Computing 4, 202—206 (1969)

Note on the Computation of Jacobi's Nome and its Inverse

By

H. E. Fettis, Ohio

(Received September 18, 1968)

Summary — Zusammenfassung

Note on the Computation of Jacobi's Nome and its Inverse. By employing the method of modulus reduction, JACOBI's nome and its inverse may both be computed to any desired accuracy without recourse to series expansions. The accuracy attainable is limited only by the accuracy to which the given parameters are known.

Bemerkung zur Berechnung des Jacobischen Nomes und seiner Inversen. Durch Anwendung der Modul-Reduktionsmethode kann der JACOBIsche Nome und seine Inverse ohne Benutzung von Reihenentwicklungen mit beliebiger Genauigkeit berechnet werden. Die erreichbare Genauigkeit ist lediglich durch die Genauigkeit beschränkt, mit der die gegebenen Parameter bekannt sind.

In the theory of elliptic and theta functions the parameter q known as the "nome" is of prime importance. It may be defined in terms of the complete elliptic integral $K(k)$ by

$$q = e^{-\pi K'/K} \tag{1}$$

where $K' = K(k')$ and $k' = \sqrt{1-k^2}$ is the complementary modulus. Its importance lies in the fact that all of the JACOBIan elliptic functions, as well as the theta functions, have simple expansions in terms of it. In particular, the modulus and the complementary modulus may both be expressed as functions of q by

$$\sqrt{k(q)} = \frac{H_1(0,q)}{\theta_1(0,q)} = \frac{2\sum\limits_{m=1}^{\infty} q^{\left(m-\frac{1}{2}\right)^2}}{1+2\sum\limits_{1}^{\infty} q^{m^2}} \tag{2}$$

$$\sqrt{k'(q)} = \frac{\theta(0,q)}{\theta_1(0,q)} = \frac{1+2\sum\limits_{1}^{\infty}(-1)^m q^{m^2}}{1+2\sum\limits_{1}^{\infty} q^{m^2}}. \tag{3}$$

By forming the quantity

$$Z = \frac{1 - \sqrt{k'}}{1 + \sqrt{k'}}, \tag{4}$$

we find that

$$Z = \frac{2 \sum_{1}^{\infty} q^{(2m-1)^2}}{1 + 2 \sum_{1}^{\infty} q^{(2m)^2}} = \frac{2 \sum_{1}^{\infty} (q^4)^{\left(m - \frac{1}{2}\right)^2}}{1 + 2 \sum_{1}^{\infty} (q^4)^{m^2}}. \tag{5}$$

Hence

$$Z = \sqrt{k(q^4)}. \tag{6}$$

Since $q < 1$, $q^4 < q$, so that more rapid convergence will result from the latter series (5).

In many problems it is desirous to have q expressed directly as a function of k. This may be done by inversion of the series (2). The first fourteen terms of this series have been given in [1]. Since Z is the same function of q^4 as $\sqrt{k}$ is of q, the latter series with Z as argument can also be used to accelerate convergence of the inverted series. However, as shown in [1], even with fourteen terms of the above series, at most seven places can be obtained correctly when $k > .9999$.

In this note we show how the method of modulus reduction can be applied to compute either $k(q)$ or $q(k)$ to as many places as is desired for any $k < 1$. The method is better adapted to computers since it requires no constants as initial input as does the series method. It is based on the fact that if two moduli k and k_1 are related according to

$$k_1 = \frac{1 - k'}{1 + k'}, \tag{7}$$

where k' is the complementary modulus of k (GAUSS' or LANDEN's Transformation). The corresponding nomes q and q_1 are related by

$$q_1 = q^2 \tag{8}$$

By noting that

$$k_2 = \frac{1 - k_1'}{1 + k_1'} = \left(\frac{1 - \sqrt{k'}}{1 + \sqrt{k'}}\right)^2 \tag{9}$$

we see that the transformation defined by Eq. (4) is actually equivalent to two applications of the GAUSS transformation. It is also clear that repeated applications of (7) and (8) will lead to sequences of k's and q's which tend to zero. Let N be that value of n for which q_n is zero to within a specified tolerance. Then from (2)

$$k_n \cong 4\, q_n^{\frac{1}{2}} = 4\, q_{n-1}, \tag{10}$$

with an error less than q_n. This leads to the following procedures:

1. Given k to find q

It is reasonable to assume that the initial values of k and k' are known accurately. Then, without appreciable loss of significant digits we can compute successive values of k_n and k'_n according to the scheme

$$\begin{aligned} k_{n+1} &= \left(\frac{k_n}{1+k'_n}\right)^2 \\ k'_{n+1} &= \frac{2\sqrt{k'_n}}{1+k'_n} \end{aligned} \tag{11}$$

until a value of k_n is obtained which is zero to within the desired tolerance. The corresponding value of q is then found from

$$q_n \cong \left(\frac{k_n}{4}\right)^2 \tag{12}$$

and the original q found by successive square roots. $\left(\text{Note that } q_{n-1} = \sqrt{q_n} \cong \frac{k_n}{4}.\right)$

2. Given q to find k

In this case we form successive q_n according to

$$q_{n+1} = q_n^2 \tag{13}$$

continuing until $q_n \cong 0$. At this point the corresponding modulus is given by

$$k_n \cong 4\, q_n^2. \tag{14}$$

The original modulus may now be determined by constructing the sequence

$$\begin{aligned} k_{n-1} &= \frac{2\sqrt{k_n}}{1+k_n} \\ k'_{n-1} &= \left(\frac{k'_n}{1+k_n}\right)^2 \end{aligned} \tag{15}$$

for $n = N, \ldots, 0$, thus arriving at the values of k and k' corresponding to the given value of q.

Advantages over the series method

1. Eqs. (11) and (15) differ only in that the roles of k and k' are interchanged. Thus $k(q)$ and $q(k)$ are obtainable by essentially the same coding.

2. When calculating $q(k)$, the accuracy attainable is unlimited, while the accuracy attainable by the series method is limited by the number of terms employed.

3. No input constants are required.

4. By combining a sine and cosine routine with the present method, all of the JACOBIan functions can be computed for a given "q" by a single program. (See further [2], [4], and [5].)

Illustrative Examples

1. Given $k = \sin 89°$, $k' = \cos 89°$, *to find* q.

Table 1. The sequence of reduced moduli calculated from Eq. (11)

n	k_n			k'_n		
0	.99984	76951	56	.17452	40643	73
1	.96569	39109	35	.25968	30190	50
2	.58770	10087	10	.80907	81742	20
3	.10553	54082	48	.99441	55457	38
4	.28000	45493	89×10^{-2}	.99999	60798	65
5	.19600	71375	70×10^{-5}	.99999	99999	98
6	.96046	99494	58×10^{-12}	1.00000	00000	00

Since $k_6 < 10^{-10}$ we have, according to Eq. (16):

$$q_5 \cong \frac{k_6}{4} = .24001 \quad 74873 \quad 64 \times 10^{-12}$$

$$q_4 \cong \frac{\sqrt{k_6}}{2} = .49001 \quad 78439 \quad 24 \times 10^{-6}$$

$$q_3 = .70001 \quad 27455 \quad 43 \times 10^{-3}$$

$$q_2 = .26457 \quad 75397 \quad 77 \times 10^{-1}$$

$$q_1 = .16265 \quad 83965 \quad 79$$

Finally

$$q = .40330 \quad 93063 \quad 38$$

2. Given $q = .345$, *to find* k *and* k'.

Successive squarings give

$$q_5 = .16225 \quad 98396 \quad 07 \times 10^{-14} < 10^{-10}$$

Hence

$$k_6 \cong 4\,q_4^2 \cong 4\,q_5 = .64903 \quad 93584 \quad 29 \times 10^{-14}$$

Table 2. Table of increasing k's constructed from Eq. (15)

n	k_n			k'_n		
6	.64903	93584	29×10^{-14}	1.00000	00000	00
5	.16112	59579	87×10^{-6}	1.00000	00000	00
4	.80280	98306	91×10^{-3}	.99999	96777	48
3	.56622	34006	27×10^{-1}	.99839	56683	12
2	.45040	59477	93	.89282	38808	37
1	.92542	76731	05	.37892	42956	73
0	.99924	97017	55	.38730	26649	43×10^{-1}

Hence for $q = .345$

$$k = .99924 \quad 97017 \quad 55$$

$$k' = .03873 \quad 02664 \quad 943.$$

References

[1] LOWAN, A. N., G. BLANCH, and W. HORENSTEIN: On the Inversion of the q-Series Associated with JACOBIan Elliptic Functions. Bull. Amer. Math. Soc. **48**, 737 (1942).

[2] CURTIS, A. R.: Tables of JACOBIan Elliptic Function Whose Arguments are Rational Fractures of the Quarter Period. Mathematical Tables (National Physical Laboratory, Teddington) **7**, (1964).

[3] ABRAMOWITZ, M., and I. STEGUN: Handbook of Mathematical Functions. Nat. Bur. Stand. Appl. Math. Ser. **53**, Washington, D. C., 1964.

[4] SALZER, H. E.: Quick Calculation of JACOBIan Elliptic Functions. J. Assn. Computing Mach. **5**, 399 (1962).

[5] FETTIS, H. E., and J. C. CASLIN: Elliptic Functions for Complex Arguments, Aerospace Res. Labs., 67.0001, January 1967.

Henry E. Fettis
Applied Mathematics Research Laboratory
Aerospace Research Laboratories
United States Air Force
Wright—Patterson Air Force Base, Ohio,
U.S.A.