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Discrete Time Semigroup Transformations with Random Perturbations

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Let $(X, \mathscr{B})$ and $(Y, \mathscr{C})$ be two measurable spaces with X being a linear space. A system is determined by two functions $f(X)$: $X \to X$ and φ: $X \times Y \to X$, a (small) positive parameter ε and a homogeneous Markov chain $\{y_n\}$ in $(Y, \mathscr{C})$ which describes random perturbations. States of the system, say $\{x_n^\varepsilon \in X, n = 0, 1,...\}$, are determined by the iteration relations: $x_{n+1}^\varepsilon = f(x_n^\varepsilon) + \varepsilon\varphi(x_n^\varepsilon, y_{n+1})$ for $n \geqslant 0$, where $x_0^\varepsilon = x_0$ is given. Here we study the asymptotic behavior of the solution x_n^ε as $\varepsilon \to 0$ and $n \to \infty$ under various assumptions on the data. General results are applied to some problems in epidemics, genetics and demographics.

KEY WORDS: Difference equations; random perturbation; averaging; diffusion approximation; randomly perturbed iterations; stability.

AMS 1980 Subject Classifications: 35R60, 60H15, 60J99.

1. INTRODUCTION

We consider a system in a linear phase space X with discrete time $n \in Z_+$ which is perturbed by a random process $\{y_n, n \in Z_+\}$ defined on a measurable space $(Y, \mathscr{C})$. The system depends on a small parameter $\varepsilon > 0$;

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if $\varepsilon = 0$, the system is nonrandom. Let $x_n^\varepsilon \in X$ denote the state of the system at time n. We suppose that x_n^ε is determined by the recurrence relations:

$$x_{n+1}^\varepsilon = f(x_n^\varepsilon) + \varepsilon\varphi(x_n^\varepsilon, y_{n+1}), \qquad x_n^\varepsilon = x_0 \tag{1}$$

where x_0 is given; $f\colon X \to X$, $\varphi\colon X \times Y \to X$ are given functions. The problem is to investigate the asymptotic behavior of the system as $\varepsilon \to 0$ and $n \to \infty$ under various assumptions on the data.

If $f(x) = x$, Eq. (1) is a perturbed difference equation. For such equations we obtain results that are analogous to results that have been obtained for randomly perturbed dynamical systems and differential equations by several authors; see Gikhman (1950, 1951, 1964), Khasminskii (1966, 1968, 1968a), Papanicolaou (1968), Papanicolaou *et al.* (1977), Pinsky (1974), Pardoux (1977), Krylov *et al.* (1979), Freidlin *et al.* (1979), Rozovskii (1985), Sarafyan *et al.* (1987), Skorokhod (1989), Hoppensteadt *et al.* (1994), and Hoppensteadt *et al.* (1995, 1996). Our results are concerned also with general limit theorems for Markov processes; see Skorokhod (1965), Stroock *et al.* (1979), Ethier *et al.* (1986), Jacod *et al.* (1987), and Protter (1990).

This work was motivated by some problems in epidemics, genetics, and demographics; see Hoppensteadt (1982). We apply our general theorems to problems from these fields.

Assumptions. We suppose that the noise process $\{y_n\}$ satisfies one of the following conditions: (NP1) $\{y_n\}$ is a stationary ergodic process with ergodic distribution $\rho(dy)$, i.e., for any function $g(y)\colon Y \to R$, for which $\int |g(y)|\, \rho(dy) < \infty$, we have

$$P\left\{\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} g(y_k) = \int g(y)\, \rho(dy)\right\} = 1$$

(NP2) $\{y_n\}$ is a homogeneous ergodic Markov process with transition probabilities

$$P_n(y, C) = P\{y_n \in C / y_0 = y\}, \qquad C \in \mathscr{C}, \quad y \in Y$$

and ergodic distribution $\rho(dy)$.

Here and below unlabeled integrals are assumed to be over Y.

Remark. It is easy to see that a stationary ergodic process $\{y_n, n \in Z\}$ generates an ergodic Markov process:

$$z_n = \{y_n, y_{n-1}, y_{n-2}, \ldots\} \qquad \text{in } Y^{Z_+}$$

(NP3) $\{y_n\}$ satisfies (NP2) and, additionally, its transition probabilities satisfy a strong mixing condition:

$$\sum_{k=1}^{\infty} \sup_{y \in Y,\, C \in \mathscr{C}} |P_k(y, C) - \rho(C)| < \infty$$

We use throughout notations: $\mathscr{F}_k$ the σ-algebra generated by $\{y_0,..., y_k\}$, $R_k(y, C) = P_k(y, C) - \rho(C)$, and

$$R(y, C) = 1_C(y) - \rho(C) + 2 \sum_{n=1}^{\infty} R_n(y, C) \tag{2}$$

We consider system (1) in the following phase spaces.

(PS1) X is a separable Banach space.

(PS2) $X = H$, where H is a separable Hilbert space.

(PS3) $X = R^d$.

The function f is assumed to be continuous, and φ measurable in y and continuous in x.

2. DIFFERENCE EQUATIONS

First, we consider Eq. (1) with $f(x) = x$. In this case, $\{x_n^\varepsilon\}$ is determined by the difference equation

$$x_{n+1}^\varepsilon - x_n^\varepsilon = \varepsilon\varphi(x_n^\varepsilon, y_{n+1}), \qquad x_0^\varepsilon = x_0 \tag{3}$$

We suppose condition (PS1) is satisfied and that $\varphi(x, y)$ is bounded in y over Y for all $x \in X$. Let $\bar{\varphi}$ be defined by

$$\bar{\varphi}(x) = \int \varphi(x, y)\, \rho(dy) \tag{4}$$

Let the nonrandom sequence $\{\bar{x}_n^\varepsilon\}$ be determined by the equation

$$\bar{x}_{n+1}^\varepsilon - \bar{x}_n^\varepsilon = \varepsilon\bar{\varphi}(\bar{x}_n^\varepsilon), \qquad \bar{x}_0^\varepsilon = x_0 \tag{5}$$

It is referred to as the *averaged system*. The following result is easy to establish (see, e.g., Hoppensteadt, 1993, Chap. 7.3).

Lemma 1. *Let $\bar{\varphi}$ satisfy the following Lipschitz condition: There exists $\bar{l} > 0$ for which $\|\bar{\varphi}(x) - \bar{\varphi}(x')\| \leqslant \bar{l}\, \|x - x'\|$. Denote by $\bar{x}(t)$ the solution of the differential equation:*

$$\frac{d\bar{x}(t)}{dt} = \bar{\varphi}(\bar{x}(t)), \qquad \bar{x}(0) = x_0 \tag{6}$$

Then for any $t_0 > 0$,

$$\lim_{\varepsilon \to 0} \sup_{\varepsilon n \leqslant t_0} \|\bar{x}(\varepsilon n) - \bar{x}_n^\varepsilon\| = 0 \tag{7}$$

This result can be generalized to randomly perturbed systems in the following way.

Theorem 1. *Let* $\{y_n\}$ *satisfy condition (NP1), and let* φ *satisfy the following conditions.*

(B) $\int \|\varphi(x, y)\| \, \rho(dy) < \infty$ *for all* $x \in X$.

(L1) There exists a measurable function $l(y)\colon Y \to R_+$ *for which* $\int l(y) \, \rho(dy) = \bar{l} < \infty$ *and for* $y \in Y$, $x, x' \in X$, $\|\varphi(x, y) - \varphi(x', y)\| \leqslant l(y) \, \|x - x'\|$.

Then for any $t_0 > 0$

$$P\{\lim_{\varepsilon \to 0} \sup_{n\varepsilon \leqslant t_0} \|x_n^\varepsilon - \bar{x}(\varepsilon n)\| = 0\} = 1 \tag{8}$$

Proof. Using (7) it suffices to prove that

$$P\{\lim_{\varepsilon \to 0} \sup_{n\varepsilon \leqslant t_0} \|x_n^\varepsilon - \bar{x}_n^\varepsilon\| = 0\} = 1 \tag{9}$$

Because of (2) and (4) we can write

$$\begin{aligned} x_n^\varepsilon - \bar{x}_n^\varepsilon &= \varepsilon \sum_{k<n} [\varphi(x_k^\varepsilon, y_{k+1}) - \bar{\varphi}(\bar{x}_k^\varepsilon)] \\ &= \varepsilon \sum_{k<n} [\varphi(x_k^\varepsilon, y_{k+1}) - \varphi(\bar{x}_k^\varepsilon, y_{k+1})] + R_\varepsilon(n) \end{aligned} \tag{10}$$

where

$$R_\varepsilon(n) = \varepsilon \sum_{k<n} [\varphi(\bar{x}_k^\varepsilon, y_{k+1}) - \bar{\varphi}(\bar{x}_k^\varepsilon)] \tag{11}$$

Therefore

$$\|x_n^\varepsilon - \bar{x}_n^\varepsilon\| \leqslant \sup_{k \leqslant n} \|R_\varepsilon(k)\| + \varepsilon \sum_{k<n} l(y_{k+1}) \, \|x_k^\varepsilon - \bar{x}_k^\varepsilon\|$$

from which we can easily deduce

$$\|x_n^\varepsilon - \bar{x}_n^\varepsilon\| \leqslant \sup_{k \leqslant n} \|R_\varepsilon(k)\| \exp\left\{\varepsilon \sum_{k<n} l(y_{k+1})\right\}$$

and hence

$$\sup_{n\varepsilon \leqslant t_0} \|x_n^\varepsilon - \bar{x}_n^\varepsilon\| \leqslant \sup_{\varepsilon n \leqslant t_0} |R_\varepsilon(n)| \exp\left\{\varepsilon \sum_{n\varepsilon \leqslant t_0} l(y_{n+1})\right\} \tag{12}$$

Note that

$$\lim_{\varepsilon \to 0} \varepsilon \sum_{n\varepsilon \leqslant t_0} l(y_{n+1}) = t_0 \int l(y)\, \rho(dy) \tag{13}$$

Let $\hat{R}_\varepsilon(n) = \varepsilon \sum_{k<n} [\varphi(\bar{x}(k\varepsilon), y_{k+1}) - \bar{\varphi}(\bar{x}(k\varepsilon))]$. Lemma 1 implies that

$$\lim_{\varepsilon \to 0} \sup_{\varepsilon n \leqslant t_0} \|R_\varepsilon(n) - \hat{R}_\varepsilon(n)\| = 0$$

To prove the theorem it suffices to show that

$$\lim_{\varepsilon \to 0} \sup_{n\varepsilon \leqslant t_0} \|\hat{R}_\varepsilon(n)\| = 0$$

Let $0 = t_1 < t_2 < \cdots < t_r = t_0$ and

$$S_n^r = \varepsilon \sum_{i=1}^{r-1} \sum_{k<n} (\varphi(\bar{x}(t_i), y_{k+1}) - \bar{\varphi}(\bar{x}(t_i)))\, 1_{\{t_i \leqslant k\varepsilon < t_{i+1}\}} \tag{14}$$

Condition (NP1) implies the relation

$$\lim_{\varepsilon \to 0} \sup_{\varepsilon n \leqslant t_0} \|S_n^r\| = 0$$

In addition, we have

$$\begin{aligned} |\hat{R}_\varepsilon(n) - S_n^r| &\leqslant \varepsilon \sum_{i=1}^{r-1} \sum_{k<n} |\varphi(\bar{x}_k^\varepsilon, y_{k+1}) - \bar{\varphi}(\bar{x}_k^\varepsilon) - \varphi(\bar{x}(t_i), y_{k+1}) \\ &\quad + \bar{\varphi}(\bar{x}(t_i))|\, 1_{\{t_i \leqslant k\varepsilon < t_{i+1}\}} \\ &\leqslant \varepsilon \sum_{i=1}^{r-1} \sum_{k \leqslant n} (\bar{l} + l(y_{k+1}))\, |\bar{x}_k^\varepsilon - \bar{x}(t_i)|\, 1_{\{t_i \leqslant k\varepsilon < t_{i+1}\}} \end{aligned}$$

Using Lemma 1, formulas (13) and (14), we can show that

$$\limsup_{\varepsilon \to 0} \sup_{n\varepsilon \leq t_0} |\hat{R}_\varepsilon(n)| \leq 2\bar{l}^2 \sum_{i=1}^{r-1} (t_{i+1} - t_i) \, |\bar{x}(t_{i+1}) - \bar{x}(t_i)|$$

The right-hand side of this inequality can be made arbitrary small. □

Now we consider deviations of the solution of the perturbed system from the solution of the averaged system. Let

$$z_n^\varepsilon = (x_n^\varepsilon - \bar{x}_n^\varepsilon)/\sqrt{\varepsilon} \tag{15}$$

We show that under certain natural conditions z_n^ε converges to a Gaussian process. This result is based on central limit theorem (CLT) for X-valued random variables. If we suppose that $X = H$, then the CLT does not require additional conditions, and we have the following theorem. In the following theorem as well as other places, the weak convergence of processes means the weak convergence of the finite dimensional distributions of the processes involved.

Theorem 2. *Let $\{y_n\}$ satisfy condition (NP3), X satisfy condition (PS2), and φ satisfy the conditions:*

(C) $\varphi(x, y)$ is bounded and there exists a positive compact operator B: $H \to H$ such that $B^{-1}\varphi(x, y)$ is bounded,

(L2) $(\partial/\partial x)\, \varphi(x, y)$ is continuous in x uniformly with respect to y and

$$\int \sup_x \left\| \frac{\partial}{\partial x} \varphi(x, y) \right\| \rho(dy) < \infty$$

We define the jump process

$$z^\varepsilon(t) = \sum_{n=1}^{\infty} z_n^\varepsilon 1_{\{n\varepsilon \leq t < (n+1)\varepsilon\}} \tag{16}$$

Then $z^\varepsilon(t)$ converges weakly as $\varepsilon \to 0$ to a process $z(t)$, which solves the equation

$$z(t) = \int_0^t \frac{\partial}{\partial x} \bar{\varphi}(\bar{x}(s))\, z(s)\, ds + \eta(t) \tag{17}$$

where $\eta(t)$ is a Gaussian process with independent increments in H having mean $E\eta(t) = 0$ and variance

$$E(\eta(t), x)^2 = \int_0^t \iint (\varphi(\bar{x}(s), y), x)(\varphi(\bar{x}(s), y'), x)\, \rho(dy)\, R(y, dy')\, ds \tag{18}$$

Proof.

(1) Let

$$\eta^{\varepsilon}(t)=\sum_{k=1}^{\infty}\frac{1}{\sqrt{\varepsilon}}R_{\varepsilon}(k)\,1_{\{k\varepsilon\leqslant t<(k+1)\varepsilon\}} \tag{19}$$

with $R_{\varepsilon}(k)$ as in (11). It follows from Theorem 2 in the Appendix that $\eta_{\varepsilon}(t)$ converges weakly to the process $\eta(t)$ described in the theorem.

(2) Note that z_n^{ε} satisfies the relation

$$\begin{aligned} z_n^{\varepsilon}&=\sqrt{\varepsilon}\sum_{k<n}(\varphi(x_k^{\varepsilon},y_{k+1})-\bar{\varphi}(\bar{x}_k^{\varepsilon}))\\ &=\sqrt{\varepsilon}\sum_{k<n}(\varphi(x_k^{\varepsilon},y_{k+1})-\varphi(\bar{x}_k^{\varepsilon},y_{k+1}))+\frac{1}{\sqrt{\varepsilon}}R_{\varepsilon}(n) \end{aligned} \tag{20}$$

Therefore

$$\begin{aligned} \|z_n^{\varepsilon}\|&\leqslant\varepsilon\sum_{k<n}l(y_k)\,\|z_k^{\varepsilon}\|+\frac{1}{\sqrt{\varepsilon}}\|R_{\varepsilon}(n)\|\\ \sup_{k\leqslant n}\|z_k^{\varepsilon}\|&\leqslant\sup_{k\leqslant n}\frac{1}{\sqrt{\varepsilon}}\|R_{\varepsilon}(k)\|\exp\left\{\varepsilon\sum_{k=1}^{n}l(y_k)\right\} \end{aligned} \tag{21}$$

Since the distribution of $\sup_{n\varepsilon\leqslant t_0}(1/\sqrt{\varepsilon})\,\|R_{\varepsilon}(n)\|$ converges to the distribution of $\sup_{t\leqslant t_0}\|\eta(t)\|$ (see the Appendix, Corollary to Theorem 2), therefore $\sup_{n\varepsilon\leqslant t_0}\|z_n^{\varepsilon}\|$ is bounded in probability as $\varepsilon\to 0$.

Set

$$\begin{aligned} \alpha_n^{\varepsilon}=\sqrt{\varepsilon}\sum_{k<n}&(\varphi(x_k^{\varepsilon},y_{k+1})-\varphi(\bar{x}_k^{\varepsilon},y_{k+1})\\ &-\frac{\partial}{\partial x}\varphi(\bar{x}_k^{\varepsilon},y_{k+1})(x_k^{\varepsilon}-\bar{x}_k^{\varepsilon})) \end{aligned}$$

$$\beta_n^{\varepsilon}=\varepsilon\sum_{k<n}\left(\frac{\partial}{\partial x}\varphi(\bar{x}_k^{\varepsilon},y_{k+1}\right)-\frac{\partial}{\partial x}\bar{\varphi}(\bar{x}_k^{\varepsilon}))\,z_k^{\varepsilon}$$

Equation (20) may be rewritten in the form

$$z_n^{\varepsilon}=\varepsilon\sum_{k<n}\frac{\partial}{\partial x}\bar{\varphi}(\bar{x}_k^{\varepsilon})\,z_k^{\varepsilon}+\varepsilon^{-1/2}R_{\varepsilon}(n)+\alpha_n^{\varepsilon}+\beta_n^{\varepsilon} \tag{22}$$

Using condition (L2) we can prove that $\sup_{n\varepsilon < t_0} \|\alpha_n^\varepsilon\| \to 0$ in probability as $n \to \infty$, $\varepsilon \to 0$.

In the same way as (21) we can show that for $0 < t_1 < t_2$,

$$\sup_{t_1 \leqslant \varepsilon k_0 < \varepsilon k < t_2} \|z_k^\varepsilon - z_{k_0}^\varepsilon\| \leqslant \sup_{t_1 \leqslant \varepsilon k_0 < \varepsilon k < t_2} \varepsilon^{-1/2} \|R_\varepsilon(k) - R_\varepsilon(k_0)\| \times \exp\left\{\varepsilon \sum_{\varepsilon n < t_2} l(y_n)\right\} \tag{23}$$

Let $0 < t_1 < \cdots < t_{r+1} = t_0$ and $0 < k_1 < \cdots < k_{r+1}$ satisfy the inequalities $t_i \leqslant \varepsilon k_i < t_i + \varepsilon$. Using the representation

$$\beta_n^\varepsilon = \varepsilon \sum_{i=1}^{r} \sum_{k<n} \left(\frac{\partial}{\partial x} \varphi(\bar{x}_k^\varepsilon, y_{k+1}) - \frac{\partial}{\partial x} \bar{\varphi}(\bar{x}_k^\varepsilon)\right) z_{k_i}^\varepsilon 1_{\{k_i \leqslant k < k_{i+1}\}}$$
$$+ \varepsilon \sum_{i=1}^{r} \sum_{k<n} \left(\frac{\partial}{\partial x} \varphi(\bar{x}_k^\varepsilon, y_{k+1}) - \frac{\partial}{\partial x} \bar{\varphi}(\bar{x}_k^\varepsilon)\right) (z_k^\varepsilon - z_{k_i}^\varepsilon)\, 1_{\{k_i \leqslant k < k_{i+1}\}}$$

inequality (23), and proof of relation (14), one can show that $\sup_{n\varepsilon < t_0} \|\beta_n^\varepsilon\| \to 0$ in probability as $\varepsilon \to 0$, $n \to \infty$.

Let $\hat{z}_n^\varepsilon$ be determined by equation

$$\hat{z}_n^\varepsilon = \varepsilon \sum_{k<n} \frac{\partial}{\partial x} \bar{\varphi}(\bar{x}_k^\varepsilon)\, \hat{z}_k^\varepsilon + \frac{1}{\sqrt{\varepsilon}} R_\varepsilon(n) \tag{24}$$

It is easy to see that

$$\sup_{\varepsilon n < t_0} \|z_n^\varepsilon - \hat{z}_n^\varepsilon\| = O(\sup_{\varepsilon n < t_0} \|\alpha_n^\varepsilon\| + \sup_{\varepsilon n < t_0} \|\beta_n^\varepsilon\|)$$

consequently

$$\sup_{\varepsilon n \leqslant t_0} \|z_n^\varepsilon - \hat{z}_n^\varepsilon\| \to 0 \qquad \text{in probability as} \quad \varepsilon \to 0 \tag{25}$$

Let

$$\hat{z}^\varepsilon(t) = \sum z_n^\varepsilon 1_{\{n\varepsilon \leqslant t < (n+1)\varepsilon\}}$$

Then it follows from (24) that

$$\hat{z}^\varepsilon(t) = \int_0^t \frac{\partial}{\partial x} \varphi(\bar{x}(s))\, \hat{z}^\varepsilon(s)\, ds + \eta^\varepsilon(t) + \delta_\varepsilon(t) \tag{26}$$

where $\delta_\varepsilon(t) \to 0$ uniformly in $t \leqslant t_0$. The proof of the theorem is now a consequence of (1), (25), and (26). □

2.1. Diffusion Approximation for Large Time

Here we consider the system of the previous section but with $\bar{\varphi}=0$, so $(\bar{x}(t)\equiv x_0)$. We introduce the stochastic jump process

$$\tilde{x}^{\varepsilon}(t)=\sum_{n=1}^{\infty} x_n^{\varepsilon} 1_{\{\varepsilon^2 n\leqslant t<\varepsilon^2(n+1)\}} \tag{27}$$

We will show that $\tilde{x}^{\varepsilon}(t)$ converges weakly to a diffusion process in X under some natural conditions. This result is based on a martingale characterization of diffusion and its application to limit theorems for weak convergence to diffusion processes (Strook *et al.*, 1979). Since such methods were developed for diffusions in R^d, we restrict our attention to iterations in R^d.

We need the following statement (see Skorokhod, 1989, p. 78, Theorem 1).

Proposition. *Let a differential operator L_t be of the form*

$$L_t f(x)=(a(t,x),f'(x))+1/2\,\mathrm{Tr}\,B(t,x)\,f''(x)$$

$f(x)$ is a twice differentiable function $R^d\to R$, $a(t,x)$: $R_+\times R^d\to R^d$, $B(t,x)$: $R_+\times R^d\to L_+(R^d)$, where $L_+(R^d)$ is the space of nonnegative symmetric operators $R^d\to R^d$. Suppose that the stochastic differential equation

$$d\eta(t)=a(t,\eta(t))\,dt+B^{1/2}(t,\eta(t))\,dw(t) \tag{28}$$

where $w(t)$ is the Wiener process in R^d and $B^{1/2}$ is the nonnegative square root of B, has a weakly unique solution.

Let D be a set of bounded functions f: $R^d\to R$ for which derivatives f' and f'' exist and are continuous bounded functions, and D is dense in the space $C_0(R^d)$ of all continuous bounded functions which tend to zero at infinity.

If a set $\{\xi_\varepsilon(t),\ \varepsilon>0\}$ of R^d-valued stochastic processes satisfy the conditions

(1) for any $f\in D$ and continuous bounded function $\phi(x_1,\dots,x_n)$: $(R^d)^n\to R$

$$\lim_{\varepsilon\to 0} E\phi(\xi_\varepsilon(t_1),\dots,\xi_\varepsilon(t_{n-1}),\xi_\varepsilon(t))$$

$$\times\left[f(\xi_\varepsilon(t+h)-\xi_\varepsilon(t)-\int_t^{t+h}L_s(f(\xi_\varepsilon(s)))\,ds\right]=0$$

uniformly for $0\leqslant t_1<\cdots<t_{n-1}\leqslant t\leqslant t+h\leqslant T$ for any $T>0$, *(2)* $\xi_\varepsilon(0) \to x$ *in probability as* $\varepsilon \to 0$, *then* $\xi_\varepsilon(t)$ *converges weakly to the solution* $\eta(t)$ *of Eqs. (28) for which* $\eta(0) = x$.

Theorem 3. *Let conditions (NP3) and (PS3) be fulfilled and suppose* φ *satisfies the condition (L3):* φ *is bounded, its derivatives* $(\partial/\partial x)\,\varphi(x, y)$, $(\partial^2/\partial x^2)\,\varphi(x, y)$ *are bounded and uniformly continuous in* x *uniformly in* y, *and* $\int \varphi(x, y)\,\rho(dy) = 0$.

Then the process $\tilde{x}^\varepsilon(t)$ *converges weakly to a diffusion process* $\tilde{x}(t)$ *having generator* $\tilde{L}$ *that is defined for any* $f \in C^2(R^d)$ *by the formula*

$$\begin{aligned}\tilde{L}f(x) = \iint & [(f''(x)\,\varphi(x, y), \varphi(x, y')) \\ & + (\varphi'(x, y')\,\varphi(x, y), f'(x))]\,\rho(dy)\,R(y, dy') \\ & - \int (\varphi'(x, y)\,\varphi(x, y), f'(x))\,\rho(dy)\end{aligned} \tag{28}$$

and $\tilde{x}(0) = x_0$.

Proof. Denote by $\mathscr{F}_k$ the σ-algebra generated by $\{y_0, \ldots, y_k\}$. To prove the theorem it suffices to show that for a class of functions dense in $C^2(R^d)$, we have

$$\lim_{\varepsilon \to 0} E\left|E\left(f(\tilde{x}^\varepsilon(t+h)) - f(\tilde{x}^\varepsilon(t)) - \int_t^{t+h} \tilde{L}f(\tilde{x}^\varepsilon(s))\,ds/\mathscr{F}_t^\varepsilon\right)\right| = 0 \tag{29}$$

here $\mathscr{F}_t^\varepsilon = \mathscr{F}_k$ for $\varepsilon^2 k \leqslant t < \varepsilon^2(k+1)$.

We need the following auxiliary result.

Lemma 2. *Let* $g(x, y)\colon R^d \times Y \to R$ *be a bounded function that is continuous in* x *and measurable in* y *for which* $g'_x(x, y)$ *and* $g''_{xx}(x, y)$ *are bounded and uniformly continuous in* x *uniformly with respect to* y. *Then if* $\varepsilon^2 n \leqslant t$,

$$\lim_{n \to \infty,\, \varepsilon \to 0} E\left(\frac{1}{n}\sum_{k=1}^{n}\left[g(x_k^\varepsilon, y_{k+1}) - \int g(x_k^\varepsilon, y')\,\rho(dy')\right]\right) = 0 \tag{30}$$

Proof. Set $\tilde{g}(x, y) = g(x, y) - \int g(x, y')\,\rho(dy')$. Then

$$E\frac{1}{n}\sum_{k=1}^{n} \tilde{g}(x_0^\varepsilon, y_{k+1}) = \frac{1}{n}\sum_{k=1}^{n}\int \tilde{g}(x_0, y')\,R_{k+1}(y_0, dy')$$

It follows from condition (NP3) that for all n,

$$\sum_{k=1}^{n} \int \tilde{g}(x_0, y') R_{k+1}(y_0, dy')$$

is bounded, and as a result,

$$\lim_{n\to\infty, \varepsilon\to 0} E\frac{1}{n}\sum_{k=1}^{n} \tilde{g}(x_0^\varepsilon, y_{k+1}) = 0$$

We have

$$\sum_{k=1}^{n} [\tilde{g}(x_k^\varepsilon, y_{k+1}) - \tilde{g}(x_0^\varepsilon, y_{k+1})]$$
$$= \sum_{k=1}^{n} \sum_{i=0}^{k-1} [\tilde{g}(x_{i+1}^\varepsilon, y_{k+1}) - \tilde{g}(x_i^\varepsilon, y_{k+1})]$$
$$= \varepsilon \sum_{0\leqslant i<k\leqslant n} (\tilde{g}'_x(x_i^\varepsilon, y_{k+1}), \varphi(x_i^\varepsilon, y_{k+1}))$$
$$+ \tfrac{1}{2}\varepsilon^2 \sum_{0\leqslant i<k\leqslant n} (g''_{xx}(x_i^\varepsilon, y_{k+1})\, \varphi(x_i^\varepsilon, y_{i+1}), \varphi(x_i^\varepsilon, y_{i+1}))(1+\theta_{i,k})$$

where $\theta_{ik} \to 0$ uniformly as $n \to \infty$, $\varepsilon \to 0$, $\varepsilon^2 n \leqslant T$, because $g''_{xx}(x, y)$ is uniformly continuous in x. We have

$$E\frac{\varepsilon}{n} \sum_{0\leqslant i<k\leqslant n} (\tilde{g}'_x(x_i^\varepsilon, y_{k+1}), \varphi(x_i^\varepsilon, y_{i+k}))$$
$$= \varepsilon/n \sum_{0\leqslant i<k\leqslant n} \int \tilde{g}'_x(x_i^\varepsilon, y')\, R_{k-i}(y_{i+1}, dy') = O(\varepsilon)$$

and

$$(\varepsilon^2/n) \sum_{0\leqslant i<k\leqslant n} E(\tilde{g}''_{xx}(x_i^\varepsilon, y_{k+1})\, \varphi(x_i^\varepsilon, y_{i+1}), \varphi(x_i^\varepsilon, y_{i+1})) = O(\varepsilon^2)$$

Therefore,

$$(\varepsilon^2/n) \sum_{0\leqslant i<k\leqslant n} E(\tilde{g}''_{xx}(x_i^\varepsilon, y_{k+1})\, \varphi(x_i^\varepsilon, y_{i+1}), \varphi(x_i^\varepsilon, y_{i+1}))\, \theta_{ik}$$
$$= O(n\varepsilon^2 E \max_{i,k} \theta_{ik})$$

□

We now return to the proof of the theorem. Note that the sequence $\{x_n^\varepsilon, y_n\}$ is a homogeneous Markov sequence, so it suffices to prove (29) for $t=0$. Let f be in $C^3(R^d)$. We have

$$f(x_n^\varepsilon)-f(x_0^\varepsilon)=\sum_{k=0}^{n-1}(f(x_{k+1}^\varepsilon)-f(x_k^\varepsilon))=\varepsilon\sum_{k=0}^{n}(f'(x_k^\varepsilon),\varphi(x_k^\varepsilon,y_{k+1}))$$
$$+(\varepsilon^2/2)\sum_{k=0}^{n}(f''(x_k^\varepsilon)\,\varphi(x_k^\varepsilon,y_{k+1}),\varphi(x_k^\varepsilon,y_{k+1}))+O(n\varepsilon^3) \tag{31}$$

Set

$$(f'(x),\varphi(x,y))=g(x,y)$$

Then

$$\varepsilon E\sum_{k=0}^{n-1}g(x_k^\varepsilon,y_{k+1})=\varepsilon E\sum_{k=0}^{n-1}g(x_0^\varepsilon,y_{k+1})$$
$$+\varepsilon E\sum_{k=1}^{n-1}\sum_{i=0}^{k-1}(g(x_{i+1}^\varepsilon,y_{k+1})-g(x_i^\varepsilon,y_{k+1}))$$
$$=\varepsilon E\sum_{k=0}^{n-1}g(x_0^\varepsilon,y_{k+1})$$
$$+\varepsilon^2E\sum_{0\leqslant i<k<n}(g_n'(x_i^\varepsilon,y_{k+1}),\varphi(x_i^\varepsilon,y_{i+1}))+O(n\varepsilon^3)$$
$$=\varepsilon^2\sum_{i=0}^{n-2}E\sum_{l=1}^{n-k}\int(g_x'(x_i^\varepsilon,y'),\varphi(x_i^\varepsilon,y_{i+1}))\,R_l(y_{i+1},dy')$$
$$+O(\varepsilon+n\varepsilon^3)$$

Let

$$G(x,y)=\sum_{l=1}^{\infty}(g_x'(x,y'),\varphi(x,y))\,R_l(y,dy')$$

It is clear that subject to $n\varepsilon^2\leqslant T$, we have

$$\varepsilon^2E\sum_{i=0}^{n-2}\left(\sum_{l=1}^{n-k}\int(g_x'(x_i^\varepsilon,y')\,\varphi(x_i^\varepsilon,y_{i+1}))\,R_l(y_{i+1},dy')-G(x_i^\varepsilon,y_{i+1})\right)$$

tends to zero as $n\to\infty$, $\varepsilon\to 0$.

Thus subject to the constraint that $n\varepsilon^2 \leqslant T$, we have

$$E\left(f(x_n^\varepsilon)-f(x_0^\varepsilon)-\varepsilon^2\sum_{i=0}^{n-2}\left[\tfrac{1}{2}(f_x''(x_k^\varepsilon)\,\varphi(x_k^\varepsilon, y_{k+1}), \varphi(x_k^\varepsilon, y_{k+1}))\right.\right.$$
$$\left.\left.+\,G(x_k^\varepsilon, y_{k+1})\right]\right)$$

tends to zero as $n\to\infty$, $\varepsilon\to 0$.

The proof of the theorem now follows from Lemma 2. □

Corollary. *Under the conditions of Theorem 3, for all $\delta>0$*

$$\lim_{\varepsilon\to 0} P\{\sup_{n\leqslant\varepsilon^2 t_\varepsilon} |x_n^\varepsilon - x_0^\varepsilon| > \delta\} = 0 \qquad \text{if} \quad t_\varepsilon\to 0 \text{ as } \varepsilon\to 0$$

3. LINEAR DIFFERENCE EQUATIONS

All results of Section 1 are valid for linear equations of the form

$$x_{n+1}^\varepsilon - x_n^\varepsilon = \varepsilon A(y_{n+1})\, x_n^\varepsilon \tag{32}$$

where $A(y)$ is a measurable function: $Y\to L(X)$, where $L(X)$ is the space of bounded linear operators: $X\to X$ [this means that $A(y)\,x$ is a measurable function: $Y\to X$ for each $x\in X$]. Here we consider some results for (32) that cannot be obtained from Theorems 1–3. First, in Theorem 4 we determine the behavior of x_n^ε as $n\to\infty$ for all ε sufficiently small. The special form of this equation enables us in Theorem 5 to extend our earlier results to cases where $A=A_\varepsilon$ is unbounded as $\varepsilon\to 0$.

3.1. Stability

We first derive some results about the stability of solutions to linear problems.

Theorem 4. *Suppose that conditions (NP3) and (PS2) are satisfied. Moreover, suppose that*

(1) $\sup_{y\in Y}\int \|A(y')\|^2 P_1(y, dy') \leqslant c$ *for some constant* c,

(2) the solution to linear equation

$$\frac{d\bar{x}(t)}{dt} = \bar{A}\bar{x}(t) \tag{33}$$

where $\bar{A}x = \int A(y)\, x\, p(dy)$ *is such that for some* $a > 0$ *and* $\alpha > 0$, $|\bar{x}(t)| \leqslant a e^{-\alpha t} |\bar{x}(0)|$. *Then there exists* $\varepsilon_0 > 0$ *for which*

$$P\{\lim_{n \to \infty} \|x_n^\varepsilon\| = 0\} = 1 \qquad \textit{for} \quad \varepsilon \leqslant \varepsilon_0$$

Proof. Let $\bar{x}_n^\varepsilon = (I + \varepsilon \bar{A})^n x_0$, $z_n^\varepsilon = x_n^\varepsilon - \bar{x}_n^\varepsilon$. Then

$$z_n^\varepsilon = \varepsilon \sum_{k=0}^{n-1} (I + \varepsilon \bar{A})^k \tilde{A}(y_{n-k+1})\, x_{n-k}^\varepsilon \tag{34}$$

where $\tilde{A}(y) = A(y) - A$. Using the representation

$$x_n^\varepsilon = x_0 + \varepsilon \sum_{i=0}^{n-1} A(y_{i+1})\, x_i^\varepsilon$$

we can rewrite (34) in the form

$$\begin{aligned} z_n^\varepsilon &= \varepsilon \sum_{k=0}^{n-1} (I + \varepsilon \bar{A})^k \tilde{A}(y_{n-k+1})\, x_0 \\ &\quad + \varepsilon^2 \sum_{i=0}^{n-1} \sum_{l=i+1}^{n-1} (I + \varepsilon \bar{A})^{n-l} \tilde{A}(y_{l+1})\, A(y_{i+1})\, x_i^\varepsilon \\ &= \varepsilon u_n^\varepsilon + \varepsilon^2 w_n^\varepsilon \end{aligned} \tag{35}$$

where u_n^ε is the first sum and w_n^ε is the second double sum of the expression for z_n^ε.

Let us evaluate $E(u_n^\varepsilon, u_n^\varepsilon)$ and $E(w_n^\varepsilon, w_n^\varepsilon)$. We have

$$\begin{aligned} E(u_n^\varepsilon, u_n^\varepsilon) &= \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} E((I + \varepsilon \bar{A})^k \tilde{A}(y_{n-k+1}))\, x_0, (I + \varepsilon \bar{A})^l \tilde{A}(y_{n-l+1})\, x_0) \\ &= \sum_{k=0}^{n-1} \|I + \varepsilon \bar{A}\|^{2k} \int \|\tilde{A}(y')\|^2 P_{n-k+1}(y_0, dy')\, \|x_0\|^2 \\ &\quad + 2 \sum_{1 \leqslant k < l \leqslant n+1} E \int ((I + \varepsilon \bar{A})^{n-k+1} \tilde{A}(y_k)\, x_0, \\ &\quad (I + \varepsilon \bar{A})^{n-l+1} \tilde{A}(y')\, x_0)\, P_{l-k}(y_k, dy') \\ &\leqslant n e^{\varepsilon 2n \|\bar{A}\|} c\, \|x_0\|^2 + 2 \sum_{1 \leqslant k \leqslant n+1} \int E\Bigg((I + \varepsilon \bar{A})^{n-k+1} \tilde{A}(y_k)\, x_0, \\ &\quad \sum_{l=1}^{n+1-k} (I + \varepsilon \bar{A})^{n-k-l+1} \tilde{A}(y')\, x_0 \Bigg) R_l(y_k, dy') \end{aligned}$$

Note that for $m > 1$

$$\int \sum_{l=1}^{m} (I+\varepsilon \bar{A})^{m-l} \tilde{A}(y')\, x_0 R_l(y, dy')$$

$$= \int \tilde{A}(y')\, x_0 \sum_{l=1}^{m} R_l(y, dy') + \varepsilon \sum_{l=1}^{m-1} \int (I+\varepsilon \bar{A})^{m-l-1}$$

$$\bar{A}\tilde{A}(y')\, x_0 \sum_{k=1}^{l} R_l(y, dy')$$

Since

$$\left\| \int \tilde{A}(y')\, x_0 \sum_{l=1}^{m} R_l(y, dy') \right\|$$

$$= \left\| \iint \tilde{A}(y'')\, x_0 \sum_{l=1}^{m-1} R_l(y, dy')\, P_1(y', dy'') \right\|$$

hence condition (1) and (NP3) imply that

$$\left\| \int \tilde{A}(y')\, x_0 R_m(y, dy') \right\| \leqslant c_1 \|x_0\|$$

for all $m \geqslant 1$, where c_1 is a constant which depends on c.

This implies the inequalities

$$\left\| \int \sum_{l=1}^{m} (I+\varepsilon \bar{A})^{m-l} \tilde{A}(y')\, x_0 R_l(y, dy') \right\|$$

$$\leqslant (c_1 + \varepsilon m e^{\varepsilon m \|\bar{A}\|} \|\bar{A}\|\, c_1)\, \|x_0\|$$

and

$$E(u_n^\varepsilon, u_n^\varepsilon) \leqslant \{n e^{\varepsilon 2n \|\bar{A}\|} c + (2n^2 \varepsilon e^{\varepsilon n \|\bar{A}\|} \|\bar{A}\|\, c_1 + c_1 \varepsilon n e^{\varepsilon n \|\bar{A}\|})$$

$$\times \sup_{0 < k \leqslant n} E \|\tilde{A}(y_k)\|^2\} \|x_0\|^2$$

Thus for $\varepsilon n \leqslant T$,

$$E(u_n^\varepsilon, u_n^\varepsilon) \leqslant n C_T \|x_0\|^2, \text{ where } C_T \text{ is a constant}$$

In the same way we can show that

$$E(w_n^\varepsilon, w_n^\varepsilon) = 2 \sum_{0 \leqslant i < j \leqslant n-1} E\Bigg(\sum_{l=i+1}^{n-1} (I+\varepsilon\bar{A})^{n-l} \tilde{A}(y_{l+1})$$

$$A(y_{i+1})\, x_i^\varepsilon, \sum_{m=j+1}^{n-1} (I+\varepsilon\bar{A})^{n-m} \tilde{A}(y_{m+1})\, A(y_{j+1})\, x_j^\varepsilon \Bigg)$$

$$+ \sum_{i=0}^{n-1} E\Bigg(\sum_{l=i+1}^{n-1} (I+\varepsilon\bar{A})^{n-l} \tilde{A}(y_{l+1})$$

$$A(y_{i+1})\, x_i^\varepsilon, \sum_{l=j+1}^{n-1} (I+\varepsilon\bar{A})^{n-l} \tilde{A}(y_{l+1})\, A(y_{j+1})\, x_j^\varepsilon \Bigg)$$

Since

$$E \left\| \sum_{l=i+1}^{n-1} (I+\varepsilon\bar{A})^{n-1} \tilde{A}(y_{l+1})\, A(y_{i+1})\, x_i^\varepsilon \right\|^2 \leqslant nC_T cE\, \|x_i^\varepsilon\|^2$$

so

$$E(w_n^\varepsilon, w_n^\varepsilon) \leqslant nC_T c \sum_{i=0}^{n-1} E\, \|x_i^\varepsilon\|^2 + 2 \sum_{0 \leqslant i < j \leqslant n-1} nC_T c \sqrt{E\, \|x_i^\varepsilon\|^2\, E\, \|x_j^\varepsilon\|^2}$$

$$\leqslant (n^2 cC_T + 2n^3 cC_T) \max_{i \leqslant n} E\, \|x_i^\varepsilon\|^2$$

We have

$$E\, \|x_n^\varepsilon\|^2 = E\, \|x_{n-1}^\varepsilon\|^2 + 2\varepsilon E(x_{n-1}^\varepsilon, A(y_n)\, x_{n-1}^\varepsilon) + \varepsilon^2 E\, \|A(y_n)\, x_{n-1}^\varepsilon\|^2$$

$$\leqslant E\, \|x_{n-1}^\varepsilon\|^2 \Bigg(1 + 2\varepsilon \int \|A(y')\|\, P(y_{n-1}, dy')$$

$$+ \varepsilon^2 \int \|A(y')\|^2\, P(y_n, dy') \Bigg)$$

Therefore

$$E\, \|x_n^\varepsilon\|^2 \leqslant e^{n\varepsilon c_2}\, \|x_0\|^2 \tag{36}$$

where $c_2 > 0$ is a constant. Thus

$$E\, \|w_n^\varepsilon\|^2 \leqslant O(n^3)\, \|x_0\|^2 \qquad \text{for} \quad \varepsilon n \leqslant T$$

$$E\, \|z_n^\varepsilon\|^2 \leqslant 2\varepsilon^2 E\, \|u_n^\varepsilon\|^2 + 2\varepsilon^4 E\, \|w_n^\varepsilon\|^2 \sup_{n\varepsilon \leqslant T} E\, \|z_n^\varepsilon\|^2 \leqslant \varepsilon b_T\, \|x_0\|^2 \tag{37}$$

where b_T is a constant.

Let $q<1$ and t_0 be chosen such that

$$\|\bar{x}(t_0)\|^2 < q\ \|x_0\|^2$$

Since

$$E\ \|x_n^\varepsilon - \bar{x}(t_0)\|^2 \to 0 \qquad \text{as} \quad \varepsilon \to 0, n\varepsilon \to t_0$$

there exists ε_0 such that

$$E\ \|x_n^\varepsilon\|^2 < q\ \|x_0\|^2 \qquad \text{if} \quad \varepsilon < \varepsilon_0 \qquad \text{and} \qquad n\varepsilon \leqslant t_0 < (n+1)\,\varepsilon \tag{38}$$

Note that

$$E\ \|x_{kn}^\varepsilon\|^2 \leqslant q^k\ \|x_0\|^2$$

and

$$E\ \|x_{kn+r}^\varepsilon\|^2 \leqslant q^k E\ \|x_r^\varepsilon\|^2$$

$$E \sum_{j=0}^{\infty} \|x_j^\varepsilon\|^2 = \sum_{k=0}^{\infty} \sum_{r=0}^{n-1} E\ \|x_{kn+r}^\varepsilon\|^2 \leqslant \frac{\sum_{r=0}^{n-1} E\ \|x_r^\varepsilon\|^2}{1-q}$$

Thus

$$P\left\{\sum_{j=0}^{\infty} \|x_j^\varepsilon\|^2 < \infty\right\} = 1$$

□

Remark. Let $X = R^d$. Then the assertion of Theorem 4 is true with (NP2) and (2) and replacing (1) with

$$(1') \qquad \sup_{y \in Y} \int \|A(y')\|\ P_1(y, dy') < \infty$$

To prove this we note that

$$\sup_{y \in Y} \sup_{\varepsilon n \leqslant T} E(|x_n^\varepsilon|/y_0 = y) \leqslant C_T\, |x_0| \tag{39}$$

and that for any $\delta > 0$, $t_0 > 0$, $x_0 \in R^d$ $\sup_{y \in Y} P_y\{|x_n^\varepsilon - \bar{x}(t_0)| > \delta\} \to 0$ as $\varepsilon n \to t_0$. P_y is the conditional probability given that $y_0 = y$. Let $\{e_1, \dots, e_d\}$ be an orthonormal basis in R^d and let $x_n^\varepsilon(x_0)$ be the solution of (32) with $x_0^\varepsilon(x_0) = x_0$. Then

$$x_n^\varepsilon(x_0) = \sum_{i=1}^{d} (x_0, e_k)\, x_n^\varepsilon(e_k)$$

We denote by $\bar{x}(x_0, t)$ the solution of Eq. (33) for which $\bar{x}(x_0, 0) = x_0$. Then for any $\delta > 0$ and t_0

$$\lim_{n\varepsilon \to t_0} \sup_{y \in Y} P_y\{ \sup_{|x_0| \leqslant 1} |x_n^\varepsilon(x_0) - \bar{x}(x_0, t_0)| > \delta\} = 0 \tag{40}$$

It follows from (39) and (40) that for any $0 < p < 1$

$$\lim_{\varepsilon n \to t_0} \sup_{y \in Y} E(|x_n^\varepsilon(x_0) - \bar{x}(x_0, t_0)|^p / y_0 = y) = 0$$

and there exist $t_0 \in R_+$, $q < 1$, and $\varepsilon_0 > 0$ for which

$$\sup_y E(|x_n^\varepsilon|^p / y_0 = y) < q\, |x_0|^p \tag{41}$$

if $\varepsilon < \varepsilon_0$ and $n\varepsilon \leqslant t_0 < (n+1)\,\varepsilon$. Inequality (41) implies the relation

$$P\left\{ \sum_{j=0}^{\infty} |x_j^\varepsilon| < \infty \right\} = 1$$

Now we consider a generalization of Eq. (32) in a Hilbert space H: Consider

$$x_{n+1}^\varepsilon - x_n^\varepsilon = \varepsilon A_\varepsilon(y_{n+1})\, x_n^\varepsilon, \qquad x_0^\varepsilon = x_0 \tag{42}$$

where $A_\varepsilon(y)$ is a measurable function: $Y \to L(H)$ for each $\varepsilon > 0$. We assume that condition (NP2) holds for the sequence $\{y_n\}$. The first condition on $A_\varepsilon(y)$ is

(A1) (a) $\int (A_\varepsilon(y')\, x, x)\, P_1(y, dy') \leqslant c_1\, \|x\|^2$,

(b) $\int \|A_\varepsilon(y')\, x\|^2\, P_1(y, dy') \leqslant c_1 \varepsilon^{-1}\, \|x\|$,

for all $y \in Y$, $x \in H$; here c_1 is some constant.

Set $\bar{A}_\varepsilon x = \int A_\varepsilon(y')\, x p(dy')$. Then $\|\bar{A}_\varepsilon\|^2 \leqslant c_1 \varepsilon^{-1}$ and $(\bar{A}_\varepsilon x, x) \leqslant c_1\, \|x\|^2$. Denote by $\mathscr{F}_n$ the σ-algebra generated by $\{y_0, ..., y_n\}$.

Lemma 3. *Under condition (A1) we have*

$$E(\|x_n^\varepsilon\|^2 / y_0 = y) \leqslant \|x_0^\varepsilon\|^2\, e^{3\varepsilon c_1 n}$$
$$\|(I + \varepsilon \bar{A}_\varepsilon)^n\| \leqslant e^{(3/2)\,\varepsilon c_n n}$$

Proof. The first inequality is a consequence of the relation

$$E(\|x_n^\varepsilon\|^2 / \mathscr{F}_{n-1}) = \|x_{n-1}^\varepsilon\|^2 + 2\varepsilon E((x_{n-1}^\varepsilon, A_\varepsilon(y_n)\, x_{n-1}^\varepsilon) / \mathscr{F}_{n-1})$$
$$+ \varepsilon^2 E(\|A(y(n)\, x_{n-1}^\varepsilon\|^2 / \mathscr{F}_{n-1}) \leqslant (1 + 3\varepsilon c_1)\, \|x_{n-1}^\varepsilon\|^2 \quad \square$$

The second condition on A is

(A2) there exists an invertible compact operator $R \in L(H)$ for which

(c) $\int \|A_\varepsilon(y')\, Rx\|^2\, P_1(y, dy') \leqslant c_2\, \|x\|^2$,

(d) $\int \|(R^{-1}A_\varepsilon(y') - A_\varepsilon(y')\, R^{-1})\, x\|^2\, P_1(y, dy') \leqslant c_2\, \|x\|^2$,

(e) $\int \|(R^{-2}A_\varepsilon(y') - A_\varepsilon(y')\, R^{-2})\, x\|^2\, P_1(y, dy') \leqslant c_2\, \|x\|^2$,

for all $y \in Y$, $x \in X$; c_2 is a constant.

Lemma 4. *Let conditions (A1) and (A2) be satisfied. Then there exists $c_3 > 0$ for which*

(I) $E(\|R^{-1}x_n^\varepsilon\|^2/\mathscr{F}_{n-1}) \leqslant e^{c_3\varepsilon n}\, \|R^{-1}x_0\|^2$,

(II) $E(\|R^{-2}x_n^\varepsilon\|^2/\mathscr{F}_0) \leqslant e^{c_3\varepsilon n}\, \|R^{-2}x_0\|^2$,

(III) $E(\|x_n^\varepsilon - x_0^\varepsilon\|^2/\mathscr{F}_0) \leqslant c_3\varepsilon n\, e^{c_3\varepsilon n}\, \|R^{-1}x_0^\varepsilon\|^2$,

(IV) $E(\|R^{-1}x_n^\varepsilon - R^{-1}x_0^\varepsilon\|^2/\mathscr{F}_0) \leqslant c_3\varepsilon n\, e^{c_3\varepsilon n}\, \|R^{-2}x_0^\varepsilon\|^2$,

(V) $\|(I + \varepsilon\bar{A}_\varepsilon)^n\, R - R\| \leqslant c_3\varepsilon n\, e^{c_3\varepsilon n}$.

Proof.

(I) We have

$$R^{-1}x_n^\varepsilon = R^{-1}x_{n-1}^\varepsilon + \varepsilon A_\varepsilon(y_n)\, R^{-1}x_{n-1}^\varepsilon + \varepsilon B_\varepsilon(y_n)\, x_{n-1}^\varepsilon$$

where $B_\varepsilon(y) = R^{-1}A_\varepsilon(y) - A_\varepsilon(y)\, R^{-1}$. This relation implies the inequality

$$E(\|R^{-1}x_n^\varepsilon\|^2/\mathscr{F}_{n-1}) \leqslant (1 + c_3\varepsilon)\, \|R^{-1}x_{n-1}^\varepsilon\|^2$$

(II) This can be proved in the same way. We omit its proof here.

(III) We have

$$x_n^\varepsilon - x_0^\varepsilon = \varepsilon \sum_{k=0}^{n-1} A_\varepsilon(y_{k+1})\, R\, R^{-1}x_k^\varepsilon$$

$$E(\|x_n^\varepsilon - x_0^\varepsilon\|^2/\mathscr{F}_0) \leqslant E\left(\varepsilon^2 n \sum_{k=0}^{n-1} E(\|A_\varepsilon(y_{k+1})\, R\, R^{-1}x_k^\varepsilon\|^2/\mathscr{F}_k\right)$$

We use (c) and I to complete the proof of III.

(IV) This can be proved in the same way. We omit its proof here.

(V) $\|(I + \varepsilon\bar{A}_\varepsilon)^n\, R - R\| \leqslant \varepsilon \sum_{k=0}^{n-1} \|(I + \varepsilon\bar{A})^k\|\, \|\bar{A}_\varepsilon R\|$

note that $\|\bar{A}_\varepsilon R\| < \infty$. □

The last auxiliary result we need is the following.

Lemma 5. *Let $B(\cdot)$ be a measurable function: $Y \to L(H)$ for which $\int \|B(y)\|\, \rho(dy) < \infty$, and let R be a compact operator from $L(H)$. Let $\bar{B} = \int B(y)\, \rho(dy)$. Then*

$$P\left(\lim_{n \to \infty} \left\| \left(\frac{1}{n} \sum_{k=1}^{n} B(y_k) - \bar{B} \right) R \right\| = 0 \right) = 1$$

Proof. It is easy to see that for all x,

$$P\left(\lim_{n \to \infty} \left\| \frac{1}{n} \sum_{k=1}^{n} B(y_k)\, x - \bar{B}x \right\| = 0 \right) = 1$$

Therefore

$$P\left(\lim_{n \to \infty} \left\| \left(\frac{1}{n} \sum_{k=1}^{n} B(y_k) - \bar{B} \right) R_1 \right\| = 0 \right) = 1$$

if R_1 is finite dimensional, i.e., if $R_1(H)$ is a finite-dimensional subspace. We note that R can be represented in the form

$$R = R_1^\delta + R_2^\delta$$

where R_1^δ is a finite-dimensional operator and $\|R_2^\delta\| \leqslant \delta$. □

The main result concerning the solution of Eq. (42) is stated in the following theorem.

Theorem 5. *Suppose conditions (NP2), (PS2), (A1), and (A2) are satisfied. In addition, suppose the following condition is satisfied.*

(A3) There exists a measurable function $B(y)$: $Y \to L(H)$ and a constant c_4 such that

(f) $\int \|B(y)\|\, \rho(dy) < \infty$,

(g) $\|\int (A_\varepsilon(y')\, R - B(y'))\, x P_1(y, dy')\| \leqslant c_4 \|x\|$,

for all $x \in H$, $y \in Y$.

Then for any $t_0 > 0$ and $\delta > 0$, we have

$$\lim_{\varepsilon \to 0} \sup_{\varepsilon n \leqslant t_0} P\{\|x_n^\varepsilon - \bar{x}_n^\varepsilon\| > \delta\} = 0 \tag{43}$$

where $\bar{x}_n^\varepsilon = (I + \varepsilon \bar{A}_\varepsilon)^n x_0$.

Proof. Let $z_n^\varepsilon = x_n^\varepsilon - \bar{x}_n^\varepsilon$, $\tilde{A}_\varepsilon(y) = A_\varepsilon(y) - \bar{A}_\varepsilon$, then

$$z_n^\varepsilon = \varepsilon \sum_{k=0}^{n-1} (I + \varepsilon \bar{A}_\varepsilon)^k \tilde{A}_\varepsilon(y_{n-k+1})\, x_{n-k}^\varepsilon \tag{44}$$

Let $0 = t_1 < t_2 < \cdots < t_{r+1} = t_0$, $\varepsilon(k_i - 1) < t_i \leqslant \varepsilon k_i$. We can rewrite (44) in the form

$$z_n^\varepsilon = \varepsilon \sum_{i=1}^{r} (I + \varepsilon \bar{A}_\varepsilon)^{k_i} \sum_{t_i \leqslant \varepsilon k < (t_{i+1} \blacktriangle \varepsilon n)} \tilde{A}_\varepsilon(y_{n-k+1})\, x_{n-k_{i+1}+1}^\varepsilon + u_n^\varepsilon + v_n^\varepsilon \tag{45}$$

where $(a \blacktriangle b) = \min[a, b]$ and

$$u_n^\varepsilon = \varepsilon \sum_{i=1}^{r} (I + \varepsilon \bar{A}_\varepsilon)^{k_i} \sum_{t_i \leqslant \varepsilon k < (t_{i+1} \blacktriangle \varepsilon n)} [I + \varepsilon A_\varepsilon)^{k-k_i} - I]\, \tilde{A}_\varepsilon(y_{n-k+1})\, x_{n-k}^\varepsilon \tag{46}$$

$$v_n^\varepsilon = \varepsilon \sum_{i=1}^{r} (I + \varepsilon \bar{A}_\varepsilon)^{k_i} \sum_{t_i \leqslant \varepsilon k < (t_{i+1} \blacktriangle \varepsilon n)} \tilde{A}_\varepsilon(y_{n-k+1})(x_{n-k}^\varepsilon - x_{n-k_{i+1}+1}^\varepsilon) \tag{47}$$

We consider $x_0 = R^2 w_0$, $w_0 \in H$. Using representations

$$\begin{aligned}\tilde{A}_\varepsilon(y_{n-k+1})\, x_{n-k_{i+1}+1}^\varepsilon &= \tilde{B}(y_{n-k+1})\, R(R^{-2} x_{n-k_{i+1}+1}^\varepsilon) \\ &\quad + (\tilde{A}_\varepsilon(y_{n-k+1})\, R - \tilde{B}(y_{n-k+1}))(R^{-1} x_{n-k_{i+1}+1}^\varepsilon)\end{aligned}$$

[here $\bar{B} = \int B(y)\, \rho(dy)$, $\tilde{B}(y) = B(y) - \bar{B}$],

$$\begin{aligned}&\{(I + \varepsilon \bar{A}_\varepsilon)^{k-k_i} - I]\, \tilde{A}_\varepsilon(y)\, x \\ &\quad = [(I + \varepsilon \bar{A})^{k-k_i} - I]\, R(R^{-1} \tilde{A}_\varepsilon(y) - \tilde{A}_\varepsilon(y)\, R^{-1})\, x \\ &\qquad + [(I + \varepsilon \bar{A})^{k-k_i} - I]\, R\, \tilde{A}_\varepsilon(y)\, R(R^{-2} x) \\ &\tilde{A}_\varepsilon(y_{n-k+1})(x_{n-k}^\varepsilon - x_{n-k_{i+1}+1}^\varepsilon) \\ &\quad = \tilde{A}_\varepsilon(y_{n-k+1})\, R(R^{-1} x_{n-k}^\varepsilon - R^{-1} x_{n-k_{i+1}+1}^\varepsilon)\end{aligned}$$

and Lemmas 3–5, we can prove the relations

$$\lim_{\varepsilon \to 0} \sup_{\varepsilon n \leqslant t_0} P\{\|z_n^\varepsilon - u_n^\varepsilon - v_n^\varepsilon\| > \delta\} = 0$$

$$\lim_{\varepsilon \to 0} \sup_{\varepsilon n \leqslant t_0} E\, \|u_n^\varepsilon\| \leqslant c_5\, \|R^{-2} x_0\| \sum_{i=1}^{r} (t_{i+1} - t_i)^2$$

$$\lim_{\varepsilon \to 0} \sup_{\varepsilon n \leqslant t_0} E\, \|v_n^\varepsilon\| \leqslant c_5\, \|R^{-2} x_0\| \sum_{i=1}^{r} (t_{i+1} - t_i)^2$$

which imply (43). □

The following example illustrates the theorem.

Example. Let $H = L_2[0, \pi]$, and for $f \in L_2[0, 2\pi]$ we let

$$A_\varepsilon(y) f = a(y) \frac{1}{2\sqrt{\varepsilon}} [\tilde{f}(\alpha + \sqrt{\varepsilon}) - \tilde{f}(\alpha - \sqrt{\varepsilon})]$$

where $\tilde{f}(\alpha)$ is the 2π-periodic extension of f, and a: $Y \to R$. Suppose that $\int |a(y)| \rho(dy) < \infty$ and $\sup_y \int a^2(y') P_1(y, dy') < \infty$.

Then $A_\varepsilon(y)$ satisfies condition (A1) since

$$\int_0^{2\pi} (A_\varepsilon(y) f(\alpha)) f(\alpha) \, d\alpha = 0$$

and

$$\int_0^{2\pi} (A_\varepsilon(y) f(\alpha))^2 \leqslant \frac{a^2(y)}{\varepsilon} \int_0^{2\pi} f^2(\alpha) \, d\alpha$$

We consider an operator $\mathbb{R}$ of the form

$$\mathbb{R}f(\alpha) = \int_0^{2\pi} R(\beta) \tilde{f}(\alpha - \beta) \, d\beta$$

where $R(\beta)$: $[0, 2\pi] \to R$ is an integrable function.

This operator is invertible if $\int_0^{2\pi} e^{in\beta} R(\beta) \, d\beta \neq 0$ for all $n \in Z$; it is compact if $\int_0^{2\pi} R^2(\beta) \, d\beta < \infty$; for all k we have $R^{-k} A_\varepsilon(y) = A_\varepsilon(y) R^{-k}$ if R is invertible; and if $R'(\beta) \in L_2[0, 2\pi]$, then

$$(A_\varepsilon(y) R - a(y) R') f(\alpha)$$
$$= a(y) \int_0^{2\pi} \left[\frac{\tilde{R}(b + \sqrt{\varepsilon}) - \tilde{R}(\beta - \sqrt{\varepsilon})}{2\sqrt{\varepsilon}} - \tilde{R}'(\beta) \right] \tilde{f}(\alpha - \beta) \, d\beta$$

where $\mathbb{R}' f(\alpha) = \int_0^{2\pi} R'(\beta) \tilde{f}(\alpha - \beta) \, d\beta$.

Therefore the condition $\lim_{\varepsilon \to 0} \int_0^{2\pi} [((\tilde{R}(\beta + \sqrt{\varepsilon}) - \tilde{R}(\beta - \sqrt{\varepsilon}))/2\sqrt{\varepsilon}) - \tilde{R}'(\beta)]^2 \, d\beta = 0$ implies conditions (A2) and (A3).

4. RANDOMLY PERTURBED ITERATIONS

We first consider linear iterations in a Hilbert space H. Let $\{x_n^\varepsilon\}$ be determined by the iterations

$$x_{n+1}^\varepsilon = B x_n^\varepsilon + \varepsilon A(y_{n+1}) x_n^\varepsilon, \qquad x_0^\varepsilon = x_0$$

where $B \in L(H)$, $\{y_n\}$ satisfies one of the conditions (NP1), (NP2), or (NP3), and $A(y)$ is a measurable mapping of Y into $L(H)$ for which (we assume) the operator

$$\bar{A}x = \int A(y)\, x p(dy)$$

exists with $\bar{A} \in L(H)$. We define

$$\bar{x}^{\varepsilon}_{n+1} = (B + \varepsilon\bar{A})^{n+1} x_0$$

It follows from Theorem 1 that for each fixed t, $\sup_{n\varepsilon \leqslant t} \|x^{\varepsilon}_n - \bar{x}^{\varepsilon}_n\| \to 0$ with probability 1 as $\varepsilon \to 0$. Our goal is to investigate the difference $x^{\varepsilon}_n - \bar{x}^{\varepsilon}_n$ for larger values of n [i.e., for n larger than $O(1/\varepsilon)$].

We have the following result.

Theorem 6. *Suppose (NP3), (PS2) and the following two conditions are satisfied:*

(1) $\|(B + \varepsilon\bar{A})^n\| \leqslant c_1 e^{n\varepsilon^2\alpha}$,

(2) $\int \|A(y')\|^2 P_1(y, dy') \leqslant c_1^2$, *where* c_1 *and* α *are positive constants. Then* $\lim_{\theta \to 0} \limsup_{\varepsilon \to 0} \sup_{n \leqslant \theta/\varepsilon^2} E\, \|x^{\varepsilon}_n - \bar{x}^{\varepsilon}_n\|^2 = 0$.

Proof. We introduce the following notations:

$$T_\varepsilon = B + \varepsilon\bar{A}, \qquad \tilde{A}(y) = A(y) - \bar{A}, \qquad \tilde{A}_k = \tilde{A}(y_k)$$

$$x^{\varepsilon}_n = \sum_{k=0}^{n} \varepsilon^k S_\varepsilon(n, k)\, x_0, \qquad \text{where} \quad S_\varepsilon(n, 0) = T^n_\varepsilon, \qquad S_\varepsilon(n, n) = \tilde{A}_n \cdots \tilde{A}_1$$

and for $k < n$,

$$S_\varepsilon(n+1, k+1) = \tilde{A}_{n+1} S_\varepsilon(n, k) + T_\varepsilon S_\varepsilon(n, k+1)$$

We then note that

$$\bar{x}^{\varepsilon}_n = S_\varepsilon(n, 0)\, x_0$$

$$E\, \|x^{\varepsilon}_n - \bar{x}^{\varepsilon}_n\|^2 = \sum_{k, l=1}^{n} \varepsilon^{k+l} E(S_\varepsilon(n, k)\, x_0, S_\varepsilon(n, l)\, x_0) \tag{48}$$

It is easy to check that

$$S_\varepsilon(n, k) = \sum_{1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n} T^{n-i_k}_\varepsilon \tilde{A}_{i_k} T^{i_k - i_{k-1} - 1}_\varepsilon \tilde{A}_{i_{k-1}} \cdots \tilde{A}_{i_1} T^{i_1 - 1}_\varepsilon \tag{49}$$

Let $R_k^*(y, dy') = \text{var } R_k(y, dy')$, and

$$R^*(y, dy') = \sum_{k=1}^{\infty} R_k^*(y, dy')$$

It follows from condition (NP3) that $\sup_y R^*(y, Y) \equiv R^* < \infty$.

Let

$$\hat{A}(y) = \int \tilde{A}(y')\, P_1(y, dy')$$

Then

$$\|\hat{A}(y)\| \leqslant 2c_1$$

We represent $E(S_\varepsilon(n, k)\, x_0, S_\varepsilon(n, k)\, x_0) = u_\varepsilon(n, k)$ in the form $u_\varepsilon(n, k) = \sum_{j=0}^{k} u_\varepsilon(n, k, j)$, where

$$u_\varepsilon(n, k, j) = \sum E(T_\varepsilon^{n-i_k} \tilde{A}_{i_k} \cdots \tilde{A}_{i_1} T_\varepsilon^{i_1 - 1} x_0,\ T_\varepsilon^{n-j_k} \tilde{A}_{j_k} \cdots \tilde{A}_{j_1} T_\varepsilon^{j_1 - 1} x_0) \tag{50}$$

and the sum is taken over the partitions $\{i_1,..., i_k\}$, $\{j_1,..., j_k\}$ for which $\text{card}\{i_1,..., i_k\} \cap \{j_1,..., j_k\} = j$.

Let $\Lambda_1 \cup \Lambda_2 \cup \Lambda_3 = \{1, 2,..., 2k-j\}$, where $\text{card } \Lambda_1 = \text{card } \Lambda_2 = k - j$, $\text{card } \Lambda_3 = j$.

Let $\{i_1,..., i_k\} \cup \{j_1,..., j_k\} = \{h_1,..., h_{2k-j}\}$ and denote by $u_\varepsilon(n, k, j, \Lambda_1, \Lambda_2, \Lambda_3)$ the sum of those terms of the right-hand side of (50) for which $h_i \in \{i_1,..., i_k\}$ if $i \in \Lambda_1 \cup \Lambda_3$ and $h_i \in \{j_1,..., j_k\}$ if $i \in \Lambda_2 \cup \Lambda_3$. Then

$$\begin{aligned} &u_\varepsilon(n, k, j, \Lambda_1, \Lambda_2, \Lambda_3) \\ &\quad = \sum_{1 \leqslant h_1 < \cdots < h_{2k-j} \leqslant n} \int \cdots \int \phi(y'_1,..., y'_{2k-j}, h_1,..., h_{2k-j}, \Lambda_1, \Lambda_2, \Lambda_3) \\ &\qquad P_{h_1}(y_0, dy'_1)\, P_{h_2 - h_1}(y'_1, dy'_2) \cdots \hat{P}_{h_{2k-j} - h_{2k-j-1}}(y'_{2k-j-1}, dy'_{2k-j}) \end{aligned}$$

where

$$\begin{aligned} &\phi(y'_1,..., y'_{2k-j}, h_1,..., h_{2k-j}, \Lambda_1, \Lambda_2, \Lambda_3) \\ &\quad = (T_\varepsilon^{n-i_1} \tilde{A}(y'_{\alpha_1}) \cdots \tilde{A}(y'_{\alpha_k})\, T_\varepsilon^{i_1 - 1} x_0,\ T_\varepsilon^{n-j_k} \tilde{A}(y'_{\beta_1}) \cdots \tilde{A}(y'_{\beta_k})\, T_\varepsilon^{j_1 - 1} x_0) \end{aligned}$$

and

$$\{\alpha_1,..., \alpha_k\} \cup \{\beta_1,..., \beta_k\} = \{1,..., 2k-j\}$$

$$\alpha_1 < \cdots < \alpha_k,\ \beta_1 < \cdots < \beta_k, \qquad \alpha_s = \beta_r \quad \text{if} \quad i_{\alpha_s} = j_{\beta_r}$$

Let

$$\hat{\phi}(y_1,\dots, y_{2k-j}) = \int \cdots \int \phi(y'_1,\dots, y'_{2k-j}) \sum_{i=1}^{2k-j} P_1(y_i, dy'_i)$$

Then $\hat{\phi}$ satisfies the conditions

(I) $\int \hat{\phi}(y_1,\dots, y_{2k-j},\dots)\, \rho(dy_i) = 0$ if $i \in \Lambda_1 \cup \Lambda_2$,

(II) $|\hat{\phi}(y_1,\dots, y_{2k-j},\dots)| \leqslant (2c_1)^{2k-j}\, \|x_0\|^2$,

(III) $u_\varepsilon(n, k, j, \Lambda_1, \Lambda_2, \Lambda_3) = \sum \int \cdots \int \hat{\phi}(y'_1,\dots, y'_{2k-j}, h_1,\dots, h_{2k-j}, \Lambda_1, \Lambda_2, \Lambda_3) \prod_{i=1}^{2k-j} [R_{h_i - h_{i-1}}(y'_{i-1}, dy'_i) + \rho(dy'_i)]$, where the sum is taken over all $0 \leqslant h_1 \leqslant h_2 \leqslant \cdots \leqslant h_{2k-j} \leqslant n-1$.

Using I–III we can show that

$$|u_\varepsilon(n, k, j, \Lambda_1, \Lambda_2, \Lambda_3)| \leqslant (2c_1)^{2k-j}\, (R^*)^{2k-j} \frac{n^k}{k!}\, \|x_0\|^2$$

so

$$|u_\varepsilon(n, k, j)| \leqslant \frac{(2k-j)!}{(k-j)!\,(k-j)!\,j!}\, (2c_1)^{2k-j}\, (R^*)^{2k-j} \frac{n^k}{k!}\, \|x_0\|^2$$

$$|u_\varepsilon(n, k)| \leqslant Q^k \frac{n^k}{k!}\, \|x_0\|^2$$

where $Q > 0$ is a constant. Therefore

$$\begin{aligned} E\, \|x_n^\varepsilon - \bar{x}_n^\varepsilon\|^2 &\leqslant \sum_{k,\, l=1}^{n} \varepsilon^{k+l}\, |E(S_\varepsilon(n, k)\, x_0, S_\varepsilon(n, l)\, x_0)| \\ &\leqslant \sum_{k=i,\, l=1}^{\infty} \frac{(\varepsilon Q^{1/2} \sqrt{n})^{k+l}}{\sqrt{k!\, l!}}\, \|x_0\|^2 \\ &\leqslant \left(\sum_{k=1}^{\infty} \frac{(Q^{1/2}\theta^{1/2})^k}{\sqrt{k!}} \right)^2 \|x_0\|^2 \end{aligned}$$

if $n < \theta/\varepsilon^2$. It is easy to see that the right-hand side of the last inequality tends to zero as $\theta \to 0$. □

Remarks.

(I) It follows from the proof of the theorem that

$$E\, \|x_n^\varepsilon - \bar{x}_n^\varepsilon\|^2 = O(\varepsilon n^2) \qquad \text{for} \quad O(\varepsilon n^2) = O(1)$$

(II) In the same way, we can prove that

$$E\,\|x_n^\varepsilon - \bar{x}_n^\varepsilon\| \leqslant (e^{Q_1\varepsilon^2 n} - 1)\,\|x_0\| \tag{51}$$

where Q_1 is a constant.

(III) Suppose that the following condition is satisfied instead of (1) in Theorem 6:

$\|(B+\varepsilon\bar{A})^n\| \leqslant 1 - \psi(\varepsilon^2 n)$, where a $\psi\colon R_+ \to R_+$ is a function for which

$$\lim_{\delta \to 0} (\psi(\delta)/\delta) = \infty \tag{1'}$$

Then there exists a positive number ε_0 and constants $c_0 > 0$ and $\alpha_0 > 0$ for which

$$E\,\|x_n^\varepsilon\|^2 \leqslant c_0 e^{-\alpha_0 \varepsilon^2 n} \tag{52}$$

for $\varepsilon < \varepsilon_0$, $n \in Z_+$.

To prove this we note that I and II imply

$$E\,\|x_n^\varepsilon\|^2 = \|\bar{x}_n^\varepsilon\|^2 + E\,\|x_n^\varepsilon - \bar{x}_n^\varepsilon\|^2 + 2(\bar{x}_n^\varepsilon, Ex_n^\varepsilon - \bar{x}_n^\varepsilon)$$
$$\leqslant (1 - \psi(\varepsilon^2 n) + R\varepsilon^2 n)\,\|x_0\|^2 \leqslant \left(1 - \left(\frac{\psi(\varepsilon^2 n)}{\varepsilon^2 n} - R\right)\varepsilon^2 n\right)\|x_0\|^2$$

where the constant R is determined by (I) and (51).

Let Δ satisfy relation $(\psi(\delta)/\delta) - R > 2$ for $\delta \leqslant \Delta$; then $E\,\|x_n^\varepsilon\|^2 \leqslant e^{-2\varepsilon^2 n}$ for $\varepsilon^2 n \leqslant \Delta$. If $\varepsilon^2 \leqslant \Delta$, then this inequality is true for all n.

4.1. Iterations of Nonlinear Function

Let $f\colon H \to H$, $\varphi\colon H \times Y \to H$ be given, and let $\{y_n\}$ satisfy condition (NP2). Let x_n^ε be determined by

$$x_{n+1}^\varepsilon = f(x_n^\varepsilon) + \varepsilon\varphi(x_n^\varepsilon, y_{n+1}), \qquad x_0^\varepsilon = x_0 \tag{53}$$

We have the following result.

***Theorem** 7. Suppose that conditions (NP2) and (PS2) are satisfied as well as*

(1) $f(x)$ and $f'(x)$ are continuous, $\|f'(x)\| \leqslant q < 1$, and there is a unique fixed point for f, say $f(\bar{x}) = \bar{x}$,

(2) $\varphi(x, y)$, $\varphi'_x(x, y)$ are bounded and continuous in x,

(3) $\lim_{n\to\infty} P_n(y, B) = \rho(B)$ for all $B \in \mathscr{B}$.

Then as $\varepsilon \to 0$ *and* $N \to \infty$, *the random sequence*

$$u^{\varepsilon}_{N,n} = \varepsilon^{-1}(x^{\varepsilon}_{N+n} - \bar{x})$$

(for $n \geqslant -N$*) converges weakly to a stationary random sequence* $\{\hat{u}_n, n \in Z\}$, *where*

$$\hat{u}_n = \sum_{k=-\infty}^{n} \Lambda^{n-k} \varphi(\bar{x}, y^*_k) \tag{54}$$

Here $\Lambda = f'(\bar{x})$, *and* $\{y^*_k, k \in Z\}$ *is a stationary Markov chain in* Y *whose* n*th step transition probability is* $P_n(y, B)$ *and whose stationary distribution is* $\rho(dy)$.

Proof. Let $z^{\varepsilon}_n = x^{\varepsilon}_n - \bar{x}$. Then

$$z^{\varepsilon}_{n+1} = f(\bar{x} + z^{\varepsilon}_n) - f(\bar{x}) + \varepsilon\varphi(\bar{x} + z^{\varepsilon}_n, y_{n+1}), \qquad z^{\varepsilon}_0 = x_0 - \bar{x}$$

Set

$$\hat{z}^{\varepsilon}_{n+1} = \Lambda \hat{z}^{\varepsilon}_n + \varepsilon\varphi(\bar{x} + \hat{z}^{\varepsilon}_n, y_{n+1}), \qquad \hat{z}^{\varepsilon}_0 = x_0 - \bar{x}$$

It is easy to see that

$$\hat{z}^{\varepsilon}_n = \varepsilon \sum_{k=1}^{n} \Lambda^{n-k} \varphi(\bar{x} + z^{\varepsilon}_{k-1}, y_k) \tag{55}$$

Using the relation

$$z^{\varepsilon}_{n+1} - \hat{z}^{\varepsilon}_{n+1} = \Lambda(z^{\varepsilon}_n - \hat{z}^{\varepsilon}_n) + \varepsilon[\tilde{\varphi}(\bar{x} + z^{\varepsilon}_n, y_{n+1}) - \varphi(\bar{x} + \hat{z}^{\varepsilon}_n, y_{n+1})] + g(\hat{z}^{\varepsilon}_n)$$

where

$$g(z) = f(\bar{x} + z) - f(\bar{x}) - \Lambda z$$

in conjunction with Conditions (1) and (2), we can show that for ε satisfying the inequality

$$\sup_{x,y} (\|f'(x)\| + \varepsilon \|\varphi'(x, y)\|) \leqslant q_1 < 1$$

we have

$$\|z^{\varepsilon}_{n+1} - \hat{z}^{\varepsilon}_{n+1}\| < q_1 \|z^{\varepsilon}_n - \hat{z}^{\varepsilon}_n\| + \|g(\hat{z}^{\varepsilon}_n)\|$$

$$\|z^{\varepsilon}_{n+1} - \hat{z}^{\varepsilon}_{n+1}\| \leqslant \sum_{k=1}^{n} q_1^{n-k} \|g(\hat{z}^{\varepsilon}_k)\|$$

Since

$$\|\hat{z}_k^\varepsilon\| = O(\varepsilon)$$

we have

$$\|g_\varepsilon(\hat{z}_k^\varepsilon)\| = o(\varepsilon) \qquad \text{and} \qquad \|z_k^\varepsilon - \hat{z}_n^\varepsilon\| = \frac{o(\varepsilon)}{\varepsilon}$$

Set

$$\hat{u}_k^\varepsilon = \frac{1}{\varepsilon}\hat{z}_k^\varepsilon \tag{56}$$

then $\|u_k^\varepsilon - \hat{u}_k^\varepsilon\| = o(\varepsilon)/\varepsilon$.

We introduce the Markov chain $\{y_n^{*N}, n \geq -N\}$, $y_n^{*N} = y_{n+N}$. It follows from Condition (3) that the distributions of $\{y_n^{*N}\}$ converge to the distributions of $\{y_n^*, n \in Z\}$ as $N \to \infty$. Using the representation

$$u_{N+n}^\varepsilon = \sum_{k=-N}^{n} A^{n-k}\varphi\left(\bar{x} + \frac{o(\varepsilon)}{\varepsilon}, y_k^{*N}\right)$$

which follows from (55) and (56), we obtain (54). □

Remarks.

(I) The assertions of Theorem 7 are valid if $\eta(x, \varepsilon, y) = f(x) + \varepsilon\varphi(x, y)$, for any sufficiently small ε and $y \in Y$, maps a closed subset $F \subset H$ into itself.

(II) Suppose that there is a finite number of closed subsets $F_i \subset H$, $i = 1,\ldots, k$ which are invariant with respect to the mapping η for any sufficiently small ε and $y \in Y$. Also suppose that the conditions of the theorem are valid for each F_i [there exists a unique fix point $\bar{x}^i \in F_i$ for the mapping $f(x)$]. Denote by $G_i = \{x \in H\colon \liminf_{n\to\infty} \|f^{(n)}(x) - \bar{x}^i\| = 0\}$, where $f^{(n+1)}(x) = f(f^{(n)}(x))$, $f^{(1)}(x) = f(x)$. Finally, suppose that the Markov chain $\{x_n^\varepsilon, y_n\}$ in $H \times Y$ is ergodic for all $\varepsilon < 1$ and its ergodic distribution $m^\varepsilon(dx, dy)$ has the property that

$$m^\varepsilon\left(\left(H - \bigcup_{i=1}^{k} G_i\right) \times Y\right) = 0$$

for all small $\varepsilon > 0$. Then

$$P\{\lim_{n\to\infty} \inf_i \|x_n^\varepsilon - \bar{x}^i\| = 0\} = 1$$

5. SOME APPLICATIONS

Although there are a great many applications for the results derived above, including problems in numerical analysis and simulation, we present here three typical applications: one from epidemics, one from genetics, and one from demographics.

5.1. An Epidemic Model

A population in which an infection is active can be divided (roughly) into three classes: those susceptible to, those infectious with, and those removed from the disease process. We take the simplest case, depicted by the graph

$$S \to I \to$$

where S denotes the susceptible population and I the infective population. Passage from S to I depends on effective contact between a susceptible and an infective. The numbers in these classes are counted at fixed sampling times, and the results are denoted by $\{S_n, I_n\}$. We suppose that these satisfy the Kermack–McKendrick model (see Hoppensteadt, 1975, 1982):

$$\begin{aligned} S_{n+1} &= S_n e^{-\alpha I_n} \\ I_{n+1} &= (1 - e^{-\alpha I_n})\, S_n + \lambda I_n \end{aligned} \tag{57}$$

where α measures the infectiousness of the pathogen and λ gives the proportion of infectives surviving as infectives over one sampling period. We see immediately that $S_n \downarrow S_\infty$ and $I_n \to 0$ as $n \to \infty$. But the interest is in whether an epidemic ensues from an initial infective, which we take to mean that I_n increases (significantly) before eventually vanishing. If the sampling times are short, then $\alpha \approx 0$ and $\lambda \approx 1$. So we write $\alpha = \varepsilon\alpha$ and $\lambda = 1 - \varepsilon\beta$. We then denote the solution of the above iteration having initial values S_0, I_0 by $\{S_n^\varepsilon, I_n^\varepsilon\}$. Lemma 1 shows that for any t_0,

$$\lim_{\varepsilon \to 0} \sup_{\varepsilon n \leqslant t_0} (|S_n^\varepsilon - S(\varepsilon n)| + |I_n^\varepsilon - I(\varepsilon n)|) = 0 \tag{58}$$

where $S(t)$ and $I(t)$ are the solutions of the system of differential equations

$$\begin{aligned} \dot{S} &= -\alpha S I, \qquad & S(0) &= S_0 \\ \dot{I} &= \alpha S I - \beta I, \qquad & I(0) &= I_0 \end{aligned} \tag{59}$$

We define a critical parameter by $\tilde{S} = \beta/\alpha$. Integrating this system of equations gives

$$I = S_0 + I_0 - S + \tilde{S} \log \frac{S}{S_0} \tag{60}$$

If $S_0 < \tilde{S}$, then $I(t) \downarrow 0$ and $S(t) \downarrow S_{\min}$ as $t \to \infty$, where $S_{\min}$ satisfies the equation

$$S_{\min} = S_0 + I_0 + \tilde{S} \log \frac{S_{\min}}{S_0} \tag{61}$$

On the other hand, if $\tilde{S} > S_0$, then I increases for $\tilde{S} < S < S_0$, and I decreases for $S_{\min} < S < \tilde{S}$. The maximum of $I(t) = I_{\max}$ is determined from the equation

$$I_{\max} = S_0 + I_0 - \tilde{S} + \tilde{S} \log \frac{\tilde{S}}{S_0}$$

Therefore, the dimensionless parameter $\tilde{S}/S_0$ indicates the vulnerability of the susceptible population to supporting an epidemic. This is referred to as the Kermack–McKendrick threshold theorem (see Hoppensteadt, 1975, 1982).

Now consider the perturbed model where α and β are perturbed by random processes. Let S_n^ε and I_n^ε be determined by the perturbed Kermack–McKendrick model

$$\begin{aligned} S_{n+1}^\varepsilon &= S_n^\varepsilon \exp\{-\varepsilon\alpha(y_{n+1}) I_n^\varepsilon\} \\ I_{n+1}^\varepsilon &= (1 - \exp\{-\varepsilon\alpha(y_{n+1}) I_n^\varepsilon\}) S_n^\varepsilon + (1 - \varepsilon\beta(y_{n+1})) I_n^\varepsilon \\ S_0^\varepsilon &= S_0 \\ I_0^\varepsilon &= I_0 \end{aligned} \tag{62}$$

Here α and β are nonnegative, bounded measurable functions defined on Y, and $\{y_n\}$ is a Markov process that satisfies condition (NP2).

Let $\bar{\alpha} = \int_Y \alpha(y)\, \rho(dy)$ and $\bar{\beta} = \int_Y \beta(y)\, \rho(dy)$, and as before, $S(t)$ and $I(t)$ are determined from the averaged differential equations. Then we have the following result which shows that small random perturbations do not significantly alter the threshold theorem for the deterministic case.

Theorem 8. *For any* $\delta > 0$,

(1) $\lim_{\varepsilon \to 0} P\{\sup_n(|S_n^\varepsilon - S(\varepsilon n)| + |I_n^\varepsilon - I(\varepsilon n)|) > \delta\} = 0$

(2) Set $S_{\min}^\varepsilon = \lim_{n\to\infty} S_n^\varepsilon$, $I_{\max}^\varepsilon = \sup_n I_n^\varepsilon$, *then*

$$\lim_{\varepsilon \to 0} P\{|S_{\min}^\varepsilon - S_{\min}| + |I_{\max}^\varepsilon - I_{\max}| > \delta\} = 0$$

Proof. It is evident that (1) implies (2). To prove (1) we note that by Theorem 1 we have

$$\lim_{\varepsilon \to 0} P\{ \sup_{n\varepsilon \leqslant t_0} |S_n^\varepsilon - S(\varepsilon n)| + |I_n^\varepsilon - I(\varepsilon n)|) > \delta\} = 0$$

for any $\delta > 0$ and t_0.

It follows from (62) that

$$I_{n+1}^\varepsilon \leqslant I_n^\varepsilon + \varepsilon(\alpha(y_{n+1}) S_n^\varepsilon - \beta(y_{n+1})) I_n^\varepsilon \leqslant I_n^\varepsilon \exp\{\varepsilon(\alpha(y_{n+1}) S_n^\varepsilon - \beta(y_{n+1}))\}$$

and therefore

$$I_{n+m}^\varepsilon \leqslant I_m^\varepsilon \exp\left\{\varepsilon \sum_{k=0}^{n-1} (S_m^\varepsilon \alpha(y_{m+k}) - \beta(y_{m+k}))\right\}$$
$$\leqslant I_m^\varepsilon \exp\{n\varepsilon(S_m^\varepsilon \alpha - \beta + \theta_n^m)\}$$

where

$$\theta_n^m = \sup_{N \geqslant n} \left(\left|(1/N) \sum_{k=0}^{N-1} \alpha(y_{m+k}) - \alpha\right| S_0 + \left|(1/N) \sum_{k=0}^{N-1} \beta(y_{m+k}) - \beta\right|\right).$$

It follows from (NP2) that $\theta_n^m \to 0$ with probability 1 as $n \to \infty$ and $\theta_{n+1}^m \leqslant \theta_n^m$. Suppose that $S_m^\varepsilon \alpha - \beta + \theta_{n_0 - m} < -\gamma$, where $\gamma > 0$ and $n_0 > m$, then

$$\varepsilon \sum_{n=n_0}^{\infty} I_n^\varepsilon \leqslant I_m^\varepsilon \frac{\varepsilon}{1 - e^{-\varepsilon\gamma}} \leqslant \frac{1}{\gamma} I_m^\varepsilon e^{\varepsilon\gamma}$$

Note that for $n > n_0$,

$$S_n^\varepsilon = S_{n_0}^\varepsilon \exp\left\{-\varepsilon \sum_{n_0 \leqslant k < n} \alpha(y_{k+1}) I_k^\varepsilon\right\}$$

and

$$|S_n^\varepsilon - S_{n_0}^\varepsilon| \leqslant \varepsilon S_{n_0}^\varepsilon \sum_{k=n_0}^{\infty} \alpha(y_{k+1}) I_k^\varepsilon \leqslant \frac{S_0 I_m^\varepsilon}{\gamma} e^{\varepsilon\gamma}$$

For $n \geqslant n_0$

$$I_n^\varepsilon \leqslant I_m^\varepsilon \exp\{\varepsilon(n-m)(S_m^\varepsilon \alpha - \beta + \theta_{n_0 - m})\} \leqslant I_m^\varepsilon e^{-\varepsilon(n-m)\gamma}$$

if $S_m^\varepsilon \alpha - \beta + \theta_{n_0 - m} < -\gamma$.

Therefore

$$\sup_{\varepsilon n \geqslant t_0} (|I_n^\varepsilon - I(\varepsilon n)| + |S_n^\varepsilon - S(\varepsilon n)|)$$

$$\leqslant I(t_0) + I_m^\varepsilon e^{-\varepsilon(n_0 - m)\gamma} + \frac{S_0 I_m^\varepsilon}{\gamma} e^{\varepsilon\gamma} + S(t_0) - S_{\min} \tag{63}$$

if $S_m^\varepsilon \alpha - \beta + \theta_{n_0 - m} < -\gamma$.

Set $\gamma = \frac{1}{2}(\beta - \alpha S_{\min})$, and let t_1 satisfy condition

$$S(t_1) - S_{\min} < \delta_1, \qquad I(t_1) < \delta_1, \qquad \delta_1 < \frac{\gamma}{4\alpha}$$

where $t_0 > t_1$, $(m-1)\varepsilon \leqslant t_1 < (m\varepsilon)$, and $(n_0 - 1)\varepsilon \leqslant t_0 < n_0\varepsilon$.

It follows from the conditions

$$|\theta_{n_0 - m}| < \frac{\gamma}{2} \qquad \text{and} \qquad \sup_{\varepsilon n \leqslant t_0} (|I_n^\varepsilon - I(\varepsilon n)| + |S_n^\varepsilon - S(\varepsilon n)|) \leqslant \delta_1$$

that

$$\sup_{\varepsilon n \geqslant t_0} (|I_n^\varepsilon - I(\varepsilon n)| + |S_n^\varepsilon - S(\varepsilon n)|) \leqslant \delta_1 \left(4 + 2S_0 \frac{e^{\varepsilon\gamma}}{\gamma}\right) \qquad \square$$

5.2. Slow Genetic Selection

Let ρ, σ, and τ be positive constants and let the sequence $\{g_n\}$ be determined by Mendelian genetics for a single locus genetic trait having two allelic forms, say A and B, in a synchronized population. Namely, let g_n denote the proportion of the gene pool that is of type A in the nth generation. Then (see Hoppensteadt (1982)).

$$g_{n+1} = f(g_n, \rho, \sigma, \tau), \qquad g_0 \text{ is given} \tag{64}$$

where

$$f(g_n, \rho, \sigma, \tau) = \frac{\rho g_n^2 + \sigma g_n(1 - g_n)}{\rho g_n^2 + 2\sigma g_n(1 - g_n) + \tau(1 - g_n)^2}$$

The parameters ρ, σ, and τ are the relative fitnesses of the genotypes AA, AB, and BB, respectively.

When selection is slow relative to reproduction, then these parameters are near 1, so in this case we write

$$\rho = 1 + \varepsilon\alpha, \qquad \sigma = 1 + \varepsilon\beta, \qquad \tau = 1 + \varepsilon\gamma$$

It is convenient to write the iteration in the form

$$g^{\varepsilon}_{n+1} - g^{\varepsilon}_n = \frac{\varepsilon Q(g^{\varepsilon}_n)}{1+\varepsilon P(g^{\varepsilon}_n)}\, g^{\varepsilon}_n, \qquad g^{\varepsilon}_0 = g_0 \tag{65}$$

where $P(x) = \alpha x^2 + 2\beta x(1-x) + \gamma(1-x)^2$ and $Q(x) = x(1-x)(ax+b)$, where $a = \alpha + \gamma - 2\beta$ and $b = \alpha - \gamma$.

Let $g(t)$ denote the solution of the differential equation

$$\begin{aligned} \dot{g} &= Q(g) \\ g(0) &= g_0 \end{aligned} \tag{66}$$

We expect the solution of the iteration to be approximated by the solution of this equation in some sense and we write

$$g^{\varepsilon}_n \approx g(\varepsilon n)$$

There are four cases of interest regarding this system.

(1) A dominant: If $g_0 > 0$, $b \geqslant 0$, $a+b \geqslant 0$, and $|a| + |b| > 0$, then $g(\infty) = 1$.

(2) B dominant: If $g_0 < 1$, $b \leqslant 0$, $a+b \leqslant 0$, and $|a| + |b| > 0$, then $g(\infty) = 0$.

(3) Polymorphism: If $1 > g_0 > 0$, $b > 0$, $a+b < 0$, then $g(\infty) = -b/\alpha$.

(4) Disruptive selection:
- If $-b/\alpha > g_0 > 0$, $b > 0$, $a+b < 0$, then $g(\infty) = 0$.
- If $-b/\alpha < g_0 < 1$, $b < 0$, $a+b > 0$, then $g(\infty) = 1$.

We next consider this model perturbed by random noise and show that, under certain conditions, these four cases carry over with convergence meaning in probability. The perturbed system has the form

$$\begin{aligned} g^{\varepsilon}_{n+1} &= f(g^{\varepsilon}_n, \rho^{\varepsilon}_n, \sigma^{\varepsilon}_n, \tau^{\varepsilon}_n) \\ g^{\varepsilon}_0 &= g_0 \end{aligned} \tag{67}$$

where

$$\begin{aligned} \rho^{\varepsilon}_n &= 1 + \varepsilon\alpha(y_{n+1}) \\ \sigma^{\varepsilon}_n &= 1 + \varepsilon\beta(y_{n+1}) \\ \tau^{\varepsilon}_n &= 1 + \varepsilon\gamma(y_{n+1}) \end{aligned}$$

(*) We suppose that the functions α, β, and τ are bounded measurable functions mapping $Y \to R$ and that the Markov process $\{y_n\}$ satisfies condition (NP2).

We denote the averages of the data by

$$\bar{\alpha} = \int \alpha(y)\, \rho(dy)$$

$$\bar{\beta} = \int \beta(y)\, \rho(dy)$$

$$\bar{\gamma} = \int \gamma(y)\, \rho(dy)$$

and

$$\bar{a} = \bar{\alpha} + \bar{\gamma} - 2\bar{\beta}$$
$$\bar{b} = \bar{\alpha} - \bar{\gamma}$$

Theorem 9. *Under the conditions (*) listed above, we have the following results:*

(1) A dominant: If $g_0 > 0$, $\bar{b} \geqslant 0$, $\bar{a} + \bar{b} \geqslant 0$, *and* $|\bar{a}| + |\bar{b}| > 0$, *then*

$$\lim_{\varepsilon \to 0} P\{\lim_{n \to \infty} g_n^\varepsilon = 1\} = 1$$

(2) B dominant: If $g_0 < 1$, $\bar{b} \leqslant 0$, $\bar{a} + \bar{b} \leqslant 0$, *and* $|\bar{a}| + |\bar{b}| > 0$, *then*

$$\lim_{\varepsilon \to 0} P\{\lim_{n \to \infty} g_n^\varepsilon = 0\} = 1$$

(3) Polymorphism: If $1 > g_0 > 0$, $\bar{b} > 0$, $\bar{a} + \bar{b} < 0$, *then*

$$\lim_{\varepsilon \to 0} P\{\lim_{n \to \infty} g_n^\varepsilon = -b/a\} = 1$$

(4) Disruptive selection:

(a) If $-\bar{b}/\bar{a} > g_0 > 0$, $\bar{b} > 0$, $\bar{a} + \bar{b} < 0$, *then*

$$\lim_{\varepsilon \to 0} P\{\lim_{n \to \infty} g_n^\varepsilon = 0\} = 1$$

(b) If $-\bar{b}/\bar{a} < g_0 < 1$, $\bar{b} < 0$, $\bar{a} + \bar{b} > 0$, *then*

$$\lim_{\varepsilon \to 0} P\{\lim_{n \to \infty} g_n^\varepsilon = 1\} = 1$$

Theorem 9 shows that if selection is slow relative to reproduction, then the genetic structure of the population proceeds in strict analogy with the deterministic case. In particular, in cases (1), (2), and (4) fixation is probable.

Proof. Consider (2). It follows from Theorem 1 that

$$\lim_{\varepsilon \to 0} P\{\sup_{\varepsilon n \leqslant t_0} |g_n^\varepsilon - g(\varepsilon n)| > \delta\} = 0$$

for any $\delta > 0$ and $t_0 > 0$.

Suppose that $g(t_0) < \delta$ and $g_{n_0}^\varepsilon \leqslant 2\delta$, where $\varepsilon n_0 < t_0 < \varepsilon(n_0+1)$ and δ is small enough. Let $b(y) = \alpha(y) - \gamma(y)$. Then there exists a constant $c_1 > 0$ for which

$$g_{n+1}^\varepsilon \leqslant g_n^\varepsilon(1 + \varepsilon b(y_{n+1}) + \varepsilon c_1 g_n^\varepsilon) \tag{68}$$

and there exists $c_2 > 0$ for which

$$g_{n+1}^\varepsilon - g_n^\varepsilon \leqslant c_2 \varepsilon$$

Let $\hat{g}_{n,m}^\varepsilon = \max_{0 \leqslant k \leqslant m} g_{n+k}^\varepsilon$. Then $\hat{g}_{n,m}^\varepsilon \leqslant g_n^\varepsilon + mc_2\varepsilon$. It follows from (68) that

$$\begin{aligned} g_{n+m}^\varepsilon &\leqslant g_n^\varepsilon \exp\left\{\varepsilon \sum_{k=1}^m b(y_{n+k}) + \varepsilon m c_1 \hat{g}_{n,m}^\varepsilon\right\} \\ &\leqslant g_n^\varepsilon \exp\{\varepsilon m(b + \theta_{n,m} + c_1 \hat{g}_{n,m}^\varepsilon)\} \end{aligned}$$

where $0 < \theta_{n,m+1} < \theta_{n,m}$, $\theta_{n,m} \to 0$ as $m \to \infty$. Suppose that $\theta_{n_0,m_0} < |b|/4$ and

$$c_1(2\delta + \varepsilon m_0 c_2) < \frac{|b|}{4} \tag{69}$$

Then

$$g_{n_0+m}^\varepsilon \leqslant 2\delta \exp\left\{\varepsilon m \frac{b}{2}\right\} \qquad \text{for} \quad m > m_0$$

therefore

$$P\{\lim_{n \to \infty} g_n^\varepsilon = 0\} \geqslant 1 - P\left\{\theta_{n_0,m_0} \geqslant \frac{|b|}{4}\right\}$$

where m_0 satisfies (69). If $2c_1\delta = |b|/8$, and

$$\frac{|b|}{10c_2} < \varepsilon m_0 < \frac{|b|}{8c_2}$$

then $m_0 \to \infty$ as $\varepsilon \to 0$ and

$$\lim_{\varepsilon \to 0} P\left\{\theta_{n_0, m_0} \geq \frac{|b|}{4}\right\} = 0$$

The proof of the other points are similar. □

5.3. Demographics

The age distribution of a population can be determined by census, and then the numbers in various age classes, say $\{v_1, \ldots, v_m\}$, define a vector whose dynamics can usefully describe how a population's age structure changes. For example, a census might be taken every 5 years, females counted, and data kept up through the end of reproductive ages, say age 55. Then 11 age classes would be monitored.

The population's dynamics are described by the system of equations

$$v_{n+1} = A v_n$$

where v_n is the vector of age classes at the nth census and A is Leslie's matrix (see Keyfitz and Flieger, 1971) given by

$$A = \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_{m-1} & 0 \\ \lambda_1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_2 & \cdots & 0 & 0 \\ \dot{0} & 0 & \ddots & 0 & 0 \\ 0 & 0 & \cdots & \lambda_{m-1} & 0 \end{pmatrix} \tag{70}$$

where $\alpha_i \geq 0$ are the fertilities of the various age groups and $\lambda_i \in (0, 1]$ are the survival probabilities of the various age groups to the next census. The first problem we consider here is to determine the asymptotic behavior of $A^n x$ as $n \to \infty$.

The theory of nonnegative matrices shows (see Harris, 1963, Chap. 2, Sect. 9) that

(1) A has a unique positive eigenvalue ξ;

(2) ξ is an eigenvalue for A^*, the adjoint of A;

(3) the corresponding eigenvectors, say b and b^*, of A and A^*, respectively, have nonnegative components;

(4) if $\theta \neq \xi$ is another eigenvalue of either A or A^*, then $|\theta| < \xi$; and

(5) the powers of Λ can be calculated asymptotically:

$$\Lambda^n x = (b^*, x)\, \xi^n b + O(\theta_1^n) \tag{71}$$

where θ_1 is some constant satisfying $0 < \theta_1 < \xi$.

The second problem is to determine the asymptotic behavior of solutions to a randomly perturbed Leslie matrix. Let $\tilde{\Lambda}(y)$ denote a matrix of the same form as Λ but having coefficients $\alpha_i(y)$ and $\lambda_i(y)$ instead of α_i and λ_i, respectively. The functions $\alpha(y)$ and $\lambda(y)$ are assumed to be bounded and measurable functions mapping: $Y \to R$.

We consider the perturbed problem

$$x_{n+1}^{\varepsilon} = (\Lambda + \varepsilon \tilde{\Lambda}(y_{n+1}))\, x_n^{\varepsilon}$$
$$x_0^{\varepsilon} = x_0$$

We suppose that $\{y_n\}$ satisfies condition (NP2).

Theorem 10. *Let the conditions listed above in this section be satisfied. Then*

$$P\left\{\lim_{\varepsilon \to 0}\ \limsup_{n \to \infty} \left|\frac{\log \|x_n^{\varepsilon}\|}{n} - \log \xi\right| = 0\right\} = 1$$

This theorem shows that for small ε, the intrinsic growth rate of the perturbed population is probably the same as the unperturbed one.

The proof of the theorem is based on the representation

$$x_n^{\varepsilon} = \prod_{k=1}^{n} (\Lambda + \varepsilon \tilde{\Lambda}(y_n))\, x_0 = \Lambda^n x_0 + \varepsilon S_{n,1} x_0 + \cdots + \varepsilon^n S_{n,n} x_0$$

where

$$S_{n,1} = \sum_{k=1}^{n} \Lambda^{n-k} \tilde{\Lambda}(y_k)\, \Lambda^{k-1}$$

$$S_{n,r} = \sum_{n \geq i_1 > i_2 > \cdots > i_r \geq 1} \Lambda^{n-i_1} \tilde{\Lambda}(y_{i_1})\, \Lambda^{i_1 - i_2 - 1} \tilde{\Lambda}(y_{i_2}) \cdots \tilde{\Lambda}(y_{i_r})\, \Lambda^{i_r - 1}$$

and formula (71). The details are not carried out here.

APPENDIX

Here we present some limit theorem for random variables which are functions of a Markov chain $\{y_n\}$ that satisfies condition (NP3).

Let

$$r_k = 2 \sup_{y \in Y,\, C \in \mathscr{C}} |R_k(y, C)| = \sup_{y \in Y} \operatorname{var} |R_k(y, \cdot)|$$

Condition (NP3) implies that $\sum_k r_k < \infty$.

We consider limit theorem for H-valued random variables of the form $\sum_{k=1}^{n} f(y_k)$, where f: $Y \to H$ is a measurable function.

Lemma 1. *Let $g(z_1,..., z_m)$: $Y^m \to R$ be $\mathscr{C}^m$-measurable bounded function for which $\int g(z_1,..., z_m)\, \rho(dz_k) = 0$ for all $k \leqslant m$. Then, with $\|g\|$ denoting the usual sup norm,*

$$\left| E\left(\sum_{1 \leqslant k_1 < \cdots < k_m \leqslant n}\right) g(y_{k_1}, y_{k_2},..., y_{k_m})/y_0 \right| \leqslant \|g\| \left(\sum_{k=1}^{\infty} r_k\right)^m$$

Proof. We have for $0 < k_1 < \cdots < k_m$,

$$\begin{aligned} &|Eg(y_{k_1}, y_{k_2},..., y_{k_m})/y_0| \\ &\quad = \left| E \int \cdots \int g(z_1, z_2,..., z_m)\, P_{k_1}(y_0, dz_1) \right. \\ &\qquad\qquad \left. P_{k_2-k_1}(z_1, dz_2) \cdots P_{k_m-k_{m-1}}(z_{m-1}, dz_m) \right| \\ &\quad = \left| E \int \cdots \int g(z_1, z_2,..., z_m)\, R_{k_1}(y_0, dz_1) \right. \\ &\qquad\qquad \left. P_{k_2-k_1}(z_1, dz_2) \cdots P_{k_m-k_{m-1}}(z_{m-1}, dz_m) \right| \\ &\quad = \left| E \int \cdots \int g(z_1,..., z_m)\, R_{k_1}(y_0, dz_1) \right. \\ &\qquad\qquad \left. R_{k_2-k_1}(z_1, dz_2) \cdots R_{k_m-k_{m-1}}(z_{m-1}, dz_m) \right| \\ &\quad \leqslant \|g\|\, r_{k_1} r_{k_2-k_1} \cdots r_{k_m-k_{m-1}} \end{aligned}$$

This inequality proves the lemma.

Condition (A1). Let $g(z)$: $Y \to R$ be a bounded measurable function for which $\int g(z)\, \rho(dz) = 0$.

Corollaries. *If g satisfies condition (A1), then uniformly in y_0,*

(1) $E(\sum_{k=1}^{n} g(y_k)/y_0) = O(1)$,

(2) $E(\sum_{1 \leqslant k_1 < k_2 < k_3 \leqslant n} g^2(y_{k_1})\, g(y_{k_2})\, g(y_{k_3})/y_0) = O(n)$.

To prove this relation we set $c = \int g^2(z)\, \rho(dz)$. *Then*

$$E\left(\sum_{1 \leqslant k_1 < k_2 \leqslant n} (g^2(y_{k_1}) - c)\, g(y_{k_2})\, g(y_{k_3})/y_0\right) = O(1)$$

and

$$cE\left(\sum_{1 \leqslant k_1 < k_2 < k_3 \leqslant n} (g(y_{k_2})\, g(y_{k_3})/y_0\right) = O(n)$$

(3) In the same way

$$E\left(\sum_{1 \leqslant k_1 < k_2 < k_3 \leqslant n} g(y_{k_1})\, g^2(y_{k_2})\, g(y_{k_3})/y_0\right) = O(n)$$

$$E\left(\sum_{1 \leqslant k_1 < k_2 < k_3 \leqslant n} g(y_{k_1})\, g(y_{k_2})\, g^2(y_{k_3})/y_0\right) = O(n)$$

(4) $E((\sum_{k=1}^{n} g(y_k))^4/y_0) = (n^2)$.

This follows from (2) and (3) and the relation

$$E\left(\sum_{1 \leqslant k_1 < \cdots < k_4 \leqslant n} g(y_{k_1}) \cdots g(y_{k_4})/y_0\right) = O(1)$$

(5) $E((\sum_{k=1}^{n} g(y_k))^2/y_0) = O(n)$.

Lemma 2. *Let* $g(y)$ *satisfy condition (A1). Then for any* $y \in Y$,

$$\lim_{n \to \infty} E\left(\frac{1}{n}\left(\sum_{k=1}^{n} g(y_k)\right)^2 \Big/ y_0 = y\right) = \iint g(z)\, g(z')\, \rho(dz)\, R(z, dz') = b$$

Proof. We have

$$\begin{aligned} &E\left(\left(\sum_{k=1}^{n} g(y_k)\right)^2 \Big/ y_0\right) \\ &\quad = E\left(\sum_{k=1}^{n} g^2(y_k) + 2 \sum_{1 \leqslant k < j \leqslant n} g(y_k) \int g(z)\, R_{j-k}(y_k, dz) \Big/ y_0\right) \\ &\quad = E\left(\sum_{k=1}^{n} \left(g^2(y_k) + 2g(y_k) \int g(z) \sum_{i=1}^{\infty} R_i(y_k, dz) \Big/ y_0\right)\right) \\ &\qquad + O\left(\sum_{k=1}^{n} \sum_{j=n+1}^{\infty} r_{j-k}\right) \end{aligned}$$

The statement of the lemma follows from ergodic theorem and the relation

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \sum_{i=n+1-k}^{\infty} r_i = \lim_{n \to \infty} \sum_{i=1}^{\infty} \frac{i \blacktriangle n}{n} r_i = 0 \qquad \square$$

Theorem 1. *Let g satisfy condition (A1). Set $\zeta_k = \sum_{i=1}^{k} g(y_i)$, $\zeta_0 = 0$, and let*

$$\zeta_n(t) = \frac{1}{\sqrt{nb}} (\theta \zeta_k + (1-\theta)\, \zeta_{k+1}) \qquad \text{if} \quad t \in [k/n, k+1/n], \quad \theta = k+1-nt$$

where b is determined in Lemma 2.

Then the distribution of $\zeta_n(t)$ converges weakly in $C_{[0,\, T]}$ to the distribution of the Wiener process $w(t)$ for any $T > 0$.

Proof.

(I) At first we note that it follows from Corollary 4 that

$$E\, |\zeta_n(t_1) - \zeta_n(t_2)|^4 \leqslant c_1\, |t_1 - t_2|^4$$

where c_1 is a constant which does not depend on n. Therefore the sequence $\{\zeta_n(t), t \in [0, T]\}$ is tight in $C[0, T]$ (see Billingsley, 1968, p. 95, Theorem 12.3).

Let a subsequence $\zeta_{n_k}(t)$ converge weakly in $C[0, T]$ to the process $\zeta_0(t)$. It follows from Corollary 1 that $\zeta_0(t)$ is a martingale and from Lemma 2 that it has the square characteristic equal to t. Therefore $\zeta_0(t)$ is the Wiener process.

Corollary. *The random variable ζ_k is asymptotically Gaussian with the mean value 0 and the variance $\sqrt{kb}$.*

Now we consider a limit theorem for H-valued functions f. We assume that the function f satisfies the following condition.

(A2) f is a bounded measurable function: $Y \to H$ for which $\int f(y)\, \rho(dy) = 0$ and there exists a positive compact symmetric operator B for which $\sup_y \|B^{-1} f(y)\| < \infty$.

We introduce the sequence of H-valued random variables:

$$\eta_k = \sum_{i=1}^{k} f(y_i)$$

and the sequence of H-valued continuous stochastic processes:

$$\eta_n(t)=\frac{1}{\sqrt{n}}((k+1-nt)\,\eta_k+(nt-k)\,\eta_{k+1})$$
$$\text{for}\quad \frac{k}{n}\leqslant t\leqslant\frac{k+1}{n},\quad k=0,1,\ldots$$

Theorem 2. *The distribution $\eta_n(t)$ converges weakly in $C_{[0,T]}(H)$ to the distribution of the homogeneous Gaussian process with independent increments $\eta(t)$ for which $E\eta(t)=0$ and*

$$E(\eta(t),z)^2=t\iint(f(y),z)(f(y'),z)\,R(y,dy')\,\rho(dy)$$
$$=(Vz,z)\qquad \textit{for any}\quad z\in H$$

Here $C_{[0,T]}(H)$ is the space of continuous H-valued functions on $[0,T]$.

Proof. It follows from Theorem 1 that the finite-dimensional distributions of $(\eta_n(t),z)$ converge to the finite-dimensional distribution of the process $(\eta(t),z)$, in particular,

$$\lim_{n\to\infty}E\exp\{i(\eta_n(t),z)\}=\exp\{-1/2t(Vz,z)\}\tag{1}$$

Let B be the operator from condition (A2). Then we have

$$\begin{aligned}\mathrm{Re}(1-Ee^{i(\eta_n(t),B^{-1}z)})&=E(1-\cos(B^{-1}\eta_n(t),z))\\&=E(1-\cos(B^{-1}\eta_n(t),z))\,1_{\{\|B^{-1}\eta_n(t)\|\leqslant r\}}\\&\quad+2P\{\|B^{-1}\eta_n(t)\|>r\}\\&\leqslant\tfrac{1}{2}E(B^{-1}\eta_n(t),z)^2\,1_{\{\|B^{-1}\eta_n(t)\|\leqslant r\}}\\&\quad+2P\{\|B^{-1}\eta_n(t)\|>r\}\end{aligned}$$

Denote by Q_r the nonnegative symmetric operator $H\to H$ for which

$$(Q_rz,z)=\tfrac{1}{2}E(B^{-1}\eta_n(t),z)^2\,1_{\{\|B^{-1}\eta_n(t)\|\leqslant r\}}$$

It is evident that $\mathrm{tr}\,Qr\leqslant\frac{1}{2}r^2$. In the same way as Corollary 5 after Lemma 1 we can show that

$$E\left(\sum_{k=1}^{n}B^{-1}f(y_k),\sum_{k=1}^{n}B^{-1}f(y_k)\right)=O(n)\tag{2}$$

and therefore

$$E\,\|B^{-1}\eta_n(t)\|^2 \leqslant c_2 t$$

where c_2 is a constant. Therefore

$$\mathrm{Re}(1 - Ee^{i(\eta_n(t),\, B^{-1}z)}) \leqslant (Q_r z, z) + \frac{c_2 t}{r^2} \tag{3}$$

Since (2) and (3) imply that the distribution $\eta_n(t)$ converges weakly in H to the distribution of $\eta(t)$ (see Gikhman *et al.*, 1974, p. 372, Theorem 1). Hence the finite-dimensional distributions of $\eta_n(t)$ converge weakly in H to the finite-dimensional distributions of $\eta(t)$.

In the same way as Corollary 4 after Lemma 2 we can show that

$$E\left(\sum_{k=1}^{n} f(y_k), \sum_{k=1}^{n} f(y_k)\right)^2 = O(n^2)$$

and therefore

$$E\,\|\eta_n(t_2) - \eta_n(t_1)\|^4 \leqslant c_4 (t_2 - t_1)^2$$

where c_4 is a constant.

Now the proof of the theorem follows from the general theorem on weak convergence in the space $C_{[0,T]}(X)$, where X is a complete separable metric space (see Gikhman *et al.*, 1974, p. 140, Theorem 2).

Corollary. *The distribution of* $(1/\sqrt{n}) \sup_{k \leqslant nT} \|\eta_k\|$ *converges to the distribution of* $\sup_{t \leqslant T} \|\eta(t)\|$.

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