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Computer-Assisted Existence Proofs for Two-Point Boundary Value Problems

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Abstract — Zusammenfassung

Computer-Assisted Existence Proofs for Two-Point Boundary Value Problems. For (scalar) nonlinear two-point boundary value problems of the form $-U'' + F(x, U, U') = 0$, $B_0[U] = B_1[U] = 0$, with Sturm-Liouville or periodic boundary operators B_0 and B_1, we present a method for proving the existence of a solution within a "close" C_1-neighborhood of an approximate solution.

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Computer-unterstützte Existenzbeweise für Zweipunkt-Randwertprobleme. Für (skalare) nichtlineare Zweipunkt-Randwertprobleme der Form $-U'' + F(x, U, U') = 0$, $B_0[U] = B_1[U] = 0$ mit Sturm-Liouville-oder periodischen Randoperatoren B_0, B_1 wird eine Methode vorgestellt, mit der die Existenz einer Lösung innerhalb einer "kleinen" C_1-Umgebung einer Näherungslösung bewiesen werden kann.

1. Introduction

The present article is concerned with scalar two-point boundary value problems of the following form:

$$\left.\begin{aligned} -U''(x) + F(x, U(x), U'(x)) = 0 \qquad (0 \le x \le 1),\\ B_0[U] = B_1[U] = 0. \end{aligned}\right\} \tag{1.1}$$

Here, F is a given smooth function on $[0,1] \times \mathbb{R} \times \mathbb{R}$, and B_0, B_1 are linear boundary operators, either of *Sturm-Liouville* type:

$$B_0[u] = -\alpha_0 u'(0) + \gamma_0 u(0), \qquad B_1[u] = \alpha_1 u'(1) + \gamma_1 u(1) \tag{1.2}$$

(where $\alpha_0^2 + \gamma_0^2 > 0, \alpha_1^2 + \gamma_1^2 > 0$), or of *periodic* type:

$$B_0[u] = u(1) - u(0), \qquad B_1[u] = u'(1) - u'(0). \tag{1.3}$$

Many other boundary conditions may be treated but will not be considered here for reasons of technical uniformity.

We will establish a method for proving the *existence* of a solution for problem (1.1), in combination with an explicit *error bound* for some approximate solution $\omega \in R$, where

$$R := \{u \in H_2(0,1): B_0[u] = B_1[u] = 0\}. \tag{1.4}$$

The main parts of the method have *algorithmical* form and may therefore be carried out *on a computer*. These parts consist in

i) the computation of an approximate solution $\omega \in R$;
ii) the estimation of the L_2-norm of its *defect* $-\omega'' + F(\cdot, \omega, \omega')$;
iii) the calculation of a positive lower bound for the minimal eigenvalue of L^*L on $\{u \in R: L[u] \in R^*\}$, where L is the operator obtained by *linearization* of the given problem at ω, i.e.,

$$L[u] := -u'' + bu' + cu, \quad b := (\partial_3 F)(\cdot, \omega, \omega'), \quad c := (\partial_2 F)(\cdot, \omega, \omega'), \tag{1.5}$$

and L^* denotes the (formally) adjoint operator; R^* is defined as in (1.4), but with the adjoint boundary operators B_0^*, B_1^* in place of B_0, B_1.

Since *safe* bounds are required in parts ii) and iii), the rounding errors made during the corresponding computations must be taken into account. For this reason, we use *interval*-arithmetic (ACRITH[7], FORTRAN-SC[9]) in these parts of the algorithm. The computation of ω (part i)), however, may be carried out in usual real floating-point arithmetic. Thus, well-established approximation methods may be used for that purpose, and there is no need for interval-arithmetical versions of these methods.

It is important to note that we do not pose any kinds of monotonicity or growth conditions on the nonlinearity F, or inverse-positivity assumptions on the linearized operator L. We only require L to be *invertible* on R (compare part iii) mentioned above). Thus, many cases are covered where the well-known method of (smooth) *comparison functions* (upper and lower solutions) or *monotonicity principles* (see [2, 17]) cannot be applied.

Other approaches avoiding monotonicity and inverse-positivity assumptions may be found, for example, in [5, 8, 10, 11, 19]. In [8], the Newton-Kantorovich Theorem is applied; under this aspect, the method is comparable to ours, but the estimation of the inverse of the linearized operator is carried out in a completely different way. Similar remarks hold true for the method presented in [10], and for the approach in [5] where the given problem is transformed by introduction of *breakpoints* and *breakpoint-functions*; the transformed problem is suitable for the application of the method of (non-smooth) comparison functions. In [11], shooting methods are used; the occurring initial value problems are treated by interval-arithmetical versions of one-step methods.

In [13, 14], a modified version of our approach presented here is applied to several examples where the nonlinearity F depends at most linearly on U' and, moreover, the linearized operator L is symmetric on R (with respect to some suitably weighted inner product), which simplifies the eigenvalue estimates required in part iii) above.

Compared with the other methods mentioned above, our way of proceeding has the advantage of being transferable to *elliptic* boundary value problems in a more or less direct manner (although with much more theoretical effort), if the nonlinearity F depends only on x and U; see [15, 16] and further forthcoming papers.

2. The Existence and Inclusion Theorem

In this section, we will describe the structure of our algorithm and prove the resulting existence and inclusion statement. The general algorithmical framework is based on J. Schröder's ideas (see [18, 19]). The main parts of the algorithm will be discussed in more detail in the following sections.

Suppose that some approximate solution $\omega \in R$ (see (1.4)) of problem (1.1) and some constant $\delta \geq 0$ satisfying

$$\|-\omega'' + F(\cdot, \omega, \omega')\|_2 \leq \delta \tag{2.1}$$

have been computed. Moreover, let constants K and K' be known such that, with L defined in (1.5),

$$\|u\|_\infty \leq K\|L[u]\|_2, \qquad \|u'\|_\infty \leq K'\|L[u]\|_2 \qquad \text{for } u \in R. \tag{2.2}$$

In particular, L is required to be *invertible* on R. It will turn out in section 4 that the main numerical work which has to be done to compute such constants K, K' consists in the calculation of an eigenvalue bound as described in section 1.

Finally, let some function G: $[0, \infty) \times [0, \infty) \to [0, \infty)$ be given which is monotonically nondecreasing with respect to both its variables and which satisfies, with b and c defined in (1.5),

$$|F(x, \omega(x) + y, \omega'(x) + p) - F(x, \omega(x), \omega'(x)) - c(x)y - b(x)p| \leq G(|y|, |p|) \tag{2.3}$$

for $x \in [0, 1]$, $y \in \mathbb{R}$, $p \in \mathbb{R}$. Since F is smooth, G may be chosen such that

$$G(\alpha, \beta) = O(\alpha^2 + \beta^2) \qquad \text{for } \alpha, \beta \to 0. \tag{2.4}$$

Usually, such a function G may easily be calculated "by hand" if (rough) constant upper and lower bounds for ω and ω' are known.

Theorem 1. *Suppose that constants α, $\beta \geq 0$ exist such that*

$$\delta \leq \min\left\{\frac{\alpha}{K}, \frac{\beta}{K'}\right\} - G(\alpha, \beta). \tag{2.5}$$

Then, there exists a solution $U \in C_2[0, 1]$ of problem (1.1) *satisfying*

$$\|U - \omega\|_\infty \leq \alpha, \qquad \|U' - \omega'\|_\infty \leq \beta. \tag{2.6}$$

Due to (2.4), the crucial condition (2.5) is satisfied for "small" α, β if δ is sufficiently small, i.e., if the approximate solution ω has been computed with *sufficient accuracy*.

Proof of Theorem 1: Consider the boundary value problem

$$L[u] + f(\cdot, u, u') = -d[\omega] \quad \text{on } (0, 1), \qquad B_0[u] = B_1[u] = 0 \tag{2.7}$$

where $f(x, y, p) := F(x, \omega(x) + y, \omega'(x) + p) - F(x, \omega(x), \omega'(x)) - c(x)y - b(x)p$ and

$d[\omega] := -\omega'' + F(\cdot, \omega, \omega')$. It suffices to prove that problem (2.7) has a solution $u^* \in R$ such that $\|u^*\|_\infty \le \alpha$, $\|(u^*)'\|_\infty \le \beta$ because then $U := \omega + u^* \in R$ is a solution of problem (1.1) satisfying the estimates (2.6). The smoothness of U follows a posteriori from the differential equation in (1.1).

(2.2) and the well-known theory of linear boundary value problems show that the inverse operator $L^{-1}: L_2(0,1) \to R \subset H_2(0,1)$ exists and is continuous. Due to the compactness of the embedding $H_2(0,1) \hookrightarrow C_1[0,1]$, the operator $T: C_1[0,1] \to C_1[0,1]$ defined by

$$Tu := -L^{-1}(d[\omega] + f(\cdot, u, u')) \tag{2.8}$$

is therefore continuous and compact. Thus, the existence of a solution of problem (2.7) with the required properties follows from Schauder's fixed-point theorem if we show that

$$TD \subset D \qquad \text{for } D := \{u \in C_1[0,1]: \|u\|_\infty \le \alpha, \|u'\|_\infty \le \beta\}.$$

For $u \in D$, (2.3) and the monotonicity of G imply

$$|f(x, u(x), u'(x))| \le G(|u(x)|, |u'(x)|) \le G(\alpha, \beta) \qquad (0 \le x \le 1)$$

and thus, $\|f(\cdot, u, u')\|_2 \le G(\alpha, \beta)$. Using (2.8), (2.1), and (2.5) we therefore obtain

$$\|L[Tu]\|_2 = \|d[\omega] + f(\cdot, u, u')\|_2 \le \delta + G(\alpha, \beta) \le \min\left\{\frac{\alpha}{K}, \frac{\beta}{K'}\right\}$$

so that (2.2) provides (regard that $Tu \in R$):

$$\|Tu\|_\infty \le K\|L[Tu]\|_2 \le \alpha, \qquad \|(Tu)'\|_\infty \le K'\|L[Tu]\|_2 \le \beta.$$

□

3. Computation of ω and δ

The computation of an approximate solution $\omega \in R$ of problem (1.1) and the estimation of the L_2-norm of its defect may be carried out by any suitable sub-algorithms. In our examples, we used a Newton-collocation method for the former, and a theorem by Ehlich and Zeller [3] for the latter task. Most of the details have been developed in joint work with M. Göhlen and J. Schröder (see [5]).

One Newton-step is carried out as follows. Given $\omega_{n-1} \in R$ we compute an approximate solution $u_n \in R$ of the linear boundary value problem

$$-u'' + (\partial_3 F)(\cdot, \omega_{n-1}, \omega'_{n-1}) \cdot u' + (\partial_2 F)(\cdot, \omega_{n-1}, \omega'_{n-1}) \cdot u = -d[\omega_{n-1}], \tag{3.1}$$
$$B_0[u] = B_1[u] = 0$$

(with $d[\omega_{n-1}] := -\omega''_{n-1} + F(\cdot, \omega_{n-1}, \omega'_{n-1})$) and define $\omega_n := \omega_{n-1} + u_n$. To determine u_n we use a *collocation* procedure for the "ansatz"

$$u(x) = \sum_{k=0}^{M} a_k P_k(x) \tag{3.2}$$

with *polynomials* $P_0, \ldots, P_M \in R$, i.e., we fix the coefficients $a_0, \ldots, a_M$ by requiring that the differential equation (3.1) is satisfied at given points $x_0, \ldots, x_M \in [0,1]$, which obviously results in a linear algebraic system for $a_0, \ldots, a_M$. This system is solved (approximately) by a Gauß-algorithm with partial pivoting.

In particular, we choose the collocation points

$$x_k := \frac{1}{2}\left[1 - \cos\left(\frac{k+1}{M+2}\pi\right)\right] \qquad (k = 0, \ldots, M)$$

and the basis functions

$$P_k(x) := x^2(1-x)^2 T_{k-2}(2x-1) \qquad \text{for } k \geq 2,$$

with T_i denoting the i-th Chebyshev-polynomial of the first kind. P_0 and P_1 are chosen to form a basis of the space of polynomials of degree ≤ 3 which satisfy the required boundary conditions. More general spline-type bases $\{P_0, \ldots, P_M\}$ are discussed in [5].

The above Newton-iteration is terminated when, for some $n \in \mathbb{N}$, the coefficients $a_k^{(n)}$ of u_n are (in modulus) below some given tolerance. Then, the approximate solution $\omega := \omega_n$ is given in the form (3.2), provided that the starting approximation ω_0 of the Newton-iteration has that form. To find such a function ω_0 we use a homotopy method in our (parameter-dependent) examples.

To compute a constant δ satisfying (2.1) one may use several approaches. For example, one may apply a quadrature formula and a remainder-term-estimate, as described in [12] for a similar situation. Here, we proceed differently. Suppose first that the nonlinearity F is a *polynomial* with respect to all its variables. Then, with ω calculated as described above, the defect $d[\omega] = -\omega'' + F(\cdot, \omega, \omega')$ is a polynomial of some known degree n. Thus, a theorem by Ehlich and Zeller [3, Theorem 2] provides the estimate

$$\|d[\omega]\|_\infty \leq C \max\{|d[\omega](\xi_j)| : j = 0, \ldots, N\} \tag{3.3}$$

where $N \in \mathbb{N}$, $N > n$, $C = C(n, N) := [\cos(\pi n/2N)]^{-1}$, and $\xi_j := \frac{1}{2}[1 + \cos(j\pi/N)]$ for $j = 0, \ldots, N$. Since $\|d[\omega]\|_2 \leq \|d[\omega]\|_\infty$, (3.3) reduces the problem of calculating δ to the evaluation of $d[\omega]$ at the points $\xi_0, \ldots, \xi_N$. In order to take rounding errors into account we use *interval*-arithmetic (ACRITH [7] and FORTRAN SC [9]) for that purpose. Practical details may be found in [5].

For non-polynomial but smooth F (see our first example) we apply a method proposed by Gärtel [4]. First we use the Ehlich/Zeller-Theorem mentioned above to compute constant lower and upper bounds $\underline{\omega}$, $\bar{\omega}$ and $\underline{\omega}'$, $\bar{\omega}'$ for the polynomials ω and ω', respectively. Suppose next that, for some given $\varepsilon > 0$, numbers n_1, n_2, $n_3 \in \mathbb{N}$ are known such that

$$|F(x, y, p) - P(x, y, p)| \leq \varepsilon \qquad (0 \leq x \leq 1, \underline{\omega} \leq y \leq \bar{\omega}, \underline{\omega}' \leq p \leq \bar{\omega}') \tag{3.4}$$

for some function P which is a polynomial in x, y and p with degrees n_1, n_2 and n_3, respectively (P need *not* be computed!). Such numbers can easily be calculated by use of well-known error estimates for polynomial approximations, if explicit formulas for higher derivatives of F are at hand (as in our first example). Now, since $\tilde{d}[\omega] := -\omega'' + P(\cdot, \omega, \omega')$ is a polynomial of some known degree n, one may use (3.3) and (3.4) to estimate $d[\omega]$ as follows:

$$\|d[\omega]\|_\infty \leq \|\tilde{d}[\omega]\|_\infty + \varepsilon \leq C \max_{j=0,\ldots,N} |\tilde{d}[\omega](\xi_j)| + \varepsilon$$

$$\leq C \max_{j=0,\ldots,N} |d[\omega](\xi_j)| + (C+1)\varepsilon. \tag{3.5}$$

It should be regarded that N usually depends on ε, so that the numerical effort increases as ε decreases. In our first example, ε is chosen such that the two summands in (3.5) are approximately equal.

4. Computation of K and K'

In order to calculate constants K and K' satisfying the estimates (2.2), with L given by (1.5), we first compute constants C_0, C_1, K_0, K_1, K_2 such that

$$\|u\|_\infty \leq C_0 \|u\|_2 + C_1 \|u'\|_2 \quad \text{for } u \in H_1(0,1), \tag{4.1}$$

$$\|u\|_2 \leq K_0 \|L[u]\|_2, \quad \|u'\|_2 \leq K_1 \|L[u]\|_2, \quad \|u''\|_2 \leq K_2 \|L[u]\|_2 \quad \text{for } u \in R.$$

Then, (2.2) obviously holds with $K := C_0 K_0 + C_1 K_1$ and, since u' may be inserted into (4.1) in place of u, with $K' := C_0 K_1 + C_1 K_2$. This approach has the advantage of being transferable to *elliptic* boundary value problems, as long as only the *first* estimate in (2.2) is considered (see [15]).

Lemma 1: *The estimate* (4.1) *holds with*

$$C_0 \geq 1 \textit{ arbitrary}, \qquad C_1 = \frac{1}{\sqrt{3C_0}}.$$

Moreover, corresponding inequalities hold also with other constants C_0, C_1 for some restricted classes of functions $u \in H_1(0,1)$, in particular, with

$$C_0 = 0, \qquad C_1 = \frac{1}{2} \qquad \textit{if } u(0) = u(1) = 0,$$

$$C_0 = 0, \qquad C_1 = 1 \qquad \textit{if } u(0) = 0 \textit{ or } u(1) = 0,$$

$$C_0 \geq 1 \textit{ arbitrary}, \qquad C_1 = \frac{1}{2\sqrt{3C_0}} \qquad \textit{if } u(0) = u(1).$$

For a proof, see [14]. Depending on the given boundary operators B_0 and B_1, one will prefer different choices of corresponding pairs (C_0, C_1), according to Lemma 1, for the computation of K and K' described above. If, for example, $B_0[u] = u(0)$ and $B_1[u] = u(1)$, one may choose $C_0 = 0$, $C_1 = \frac{1}{2}$ for K, and $C_0 = 1$, $C_1 = 1/\sqrt{3}$ for K'. If $B_0[u] = u(0)$ and $B_1[u] = u'(1)$, one may use $C_0 = 0$, $C_1 = 1$ for both K and K'.

4.1. Computation of K_0

By variational arguments, it is easy to reduce the calculation of a constant K_0 satisfying

$$\|u\|_2 \leq K_0 \|L[u]\|_2 \qquad \text{for } u \in R \tag{4.2}$$

to the estimation of *eigenvalues* of a symmetric differential operator. In [14], where L itself is assumed to be symmetric on R (with respect to some suitably weighted inner product), (4.2) is proved to hold for $K_0 := \sigma^{-1}B$, with σ denoting a lower bound for the distance between $0 \in \mathbb{R}$ and the spectrum of L on R, and with some constant B depending on the weight function. This approach has the advantage of being related to a *second*-order eigenvalue problem, compared with the method described below. However, it is not applicable if L is not symmetric on R (as partly in our first example presented here); if the constant B is "large" (as partly in our second example), it is applicable but not recommendable.

Here, we assume that the coefficients b and c defined in (1.5) are sufficiently smooth (which is certainly true if F is smooth and ω is computed as described in section 3) and consider the fourth-order eigenvalue problem

$$L^*L[u] = \lambda u \quad \text{on } (0,1), \qquad B_0[u] = B_1[u] = B_0^*[L[u]] = B_1^*[L[u]] = 0, \tag{4.3}$$

with L^*, B_0^*, B_1^* denoting the operators adjoint to L, B_0, B_1 with respect to the canonical inner product $\langle\ ,\ \rangle$ on $L_2(0,1)$. The variational characterization of the minimal eigenvalue λ_1 of problem (4.3) reads

$$\lambda_1 = \min_{u \in R} \frac{\langle L[u], L[u]\rangle}{\langle u, u\rangle}$$

(regard that only the boundary conditions $B_0[u] = B_1[u] = 0$ are "essential"). Thus, (4.2) holds with

$$K_0 := \lambda_1^{-1/2}.$$

In order to compute a lower bound for λ_1 we use a numerical *homotopy method* which is described in [12] for second-order eigenvalue problems. Here, we give a brief description of the modifications which have to be made for the fourth-order problem (4.3). The "elementary" eigenvalue estimates stated in Theorems 1, 2, 4, and 5 in [12] may be transferred to problem (4.3) without any significant changes. We are left to construct a homotopy starting at some "simple" problem with known eigenvalues and ending at the given problem, such that all eigenvalues are monotonically nondecreasing with respect to the homotopy parameter. Then, the eigenvalue estimation algorithm described in [12] may be applied to problem (4.3).

Integration by parts shows that the bilinear form $\mathscr{B}$ associated with problem (4.3) is given by

$$\mathscr{B}[u,v] = \langle L[u], L[v]\rangle = \int_0^1 [u''v'' + pu'v' + quv]\,dx + \beta[u,v] \tag{4.4}$$

for $u, v \in R$, where $p := b^2 + b' + 2c$, $q := c^2 - c'' - (bc)'$, and $\beta[u,v] := [-bu'v' - c(u'v + uv') + (c' + bc)uv]_0^1$. Let p_0 and q_0 denote constant lower bounds for p and q, respectively, and define, for $s \in [0,1]$ and $u, v \in R$,

$$p_s := sp + (1-s)p_0, \qquad q_s := sq + (1-s)q_0,$$

$$\mathscr{B}_s[u,v] := \int_0^1 [u''v'' + p_s u'v' + q_s uv]\,dx + \beta[u,v].$$

Integrating by parts one derives that, for $s \in [0,1]$, the eigenvalue problem associated with $\mathscr{B}_s$ is given by

$$\begin{aligned} u^{(4)} - (p_s u')' + q_s u &= \lambda u \qquad \text{on } (0,1), \\ B_0[u] = B_1[u] &= \hat{B}_0^{(s)}[u] = \hat{B}_1^{(s)}[u] = 0 \end{aligned} \tag{4.5$_s$}$$

where, in the Sturm-Liouville-case (1.2),

$$\hat{B}_0^{(s)}[u] := -\alpha_0(L[u])'(0) + (\gamma_0 - \alpha_0 b(0))L[u](0) + \gamma_0(1-s)(p(0)-p_0)u(0),$$
$$\hat{B}_1^{(s)}[u] := \alpha_1(L[u])'(1) + (\gamma_1 + \alpha_1 b(1))L[u](1) + \gamma_1(1-s)(p(1)-p_0)u(1),$$

and in the periodic case (1.3),

$$\hat{B}_0^{(s)}[u] := L[u](1) - L[u](0), \hat{B}_1^{(s)}[u] := [(L[u])' + b \cdot L[u] - (1-s)(p-p_0)u']_0^1.$$

It is easy to see that problem $(4.5)_1$ is the given problem (4.3), while problem $(4.5)_0$ has constant coefficients and may therefore be solved explicitly, possibly up to the solution of a real transcendental equation. Moreover, for fixed $u \in R$, $\mathscr{B}_s[u,u]$ is monotonically nondecreasing with respect to s, so that Courant's Maximum-Minimum-principle implies the required monotonicity of all eigenvalues of the problems $(4.5)_s$ with respect to $s \in [0,1]$.

As explained in [12], certain parts of the homotopy algorithm have to be carried out in interval-arithmetic, so that rounding errors are taken into account. For these parts, we use ACRITH [7] and FORTRAN-SC [9] subroutines.

It should be noted that, in place of our homotopy algorithm, other eigenvalue estimation methods may be used to compute a lower bound for λ_1. In particular, the approach developed by Bazley and Fox (e.g., [1]) or the method by Goerisch and Albrecht (e.g., [6]) are reasonable.

4.2. Computation of K_1 and K_2

While the calculation of a constant K_0 satisfying (4.2) requires a good deal of numerical work, constants K_1 and K_2 satisfying

$$\|u'\|_2 \le K_1 \|L[u]\|_2, \qquad \|u''\|_2 \le K_2 \|L[u]\|_2 \qquad \text{for } u \in R \tag{4.6}$$

may be computed in a much more direct way, as shown in the following lemmata. The first of them constitutes an alternative to Lemma 1 in [14], where the weight function mentioned at the beginning of the preceding subsection is used.

Lemma 2: *Suppose that* (4.2) *holds for some* K_0. *Let A denote a real constant such that, in the Sturm-Liouville-case* (1.2),

$$A \ge -\frac{\gamma_0}{\alpha_0} + \frac{1}{2}b(0) \qquad \text{if } \alpha_0 \ne 0, \qquad A \ge -\frac{\gamma_1}{\alpha_1} - \frac{1}{2}b(1) \qquad \text{if } \alpha_1 \ne 0,$$

and $A \ge \frac{1}{4}[b(0) - b(1)]$ *in the periodic case* (1.3). *Moreover, let $\underline{c}$ be a constant satisfying*

$$\underline{c} \le c(x) - \frac{1}{2}b'(x) - A\{2 + (1-2x)[2A(1-2x) - b(x)]\} \qquad \textit{for } x \in [0,1],$$

and define $E := \exp(\frac{1}{4}|A|)$. *Then, the first estimate in* (4.6) *holds with*

$$K_1 := \begin{cases} E \cdot [K_0(1 - \underline{c}K_0)]^{1/2} & \textit{if } \underline{c}K_0 \le \frac{1}{2} \\ E/(2\sqrt{\underline{c}}) & \textit{otherwise}. \end{cases}$$

Proof: Let $w(x) := \exp[-2Ax(1-x)]\,(0 \le x \le 1)$. For fixed $u \in R\backslash\{0\}$ we obtain

$$\begin{aligned}
&\int_0^1 wuL[u]\,dx \\
&= \int_0^1 u[-(wu')']\,dx + \frac{1}{2}\int_0^1 (w' + wb)(u^2)'\,dx + \int_0^1 wcu^2\,dx \\
&= \left[-wuu' + \frac{1}{2}(w' + wb)u^2\right]_0^1 + \int_0^1 w(u')^2\,dx \\
&\quad + \int_0^1 \left[wc - \frac{1}{2}(w' + wb)'\right]u^2\,dx \\
&= u(1)\left[-u'(1) + \left(A + \frac{1}{2}b(1)\right)u(1)\right] \\
&\quad + u(0)\left[u'(0) + \left(A - \frac{1}{2}b(0)\right)u(0)\right] + \int_0^1 w(u')^2\,dx \\
&\quad + \int_0^1 w\left[c - \frac{1}{2}(4A + 4A^2(1-2x)^2) + A(1-2x)b - \frac{1}{2}b'\right]u^2\,dx.
\end{aligned}$$

The sum of the boundary terms is nonnegative due to our choice of A. Moreover, the term in brackets in the last integral is bounded from below by $\underline{c}$. Consequently, the above equality implies, with w_0 denoting the minimum of w,

$$\int_0^1 wuL[u]\,dx \ge w_0\|u'\|_2^2 + \underline{c}\|\sqrt{w}u\|_2^2.$$

Dividing by w_0, observing that $\sqrt{w/w_0} \le E$, and defining

$$\mu := \|\sqrt{w/w_0}\,u\|_2/\|L[u]\|_2$$

we obtain

$$\|u'\|_2^2 \le \mu(E - \underline{c}\mu)\|L[u]\|_2^2. \tag{4.7}$$

Moreover, (4.2) implies $\mu \le E \cdot K_0$. Calculation of the maximum of the quadratic expression in μ in (4.7) on $[0, E \cdot K_0]$ provides the assertion. □

Lemma 3: *Suppose that* (4.2) *and the first estimate in* (4.6) *hold with constants* K_0 *and* K_1, *respectively. Let* b_0, c_0 *denote real constants which satisfy, in the Sturm-Liouville case* (1.2),

$$\gamma_0^2 b_0 + 2\alpha_0\gamma_0 c_0 - \alpha_0^2 b_0 c_0 \ge 0, \qquad -\gamma_1^2 b_0 + 2\alpha_1\gamma_1 c_0 + \alpha_1^2 b_0 c_0 \ge 0$$

(and which are arbitrary in the periodic case (1.3)*). Then, the second estimate in* (4.6) *holds with*

$$K_2 := [(1 + \|b - b_0\|_\infty K_1 + \|c - c_0\|_\infty K_0)^2 + \max\{0, -b_0^2 - 2c_0\} K_1^2]^{1/2}.$$

Proof: For $u \in R$, integration by parts shows that

$$\|-u'' + b_0 u' + c_0 u\|_2^2 = \|u''\|_2^2 + (b_0^2 + 2c_0)\|u'\|_2^2 + c_0^2\|u\|_2^2 + [-b_0(u')^2 - 2c_0 uu' + b_0 c_0 u^2]_0^1.$$

The boundary term vanishes in the periodic case (1.3). In the Sturm-Liouville case (1.2), it is nonnegative due to our conditions on b_0 and c_0. Consequently,

$$\|u''\|_2^2 \le \|-u'' + b_0 u' + c_0 u\|_2^2 - (b_0^2 + 2c_0)\|u'\|_2^2$$
$$\le (\|L[u]\|_2 + \|b - b_0\|_\infty \|u'\|_2 + \|c - c_0\|_\infty \|u\|_2)^2 - (b_0^2 + 2c_0)\|u'\|_2^2$$

so that (4.2) and the first estimate in (4.6) provide the assertion. □

The inequalities required for b_0 and c_0 in Lemma 3 are always satisfied for $b_0 = c_0 := 0$. However, a smaller constant K_2 may often be obtained by other choices of b_0 and c_0.

Lemma 3 may be generalized by consideration of *functions* b_0, c_0 in place of constants. Here, we will not discuss such choices in general, but only formulate a lemma concerned with the particular case $b_0 \equiv b$, $c_0 \equiv c$.

Lemma 4: *Let* (4.2) *and the first estimate in* (4.6) *hold with constants* K_0 *and* K_1, *respectively, and let* p, q, β, p_0, q_0 *denote the terms defined after* (4.4). *Suppose that*

$$\beta[u, u] \ge 0 \qquad \text{for } u \in R. \tag{4.8}$$

Then, the second estimate in (4.6) *holds with*

$$K_2 := [1 + \max\{0, -p_0\} K_1^2 + \max\{0, -q_0\} K_0^2]^{1/2}.$$

Proof: (4.4), (4.8) and the choice of p_0 and q_0 imply, for $u \in R$,

$$\|L[u]\|_2^2 \ge \|u''\|_2^2 + p_0\|u'\|_2^2 + q_0\|u\|_2^2.$$

The assertion now follows by use of (4.2) and the first estimate in (4.6). □

In our examples, where condition (4.8) is satisfied, we computed two constants K_2 according to Lemmata 3 and 4, and chose the minimum of both values.

5. Numerical Examples

We applied our existence and inclusion method to two parameter-dependent nonlinear boundary value problems. Starting at some "trivial" solutions we combined the Newton-collocation procedure described in section 3 with a step-wise homotopy along the solution branches. We used $M := 80$ in (3.2), i.e., 81 basis functions for the collocation. The estimate (3.3) was applied with $N := 2n$, so that $C = \sqrt{2}$.

In our first example, we looked for 2π-periodic solutions of the forced pendulum equation

$$-U'' - U'(\mu + \nu(U')^2) - \sin U = \lambda \cos x. \tag{5.1}$$

We treated three cases: a) $\mu = \nu = 0$ (no damping), b) $\mu = 1, \nu = 0$ (linear damping), c) $\mu = 1, \nu = 0.1$ (nonlinear damping). For each of them, we computed approximate solutions ω for several values of λ, after transformation of the interval $[0, 2\pi]$ onto $[0, 1]$. The corresponding bifurcation diagrams (in the $(\lambda, \omega(0))$-plane) are plotted in Figures 1a, 1b and 1c below. The branches drawn in heavy lines are formed by significantly different (approximate) solutions; all other branches (drawn in dotted lines) are obtained from them by the simple transformations $\omega \to \omega + 2\pi k$ ($k \in \mathbb{Z}$),

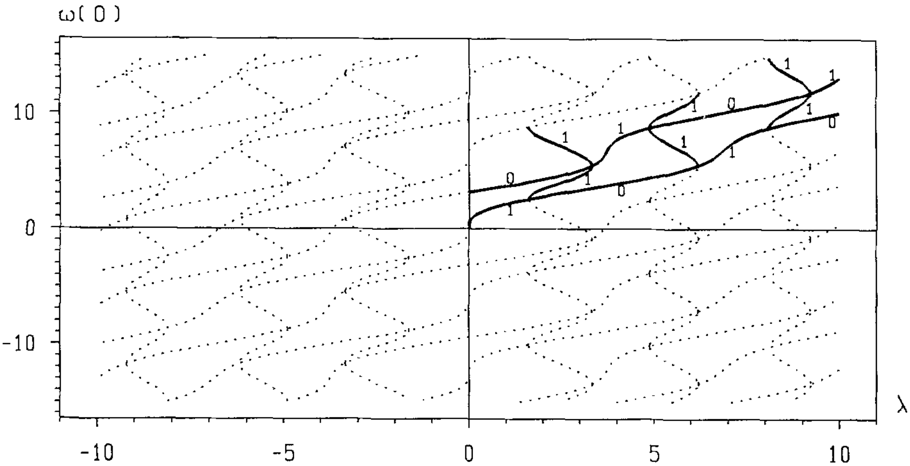

Figure 1a. The pendulum equation, no damping ($\mu = \nu = 0$)

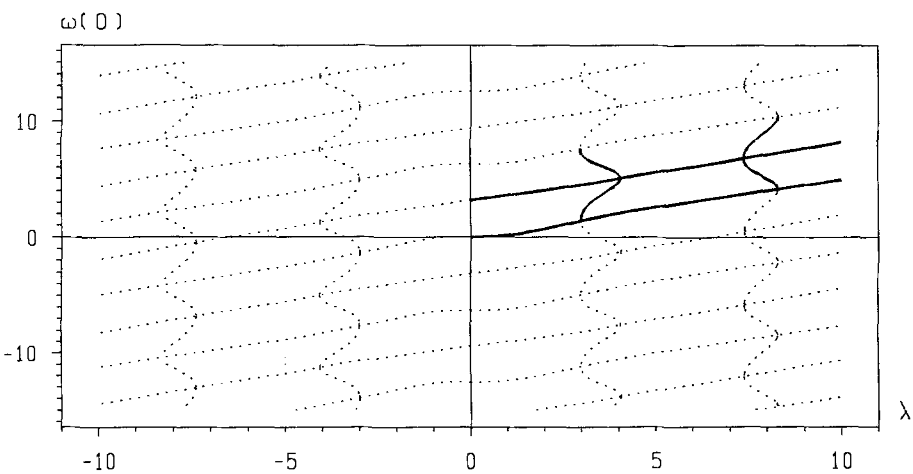

Figure 1b. Linear damping ($\mu = 1, \nu = 0$)

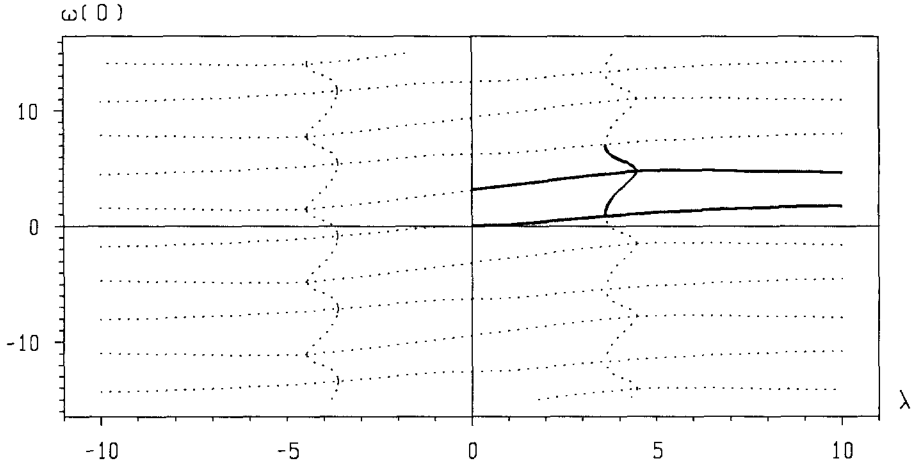

Figure 1c. Nonlinear damping ($\mu = 1, \nu = 0.1$)

which obviously provide physically identical solutions, and $\omega \to -\omega$, $\lambda \to -\lambda$. In case a), all approximate solutions are (approximately) symmetric with respect to reflection at π, but not in cases b) and c).

Our existence and inclusion method could be applied successfully to prove the existence of a 2π-periodic solution of problem (5.1) in a close neighborhood of ω, for all approximate solutions ω we computed, except for those which are close to a bifurcation point or, in case a), to the origin ($\lambda = 0$, $\omega \equiv 0$), since the operator L is not invertible at these points and thus, the constants K and K' are "large" in their neighborhood. A selection of pairs $(\lambda, \omega(0))$ on the "main" branches (passing through the origin) and of the computed constants δ, K, K', α, β, satisfying (2.1), (2.2) and (2.6), respectively, is listed in Table 1 below. The results are similar on the other branches.

In case a), where the operator L is symmetric on R, we computed a second constant K_0 according to the method presented in [14], besides the constant calculated by the algorithm described in subsection 4.1. The results are nearly identical. However, the method in [14] has the advantage of providing, as a by-product, the number of negative eigenvalues of L, which may serve as a tool for discovering bifurcation points in the course of a branch, where this number (usually) *changes*. These numbers are written down at the respective parts of the branches in Fig. 1a. It should be noted that the "classical" existence and inclusion method by smooth comparison functions can only be applied on the branch-parts with *no* negative eigenvalue. In cases b) and c), the operator L is not symmetric on R (with respect to any weight function), so that the method in [14] is not applicable.

In our second example, we treated the boundary value problem

$$-U'' = \lambda U\left[1 - \frac{1}{2}(U')^2 - \frac{1}{8}(U')^4\right] \quad \text{on } (0,1), \qquad U(0) = U'(1) = 0 \quad (5.2)$$

Table 1. The forced pendulum equation ("main" branches)

λ	$\omega(0)\approx$	δ	K	K'	α	β
			$\mu = \nu = 0$ (no damping)			
1.E-10	0.001	.990E-14	633375.962	3979618.751	.627E-08	.394E-07
1.E-05	0.043	.463E-12	294.013	1847.551	.136E-09	.855E-09
1.	2.106	.134E-10	0.342	2.363	.457E-11	.316E-10
1.61	2.532	.140E-09	60.708	381.653	.845E-08	.531E-07
1.6139	2.535	.143E-09	5457.650	34291.640	.912E-06	.573E-05
			Bifurcation point			
1.613945	2.535	.143E-09	751022.981	4718816.769	—	—
1.614	2.535	.144E-09	4331.577	27216.315	.686E-06	.431E-05
4.	3.852	.422E-06	0.208	1.520	.876E-07	.641E-06
6.2	5.460	.198E-04	21.359	134.414	.573E-03	.360E-02
6.206	5.467	.200E-04	153.585	965.212	—	—
			Bifurcation point			
6.208	5.470	.201E-04	223.770	1406.203	—	—
6.3	5.586	.237E-04	1.552	9.964	.368E-04	.237E-03
8.	8.582	.199E-03	2.743	17.450	.578E-03	.368E-02
8.1	8.675	.318E-03	45.792	287.928	—	—
			Bifurcation point			
8.13	8.701	.419E-03	36.771	231.253	—	—
8.2	8.763	.686E-03	4.072	25.794	—	—
8.3	8.847	.116E-02	1.864	11.926	.250E-02	.160E-01
10.	10.027	.137E-01	0.338	2.333	.489E-02	.339E-01
			$\mu = 1, \nu = 0$ (linear damping)			
0.1	1.E-4	.297E-11	0.082	0.626	.242E-12	.186E-11
1.	0.116	.324E-10	0.102	0.723	.330E-11	.234E-10
2.9	1.311	.127E-08	2.784	16.685	.353E-08	.212E-07
2.98	1.366	.167E-08	28.908	172.560	.481E-07	.287E-06
2.988	1.371	.170E-08	388.407	2317.523	.665E-06	.397E-05
			Bifurcation point			
2.989	1.372	.171E-08	703.426	4197.383	.124E-05	.738E-05
3.	1.379	.175E-08	22.110	132.288	.386E-07	.231E-06
6.	3.050	.238E-05	0.203	1.488	.483E-06	.354E-05
8.27	4.122	.150E-03	7.498	47.322	.150E-02	.942E-02
8.29	4.132	.154E-03	22.281	140.206	—	—
			Bifurcation point			
8.31	4.142	.159E-03	26.997	169.840	—	—
8.33	4.151	.164E-03	7.880	49.723	.181E-02	.114E-01
10.	4.937	.145E-02	0.228	1.646	.332E-03	.240E-02
			$\mu = 1, \nu = 0.1$ (nonlinear damping)			
0.1	1.E-4	.297E-11	0.082	0.627	.242E-12	.186E-11
1.	0.089	.324E-10	0.110	0.803	.354E-11	.260E-10
3.6	0.855	.993E-08	23.049	243.646	.230E-06	.243E-05
3.621	0.860	.106E-07	285.192	3021.080	.353E-05	.374E-04
3.622	0.861	.106E-07	618.519	6552.720	—	—
			Bifurcation point			
3.623	0.861	.106E-07	3716.354	39375.776	—	—
3.624	0.861	.107E-07	463.224	4908.466	—	—
3.625	0.861	.107E-07	247.179	2619.443	.299E-05	.316E-04
3.7	0.881	.132E-07	6.996	74.677	.917E-07	.979E-06
5.	1.172	.252E-06	0.528	6.235	.133E-06	.158E-05
8.	1.616	.270E-04	0.312	4.248	.840E-05	.115E-03
10.	1.787	.670E-04	0.338	5.258	.227E-04	.354E-03

which has an infinite number of nontrivial branches bifurcating from the trivial solution $U \equiv 0$ at the eigenvalues $\lambda_k = \frac{1}{4}(2k-1)^2\pi^2$ of the linearized problem.

For several values of λ, we computed approximate solutions ω on the first and on the second branch, and applied our existence and inclusion method. It was successful for λ up to 28 on the first, and for λ up to 39 on the second branch. The computed branch-parts are plotted in Fig. 2 below, where $\omega(1)$ serves as bifurcation parameter. Several pairs $(\lambda, \omega(1))$ and the calculated constants δ, K, K', α, β satisfying (2.1), (2.2) and (2.6), respectively, are contained in Table 2 below.

As in case a) of our first example, we applied also the alternative method described in [14] to compute a second constant K_0. The results are *worse* than those obtained from subsection 4.1. The reason is the rough estimation of the weight function in [14] (which, however, is unavoidable in that approach). Nevertheless, this alternative method provides the number of negative eigenvalues of L, which is written down at the two branches in Fig. 2.

In [5], problem (5.2) is treated by the "breakpoint"-approach mentioned in section 1. The obtained error bounds are similar to the results presented here.

Table 2. Second example

λ	$\omega(1)\approx$	δ	K	K'	α	β
			First Branch			
2.4675	0.011	.410E-14	7941.320	12478.997	.326E-10	.511E-10
2.468	0.028	.101E-13	1311.363	2064.686	.132E-10	.208E-10
2.48	0.128	.462E-13	62.312	102.712	.288E-11	.474E-11
2.5	0.204	.735E-13	24.084	42.714	.177E-11	.314E-11
3.	0.685	.274E-12	1.627	6.118	.445E-12	.168E-11
5.	1.004	.564E-12	0.541	4.062	.305E-12	.229E-11
10.	1.123	.295E-11	0.399	6.290	.118E-11	.186E-10
15.	1.154	.466E-08	0.360	8.705	.168E-08	.406E-07
20.	1.169	.351E-06	0.336	10.975	.118E-06	.387E-05
25.	1.177	.401E-05	0.318	13.125	.136E-05	.562E-04
28.	1.181	.149E-04	0.309	14.369	.612E-05	.285E-03
29.	1.182	.266E-04	0.307	14.777	—	—
			Second Branch			
22.207	0.003	.525E-14	6041.244	28473.542	.317E-10	.150E-09
22.21	0.007	.157E-13	695.012	3280.017	.109E-10	.514E-10
22.3	0.039	.814E-13	25.220	123.725	.206E-11	.101E-10
23.	0.109	.245E-12	2.981	19.118	.728E-12	.467E-11
24.	0.157	.380E-12	1.344	10.957	.510E-12	.416E-11
26.	0.210	.522E-11	0.668	6.951	.348E-11	.363E-10
28.	0.243	.258E-10	0.557	6.763	.144E-10	.175E-09
30.	0.266	.603E-09	0.709	9.818	.428E-09	.592E-08
32.	0.282	.535E-08	0.917	14.196	.490E-08	.759E-07
34.	0.295	.965E-07	1.196	20.422	.116E-06	.198E-05
36.	0.306	.566E-06	1.567	29.203	.905E-06	.169E-04
38.	0.314	.189E-05	2.060	41.497	.447E-05	.900E-04
39.	0.318	.330E-05	2.362	49.353	.108E-04	.225E-03
40.	0.321	.518E-05	2.710	58.614	—	—

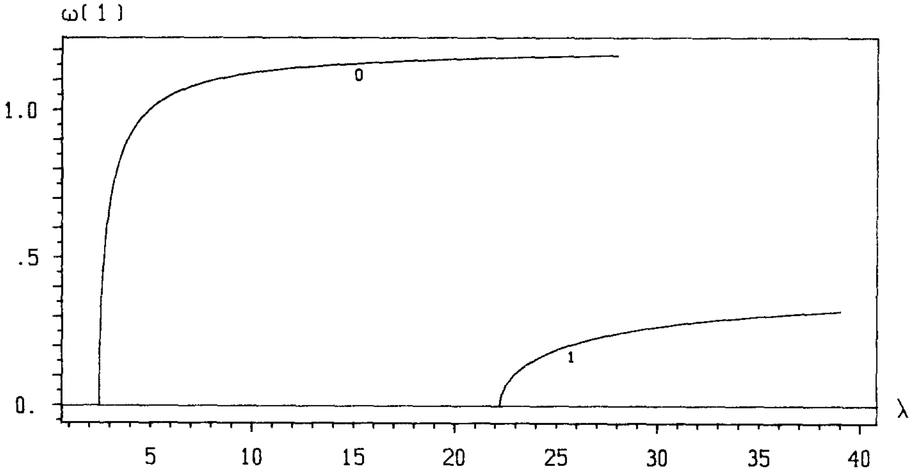

Figure 2. Bifurcation diagram for the second example

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