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Finite Element Method for Domains with Corners[1]

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Summary — Zusammenfassung

Finite Element Method for Domains with Corners. The rate of convergence of the finite element method is greatly influenced by the existence of corners on the boundary. The paper shows that proper refinement of the elements around the corners leads to the rate of convergence which is the same as it would be on domain with smooth boundary.

Die Methode der endlichen Elemente für Gebiete mit Ecken. Die Konvergenzgeschwindigkeit der Methode der endlichen Elemente wird grundsätzlich durch die Ecken der Grenze beeinflußt. In der Arbeit wird gezeigt, daß man durch geeignetes Verfeinern in der Umgebung der Ecken dieselbe Konvergenz der Methode der endlichen Elemente erzielen kann, wie im Falle eines Gebietes mit glatter Grenze.

1. Introduction

This paper deals with the problem of proper refinement in the finite element method to get the highest possible rate of convergence on domains with corners. It is known for finite difference methods that the rate of convergence strongly depends on the angles of the corners. For regular meshes lower (see e. g. [1]) and upper (see e. g. [2], [3]) bounds for the rate of convergence are known. If we solve the nonhomogenous problem $L u = f$ with homogenous boundary conditions then in the case of regular meshes and a domain with corners, the rate of convergence primarily depends on the domain and not on the smoothness of f. In the case that the domain has very smooth boundary, the rate of convergence depends only on the smoothness of f.

So there is the question. Is it possible to refine the net in a proper manner such that the number of unknowns (that we have to solve) will not increase and so that the rate of convergence will be the highest possible given only by the smoothness of f? The answer is affirmative in many cases. We shall show this approach on a model problem by solving the Neumann problem for Laplace equation. The approach can be generalized for general elliptic equations and natural boundary conditions. The case of boundary condition which not natural will be treated in special paper.

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The problem of solving the linear system of algebraic equations will be studied in a later paper also. Refinement of the meshsize or elements in the neighbourhood of the corners is made very often in practice. The only mathematical analysis connected with this idea seems to have been made by VOLKOV (see [1] and related papers there). He is interested only in the h^2 rate of convergence in C norm. This paper deals with the general case in the energy norm.

2. Some Notions and Lemmas

Let R_n be the n-dimensional EUCLIDian space, $x = (x_1, \ldots, x_n)$, $\| x \|^2 = \sum x_i^2$, $| x | = \max \{| x_i |\}$. A bounded domain $\Omega \subset R_n$ will be said to be an L-domain [LIPSCHITZ domain] if there exist numbers $\alpha > 0$, $\beta > 0$, systems of coordinates $(x_{r,1}, \ldots, x_{r,n}) = (x'_r, x_{r,n})$, $r = 1, \ldots, M$ and LIPSCHITZ functions a_r defined on the cubes $| x_{r,i} | < \alpha, i = 1, \ldots, n-1$ $r = 1, \ldots, M$, so that each point x of the boundary $\Omega^{\cdot}$ can be expressed in at least one system in the form $(x'_r, a_r(x'_r))$. For each system the points $(x'_r, x_{r,n})$ are inside (resp. outside) of Ω for

$$a_r(x'_r) < x_{r,n} < a_r(x'_r) + \beta$$

$$[\text{resp. } a_r(x'_r) - \beta < x_{r,n} < a_r(x'_r)].$$

In the paper we shall assume that all domains Ω will be LIPSCHITZ domains.

Further let us define for $h > 0$

$$\Omega^h = E\,[x \in \Omega, \quad \varrho(x, \Omega^{\cdot}) > h]$$

$$\Omega^{\cdot h} = E\,[x \in \Omega, \quad \varrho(x, \Omega^{\cdot}) \leqslant h]$$

where $\varrho(x, \Omega^{\cdot})$ is the distance of the point x to $\Omega^{\cdot}$. Frequently we shall study a special type of a domain. Let $\chi(x')$, for $x' \in R_{n-1}$, $\chi(0) = 0$ be a LIPSCHITZ function (with the LIPSCHITZ constant M). Then for $h > 0$ let us define

$$G_{M,\chi} = E\,[(x', x_n); \quad x_n > \chi(x')],$$

$$G^h_{M,\chi} = E\,[(x', x_n); \quad x_n > \chi(x') + h],$$

$$G^{\cdot h}_{M,\chi} = E\,[(x', x_n); \quad \chi(x') < x_n \leqslant \chi(x') + h].$$

The domain $G_{M,\chi}$ will be called M-LIPSCHITZ domain. Let $W_2^s(\Omega)$, resp. $W_2^s(G_{M,\chi})$, $s \geqslant 0$ real (not in general an integer) is the usual SOBOLEV fractional space, (see e. g. [4], [5]). Further let $\mathbf{y} = (y_1, \ldots, y_p)$, $y_i \in \Omega^{\cdot}$ $y_i \neq y_j$ for $i \neq j$, $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_p)$, $\alpha_i \geqslant 0$, $k \geqslant 0$ an integer.

Let us introduce the space $W^{k,s}_{2,y,\alpha}(\Omega)$, $k \geqslant s \geqslant 0$, k integral

$$\|f\|^2_{W^{k,s}_{2,y,\alpha}} = \|f\|^2_{W^s_2(\Omega)} +$$

$$+ \sum_{\Sigma i_j = k} \int_\Omega \left[\min\left\{1, \prod_{j=1}^p \varrho^{\alpha_j}(x, y_j)\right\}\right]^2 \left[\frac{\partial^k f}{\partial x_1^{i_1} \partial x_2^{i_2} \ldots \partial x_n^{i_n}}\right]^2 dx \qquad (2.1)$$

where $\varrho(x, y_j)$ is the distance of the points x and y_j. Analogously let us define the space $W^{k,s}_{2,y,\alpha}(G_{M,\chi})$.

Lemma 2.1. *Let $f \in W^s_2(\Omega)$, $s \geqslant 0$. Let $f = 0$ on Ω^h. Then*

$$\|f\|_{L_2(\Omega)} \leqslant C\, h^s \,\|f\|_{W^s_2(\Omega)} \tag{2.2}$$

where C does not depend on h and f.

Proof.

(1) Using the usual method of partition of unity it is sufficient to prove the theorem for $G_{M,\chi}$.

(2) Because of the well known theory of interpolated spaces (see e. g. [4], [5]) it is enough to prove the result for s an integer, $s \geqslant 1$.

(3) We may assume that for almost every x', $\frac{\partial f^{s-1}(x', x_n)}{\partial x_n}$ is an absolutely continuous function in x_n. (See e. g. [6], p. 313).

Obviously we have

$$f(x', x_n) = -\frac{1}{\Gamma(s)} \int\limits_{x_n}^{\infty} (t - x_n)^{s-1} \frac{\partial^s f}{\partial t^s}(x', t)\, dt,\ y_n > \chi(x'). \tag{2.3}$$

Using HARDY inequality (see [7] Theorem 329), we get

$$\int\limits_{\chi(x)}^{\infty} (f(x', x_n))^2\, dx_n \leqslant C(s)\, h^{2s} \int\limits_{\chi(x)}^{\infty} \left(\frac{\partial^s f}{\partial t^s}(x', t)\right)^2 dt. \tag{2.4}$$

Integrating by x', we get our result.

Lemma 2.2. *Let $G_{M,\chi}$ be a M-LIPSCHITZ domain. There exists a mapping T of $W^0_2(G_{M,\chi})$ into $W^0_2(R_n)$, such that*

$$(Tf)(x) = f(x) \quad \text{for} \quad x \in G_{M,\chi} \tag{2.5}$$

and for $f \in W^s_2(G_{M,\chi})$, $S \geqslant s \geqslant 0$ we have

$$\|Tf\|_{W^s_2(R_n)} \leqslant D(M, S)\, \|f\|_{W^s_2(G_{M,\chi})}. \tag{2.6}$$

where $D(M, S)$ does not depend on χ.

Lemma 2.2 is a form of extension theorem which follows immediately from the theorem for s integral by using the theory of interpolated spaces. (See e. g. [4], [5]). For s an integer the lemma is known as the CALDERON theorem or E. STEIN universal extension theorem (see [8]).

Lemma 2.3. *Let $G_{M,\chi}$ be a M-LIPSCHITZ domain. Further let $\varkappa > 1$, $q > 0$, $Z^{\varkappa,q}_j = E[\|x\| > \varkappa^{-j} q]$, $K^{\varkappa,q}_j = E[\|x\| \leqslant \varkappa^{-j} q]$, $\mathbf{y} = (0)$, $\boldsymbol{\alpha} = \alpha$, $f \in W^{k,s}_{2,y,\alpha}(G_{M,\chi})$ and $f = 0$ on $G_{M,\chi} \cap Z^{\varkappa,q^*}_0$ $q^* = \frac{q}{8 \cdot \max(1, M)}$.*

[2] C is a generalic constant with different values on different places.

Then

$$f = \sum_{j=0}^{N} f_j + g \tag{2.7}$$

where

(1) $$f_j = 0 \quad on \quad G_{M,\chi} \cap Z_j^{\varkappa, q}, \tag{2.8}$$

(2) $$f_j \in W_2^k(G_{M,\chi}) \quad and \tag{2.9}$$

$$\| f_j \|_{W_2^k(G_{M,\chi})} \leqslant C_1 \varkappa^{j\alpha} \| f \|_{W^{k,s}_{2,y,\alpha}(G_{M,\chi})}. \tag{2.10}$$

(3) $$g = 0 \ on \ G_{M,\chi} \cap Z^{\varkappa,q}_{N+1}, \tag{2.11}$$

$$\| g \|_{W_2^s(G_{M,\chi})} \leqslant C_2 \| f \|_{W^{k.s}_{2,y,\alpha}(G_{M,\chi})}. \tag{2.12}$$

In the above inequalities, C_1 and C_2 do not depend on j.

Proof.

(1) Let us put $\Lambda^M = E\,[(x', x_n),\ x_n > -3 \cdot \max(1, M) \| x' \|]$.

$$\Lambda^M(t) = \Lambda^M + \underline{t}, \quad \underline{t} = (\underline{0}, t), \quad t > 0$$

(i. e. $\Lambda^M(t) = E\,[(x', x_n), \quad x_n - t \geqslant -3 \max(1, M) \| x' \|]$).

Further let us define $G^t_{M,\chi} = G_{M,\chi} \cap (\Lambda^M + t)$. Obviously $G^t_{M,\chi} - \underline{t}$ is $3 \max(1, M)$-Lipschitz domain.

(2) Let φ_j be the extension of the function f from $G^{t_j}_{M,\chi}$, $t_j = q\, \varkappa^{-j}$ into $W_2^k(R_n)$. Because $\| f \|_{W_2^k(G^{t_j}_{M,\chi})} \leqslant C_1 t_j^{-\alpha} \| f \|_{W^{k,s}_{2,0,\alpha}(G_{M,\chi})}$ we have

$$\| \varphi_j \|_{W_2^k(R_n)} \leqslant C_2 \varkappa^{j\alpha} \cdot q \| f \|_{W^{k,s}_{2,0,\alpha}(G_{M,\chi})} \tag{2.13}$$

where C_2 does depend in general on q. We have

$$G^{t_j}_{M,\chi} \supset Z_j^{\varkappa,q} \cap G_{M,\chi} \text{ for all } j. \tag{2.14}$$

Let us now put

$$\varphi_{j+1} - \varphi_j = f_j. \tag{2.15}$$

On $G^{t_j}_{M,\chi}$ we have $\varphi_{j+1} = \varphi_j = f$. Because $f_j = 0$ on $G^{t_j}_{M,\chi}$, (2.8) holds and because of (2.13) we have proved also (2.10).

Because $G^{t_0}_{M,\chi} \subset Z_0^{\varkappa,q^*} \cap G_{M,\chi}$ we have $\varphi_0 = 0$ and so

$$\sum_{j=0}^{N} f_j = \varphi_{N+1} \tag{2.16}$$

and $f - \varphi_{N+1} = 0$ on $G^{t_{N+1}}_{M,\chi} \supset Z^{\varkappa,q}_{N+1} \cap G_{M,\chi}$ so (2.11) is proved. Because of the extension theorem — Lemma 2.2 — (2.12) holds also. So the theorem is completely proved.

Let us introduce one definition.

Definition 2.1. *We shall say that the function* $\omega(x)$, $x \in R_n$ *is a k-proper function* $(k > 1)$ *if*

(a) $\omega(x) \in W_2^1(R_n)$.

(b) $\omega(x)$ *has compact support.*

(c) *There exists a constant* C *such that for every* $f \in W_2^\beta(R_n)$ *and every* $h > 0$ *there exists* $C(k, h)$, $k = (k_{,1} \ldots, k_n)$, k_i, $i = 1, \ldots, k_n$, *integers, such that*

$$\left\| f - \sum C(k, h)\, \omega\left(\frac{x - h k}{h}\right)\right\|_{W_2^\alpha(R_n)} \leqslant C \|f\|_{W_2^\beta(R_n)} h^\mu \tag{2.17}$$

where $0 \leqslant \alpha \leqslant 1$, $\alpha \leqslant \beta$ *and* $\mu = \min[\beta - \alpha, k - \alpha]$.

(d) *There exits a constant* L, *such that if* S *is the compact support of* $f \in W_2^\beta(R_n)$ *then the function* $\sum C(k, h)\, \omega\left(\frac{x - h k}{h}\right)$ *in* (2.17) *has the support in an* $L \cdot h$ *neighborhood of* S.

Such functions have been studied in [9].

We have introduced the space $W_{2, y, \alpha}^{k, s}(\Omega)$. Let us introduce now the space $V_{2, y, \alpha}^{k, s}(\Omega)$, $s > 1$, $k \geqslant 2$, k integer. Let

$$V_{2, y, \beta}^{k, s}(\Omega) \subset \prod_{j=2}^{k} W_{2, y, \beta+j-2}^{j, s}(\Omega), \quad \beta = (\beta_1, \ldots, \beta_p), \beta_i \geqslant 0,$$

with the norm

$$\|f\|_{V_{2, y, \beta}^{k, s}(\Omega)} = \max_{j = 2, \ldots, k} [\|f\|_{W_{2, y, \beta + j - 2}^{j, s}(\Omega)}].$$

3. The Neumann Problem

As a model problem we shall investigate the NEUMANN problem for the equation

$$-\Delta u + a(x)\, u = f \quad \text{on} \quad \Omega \tag{3.1}$$

with $a(x) \geqslant 0$, $a(x) \in C^\infty$, $a(x)$ non equal identically zero with the boundary condition

$$\frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \Omega^{\cdot}. \tag{3.2}$$

We shall be interested in the weak solution.

Definition 3.1. *Let* Ω *be a* LIPSCHITZ *domain. The function* $u \in W_2^1(\Omega)$ *will be a weak solution of the* NEUMANN *problem* (3.1) *and* (3.2) *if and only if*

$$\sum_{i=1}^{n} \int_\Omega \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i}\, dx + \int_\Omega a(x)\, u\, v\, dx = \int_\Omega f\, v\, dx \quad \textit{for every} \quad v \in W_2^1(\Omega).$$

Theorem 3.1. *Let $f \in L_2(\Omega)$. Then there exists exactly one weak solution of the* NEUMANN *problem.*

This is well known theorem. (See e. g. [10]).

Definition 3.2. *Let Ω be a* LIPSCHITZ *domain, $\mathbf{y} = (y_1, \ldots, y_p)$ $y_i \in \Omega^{\cdot}$, $y_i \neq y_j$, $i \neq j$, $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_p)$, $\beta_i \geqslant 0$. The* NEUMANN *problem will be said to be $(k, s, \mathbf{y}, \boldsymbol{\beta})$ regular if its weak solution $u \in W_2^1(\Omega)$ is such that*

$$\| u \|_{V^{k,s}_{2,y,\beta}(\Omega)} \leqslant C \| f \|_{W_2^{k-2}(\Omega)}, \quad k \geqslant 2, \quad s > 1. \tag{3.3}$$

4. The Numerical Solution of the Neumann Problem

Let us propose a method for the approximative solution of the NEUMANN problem in the sense of the *Definition* 3.1. Put $\lambda > 1$, $1 > h > 0$, $\varkappa > 1$, $q > 0$, $q_1 > 0$, $k > 0$, $\varrho > 0$ (k, ϱ integrals). Further let (see *Definition* 2.1).

$$u_h^\varrho = \sum_{j=0}^{\varrho} \sum_{k} C_j(k, h)\, \omega\left(\frac{x - h\,\lambda^{-j}\, k\, q_1}{h\,\lambda^{-j}\, q_1}\right) \tag{4.1}$$

where ω is k-proper function and the second sum is taken in following manner

(1) $C_0(k, h) = 0$ for k such that

$$\operatorname{supp}\left[\omega\left(\frac{x - h\,k\,q_1}{h\,q_1}\right)\right] \cap \Omega \neq \emptyset^3. \tag{4.2}$$

(2) $C_j(k, h) = 0$, $\varrho \geqslant j > 0$ for all $\underline{k}$ such that

$$\operatorname{supp}\left[\omega\left(\frac{x - h\,\lambda^{-j}\,k\,q_1}{h\,\lambda^{-j}\,q_1}\right)\right] \cap Q_j \neq \emptyset \tag{4.3}$$

where

$$Q_j = \bigcup_{r=1}^{p} K_{j,r}^{\varkappa, 2q}, \quad K_{j,r}^{\varkappa, q} = E\,[x \in \Omega, \quad \| x - y_r \| \leqslant \varkappa^{-j} q]$$

$$r = 1, \ldots, p, \quad y_r \in \Omega^{\cdot}.$$

The value $C_j(k, h)$ are to be determined so that

$$(u_h^\varrho, v)_A = (v, f)_{L_2} \tag{4.4}$$

holds for all $v = \omega\left(\dfrac{x - h\,\lambda^{-j}\,k}{\lambda^{-j}\,h}\right)$ that occur in the sum (4.1), with

$$(u_h^\varrho, v)_A = \int_\Omega \left(\sum_{i=1}^{n} \frac{\partial u_h^\varrho}{\partial x_i} \frac{\partial v}{\partial x_i} + a\, u_h^\varrho\, v\right) dx \tag{4.5}$$

[3] supp $[f(x)]$ means support of $f(x)$.

and

$$(v, f)_{L_2} = \int_\Omega v\, f\, dx. \tag{4.6}$$

So for $C_j(k, h)$ we get a system of linear algebraic equations. Provided that all functions are linear independent we get a positive definite matrix. Hence the solution exists in this case. If the functions were linear dependent we may exclude some of them and we get a positive definite matrix also. So u_h^ϱ is determined uniquely by (4.4) (may be $C_j(k, h)$ are not).

Now our goal in general will be following. To select λ, $\varkappa$, q, q_1, k, ϱ as functions of the parameter $1 > h > 0$, to construct the functions $u_h^{\varrho(h)} = w_h$ and

(1) to estimate the error

$$\varepsilon(h) = \| u - u_k^\varrho \|_{W_2^1}.$$

(2) to determine number $P(h)$ of the unknowns which we have to solve by the system of linear algebraic equations (4.4).

(3) to get the estimate in the form

$$\varepsilon(h) \leqslant C\, [[P(h)]^{-\frac{1}{n}}]^\mu \, \| f \|_{W_2^k}.$$

Let us prove the following theorem.

Theorem 4.1. *Let the* NEUMANN *problem be* $(k, s, \mathbf{y}, \boldsymbol{\beta})$ *regular with* $k \geqslant 2$, $\beta = (\beta_1, \ldots, \beta_p)$, $\beta_i \leqslant \beta_0 < 1$, $s > 1$ *as in Definition* (3.2). *Let* ω *be* k*-proper function and*

$$\varkappa > 1, \quad \lambda = \varkappa\, \xi > 1, \quad \xi < 1. \tag{4.7}$$

$$|\lg \xi|^{-1} > 2(k-1) \max\left[1, \frac{1}{\lg \varkappa}\right] \max\left[\frac{1}{s-1}, \frac{1}{1-\beta_0}\right] \tag{4.8}$$

$$\varrho = \left| \left[\frac{k-1}{s-1} \left| \frac{\lg h}{\lg \varkappa} \right| \right] \right| + 1^4 \tag{4.9}$$

$$q_1 \leqslant \frac{q}{2\, L\, n}. \tag{4.10}$$

Then

$$\| u_h^\varrho - u \|_{W_2^1(\Omega)} \leqslant C\, h^{k-1} \| f \|_{W_2^{k-2}(\Omega)}. \tag{4.11}$$

Proof.

(1) Let us take $q > 0$ such that the sets $\overline{K}_{o,r}^{\varkappa, 2q}$ $r = 1, \ldots, \varrho$ are disjoint.

(2) By the assumption $u \in V_{2,y,\beta}^{k,s}(\Omega)$, and using Lemma 2.3 we may

[4] $|[\xi]|$ means the integral part of ξ.

write

$$u = u_0 + \sum_{l=1}^{p} \sum_{j=0}^{\varrho} u_{l,j} + u_{0,0} \tag{4.12}$$

where

$$\| u_0 \|_{W_2^k(\Omega)} \leqslant C \| f \|_{W_2^{k-2}(\Omega)}, \tag{4.13}$$

$$\| u_{l,j} \|_{W_2^k(\Omega)} \leqslant C\, \varkappa^{j(\beta_l + k - 2)} \| f \|_{W_2^{k-2}(\Omega)}, \tag{4.14}$$

$$u_{l,j} = 0 \text{ outside of } K_{j,l}^{\varkappa,q},$$

$$\| u_{0,0} \|_{W_2^s(\Omega)} \leqslant C \| f \|_{W_2^{k-2}(\Omega)}, \tag{4.15}$$

$$u_{0,0} = 0 \text{ outside } \bigcup_{l=1}^{p} K_{p+1,l}^{\varkappa,q}. \tag{4.16}$$

(3) Let $1 > h > 0$. Because of continuation theorem (see Lemma 2.2) and *Definition* 2.1 there exist constants $C_0(k, h)$ such that

$$\left\| u_0 - \sum C_0(k,h)\, \omega\left(\frac{x - h k q_1}{h q_1}\right) \right\|_{W_2^1(\Omega)} \leqslant C h^{k-1} \| f \|_{W_2^{k-2}(\Omega)}. \tag{4.17}$$

Similarly we may find $C_{j,l}(k, h)$ such that

$$\begin{aligned} &\left\| u_{l,j} - \sum_k C_{j,l}(k,h)\, \omega\left(\frac{x - q_1 h (\varkappa\xi)^{-j} k}{q_1 h (\varkappa\xi)^{-j}}\right) \right\|_{W_2^1(\Omega)} \\ &\leqslant C h^{k-1} \cdot \varkappa^{j(\beta_l + k - 2)} (\varkappa\xi)^{-(k-1)j} \| f \|_{W_2^{k-2}(\Omega)} = \\ &= C h^{k-1} \varkappa^{j(\beta_l - 1)} \xi^{-(k-1)j} \| f \|_{W_2^{k-2}(\Omega)}. \end{aligned} \tag{4.18}$$

The function

$$\sum_k C_{j,l}(k,h)\, \omega\left(\frac{x - q_1 h (\varkappa\xi)^{-j} k}{q_1 h (\varkappa\xi)^{-j}}\right) \tag{4.19}$$

has support inside the $L \cdot h (\varkappa\xi)^{-j} q_1$ neighbourhood of $K_{j,l}^{\varkappa,q}$. Provided that $\xi^{-j} h \leqslant 1$ we have

$$\operatorname{supp}\left[\omega\left(\frac{x - q_1 h (\varkappa\xi)^{-j} k}{q_1 h (\varkappa\xi)^{-j}}\right)\right] \cap K_{j,l}^{\varkappa,2q} \neq \emptyset$$

for every k that occurs in (4.19). But from (4.8) it follows immediately that for $h < 1$, $h\,\xi^{-\varrho} \leqslant 1$. Because of the Lemma 2.1 we have

$$\| u_{0,0} \|_{W_2^1(\Omega)} \leqslant C\, [q\, \varkappa^{-(\varrho+1)}]^{s-1} \| f \|_{W_2^{k-2}(\Omega)}.$$

By assumption we have

$$\varkappa^{-(\varrho+1)(s-1)} \leqslant h^{k-1}.$$

(4) Because $(\varkappa^{\beta_0-1}\,\xi^{-(k-1)}) < 1$ by assumption, we have constructed a function $\hat{u}^{\varrho(h)}$ such that

$$\| \hat{u}_h^{\varrho(h)} - u \|_{W_2^1(\Omega)} \leqslant C\, h^{k-1}\, \| f \|_{W_2^{k-2}(\Omega)}.$$

From this we get immediately that

$$\| u_h^{\varrho(h)} - u \|_{W_2^1(\Omega)} \leqslant C\, h^{k-1}\, \| f \|_{W_2^{k-2}(\Omega)}$$

(see e. g. [11]).

Let us now count the number P of unknowns that we have to find under conditions of *Theorem 4.1.* Denoting by P_j, the number of $C_j(k, h)$ in (4.1) we have

$$P_0 \leqslant C\,\frac{1}{h^n},$$

$$P_j \leqslant C\,\frac{1}{h^n}\,[1 + \xi^{-j}\,h\,L']^n\,\xi^j.$$

Because $\xi^{-\varrho}\,h < 1$ and $0 < \xi < 1$ we have

$$P \leqslant \frac{1}{h^n}\,C.$$

Now let us introduce the quantity

$$H = \left[\frac{|\,\Omega\,|}{P\,(h)}\right]^{1/n}$$

where $|\,\Omega\,|$ is the volume of the domain Ω. Let us call H an efficient meshsize. It is obvious that in the case of regular net $H \sim h$. So we have proved

$$\| u - u_h^{\varrho} \|_{W_2^1(\Omega)} \leqslant C\, H^{\mu}\, \| f \|_{W_2^{k-2}}$$

with

$$\mu = k - 1.$$

5. Applications and Conclusions

The basic notion which we have used is the notion about $(k, s, \mathbf{y}, \boldsymbol{\beta})$ regularity. This is a typical case for the boundaries with cones. The singularity of the solution in the neighborhood of a cone is studied by KONDRAT'EV (see [12]). These results are especially important in the case $n = 2$, i. e. in the plane when the domain has piecewise smooth boundary, particularly if the boundary is composed from straight lines in the neighborhood of the corner. In this case and cases that are close to that, the behavior of the solution is described by KONDRAT'EV in a sufficient manner for our purposes. In this case we have $\beta_0 = \frac{1}{2} + \varepsilon$, $\varepsilon > 0$, ε arbitrary. In the case that $a(x) = 0$ in (3.1) in the neighborhood of the corner we may show the regularity by approach similar to [13]. From these results and [14] we see also that the solution $u \in W_2^s(\Omega)$, $s > 1$.

The conclusion is that we are able to solve the problem in the plane on domains with corners with the same rate of convergence as on infinite smooth domain provided that we refine the meshsize in the proper way in the neighborhood of the corners. The rate we mean with respect to the number of unknowns or — it is equivalent — with respect to the efficient meshsize H. It is easy to see by the theory of the n-width that it is the highest possible order. See also [15], [16].

We did not discuss the problems connected with the use of this technique in the numerical approach. These problems will be discussed in a subsequent paper.

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