Computing 12, 75—89 (1974)


# Stability of Numerical Methods for Volterra Integro-Differential Equations*

By

**H. Brunner**, Halifax, and **J. D. Lambert**, Dundee

With 1 Figure

Received June 4, 1973

**Abstract — Zusammenfassung**

**Stability of Numerical Methods for Volterra Integro-Differential Equations.** A theory of weak stability for linear multistep methods for the numerical solution of Volterra integro-differential equations is developed, and a connection between this theory and the corresponding theory for ordinary differential equations is established. In addition, the order of such methods is discussed, and a new starting procedure is proposed and analyzed.

**Stabilität numerischer Verfahren zur Lösung Volterrascher Integrodifferentialgleichungen.** Die Theorie der schwachen Stabilität linearer Mehrschrittverfahren zur numerischen Lösung von Volterraschen Integrodifferentialgleichungen wird beschrieben, wobei ein enger Zusammenhang mit der entsprechenden Theorie für gewöhnliche Differentialgleichungen aufgezeigt wird. Ferner wird die Ordnung solcher Verfahren untersucht, und eine neue Methode zur Erzeugung der notwendigen Startwerte wird hergeleitet.

## 1. Introduction

We consider a Volterra integro-differential equation of the form

$$\begin{aligned} y'(x) &= F\left(x, y(x), z(x)\right), \quad 0 \leqq x \leqq a, \\ y(0) &= y_0, \end{aligned} \tag{1.1}$$

where

$$z(x) := \int_0^x K\left(x, t, y(t)\right) dt.$$

Let

$$S = \{(x, t, u): 0 \leqq t \leqq x \leqq a, \ |u| < \infty\},$$

$$T = \{(x, y, z): 0 \leqq x \leqq a, \ |y| < \infty, \ |z| < \infty\}.$$

We shall assume that $F$ and $K$ in (1.1) satisfy the following conditions:

(i) $F(x, y, z) \in C(T)$, $K(x, t, u) \in C(S)$.

* This research was supported by the National Research Council of Canada (Grant No. A-4805).

(ii) $|F(x, y, z)-F(x, \tilde{y}, z)| \leqq L_1 |y-\tilde{y}|$, all $(x, y, z), (x, \tilde{y}, z) \in T$,
$|F(x, y, z)-F(x, y, \tilde{z})| \leqq L_2 |z-\tilde{z}|$, all $(x, y, z), (x, y, \tilde{z}) \in T$.

(iii) $|K(x, t, u)-K(x, t, \tilde{u})| \leqq L_3 |u-\tilde{u}|$, all $(x, t, u), (x, t, \tilde{u}) \in S$.

It is well-known that, under these conditions, (1.1) possesses a unique solution $y(x) \in C^1[0, a]$.

Equation (1.1) may be regarded as a generalization of the initial value problem for an ordinary differential equation,

$$y'(x)=f(x, y(x)), \quad y(0)=y_0. \tag{1.2}$$

Linz [4] has proposed algorithms for the numerical solution of (1.1) which consist of linear multistep methods, of the type commonly used for the numerical solution of (1.2), combined with a class of quadrature formulae. The convergence of such algorithms has been studied in [4] and by Mocarsky [5].

It is the purpose of this paper to examine the order of such algorithms and to develop a theory of weak stability, analogous to that which exists for linear multistep methods applied to equations of the form (1.2) (see, for example, Lambert [3], p. 64). Such a theory will enable the user to make a sensible choice for the steplength of the algorithm. In addition, a new starting procedure is proposed.

Let $x_n=nh$, $n=0, 1, \ldots, N$ $(h>0, x_N=a)$. Let $y_n, z_n$ denote approximations to the exact values $y(x_n)$, $z(x_n)$, respectively. We indicate by $(\varrho, \sigma; \mathcal{Q})$ the application to (1.1) of a linear multistep method $(\varrho, \sigma)$ and a class of appropriate quadrature formulae $\mathcal{Q}$ in the following manner:

$(\varrho, \sigma)$:

$$\sum_{\nu=0}^{k} \alpha_\nu y_{n+\nu}=h \sum_{\nu=0}^{k} \beta_\nu F(x_{n+\nu}, y_{n+\nu}, z_{n+\nu}), \tag{1.3}$$

where

$$\varrho(r):=\sum_{\nu=0}^{k} \alpha_\nu r^\nu, \quad \sigma(r):=\sum_{\nu=0}^{k} \beta_\nu r^\nu, \quad \alpha_\kappa=1, \quad |\alpha_0|+|\beta_0| \neq 0;$$

$Q_m \in \mathcal{Q}$:

$$z_m=h \sum_{j=0}^{m} \gamma_{m,j} K(x_m, x_j, y_j) \quad (\text{with } z_0=0). \tag{1.4}$$

In practice, the quadrature stage is nearly always accomplished by one of the following two methods:

Method I: $\mathcal{Q}$ consists of repeated applications of a $\tilde{k}$-step quadrature formula of the form

$$\int_0^{x_{\tilde{k}}} \phi(t)\, dt \approx h \sum_{\nu=0}^{\tilde{k}} \tilde{\beta}_\nu \phi(t_\nu). \tag{1.5}$$

Method II: $\mathcal{Q}$ consists of a single application of a Gregory quadrature formula of the form

$$\int_0^{x_m} \phi(t)\,dt \approx h\left[\tfrac{1}{2}\phi_0+\phi_1+\ldots+\phi_{m-1}+\tfrac{1}{2}\phi_m\right] - h\left[c_1(\nabla\phi_m-\Delta\phi_0)+c_2(\nabla^2\phi_m+\Delta^2\phi_0)+\ldots + c_q(\nabla^q\phi_m+(-1)^q\Delta^q\phi_0)\right], \tag{1.6}$$

where $m \geqq q$. Here, the first three values for $c_q$ are given by

$$c_1 = 1/12,\ c_2 = 1/24,\ c_3 = 19/720 \tag{1.7}$$

(see, for example, Steinberg [6]).

These quadrature formulae are implemented as follows. In addition to the starting values $\{y_0, y_1, \ldots, y_{k-1}\}$ required for the application of (1.3), it will be necessary to obtain a sufficient number of starting values $\{y_\mu\}$ ($\mu$ not necessarily an integer) in order that

$$\int_0^{x_v} K(x_m, t, y(t))\,dt \approx J(v, m),\ m=1,\ldots,N; \qquad v = \begin{cases} 1,\ldots,\tilde{k}-1 & \text{(Method I)} \\ 1,\ldots,q-1 & \text{(Method II)} \end{cases} \tag{1.8}$$

can be computed to adequate accuracy. (A starting procedure having these properties will be given in section 5.)

Method I: For $m \leqq \tilde{k}-1$, set $z_m = J(m, m)$. For $m \geqq \tilde{k}$, let $m \equiv s(m) \bmod \tilde{k}$, and set

$$z_m = J(s(m), m) + h \sum_{i=0}^{M-1} \sum_{\nu=0}^{\tilde{k}} \tilde{\beta}_\nu K(x_m, x_{i(m)}, y_{i(m)}), \tag{1.9}$$

where $i(m) = s(m) + i\tilde{k} + \nu$, $M = (m - s(m))/\tilde{k}$.

Method II: For $m \leqq q-1$, set $z_m = J(m, m)$.

For $m \geqq q$, set $z_m$ equal to the right-hand side of (1.6), with $\phi_i$ replaced by $K(x_m, x_i, y_i)$, $i=0,\ldots,m$. (1.10)

## 2. The Order of the Multistep Method

Let $\mathcal{Q} = \{Q_j : j = l, \ldots, N\}$, $l \geqq 1$, denote a class of quadrature formulae corresponding to method I or II of Section 1:

Method I:

$$Q_m \in \mathcal{Q}:\ z(x_m) \to Z(x_m) := \hat{J}(s(m), m) + h \sum_{i=0}^{M-1} \sum_{\nu=0}^{\tilde{k}} \tilde{\beta}_\nu\, K(x_m, x_{i(m)},\ y(x_{i(m)})),$$

$i(m) = s(m) + i\tilde{k} + \nu$, $m \geqq \tilde{k} = l$. Here, $\hat{J}(s(m), m)$ is as in (1.8), but with the approximate values $\{y_\mu\}$ replaced by $\{y(x_\mu)\}$.

Method II:

$$Q_m \in \mathcal{Q}:\ z(x_m) \to Z(x_m) := h \sum_{i=0}^{m} \gamma_{m,i}\, K(x_m, x_i, y(x_i)),$$

$m \geqq q = l$. Here, the $\{\gamma_{m,i}\}$ denote the weights of the Gregory formula (1.6).

**Definition:** The class $\mathcal{Q}$ is said to be of order $q^*$ if $q^*$ is the largest integer such that

$$|E_m| = |E_m(K)| := |z(x_m) - Z(x_m)| \leqq C h^{q^*} \quad (C < \infty),$$

for all $Q_m \in \mathcal{Q}$ and for all sufficiently smooth $K$.

We note for later reference that, for method I, $\mathcal{Q}$ is of order $q^* = \tilde{k} + 2$ if (1.5) is interpolatory (see, for example, [1], p. 307), and if

$$\left| \int_0^{x_v} K(x_m, t, y(t))\, dt - \hat{J}(s(m), m) \right| \leqq C_1 h^{q^*} \quad (C_1 < \infty), \tag{2.1}$$

$v = 1, \ldots, \tilde{k} - 1$ for all sufficiently smooth $K$. In section 5 we shall discuss how to generate the starting values $J(v, m)$ required in (1.7) for which (2.1) holds.

For method II the order of $\mathcal{Q}$ is $q^* = q + 2$ (see [6]).

The order of the linear $k$-step method $(\varrho, \sigma; \mathcal{Q})$ will obviously depend on the order of $(\varrho, \sigma)$ as well as on the order of $\mathcal{Q}$. To be precise, we state the following

**Definition:**

(i) The difference operators $\mathcal{L}$ and $\mathcal{M}$ associated with $(\varrho, \sigma; \mathcal{Q})$ are given by

$$\mathcal{L}[y(x_m); h] := \sum_{v=0}^{k} (\alpha_v y(x_{m+v}) - h \beta_v y'(x_{m+v})) \qquad (m = 0, \ldots, N-k),$$

and

$$\mathcal{M}[y(x_m); h] := \sum_{v=0}^{k} (\alpha_v y(x_{m+v}) - h \beta_v F(x_{m+v}, y(x_{m+v}), Z(x_{m+v}))).$$

(ii) The order of $\mathcal{L}$ is defined as the order of $(\varrho, \sigma)$ for (1.2) (see [3], p. 23).

It is easily verified that, for all sufficiently smooth functions $F$, the operators $\mathcal{L}$ and $\mathcal{M}$ are related by

$$\mathcal{M}[y(x_m); h] = \mathcal{L}[y(x_m); h] + h \sum_{v=0}^{k} \beta_v \frac{\partial F(x_{m+v}, y(x_{m+v}), z^*_{m+v})}{\partial z} E_{m+v}$$

$$(m = l, \ldots, N-k),$$

where $z^*_{m+v}$ lies between $z(x_{m+v})$ and $Z(x_{m+v})$.

The following definition is now suggestive.

**Definition:** Let $\mathcal{L}$ be of order $p^*$, and let $\mathcal{Q}$ have order $q^*$. Then we define the order $r^*$ of the linear $k$-step method $(\varrho, \sigma; \mathcal{Q})$ (or, the order of the operator $\mathcal{M}$) by $r^* = \min(p^*, q^*)$.

**Definition:** The linear $k$-step method $(\varrho, \sigma; \mathcal{Q})$ is said to be convergent if, for all equations (1.1) subject to the conditions stated in section 1, we have that

$\lim_{\substack{h\to 0 \\ mh=x}} y_m = y(x)$ holds for all $x \in [0, a]$, and for all solutions $\{y_m\}$ of (1.3) satisfying starting conditions $y_j = y_j(h)$ for which $\lim_{h\to 0} y_j(h) = y_0$, $j = 0, \ldots, k-1$, and $J(v, v)$ for which $\lim_{h\to 0} J(v, v) = 0$, $v = 1, \ldots, l-1$.

Here, the last condition essentially requires that the weights in the quadrature formulae used to compute the starting values $\{J(v, v)\}$ remain bounded as $h \to 0$.

**Theorem 1:** *Assume*:

(*i*) *F and K in* (1.1) *satisfy the conditions* (*i*), (*ii*), *and* (*iii*) *of section* 1.

(*ii*) *The k-step method* $(\varrho, \sigma)$ *is zero-stable* (*see* [3], *p.* 33).

(*iii*) *The k-step method* $(\varrho, \sigma; \mathcal{Q})$ *is of at least order one.*

*Then the method* $(\varrho, \sigma; \mathcal{Q})$ *is convergent.*

The above result was proved in [4] (using a different terminology) for methods essentially of Class II (see also [5]). A simple modification of this proof shows that Theorem 1 is valid also for methods of Class I. We shall omit these details.

## 3. A Weak Stability Theory

A standard argument in weak stability theory for the linear multistep method $(\varrho, \sigma)$ applied to the ordinary differential equation (1.2) (see, for example, [3], p. 64) shows that, under the simplifying conditions that

(i) $\partial f / \partial y = \lambda = \text{constant}$, and

(ii) the local truncation error of $(\varrho, \sigma)$ is constant,
the global error $e_n := y(x_n) - y_n$ satisfies the linearized equation

$$\sum_{v=0}^{k} (\alpha_v - h\lambda\beta_v)\, e_{n+v} = \text{constant}.$$

By a straightforward extension of this argument, it is easily shown that if the $k$-step method $(\varrho, \sigma; \mathcal{Q})$, defined by (1.3) and (1.4), is applied to the equation (1.1), and it is assumed that

(i) $\dfrac{\partial F}{\partial y} = \xi = \text{constant}, \quad \dfrac{\partial F}{\partial z} \cdot \dfrac{\partial K}{\partial y} = \eta = \text{constant},$ (3.1)
and

(ii) the local truncation errors of $(\varrho, \sigma)$ and of $Q_n \in \mathcal{Q}$ are constant,

then the global error $e_n$ satisfies the linearized difference equation

$$\sum_{v=0}^{k} \left[ (\alpha_v - h\xi\beta_v)\, e_{n+v} - h^2 \eta \sum_{j=0}^{n+v} \gamma_{n+v,\,j}\, e_j \right] = \text{constant}. \tag{3.2}$$

Since the behaviour of solutions of (3.2) as $n\to\infty$ is completely determined by the behaviour of solutions of the corresponding homogeneous equation, it is sufficient to study the equation

$$\sum_{\nu=0}^{k}\left[(\alpha_\nu - h\,\xi\,\beta_\nu)\,y_{n+\nu} - h^2\,\eta\sum_{j=0}^{n+\nu}\gamma_{n+\nu,j}\,y_j\right]=0. \tag{3.3}$$

The following definitions are analogues of standard definitions in the case of ordinary differential equations (see [3], p. 66).

**Definition:** For given values of $h$, $\xi$, $\eta$, the method $(\varrho, \sigma; \mathcal{L})$ is said to be absolutely stable if all solutions of (3.3) tend to zero as $n\to\infty$.

**Definition:** A region $\mathcal{R}$ of the $(h\,\xi, h^2\,\eta)$-plane is said to be a region of absolute stability of the method $(\varrho, \sigma; \mathcal{L})$ if, for all $(h\,\xi, h^2\,\eta)\in\mathcal{R}$, the method $(\varrho, \sigma; \mathcal{L})$ is absolutely stable.

In practical applications, once the region $\mathcal{R}$ for a particular method has been established, estimates for $\xi$ and $\eta$ are computed from (3.1) (such estimates can be re-evaluated from time to time as the numerical solution proceeds), and $h$ is chosen such that $(h\,\xi, h^2\,\eta)\in\mathcal{R}$. This ensures that the global error $e_n$, assumed to satisfy (3.3), will decay as $n\to\infty$.

It is readily seen that equation (3.3) arises if the method $(\varrho, \sigma; \mathcal{L})$ is applied to the linear test equation

$$y'(x)=\xi\,y(x)+\eta\int_0^x y(t)\,dt. \tag{3.4}$$

All solutions of (3.4) tend to zero as $x\to\infty$ if and only if

$$\xi<0,\ \eta<0. \tag{3.5}$$

The following definition is again an analogue of a standard definition for ordinary differential equations (see [3], p. 253):

**Definition:** The method $(\varrho, \sigma; \mathcal{L})$ is said to be $A$-stable if its region $\mathcal{R}$ of absolute stability contains the quarter plane $h\,\xi<0$, $h^2\,\eta<0$.

All solutions of the difference equation, which results from the application of an $A$-stable method $(\varrho, \sigma; \mathcal{L})$ to the test equation (3.4) for which (3.5) holds, will tend to zero as $x\to\infty$, for all positive values of $h$.

It will occasionally be convenient to employ the following alternative form of (3.4), obtained by setting $\xi=\lambda+\mu$, $\eta=-\lambda\,\mu$:

$$y'(x)=(\lambda+\mu)\,y(x)-\lambda\,\mu\int_0^x y(t)\,dt. \tag{3.6}$$

All solutions of (3.6) tend to zero as $x\to\infty$ if and only if

$$\begin{aligned}&\lambda<0,\ \mu<0, \text{ when } \lambda \text{ and } \mu \text{ are real,}\\ &Re\,(\lambda)<0, \text{ when } \lambda\,(=\bar{\mu}) \text{ is complex.}\end{aligned} \tag{3.7}$$

Indeed, the unique solution of (3.6) satisfying $y(0)=y_0$ is given by

$$y(x)=\frac{y_0}{\lambda-\mu}(\lambda \exp(\lambda x)-\mu \exp(\mu x)). \tag{3.8}$$

When method $(\varrho, \sigma; 2)$ is applied to the test equation (3.4), equation (1.3) becomes

$$\sum_{\nu=0}^{k} \alpha_\nu y_{n+\nu}=h \sum_{\nu=0}^{k} \beta_\nu(\xi y_{n+\nu}+\eta z_{n+\nu}). \tag{3.9}$$

For Method I, $m \geqq \tilde{k}$, equation (1.4) becomes (using (1.9)),

$$z_m=J(s(m), m)+h \sum_{i=0}^{M-1} \sum_{\nu=0}^{\tilde{k}} \tilde{\beta}_\nu y_{i(m)}, \quad i(m)=s(m)+i\tilde{k}+\nu.$$

Since $s(m+\tilde{k})=s(m)$, it follows that

$$z_{m+\tilde{k}}-z_m=h \sum_{\nu=0}^{\tilde{k}} \tilde{\beta}_\nu y_{m+\nu}. \tag{3.10}$$

For Method II, $m \geqq q$, equation (1.4) becomes (using (1.10) and (1.6)),

$$z_m=h[\tfrac{1}{2} y_0+y_1+\ldots+y_{m-1}+\tfrac{1}{2} y_m]$$
$$-h[c_1(\nabla y_m-\Delta y_0)+c_2(\nabla^2 y_m+\Delta^2 y_0)+\ldots+c_q(\nabla^q y_m+(-1)^q \Delta^q y_0)].$$

Hence,

$$z_{m+1}-z_m=h[1-\tfrac{1}{2}\nabla-c_1 \nabla^2-c_2 \nabla^3-\ldots-c_q \nabla^{q+1}] y_{m+1}.$$

On replacing $m$ by $m+q$ and expressing differences in terms of function values, we may write this relation in the form

$$z_{m+q+1}-z_{m+q}=h \sum_{\nu=0}^{q+1} \hat{\beta}_\nu y_{m+\nu}, \tag{3.11}$$

where the coefficients $\{\hat{\beta}_\nu: \nu=0, \ldots, q+1\}$ can easily be computed from the known values of $c_1, c_2, \ldots, c_q$. Indeed, the coefficients $\{\hat{\beta}_\nu\}$ are precisely those appearing on the right-hand side of the $(q+1)$-step Adams-Moulton method of order $q+2$,

$$y_{n+q+1}-y_{n+q}=h \sum_{\nu=0}^{q+1} \hat{\beta}_\nu f_{n+\nu}$$

for the numerical solution of (1.2) (see equation (1.7)). Since Methods I and II are alternatives, there is no loss of generality in setting $q+1=\tilde{k}$, $\hat{\beta}_\nu=\tilde{\beta}_\nu$, $\nu=0,1,\ldots,\tilde{k}$, whereupon, on introducing the forward shift operator $E$, (3.10) and (3.11) can be combined in the single equation

$$\tilde{\varrho}(E) z_m=h \tilde{\sigma}(E) y_m, \quad m \geqq \tilde{k}, \tag{3.12}$$

where

$$\tilde{\varrho}(r):=\begin{cases} r^{\tilde{k}}-1 & \text{for Method I,} \\ r^{\tilde{k}}-r^{\tilde{k}-1} & \text{for Method II} \end{cases}; \quad \tilde{\sigma}(r):=\sum_{\nu=0}^{\tilde{k}} \tilde{\beta}_\nu r^\nu. \tag{3.13}$$

In Method I, $\tilde{\sigma}$ will be determined by the $k$-step quadrature formula, whereas in Method II, $\tilde{\sigma}$ will always be the second characteristic polynomial of a $\tilde{k}$-step

Adams-Moulton method of order $\tilde{k}+1$. Note that the two methods coincide if $\tilde{k}=1$ and, in Method I, the quadrature formula is chosen to be the Trapezoidal Rule.

To any method $(\varrho, \sigma; \mathcal{Q})$ for the numerical solution of (1.1) there correspond unique polynomials $\varrho$, $\sigma$, $\tilde{\varrho}$, and $\tilde{\sigma}$. We may therefore refer to such a method as $[(\varrho, \sigma); (\tilde{\varrho}, \tilde{\sigma})]$.

Equation (3.9) can also be written in the form

$$\varrho(E)\, y_n = h\,\sigma(E)\cdot(\xi\, y_n + \eta\, z_n). \tag{3.14}$$

If we eliminate $z_n$ between (3.12) and (3.14), we obtain the difference equation

$$[\tilde{\varrho}(E)\,\varrho(E) - h\,\xi\,\tilde{\varrho}(E)\,\sigma(E) - h^2\,\eta\,\tilde{\sigma}(E)\,\sigma(E)]\, y_n = 0.$$

It follows that the region $\mathcal{R}$ of absolute stability of $[(\varrho, \sigma); (\tilde{\varrho}, \tilde{\sigma})]$ is the set of points $(h\,\xi, h^2\,\eta)$ for which all zeros of the stability polynomial

$$\pi(r, h\,\xi, h^2\,\eta) := \tilde{\varrho}(r)\,[\varrho(r) - h\,\xi\,\sigma(r)] - h^2\,\eta\,\tilde{\sigma}(r)\,\sigma(r) \tag{3.15}$$

lie in the interior of the unit disk.

By considering the alternative form (3.6) of the test equation it is possible to establish a relationship between the region $\mathcal{R}$ of absolute stability of $[(\varrho, \sigma); (\varrho, \sigma)]$ and the complex region $\hat{\mathcal{R}}$ of absolute stability of the linear multistep method $(\varrho, \sigma)$ (see, for example, [3], p. 222).

**Theorem 2:** *Let $\hat{\mathcal{R}}$ be the (complex) region of absolute stability of the linear multistep method $(\varrho, \sigma)$ for* (1.2).
*Then the region of absolute stability of $[(\varrho, \sigma); (\varrho, \sigma)]$ is that region of the $(h\,\xi, h^2\,\eta)$-plane for which $h\,\lambda \in \hat{\mathcal{R}}$, $h\,\mu \in \hat{\mathcal{R}}$, where $\lambda + \mu = \xi$, $\lambda\,\mu = -\eta$; $\lambda, \mu \in \mathbb{C}$.*

*Proof*: On setting $\tilde{\varrho} = \varrho$, $\tilde{\sigma} = \sigma$, $\xi = \lambda + \mu$, $\eta = -\lambda\,\mu$, (3.15) takes the form

$$\begin{aligned}\pi(r, h\,\xi, h^2\,\eta) &= \varrho^2(r) - h\,(\lambda+\mu)\,\varrho(r)\,\sigma(r) + h^2\,\lambda\,\mu\,\sigma^2(r)\\ &= [\varrho(r) - h\,\lambda\,\sigma(r)]\,[\varrho(r) - h\,\mu\,\sigma(r)].\end{aligned}$$

The result follows immediately, since for $h\,\lambda \in \hat{\mathcal{R}}$, $h\,\mu \in \hat{\mathcal{R}}$, all zeros of $\varrho(r) - h\lambda\sigma(r)$ and of $\varrho(r) - h\,\mu\,\sigma(r)$ lie inside the unit disk.

Since $\tilde{\varrho}$ is constrained by (3.13), Theorem 2 is applicable only to a subclass of methods $(\varrho, \sigma; \mathcal{Q})$. It is of interest to note that, in view of the remarks following (3.11), this subclass contains the class of methods for which $(\varrho, \sigma)$ is an Adams-Moulton method, and the quadrature formula is a Gregory formula of appropriate order.

It is an immediate consequence of Theorem 2 that, if the linear multistep method $(\varrho, \sigma)$ is $A$-stable and is such that $\varrho\,(=\tilde{\varrho})$ is of the form (3.13), then the method $[(\varrho, \sigma); (\varrho, \sigma)]$ is also $A$-stable. The most obvious example of an $A$-stable method for (1.1) arises when $(\varrho, \sigma)$ and $(\varrho, \sigma)$ are both the Trapezoidal Rule (in which case Methods I and II are identical).

The effects of the individual stability properties of $(\varrho, \sigma)$ and $(\tilde{\varrho}, \tilde{\sigma})$ on the stability properties of $[(\varrho, \sigma); (\tilde{\varrho}, \tilde{\sigma})]$ can be illustrated, in the case $k = \tilde{k} = 1$, by plotting the region $\mathscr{R}$ of absolute stability of $[(\varrho, \sigma); (\tilde{\varrho}, \tilde{\sigma})]$ for all nine combinations of:

$$(\varrho, \sigma) = \begin{cases} \varrho = r - 1, \sigma = 1 \text{ (Euler's Rule)} \\ \varrho = r - 1, \sigma = \frac{1}{2}(r+1) \text{ (Trapezoidal Rule)} \\ \varrho = r - 1, \sigma = r \text{ (Backward Euler Rule)}, \end{cases}$$

$$\mathscr{Q} = \begin{cases} \text{repeated application of Euler's Rule (open formula)} \\ \text{repeated application of Trapezoidal Rule (closed formula)} \\ \text{repeated application of Backward Euler Rule (open formula).} \end{cases}$$

Note that the repeated application of the Trapezoidal Rule is equivalent to one application of the second-order Gregory formula (Method II).

The results are summarized in Fig. 1.

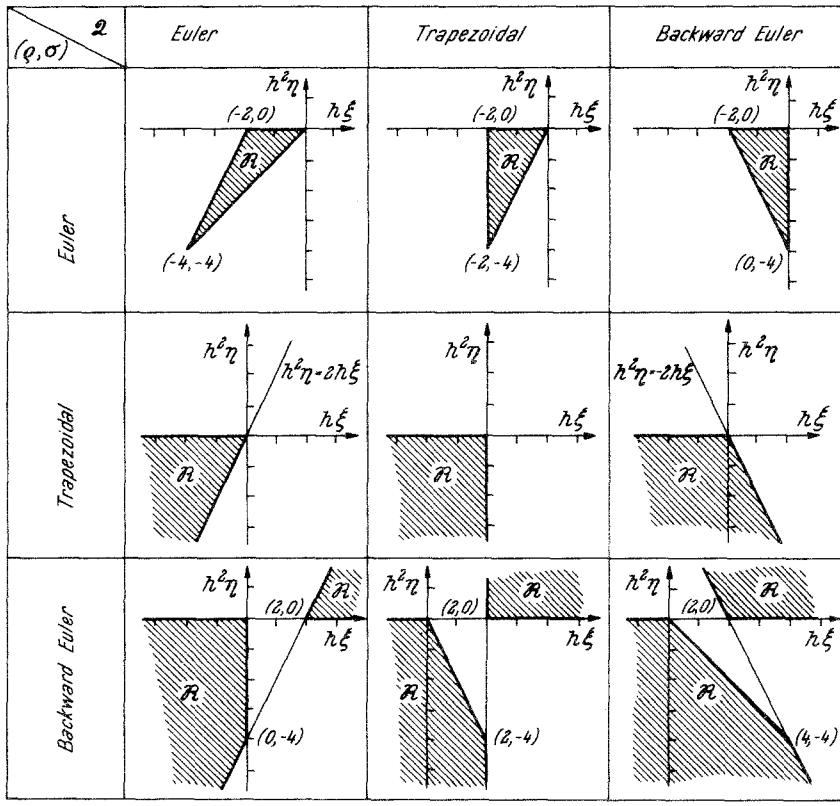


Fig. 1

The (complex) stability regions $\hat{\mathscr{R}}$ of Euler's Rule, the Trapezoidal Rule, and the Backward Euler Rule are, respectively, the interior of the circle with centre at $-1$, radius 1, the whole left half-plane, and the exterior of the circle with centre

at $+1$, radius 1. A straightforward calculation shows that, if $h\lambda$ and $h\mu$ lie in one of these regions, then $(h(\lambda+\mu), -h^2\lambda\mu)$ lies in the corresponding region $\mathscr{R}$ shown on the main diagonal of Fig. 1; thus Theorem 2 is corroborated. It is also of interest to note from Fig. 1 that only combinations of $A$-stable $(\varrho, \sigma)$ with $A$-stable $(\tilde{\varrho}, \tilde{\sigma})$ result in $A$-stable methods $[(\varrho, \sigma); (\tilde{\varrho}, \tilde{\sigma})]$.

The case where $(\varrho, \sigma)$ and $(\tilde{\varrho}, \tilde{\sigma})$ are both Euler's Rule is of particular interest. Let $\xi<0$ and $\eta<0$; then all solutions of (3.4) tend to zero as $x\to\infty$. From the slope of the relevant stability region $\mathscr{R}$ shown in Fig. 1, it is clear that there exist values of $h$ such that the method $[(\varrho, \sigma); (\varrho, \sigma)]$, with $(\varrho, \sigma)$ given by Euler's Rule, will produce stable numerical solutions, whereas, for the same value of $h\xi$, Euler's Rule, applied to the ordinary differential equation $y'=\xi y$, will produce unstable solutions. Thus, in this case, the back information inherent in a Volterra integro-differential equation has a stabilizing effect.

## 4. A Starting Procedure

If $k\geqq 2$, $l\geqq 2$ in $(\varrho, \sigma; \mathscr{Q})$, it will become necessary, as pointed out in section 1 and section 2, to generate the following sets of starting values: $\{y_1, \ldots, y_{k-1}\}$, and $\{J(v, v): v=1, \ldots, \tilde{k}-1\}$, $\{J(v, m): v=1, \ldots, \tilde{k}-1, m\geqq\tilde{k}\}$ for Method I; $\{J(v, v): v=1, \ldots, q-1\}$ for Method II.

In this section a starting procedure will be described which, in addition, satisfies the accuracy requirements (2.1).

Define

$$u(x):=\sum_{\nu=0}^{\omega}\frac{a_\nu}{\nu!}x^\nu \quad (\omega\geqq 1), \tag{4.1}$$

where the (real) coefficients $\{a_\nu\}$ are determined by requiring that $u(x)$ satisfy the following conditions:

(i) $u(x_0)=y_0$,

(ii) $u'(x_0)=F(x_0, y_0, 0)$,

(iii) $u'(x_j)=F(x_j, u(x_j), v(x_j))$, $j=1, \ldots, \omega-1$, (4.2)

with

$$v(x_j):=\int_0^{x_j} K(x_j, t, u(t))\,dt. \tag{4.3}$$

Clearly $a_0=y_0$, $a_1=F(x_0, y_0, 0)$, and (4.2) constitutes a system of $(\omega-1)$ non-linear equations for the remaining coefficients $\{a_2, \ldots, a_\omega\}$. In general, this system will be solved by fixed-point iteration, and it is easily verified that the well-known fixed-point principle may be used to show that, for all sufficiently small $h$ and for all functions $F$ and $K$ satisfying the conditions stated in section 1, there exists a unique solution $\{a_2, \ldots, a_\omega\}$ of (4.2). Once the coefficients in (4.1) are known, then:

(a) $y_j=u(x_j)$, $j=1, \ldots, \omega-1$,

(b) $J(v, v) = \int_0^{x_v} K(x_v, t, u(t))\, dt$, and

(c) $J(v, m) = \int_0^{x_v} K(x_m, t, u(t))\, dt$.

In (b), $v = 1, \ldots, \tilde{k} - 1$ for Method I; $v = 1, \ldots, q - 1$ for Method II.
In (c), $v = 1, \ldots, \tilde{k} - 1$; $m = \tilde{k}, \ldots, N$.

For the following discussion we shall assume that $\tilde{k} \leqq k$ (Method I), and $q \leqq k$ (Method II). To justify this assumption we first observe (as mentioned in section 2) that the order of (1.5) is at least $(\tilde{k}+2)$ if the quadrature formula is interpolatory, whereas the Gregory formula (1.6) is known to have order $(q+2)$ (see, for example, [6]). On the other hand, the order of a convergent linear $k$-step method $(\varrho, \sigma)$ cannot exceed $(k+1)$ if $k$ is odd, and $(k+2)$ if $k$ is even (see [3], p. 38).

**Theorem 3:** *Assume*:

(*i*) *In* (4.1), $a_0$ *and* $a_1$ *are given by* $a_0 = y_0$, $a_1 = F(x_0, y_0, 0)$.

(*ii*) *In the computation of* $\{a_2, \ldots, a_\omega\}$ *from* (4.2), *the occuring integrals are computed exactly.*

*Then*: *if* $F$, $K$, *and* $y$ *in* (1.1) *are sufficiently smooth, there exists a* $C < \infty$ *such that for all* $h \in (0, H_0)$ $(H_0 > 0)$:

$$|y(x) - u(x)| \leqq C h^{\omega+1}, \quad x \in [0, x_{\omega-1}], \tag{4.4}$$

$$|z(x_v) - J(v, v)| \leqq (\omega - 1)\, C\, L_3\, h^{\omega+2}, \quad v = 1, \ldots, \omega - 1, \tag{4.5}$$

*and*

$$|z_m(x_v) - J(v, m)| \leqq (\omega - 1)\, C\, L_3\, h^{\omega+2}, \quad v = 1, \ldots, \omega - 1;\ m = \omega, \ldots, N. \tag{4.6}$$

*Here*,

$$z_m(x_v) := \int_0^{x_v} K(x_m, t, y(t))\, dt.$$

*Remark*:

In view of the remarks preceding Theorem 3, the usual choice for $\omega$ will be $\omega = k$, except in the case of an optimal $(\varrho, \sigma)$ (whose order is $k+2$), when we shall set $\omega = k+1$ (assuming that the multistep method $(\varrho, \sigma; \mathcal{Z})$ is of order $k+1$ and $k+2$, respectively).

*Proof*: Let $y(x) \in C^{\omega+1}[0, a]$, and set

$$a_j = y^{(j)}(0) + b_j h^{\omega+1-j}, \quad j = 2, \ldots, \omega.$$

Hence, with $e(x) := y(x) - u(x)$, and by using (4.1), we get

$$e(x) = h^{\omega+1} \left( -\sum_{j=2}^{\omega} \frac{b_j}{j!} \left(\frac{x}{h}\right)^j + \frac{y^{(\omega+1)}(\xi_0(x))}{(\omega+1)!} \left(\frac{x}{h}\right)^{\omega+1} \right),$$

and

$$e'(x)=h^{\omega}\left(-\sum_{j=2}^{\omega}\frac{b_j}{(j-1)!}\left(\frac{x}{h}\right)^{j-1}+\frac{y^{(\omega+1)}(\xi_1(x))}{\omega!}\left(\frac{x}{h}\right)^{\omega}\right),$$

with $0<\xi_0(x)<x$, $0<\xi_1(x)<x$.

Here we have used $e(0)=0$, $e'(0)=0$.

We shall first show that the coefficients $\{b_j: j=2,\ldots,\omega\}$ are uniformly bounded for $h\in(0,H_0)$, $H_0>0$. Clearly, by (4.2),

$$e'(x_v)=F(x_v, y(x_v), z(x_v))-F(x_v, u(x_v), v(x_v))=$$

$$=e(x_v)\frac{\partial}{\partial y}F(x_v, y_v^*, z_v^*)+\Delta(x_v)\frac{\partial}{\partial z}F(x_v, y_v^*, z_v^*),\ v=1,\ldots,\omega-1,$$

with

$$\Delta(x_v):=\int_0^{x_v} e(t)\frac{\partial}{\partial y}K(x_v, t, y^*(t))\,dt.$$

Here, $y_v^*$ is between $y(x_v)$ and $u(x_v)$, $y^*(t)$ is between $y(t)$ and $u(t)$. Using the above expression for $e(x)$ and $e'(x)$ we obtain (after division by $h^{\omega}$),

$$\sum_{j=2}^{\omega}\frac{b_j}{(j-1)!}v^{j-1}=\frac{y^{(\omega+1)}(\xi_1(x_v))}{\omega!}v^{\omega}-h\left[\frac{\partial}{\partial y}F(x_v, y_v^*, z_v^*)\sum_{j=2}^{\omega}\frac{b_j}{j!}v^j-\right.$$

$$\left.-\frac{\partial}{\partial z}F(x_v, y_v^*, z_v^*)\int_0^{x_v}\frac{\partial}{\partial y}K(x_v, t, y^*(t))\sum_{j=2}^{\omega}\frac{b_j}{j!}\left(\frac{t}{h}\right)^j dt\right]+O(h),\ v=1,\ldots,\omega-1. \tag{4.7}$$

If $\{b_2^{(0)},\ldots,b_{\omega}^{(0)}\}$ denotes the (unique) solution of the linear system

$$\sum_{j=2}^{\omega}\frac{b_j}{(j-1)!}v^{j-1}=\frac{y^{(\omega+1)}(\xi_1(x_v))}{\omega!}v^{\omega},\ v=1,\ldots,\omega-1,$$

then, by continuity, there exists a $H_0>0$ such that, for all $h\in(0,H_0)$, the system (4.7) possesses a unique solution $\{b_2,\ldots,b_{\omega}\}$ for which

$$b_j=b_j^{(0)}+O(h),\ j=2,\ldots,\omega.$$

Hence,

$$|e(x)|\leqq(B+M_{\omega+1})h^{\omega+1},\ x\in[0,x_{\omega-1}],$$

where

$$B:=\sum_{j=2}^{\omega}\frac{|b_j|}{j!}(\omega-1)^j<\infty,\ h\in(0,H_0),$$

$$M_{\omega+1}:=\max_{x\in[0,x_{\omega-1}]}|y^{(\omega+1)}(x)|\frac{(\omega-1)^{\omega+1}}{(\omega+1)!}.$$

The assertions (4.5) and (4.6) of Theorem 3 now follow from the Lipschitz condition on $K$:

$$|z(x_v)-J(v,v)|\leqq\int_0^{x_v}|K(x_v, t, y(t))\,dt-K(x_v, t, u(t))|\,dt$$

$$\leqq C L_3 h^{\omega+2}(\omega-1),\ v=1,\ldots,\omega-1,$$

and similarly,

$$|z_m(x_v) - J(v,m)| \leqq C L_3 h^{\omega+2}(\omega-1),\ v=1,\ldots,\omega-1;\ m=\omega,\ldots,N.$$

Here, $L_3$ denotes the Lipschitz constant for $K$, as defined in section 1. This completes the proof of Theorem 3.

In practical applications, the integrals occuring in (4.2) as well as the integrals $J(v,v)$ and $J(v,m)$ will not be computed exactly but will be approximated by numerical quadrature. An argument similar to the one used above shows that Theorem 3 remains valid (with an appropriate modification of $C$ and $H_0$) if the quadrature formulae used to evaluate these integrals have at least order $\omega$. We shall omit the corresponding proof.

## 5. Practical Applications

For methods of reasonably high order, the derivation of stability regions in the $(h\xi,\ h^2\eta)$-plane becomes prohibitively complicated. It is, of course, feasible to calculate estimates for $\xi$ and $\eta$ from the numerical solution, and to check computationally whether, with the corresponding values of $h\xi$ and $h^2\eta$, the zeros of the stability polynomial lie within the unit disk. However, such an approach leads to a somewhat complicated algorithm. A much simpler procedure can be devised if we choose methods which satisfy the hypotheses of Theorem 2, that is, methods $[(\varrho,\sigma);(\varrho,\sigma)]$ for which $\varrho\,(=\tilde{\varrho})$ satisfies (3.13). In particular, such a method always results if we choose the linear multistep method to be an Adams-Moulton method, and the quadrature formula to be a Gregory formula of appropriate order. From time to time, as the numerical solution proceeds, estimates for $\partial F/\partial y$ and $(\partial F/\partial z)(\partial K/\partial y)$ can be computed, whence values of $\xi$ and $\eta$ are obtained from (3.1). A simple program allows $\lambda$ and $\mu$ to be calculated from the relations $\lambda+\mu=\xi,\ \lambda\mu=-\eta$.

For absolute stability, the steplength $h$ must be such that $h\lambda$ and $h\mu$ both lie within the (complex) region of absolute stability of the method $(\varrho,\sigma)$ for ordinary differential equations. Not only are such regions more easily computed than are the corresponding regions in the $(h\xi,\ h^2\eta)$-plane (see [3], p. 77), but the existing literature already contains much information on such regions. In the particular case when $(\varrho,\sigma)$ is an Adams-Moulton method, plots of the regions for methods of order 4, 5, 6, 7, and 8 may be found in Krogh [2].

As an illustration we consider the integro-differential equation

$$y'(x) = g(x) + f(x)\cdot\frac{1}{y(x)} + \int_0^x \frac{dt}{1+(1+x)\,y(t)},\quad 0\leqq x\leqq 10,$$

with

$$y(0)=1,$$

$$g(x) = -\frac{1+x(1+x)^2}{(1+x)^2},\quad f(x)=\log\left(\frac{2+2x}{2+x}\right).$$

Its exact solution is given by $y(x)=1/(1+x)$.

Here, we have

$$\frac{\partial F}{\partial y} = -f(x) \cdot \frac{1}{(y)^2} < 0 \quad \text{for } x > 0,$$

and

$$\frac{\partial F}{\partial z} \cdot \frac{\partial K}{\partial y} = -\frac{1+x}{(1+(1+x)\,y)^2} < 0, \quad x \geqq 0 \,(y \geqq 0).$$

The numerical solution was computed by using the Adams-Moulton method of order four,

$$y_{n+1} - y_n = \tfrac{h}{24}\,(9\,F_{n+1} + 19\,F_n - 5\,F_{n-1} + F_{n-2}),$$

combined with the Gregory formula (1.6) with $q=2$. The required starting values were computed by means of the starting procedure described in section 4, with $\omega = 3$ in (4.1).

A selection of numerical results is contained in Table 1.

*Table 1*

| $x_n$ | $e_n = y(x_n) - y_n$ $h=0.1$ | $e_n$ $h=0.2$ | $e_n$ $h=0.4$ |
|---|---|---|---|
| 0.0 | 0. | 0. | 0. |
| 0.1 | $6.67 \cdot 10^{-5}$ | | |
| 0.2 | $6.93 \cdot 10^{-6}$ | $7.42 \cdot 10^{-4}$ | |
| 0.3 | $2.03 \cdot 10^{-5}$ | | |
| 0.4 | $3.05 \cdot 10^{-5}$ | $1.15 \cdot 10^{-4}$ | $6.15 \cdot 10^{-3}$ |
| 0.8 | $4.04 \cdot 10^{-5}$ | $3.55 \cdot 10^{-4}$ | $7.06 \cdot 10^{-4}$ |
| 1.2 | $2.87 \cdot 10^{-5}$ | $2.68 \cdot 10^{-4}$ | $1.82 \cdot 10^{-3}$ |
| ⋮ | | | |
| 4.8 | $-5.60 \cdot 10^{-7}$ | $-5.39 \cdot 10^{-6}$ | $-4.02 \cdot 10^{-5}$ |
| 5.2 | $-4.56 \cdot 10^{-7}$ | $-4.37 \cdot 10^{-6}$ | $-3.67 \cdot 10^{-5}$ |
| ⋮ | | | |
| 7.2 | $-1.90 \cdot 10^{-7}$ | $-1.78 \cdot 10^{-6}$ | $1.42 \cdot 10^{-5}$ |
| 7.6 | $-1.63 \cdot 10^{-7}$ | $-1.52 \cdot 10^{-6}$ | $-6.14 \cdot 10^{-5}$ |
| 8.0 | $-1.41 \cdot 10^{-7}$ | $-1.31 \cdot 10^{-6}$ | $7.90 \cdot 10^{-5}$ |
| 8.4 | $-1.23 \cdot 10^{-7}$ | $-1.14 \cdot 10^{-6}$ | $-1.73 \cdot 10^{-4}$ |
| ⋮ | | | |
| 9.6 | $-8.25 \cdot 10^{-8}$ | $-7.58 \cdot 10^{-7}$ | $1.11 \cdot 10^{-3}$ |
| 10.0 | $-7.29 \cdot 10^{-8}$ | $-6.67 \cdot 10^{-7}$ | $-2.26 \cdot 10^{-3}$ |

We observe that $\xi := \partial F/\partial y$ (computed along the exact solution of (5.1)) behaves asymptotically like $-(1+x)^2$, whereas $\eta := (\partial F/\partial z)\,(\partial K/\partial y)$ changes more slowly; it behaves like $-(1+x)$. In all cases, the magnitudes of $\xi$ and $\eta$, computed from the numerical solutions, increase monotonically with $x$. The region of absolute stability of the fourth-order Adams-Moulton method has an intercept of $(-3, 0)$ on the real axis (see [2]). In the case when $h=0.4$, for $x=0.0$, 0.4, 0.8, ..., 2.8, the computed values of $h\,\xi$ and $h^2\,\eta$ are such that $h\,\lambda$ and $h\,\mu$ lie within this region. However, for $x=3.2$, we find that $\xi \approx -8.46$, $\eta \approx -1.05$, whence

$h\lambda \approx -3.33$, $h\mu \approx -0.05$, indicating that a reduction of steplength should be made at this point. For the numerical results displayed in Table 1, no such reduction was made, and, indeed, it can be seen that for $h=0.4$, the error oscillates and eventually grows in magnitude. Similarly, the stability analysis recommends a reduction of steplength at $x=4.4$ and $x=6.3$ in the cases $h=0.2$ and $h=0.1$, respectively. Further computation indicated that the error does indeed begin to grow in the neighbourhood of $x=10.8$ and $x=11.5$, respectively. For this highly nonlinear example, the bounds on the steplength recommended by the (linearized) stability analysis are seen to be rather conservative in practice. That the same is frequently true of stability analysis for ordinary differential equations is well-known.

**Acknowledgement**

The work of this paper was carried out while the second author was visiting the Department of Mathematics, Dalhousie University, during April and May 1973.

**References**

[1] Isaacson, E., and H. B. Keller: Analysis of Numerical Methods. New York: Wiley. 1966.
[2] Krogh, F. T.: Predictor-corrector methods of high order with improved stability characteristics. J. Assoc. Comput. Mach. **13,** 374 (1966).
[3] Lambert, J. D.: Computational Methods in Ordinary Differential Equations. London: Wiley. 1973.
[4] Linz, P.: Linear multistep methods for Volterra integro-differential equations. J. Assoc. Comput. Mach. **16,** 295 (1969).
[5] Mocarsky, W. L.: Convergence of step-by-step methods for nonlinear integro-differential equations. J. Inst. Math. Appl. **8,** 235 (1971).
[6] Steinberg, J.: Numerical solution of Volterra integral equations. Numer. Math. **19,** 213 (1972).

H. Brunner
Department of Mathematics
Dalhousie University
Halifax, Nova Scotia, Canada

J. D. Lambert
Department of Mathematics
University of Dundee
Dundee, Scotland