin{pmatrix}0\\0\\0\\0\\0\\1\\0\end{pmatrix} \quad u^2=\begin{pmatrix}0\\b-a\\0\\0\\0\\1\\0\end{pmatrix} \quad u^3=\begin{pmatrix}0\\0\\0\\0\\1\\0\\0\end{pmatrix} \quad u^4=\begin{pmatrix}-1\\0\\-1\\-1\\0\\0\\-1\end{pmatrix}$$

$$u^5=\begin{pmatrix}-1\\0\\1\\1\\0\\0\\-1\end{pmatrix}\quad u^6=\begin{pmatrix}1\\0\\1\\-1\\0\\0\\-1\end{pmatrix}\quad u^7=\begin{pmatrix}1\\0\\-1\\1\\0\\0\\-1\end{pmatrix}$$

We slightly modify this matrix to $\tilde{A}$ defined by

$$\tilde{A}=\begin{pmatrix}-6 & 0 & -4 & -1 & 0 & 0 & -4\\ 0 & \underline{b} & 0 & 0 & -1E-10 & -1E-10 & 0\\ -4 & 0 & -6 & -4 & 0 & 0 & -1\\ -1 & 0 & -4 & -6 & 0 & 0 & -4\\ 0 & -1E-10 & 0 & 0 & \underline{c} & 1E-10 & 0\\ 0 & 1 & 0 & 0 & 1E-10 & \underline{a} & 0\\ -4 & 0 & -1 & -4 & 0 & 0 & -6\end{pmatrix}$$

and choose

$$a=5.000\,001$$
$$b=4.999\,999$$
$$c=5.001\,.$$

In (1.3) we choose

$$\begin{pmatrix}m_{11} & m_{12}\\ & m_{22}\end{pmatrix}=\begin{pmatrix}5.000\,001 & 0\\ 0 & 4.999\,999\end{pmatrix},$$

$$x^1=\begin{pmatrix}0\\0\\0\\0\\0\\1\\0\end{pmatrix}\qquad x^2=\begin{pmatrix}0\\2E-6\\0\\0\\0\\1\\0\end{pmatrix}$$

The inclusions computed by (V) are

$$\begin{pmatrix}m_{11}+\mu_{11} & m_{12}+\mu_{12}\\ \mu_{21} & m_{22}+\mu_{22}\end{pmatrix}\in$$

$$\begin{pmatrix}[5.000\,050\,999\,99;\ 5.000\,051\,000\,00] & [5.399\,999\,498\,54\,E-5;\ 5.399\,999\,498\,55\,E-5]\\ [-4.999\,999\,499\,55\,E-5;\ -4.999\,999\,499\,54\,E-5] & [4.999\,949\,000\,000;\ 4.999\,949\,000\,001]\end{pmatrix}$$

$$x^1+\tilde{y}^1 \in \begin{pmatrix} [-2E-99;\ \ 2E-99] \\ [0;\ \ 0] \\ [-2E-99;\ \ 2E-99] \\ [-2E-99;\ \ 2E-99] \\ [-1.000\,901\,010\,82E-7; \\ \ -1.000\,901\,010\,79E-7] \\ [1;\ \ 1] \\ [-2E-99;\ \ 2E-99] \end{pmatrix} \qquad x^2+\tilde{y}^2 \in \begin{pmatrix} [-2E-99;\ \ 2E-99] \\ [2E-6;\ \ 2E-6] \\ [-2E-99;\ \ 2E-99] \\ [-2E-99;\ \ 2E-99] \\ [-1.002\,898\,815\,04E-7; \\ \ -1.002\,898\,815\,02E-7] \\ [1;\ \ 1] \\ [-2E-99;\ \ 2E-99] \end{pmatrix}$$

It is interesting to note that the (2, 2) matrix

$$\begin{pmatrix} m_{11}+\mu_{11} & m_{12}+\mu_{12} \\ \mu_{21} & m_{22}+\mu_{22} \end{pmatrix}$$

now has a pair of complex eigenvalues.

All computation was done on an APPLE IIe using the programming language PASCAL SC (see [4]). This system uses a decimal number system which has 12 digits in the mantissa of a floating point number. Note that all rounding errors are taken into account using this system. Therefore the bounds computed in the preceding examples are absolutely safe.

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