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Clenshaw-Curtis Quadrature with a Weighting Function

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Summary — Zusammenfassung

Clenshaw-Curtis Quadrature with a Weighting Function. An extension of the Clenshaw-Curtis quadrature method is described for integrals involving absolutely integrable weight functions. The resulting quadrature rules turn out to be slightly lower in accuracy than the corresponding Gaussian rules. This, however, seems to be paid off by the use of preassigned nodes and by the applicability of Fast Fourier Transform techniques. Some specific formulae are derived explicitly and several numerical examples are given.

Clenshaw-Curtis-Integration mit einer Gewichtsfunktion. Das Quadraturverfahren von Clenshaw und Curtis wird auf Integrale übertragen, die absolut integrierbare klassische Gewichtsfunktionen enthalten. Es stellt sich heraus, daß die entstehenden Quadraturformeln in der Genauigkeit den entsprechenden Gaußschen Formeln nur wenig nachstehen, jedoch wegen der vorgegebenen Stützstellen und wegen der Anwendbarkeit des Algorithmus von Cooley und Tukey numerische Vorteile besitzen. Einige Formeln werden zusammen mit numerischen Beispielen explizit angegeben.

1. Introduction

The Clenshaw-Curtis quadrature scheme [2] for the evaluation of

$$\int_{-1}^{1} f(x)\,dx \tag{1}$$

is obtained by replacing $f(x)$ with the polynomial $P_N(x)$ of degree N which interpolates $f(x)$ at the points

$$x_j = \cos(\pi j/N) \quad (j = 0, 1, \ldots, N), \tag{2}$$

and by evaluating the resulting integral exactly. In [2] it is shown that $P_N(x)$ can be represented in the form

$$P_N(x) = \sum_{j=0}^{N}{}'' a_j T_j(x), \tag{3}$$

where

$$a_j = \frac{2}{N} \sum_{k=0}^{N}{}'' f(x_k)\, T_j(x_k). \tag{4}$$

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Here $T_j(x)$ is the jth Chebyshev polynomial on $(-1, 1)$ and $\sum''$ indicates that the first and last term of the sum are to be halved.

The problem of how to choose an appropriate value of N is discussed extensively in [2] and by FRASER and WILSON in [6]. The number N is determined from a check on the smallness of several of the highest coefficients a_j in formula (3). Bounds for the error of the approximation $P_N(x)$ to $f(x)$ and for the Clenshaw-Curtis quadrature scheme are given by ELLIOT [5] and by CHAWLA [1].

For sufficiently smooth integrands the Clenshaw-Curtis rules yield results which are almost as good as those obtained by the corresponding Gaussian rules. This is due to the favorable choice of nodes in the interpolation process (3) and (4).

In this note we extend the principle used by Clenshaw and Curtis to integrals of the form

$$I(f) = \int_{-1}^{1} f(x)\, w(x)\, dx, \tag{5}$$

where $f(x)$ is a "well-behaved" function and $w(x)$ an absolutely integrable classical weight function, i.e. a function for which the moments are known.

The rules discussed in 2. are obtained by replacing $f(x)$ in (5) with the polynomial (3) and by integrating the resulting expression exactly. It follows immediately that the error criteria mentioned above for $w(x) \equiv 1$ carry over with slight modifications to the general case. Some well known quadrature formulae are found among those derived in 3. for specific functions $w(x)$. Numerical examples for the weight function $w(x) = (1-x)^{-1/2}$ are given in 4.

We point out that because of (4) the Fast Fourier Transform techniques described in [3, 4] can be used in the computation of definite integrals. On the other hand, the quadrature coefficients given explicitly in 2. and 3. are useful if the corresponding quadrature rules serve to discretize weakly singular integral or integro-differential equations.

2. Derivation of the Quadrature Formulae

With $P_N(x)$ from (3) we consider

$$I_N(f) = \int_{-1}^{1} P_N(x)\, w(x)\, dx \tag{6}$$

an approximation to $I(f)$ in (5). Substitution of (3) and (4) into (6) and rearrangement of the terms yields the quadrature rule

$$I_N(f) = \sum_{j=0}^{N}{}'' A_j^{(N)} f(x_j), \tag{7}$$

where the nodes x_j are defined in (2) and the weights are

$$A_j^{(N)} = \frac{2}{N} \sum_{k=0}^{N}{}'' \left\{ \int_{-1}^{1} T_k(x)\, w(x)\, dx \right\} T_k(x_j). \tag{8}$$

We note that the integrals in (8) can be written as

$$J_k(w) = \int_0^\pi \cos kt\, w(\cos t) \sin t\, dt. \tag{9}$$

Theorem: *The quadrature formula* (7) *has the properties*:

i) *It is exact for all polynomials $f(x)$ of degree $\leq N$.*

ii) *The weights $A_j^{(N)}$ $(j=0,1,\ldots,N)$ are bounded by*

$$2\int_{-1}^{1} |w(x)|\, dx$$

for all N.

iii) *The weights $A_j^{(N)}$ $(j=0,1,\ldots,N)$ tend to zero as $N\to\infty$.*

Proof: Property i) follows from the interpolatory character of our quadrature formula. The boundedness condition ii) is deduced from

$$|J_k(w)| \leq \int_{-1}^{1} |w(x)|\, dx \tag{10}$$

and from (8). To show iii) we note that the function $w(\cos t)\sin t$ in (9) is absolutely integrable on $[0,\pi]$. Thus $J_k(w)\to 0$ as $k\to\infty$ by the Riemann-Lebesgue lemma. But since the right-hand side of the inequality

$$|A_j^{(N)}| \leq \frac{2}{N} \sum_{k=0}^{N} |J_k(w)| \tag{11}$$

is twice the Cesaro limit of $|J_k(w)|$ we obtain iii).

3. Specific Weight Functions

a) $w(x)\equiv 1$: $J_{2k}(w) = -4/(4k^2-1)$, $J_{2k+1}(w)=0$.

This is the Clenshaw-Curtis quadrature scheme in the form given by FRASER and WILSON [6].

b) $w(x)=(1-x^2)^{-1/2}$: $J_0(w)=\pi$, $J_k(w)=0$ otherwise.

This yields

$$\int_{-1}^{1} f(x)\,(1-x^2)^{-1/2}\, dx \approx \frac{\pi}{2N}\left[f(-1)+f(1)+2\sum_{j=1}^{N-1} f(\cos(\pi j/N))\right] \tag{12}$$

which is the Gauss-Chebyshev formula of closed type, i.e. the quadrature rule of highest algebraic degree of precision for preassigned nodes $x=\pm 1$. It is exact for all polynomials $f(x)$ of degree $\leq 2N-1$ (see KRYLOV [7], where also error estimates are given).

c) $w(x)=(1-x^2)^{m-1/2}$, m positive integers:

We obtain

$$J_{2k}(w) = (-1)^k\, 2^{-2m}\, \pi \binom{2m}{m+k},\; J_{2k+1}(w)=0 \quad (k=0,1,\ldots). \tag{13}$$

For $m=1$ only $J_0(w)$ and $J_2(w)$ are different from zero and the weights are found to be

$$A_j^{(N)}=\frac{\pi}{N}\sin^2(\pi j/N), \tag{14}$$

i.e. $A_0^{(N)}$ and $A_N^{(N)}$ are equal to zero and the other weights are positive. On setting $N=M+1$ we obtain the Gauss-type formula

$$\int_{-1}^{1} f(x)(1-x^2)^{1/2}\,dx\approx\frac{\pi}{M+1}\sum_{j=1}^{M}\sin^2(\pi j/(M+1))\,f(\cos(\pi j/(M+1))) \tag{15}$$

discussed in [7] which is exact for all polynomials $f(x)$ of degree $\leq 2M-1$.

d) $w(x)=(1-x)^{1/2}(1+x)^{-1/2}$: $J_0(w)=-2J_1(w)=\pi$; $J_k(w)=0, k\geq 2$.

The weights are

$$A_j^{(N)}=\frac{2\pi}{N}\sin^2(\pi j/2N)\geq 0, \tag{16}$$

thus yielding a semi-open formula because of weight zero at $x=1$. A check of the general criteria developed in [7] indicates that we have derived the quadrature rule of highest algebraic degree of precision among those having a preassigned node at $x=-1$. The formula is exact for all polynomials of degree $\leq 2N-2$.

e) $w(x)=(1-x)^{-1/2}$: $J_k(w)=2\sqrt{2}(-1)^{k+1}/(4k^2-1)$.

It follows that

$$A_j^{(N)}=\frac{4\sqrt{2}}{N}\sum_{k=0}^{N}{}'' \frac{(-1)^{k+1}\cos(\pi kj/N)}{4k^2-1}. \tag{17}$$

It is easy to show that the $A_j^{(N)}$ are positive for all j and N.

In the case of $w(x)=\ln(1-x)$ the formula for the $J_k(w)$ is somewhat more complicated and we do not present it here.

4. Numerical Examples

We discuss some numerical values for integrals involving the weight function $w(x)=(1-x)^{-1/2}$. We emphasize that the material presented is typical of the results obtained for the other weight functions discussed in 3.

Table 1 shows the nodes x_j and weights $A_j^{(N)}$ for $N=2, 4, 8$. For each N the $N+1$ nodes x_j are found in the rows in which the corresponding $N+1$ weights $A_j^{(N)}$ are listed. We give only 6 decimals although 15 were used in our computations. The weights $A_0^{(N)}$ and $A_N^{(N)}$ have been divided by 2 so that $\sum''$ in (7) becomes $\sum$.

Table 1

x_j	$A_j^{(N)}$		
	$N=2$	$N=4$	$N=8$
−1.000 000	0.188 562	0.044 896	0.011 092
−0.923 880	. . .	. . .	0.105 695
−0.707 107	. . .	0.413 569	0.213 496
−0.382 683	. . .	. . .	0.308 031
0.000 000	1.508 494	0.790 164	0.393 028
0.382 683	. . .	. . .	0.461 522
0.707 107	. . .	1.023 093	0.513 285
0.923 880	. . .	. . .	0.544 511
1.000 000	1.131 371	0.556 706	0.277 766

Table 1 shows that the weights $A_j^{(N)}$ do not vary widely in size, in contrast to the Newton-Cotes type formulae for large N. Because of the interpolatory character of the quadrature rules the weights add up to $2\sqrt{2}$, the value of the definite integral of $w(x)$ over $[-1, 1]$.

In the following examples we give approximating values of the integral in question for those values of N which show typical stages of accuracy for each specific function $f(x)$. Normally one will pick only values $N=N_0 \cdot 2^j$ $(j=0, 1, \ldots)$ to make used of the previously calculated values of $f(x)$.

Example 1: $\int_{-1}^{1} e^x (1-x)^{-1/2}\, dx \approx 4.598\,499.$

Table 2

N	3	4	6	8	10	11
$I_N - I$	$-.16_{10}-3$	$.25_{10}-4$	$.39_{10}-7$	$.57_{10}-10$	$.69_{10}-13$	$.10_{10}-15$

The very accurate results are due to the smoothness of $f(x)$.

Example 2: $\int_{-1}^{1} \cos 10x\, (1-x)^{-1/2}\, dx \approx -0.585\,092.$

Table 3

N	8	12	16	20	24	32
$I_N - I$	$.14_{10}-1$	$.62_{10}-4$	$.37_{10}-6$	$.91_{10}-9$	$.12_{10}-11$	$.44_{10}-15$

In this example $f(x)$ is an oscillating function, and it takes more nodes to reasonably approximate $f(x)$ and thus $I(f)$.

Example 3: $\int_{-1}^{1} \sin\sqrt{1-x}\, dx = \int_{-1}^{1} [\sqrt{1-x}\, \sin\sqrt{1-x}]\, (1-x)^{-1/2}\, dx \approx$
$\approx 1.534\,456.$

Table 4

N	2	3	4	5	6	7
$I_N - I$	$-.17_{10}-2$	$.32_{10}-5$	$-.84_{10}-8$	$.19_{10}-10$	$-.34_{10}-13$	$.22_{10}-15$

The integrand $\sin\sqrt{1-x}$ has a singular first derivative. After splitting off the part causing this trouble we get the analytic function $f(x)=\sqrt{1-x}\,\sin\sqrt{1-x}$, and our quadrature process yields astonishingly good values.

If compared experimentally with Gaussian quadrature rules in the case where they differ from those, the above rules turn out to be somewhat lower in accuracy. This seems to be paid off, however, by the use of preassigned nodes which make computations for several N more economical.

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