Aequationes Mathematicae 41 (1991) 192–211
University of Waterloo

0001-9054/91/030192-20 $1.50 + 0.20/0

Cauchy's equation on Δ^+

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Summary. Recently R. C. Powers characterized the order automorphisms of the space of nondecreasing functions from one compact real interval to another [6, 7]. In this paper we show how his results, as well as the lattice-theoretic techniques which he employed, can be used to obtain solutions of Cauchy's equation for certain classes of semigroups (triangle functions) on the space Δ^+ of probability distribution functions of nonnegative random variables.

1. Introduction

This paper is divided into six sections, Section 1 being this introduction. The next two sections are preliminary in nature. In Section 2 we give the complete solution of Cauchy's equation for continuous Archimedean t-norms and in Section 3 we introduce and study a subspace of Δ^+ which plays a central role in our study. Some general properties of solutions of Cauchy's equation for triangle functions are presented in Section 4. In Section 5 we use these results as well as recent results by R. C. Powers to find all order automorphism solutions of Cauchy's equation for semigroups of the form (Δ^+, τ_T), where T is a continuous Archimedean t-norm. Finally, in Section 6, the central section of this paper, we obtain a representation for all sup-continuous (equivalently, residuated) solutions of Cauchy's equation for semigroups of the form (Δ^+, τ_T), where T is a strict t-norm.

AMS (1980) subject classification: Primary 39B50. Secondary 60B99.

Manuscript received September 7, 1989, and in final form, August 21, 1990.

2. Cauchy's equation for t-norms

We begin by recalling several well-known definitions and facts (for details see [8]).

A *t-norm* T is a binary operation on the interval $I = [0, 1]$ which is commutative, associative, nondecreasing in each place, and has 1 as identity. A t-norm T is *Archimedean* if $T(x, x) < x$ for all x in $(0, 1)$; and T is *strict* if it is continuous and strictly increasing on $(0, 1] \times (0, 1]$. It is well-known that every strict t-norm is Archimedean and that a continuous Archimedean t-norm admits the representation

$$T(x, y) = g^{(-1)}(g(x) + g(y)), \qquad \text{for all } x, y \text{ in } I, \tag{2.1}$$

where g is a continuous, strictly decreasing function from I into $R^+ = [0, \infty]$, $g(1) = 0$, and $g^{(-1)}$ is the *pseudo-inverse* of g, namely the function determined by

$$\begin{aligned} &g^{(-1)}(g(x)) = x && \text{for all } x \text{ in } I, \\ &g(g^{(-1)}(y)) = \min(y, g(0)) && \text{for all } y \text{ in } R^+. \end{aligned} \tag{2.2}$$

Furthermore, T is strict if and only if $g(0) = \infty$, in which case $g^{(-1)}$ is the ordinary inverse of g and T admits the representation

$$T(x, y) = g^{-1}(g(x) + g(y)) \qquad \text{for all } x, y \text{ in } I. \tag{2.3}$$

The function g is called an *inner additive generator* (briefly, a generator); and it is well-known that g and h generate the same t-norm if and only if there is a $k > 0$ such that

$$g(x) = k \cdot h(x) \qquad \text{for all } x \text{ in } I. \tag{2.4}$$

DEFINITION 2.1. Let T be a t-norm and θ a function from I into I. Then θ is a solution of Cauchy's equation for T if and only if

$$\theta(T(x, y)) = T(\theta(x), \theta(y)) \qquad \text{for all } x, y \text{ in } I. \tag{2.5}$$

There are two obvious solutions of this equation, namely:

$$\theta(x) = 0 \qquad \text{for all } x \text{ in } I, \tag{2.6}$$

and

$$\theta(x) = 1 \qquad \text{for all } x \text{ in } I. \tag{2.7}$$

We will refer to these solutions as trivial solutions and, henceforth, unless otherwise stated, "solution" will mean "nontrivial solution". Note that, if T has no interior idempotents, i.e., if T is Archimedean, then (2.6) and (2.7) are the only constant solutions of (2.5).

We now find all continuous solutions of (2.5) for strict and for continuous Archimedean t-norms.

THEOREM 2.2. *Let T be a strict t-norm, g an inner additive generator of T, and θ a mapping from I into I. Then θ is a solution of Cauchy's equation for T if and only if there is a $k > 0$, such that*

$$\theta(x) = g^{-1}(k \cdot g(x)) \qquad \text{for all } x \text{ in } I. \tag{2.8}$$

Proof. Using (2.3) we have that θ satisfies Cauchy's equation for T if and only if $\theta(g^{-1}(g(x) + g(y))) = g^{-1}(g(\theta(x)) + g(\theta(y)))$, for all x, y in I. Letting $a = g(x)$, $b = g(y)$ and applying g to both sides, we obtain

$$(g\theta g^{-1})(a + b) = (g\theta g^{-1})(a) + (g\theta g^{-1})(b), \tag{2.9}$$

for all a, b in R^+. Thus θ satisfies Cauchy's equation for T if and only if $(g\theta g^{-1})$ satisfies Cauchy's equation on R^+. Since $(g\theta g^{-1}) \geqslant 0$ on R^+, it follows that there is a $k > 0$ such that

$$(g\theta g^{-1})(s) = k \cdot s \qquad \text{for all } s \text{ in } R^+, \tag{2.10}$$

which is equivalent to (2.8). □

Note that the solution (2.8) is independent of the choice of generator and that all these solutions are order automorphisms of I (see Definition 5.1).

To find corresponding solutions for continuous, non-strict, Archimedean t-norms we need the following:

LEMMA 2.3. *Let T be a continuous Archimedean t-norm and θ a continuous (nonconstant) solution of* (2.5). *Then*

$$\theta(0) = 0 \quad \text{and} \quad \theta(1) = 1.$$

Proof. Since T is Archimedean, it has only 0 and 1 as idempotents; and since θ has to map idempotents to idempotents, we have

$$T(\theta(0), \theta(0)) = \theta(0) \quad \text{and} \quad T(\theta(1), \theta(1)) = \theta(1).$$

If $\theta(0) = 1$ then, for all x in I, $\theta(x) = T(\theta(x), \theta(0)) = \theta(T(x, 0)) = \theta(0) = 1$. Thus θ is constant, which cannot be, so that $\theta(0) = 0$. Similarly if $\theta(1) = 0$, then, for all x in I, $T(\theta(x), \theta(1)) = \theta(x) = 0$, which cannot be, so that $\theta(1) = 1$. □

THEOREM 2.4. *Let T be a continuous, non-strict, Archimedean t-norm and let g be an inner additive generator of T. Then θ is a continuous solution of Cauchy's equation for T if and only if there is a $k \geq 1$ such that*

$$\theta(x) = g^{(-1)}(k \cdot g(x)) \qquad \text{for all } x \text{ in } I. \tag{2.11}$$

Proof. Using (2.1) and (2.2) we have that θ satisfies Cauchy's equation for T if, and only if, for all x, y in I,

$$(g\theta g^{(-1)})(g(x) + g(y)) = \min(g\theta(x) + g\theta(y), g(0)). \tag{2.12}$$

Now, let $g(x) = a$ and $g(y) = b$. Then a, b are in $\operatorname{Ran} g = [0, g(0)]$ and, in view of (2.2), $x = g^{(-1)}(a)$ and $y = g^{(-1)}(b)$. Thus (2.12) becomes

$$(g\theta g^{(-1)})(a + b) = \min((g\theta g^{(-1)})(a) + (g\theta g^{(-1)})(b), g(0)) \tag{2.13}$$

for all a, b in $[0, g(0)]$.

Suppose θ is a continuous solution of (2.5). Then, using Lemma 2.3 and (2.2), we have

$$(g\theta g^{(-1)})(0) = g(\theta(1)) = g(1) = 0 \quad \text{and} \quad (g\theta g^{(-1)})(g(0)) = g(\theta(0)) = g(0),$$

whence there exists an x_1 in $(0, g(0))$ such that $0 < (g\theta g^{(-1)})(x_1) < g(0)$. Let $x_0 = \sup\{x_1 \mid (g\theta g^{(-1)})(x_1) < g(0)\}$. Then $0 < x_0 \leq g(0)$, and $(g\theta g^{(-1)})(x_0) = g(0)$. Furthermore, if $t \geq 0$ is such that $(g\theta g^{(-1)})(t) < g(0)$ then, by (2.13),

$$(g\theta g^{(-1)})(t) = (g\theta g^{(-1)})(t - u) + (g\theta g^{(-1)})(u), \qquad \text{for all } u \text{ in } [0, t],$$

which implies that $(g\theta g^{(-1)})(u) < g(0)$ for all u in $[0, t]$. Consequently, we must have

$$(g\theta g^{(-1)})(a + b) = (g\theta g^{(-1)})(a) + (g\theta g^{(-1)})(b), \tag{2.14}$$

for all a, b and $a + b$ in $[0, x_0]$, and

$$(g\theta g^{(-1)})(s) = g(0), \tag{2.15}$$

for $x_0 \leqslant s \leqslant g(0)$. The first equation yields that there exists a $k > 0$ such that

$$(g\theta g^{(-1)})(a) = k \cdot a, \tag{2.16}$$

for all a in $[0, x_0]$ (see [2]); and, if we let $a = x_0$, we obtain

$$(g\theta g^{(-1)})(x_0) = k \cdot x_0 = g(0).$$

Thus $k = g(0)/x_0 \geqslant 1$; and, combining (2.14) and (2.15), we have that

$$(g\theta g^{(-1)})(a) = \begin{cases} (g(0)/x_0) \cdot a & \text{for } a \text{ in } [0, x_0], \\ g(0) & \text{for } a \text{ in } [x_0, g(0)]. \end{cases}$$

Finally, letting $a = g(x)$ and using (2.2) we obtain (2.11).

In the other direction, suppose θ is given by (2.11). Then, using (2.2) and again letting $g(x) = a$ and $g(y) = b$, the left-hand side of (2.12) is equal to

$$\min(k \cdot \min(a + b, g(0)), g(0)), \tag{2.17}$$

whereas the right-hand side is equal to

$$\min\{\min[k \cdot \min(a, g(0)), g(0)] + \min[k \cdot \min(b, g(0)), g(0)], g(0)\}. \tag{2.18}$$

It is readily verified that both of these expressions are equal to

$$\begin{cases} k \cdot (a + b) & \text{for } 0 \leqslant a + b < g(0)/k, \\ g(0) & \text{for } g(0)/k \leqslant a + b, \end{cases}$$

whence θ satisfies Cauchy's equation for T. □

Note that even though the x_0 in the above proof depends on the choice of generator, the function θ in (2.11) does not.

Note further that, if T and θ are as in Theorem 2.4, then θ is an order automorphism of I (see Definition 5.1) if and only if $k = 1$ in (2.11), i.e., if and only if θ is the identity map on I.

To illustrate the above, let $T(x, y) = \text{Max}(x + y - 1, 0)$. Then $g(x) = 1 - x$ is a generator with pseudo-inverse $g^{(-1)}(y) = \text{Max}(1 - y, 0)$ and, using (2.11), we obtain

$$\theta(x) = \begin{cases} 0 & \text{for } 0 \leqslant x \leqslant 1 - 1/k, \\ k \cdot x + (1 - k) & \text{for } 1 - 1/k < x \leqslant 1. \end{cases}$$

3. Preliminaries from Δ^+

We will denote by Δ^+ the space of all nondecreasing functions F from R^+ into I that satisfy $F(0) = 0$, $F(\infty) = 1$ and that are left-continuous on $(0, \infty)$.

The following elements of Δ^+ are of particular importance and therefore merit special symbols:

(i) For any a in R^+, ε_a is the function in Δ^+ defined by

$$\varepsilon_a(x) = \begin{cases} 0 & \text{for } 0 \leqslant x \leqslant a, \\ 1 & \text{for } a < x \leqslant \infty, \end{cases} \quad \text{if } 0 \leqslant a < \infty, \tag{3.1}$$

and

$$\varepsilon_\infty(x) = \begin{cases} 0 & \text{for } 0 \leqslant x < \infty, \\ 1 & \text{for } x = \infty. \end{cases} \tag{3.2}$$

(ii) For any a in R^+ and b in I, $\delta_{a,b}$ is the function in Δ^+ defined by

$$\delta_{a,b}(x) = \begin{cases} 0 & \text{for } 0 \leqslant x \leqslant a, \\ b & \text{for } a < x < \infty, \\ 1 & \text{for } x = \infty, \end{cases} \quad \text{if } 0 \leqslant a < \infty, \tag{3.3}$$

and

$$\delta_{\infty,b} = \varepsilon_\infty. \tag{3.4}$$

Note that, for all a in R^+, $\delta_{a,1} = \varepsilon_a$ and $\delta_{a,0} = \varepsilon_\infty$, and that we also have:

LEMMA 3.1. *For $0 \leqslant a, c < \infty$ and $0 < b, d \leqslant 1$, $\delta_{a,b} = \delta_{c,d}$ if and only if $a = c$ and $b = d$.*

In the sequel we let

$$\Delta_\delta^+ = \{\delta_{a,b} \mid a \text{ in } R^+, b \text{ in } I\}.$$

The elements of Δ^+ are partially ordered by

$$F \leqslant G \quad \text{if and only if} \quad F(x) \leqslant G(x) \qquad \text{for all } x \text{ in } R^+. \tag{3.5}$$

In particular,

$$\varepsilon_a \leqslant \varepsilon_b, \qquad \text{whenever } a \geqslant b,$$

and

$$\delta_{a,b} \leqslant \delta_{c,d}, \qquad \text{whenever } a \geqslant c \text{ and } b \leqslant d.$$

Moreover, ε_∞ and ε_0 are, respectively, the least and greatest elements in this partial order.

DEFINITION 3.2. A triangle function τ is a binary operation on Δ^+ that is commutative, associative, nondecreasing in each place, and has ε_0 as identity.

As an immediate consequence we have that ε_∞ is a zero for τ, i.e., that

$$\tau(\varepsilon_\infty, F) = \varepsilon_\infty \qquad \text{for all } F \text{ in } \Delta^+, \tag{3.6}$$

whence (Δ^+, τ) is a semigroup with identity and zero.

DEFINITION 3.3. Let τ be a triangle function. Then for any sequence $F_1, \ldots, F_n$ in Δ^+, we write

$$\tau^2(F_1, F_2) = \tau(F_1, F_2),$$

and

$$\tau^n(F_1, \ldots, F_n) = \tau(\tau^{n-1}(F_1, \ldots, F_{n-1}), F_n),$$

for $n > 2$. Furthermore, for any F in Δ^+ and any $n \geqslant 2$, the n-th τ-power of F is denoted by

$$F_\tau^n = \tau^n(F, \ldots, F); \tag{3.7}$$

when it is clear from the context, we often omit the subscript and just write F^n.

We will be principally concerned with the class of triangle functions τ_T that are induced by left-continuous t-norms via

$$\tau_T(F, G)(x) = \sup_{u+v=x} \{T(F(u), G(v))\} \tag{3.8}$$

for all F, G in Δ^+ and x in R^+.

A simple calculation yields that, for all a, b in R^+ and all c, d in I,

$$\tau_T(\delta_{a,c}, \delta_{b,d}) = \delta_{a+b,T(c,d)}. \tag{3.9}$$

Thus we have:

LEMMA 3.4. *$(\Delta_\delta^+, \tau_T)$ is a subsemigroup of (Δ^+, τ_T).*

Moreover, it follows from (3.9) that, for any $\delta_{a,b}$ in Δ_δ^+,

$$\delta_{a,b} = \tau_T(\varepsilon_a, \delta_{0,b}). \tag{3.10}$$

Thus the semigroup $(\Delta_\delta^+, \tau_T)$ is generated by the family of elements $\{\varepsilon_a, \delta_{0,b}\}$.

With the partial order defined in (3.5), Δ^+ is a *complete lattice*, i.e., a partially ordered set in which every subset has a supremum and an infimum. Here, for any subset S of Δ^+, the supremum of S is the pointwise supremum of all functions in S and the infimum of S is the supremum of the set of all lower bounds of S. The latter refinement is necessary since the pointwise infimum of left-continuous functions need not be left-continuous.

We also have the following basic lemma, which is due to R. C. Powers [7]:

LEMMA 3.5. *Let F be in Δ^+, then*

$$F = \sup_{a \in R^+} \delta_{a,F(a)}. \tag{3.11}$$

DEFINITION 3.6. A function φ from Δ^+ into Δ^+ is said to be sup-continuous if, for any index set J and any collection $\{F_j\}$ such that F_j is in Δ^+ for all j in J, we have

$$\varphi(\sup_{j \in J} F_j) = \sup_{j \in J} \varphi(F_j). \tag{3.12}$$

It follows at once from Lemma 3.5 that a sup-continuous function on Δ^+ is completely determined by its value on Δ_δ^+.

Furthermore, letting J be the empty set yields $\varphi(\varepsilon_\infty) = \varepsilon_\infty$. Thus in lattice-theoretic terminology a function $\varphi: \Delta^+ \to \Delta^+$ is sup-continuous if and only if it is *residuated* [4].

DEFINITION 3.7. Let τ be a triangle function. Then τ is sup-continuous if, for any index set J and any collection $\{F_j\}$ such that F_j is in Δ^+ for all j in J, and for all G in Δ^+,

$$\tau(\sup_{j \in J} F_j, G) = \sup_{j \in J} \tau(F_j, G). \tag{3.13}$$

The next lemma is due to R. M. Tardiff [9] (see also [8, Sec. 12.9]).

LEMMA 3.8. *If T is a continuous t-norm, then τ_T is sup-continuous.*

However as pointed out by Tardiff, convolution is not sup-continuous.

4. Cauchy's equation for τ

We now turn to Cauchy's equation for triangle functions τ on Δ^+. We begin with some properties which do not depend on the specific choice of τ.

DEFINITION 4.1. Let τ be a triangle function and φ a function from Δ^+ into Δ^+. Then φ is a solution of Cauchy's equation for τ if and only if

$$\varphi(\tau(F, G)) = \tau(\varphi(F), \varphi(G)) \qquad \text{for all } F, G \text{ in } \Delta^+. \tag{4.1}$$

There are two trivial solutions, namely $\varphi = \varepsilon_\infty$ and $\varphi = \varepsilon_0$; and, as before, unless stated otherwise, "solution" will mean "nontrivial solution". As in the case of t-norms, we note that, if τ has no interior idempotents, i.e., if $\tau(F, F) = F$ implies that $F = \varepsilon_0$ or $F = \varepsilon_\infty$, then ε_∞ and ε_0 are the only constant solutions. Furthermore, for any such τ, a non-constant solution φ has to satisfy

$$\varphi(\varepsilon_\infty) = \varepsilon_\infty \quad \text{and} \quad \varphi(\varepsilon_0) = \varepsilon_0;$$

the proof is analogous to that of Lemma 2.3.

LEMMA 4.2. *Let φ satisfy Cauchy's equation for τ. Then*
(i) *$\varphi(\varepsilon_0)$ is the identity on Ran φ,*
(ii) *$\varphi(\varepsilon_\infty)$ is the zero on Ran φ,*
(iii) *φ maps idempotents to idempotents,*
(iv) *φ preserves n-th powers, i.e., $\varphi(H^n) = \varphi(H)^n$, for all H in Δ^+,*
(v) *φ maps any element with an n-th root to an element with an n-th root.*

Proof. (i) For all F in Δ^+, $\varphi(F) = \varphi(\tau(F, \varepsilon_0)) = \tau(\varphi(F), \varphi(\varepsilon_0))$. (ii) For all F in Δ^+, $\varphi(\varepsilon_\infty) = \varphi(\tau(F, \varepsilon_\infty))$. (iii) If H is an idempotent of τ, then $\varphi(H) = \varphi(\tau(H, H)) = \tau(\varphi(H), \varphi(H))$. Lastly, (iv) is an easy induction and (v) is a direct consequence of (iv). □

The next result is also immediate.

THEOREM 4.3. *Let τ be a triangle function. Then the following functions φ satisfy Cauchy's equation for τ:*

(i) *The identity map $\varphi(F) = F$.*
(ii) *The power function $\varphi(F) = F^n$.*
(iii) *If τ has an interior idempotent H, the function φ_H given by $\varphi_H(F) = \tau(F, H)$, for all F in Δ^+.*

The next lemma shows that for sup-continuous φ and τ it suffices to consider Cauchy's equation on the subspace Δ_δ^+.

LEMMA 4.4. *Let φ be a sup-continuous function from Δ^+ into Δ^+ and let τ be a sup-continuous triangle function. Then φ satisfies Cauchy's equation for τ if and only if*

$$\varphi(\tau(\delta_{a,c}, \delta_{b,d})) = \tau(\varphi(\delta_{a,c}), \varphi(\delta_{b,d})) \tag{4.2}$$

for all $\delta_{a,c}, \delta_{b,d}$ in Δ_δ^+.

Proof. The necessity is obvious. In the other direction, assume that (4.2) holds. Then, using Lemmas 3.5 and 3.8, we obtain

$$\begin{aligned}\varphi(\tau(F, G)) &= \varphi(\tau(\sup_a \delta_{a,F(a)}, \sup_b \delta_{b,G(b)})) = \sup_a \sup_b \varphi(\tau(\delta_{a,F(a)}, \delta_{b,G(b)}))\\ &= \sup_a \sup_b \tau(\varphi(\delta_{a,F(a)}), \varphi(\delta_{b,G(b)})) = \tau(\varphi(\sup_a \delta_{a,F(a)}), \varphi(\sup_b \delta_{b,G(b)}))\\ &= \tau(\varphi(F), \varphi(G)).\end{aligned}$$

□

5. Order automorphism solutions of Cauchy's equation for τ_T

DEFINITION 5.1. A mapping ϕ from a lattice L_1 into a lattice L_2 is called

(i) an order isomorphism if ϕ is a bijection and ϕ and ϕ^{-1} are order-preserving;
(ii) an order automorphism if (i) holds and $L_1 = L_2$;
(iii) a dual isomorphism if ϕ is a bijection and ϕ and ϕ^{-1} are order-reversing.

The set of order automorphisms of a lattice L is usually denoted by $\mathrm{Aut}(L)$; and it is a well-known lattice-theoretic fact that all order automorphisms are sup-continuous (see, e.g., [3] or [4]).

In [7] R. C. Powers showed that a mapping φ is an order automorphism of Δ^+ if and only if, for all F in Δ^+, either

$$\varphi(F) = \theta \circ F \circ \gamma, \tag{5.1}$$

where θ is in $\mathrm{Aut}(I)$ and γ is in $\mathrm{Aut}(R^+)$, or

$$\varphi(F) = \alpha \circ F^{\vee} \circ \beta, \tag{5.2}$$

where α and β are dual isomorphisms from R^+ to I, i.e., continuous, strictly decreasing functions from R^+ to I, such that $\alpha(0) = \beta(0) = 1$ and $\alpha(\infty) = \beta(\infty) = 0$, and $F^{\vee}$ is the right-continuous *quasi-inverse* of F which is given by

$$F^{\vee}(y) = \begin{cases} 0 & \text{for } y = 0, \\ \inf\{x \mid F(x) > y\} & \text{for } 0 < y < 1, \\ \infty & \text{for } y = 1. \end{cases} \tag{5.3}$$

In this section we determine all the order automorphisms of Δ^+ that satisfy Cauchy's equation for triangle functions of the form τ_T (see (3.8)) when T is a continuous Archimedean t-norm. We begin with order automorphisms of the form (5.1).

LEMMA 5.2. *Let T be a continuous t-norm and let φ be given by* (5.1). *Then φ satisfies Cauchy's equation for τ_T if and only if*

$$\gamma(a + b) = \gamma(a) + \gamma(b) \qquad \textit{for all } a, b \textit{ in } R^+, \tag{5.4}$$

and

$$\theta(T(c, d)) = T(\theta(c), \theta(d)) \qquad \textit{for all } c, d \textit{ in } I. \tag{5.5}$$

Thus φ satisfies Cauchy's equation for τ_T if and only if γ satisfies Cauchy's equation on R^+ and θ satisfies Cauchy's equation for T.

Proof. Note that, if φ is given by (5.1), then

$$\varphi(\delta_{a,b}) = \delta_{\gamma^{-1}(a),\theta(b)}, \tag{5.6}$$

and that, in view of Lemmas 3.5 and 4.4, it suffices to consider Cauchy's equation on Δ_δ^+. Thus, using (3.9) and (5.6), we have on the one hand that

$$\varphi(\tau_T(\delta_{a,c}, \delta_{b,d})) = \varphi(\delta_{a+b,T(c,d)}) = \delta_{\gamma^{-1}(a+b),\theta(T(c,d))},$$

and on the other hand that

$$\begin{aligned}\tau_T(\varphi(\delta_{a,c}), \varphi(\delta_{b,d})) &= \tau_T(\delta_{\gamma^{-1}(a),\theta(c)}, \delta_{\gamma^{-1}(b),\theta(d)}) \\ &= \delta_{\gamma^{-1}(a)+\gamma^{-1}(b),T(\theta(c),\theta(d))}.\end{aligned}$$

Hence φ satisfies Cauchy's equation for τ_T if and only if these two expressions are equal. By Lemma 3.1, this is so if and only if (5.5) holds and γ^{-1} satisfies Cauchy's equation on R^+. Since γ is in Aut(R^+), the latter holds if and only if (5.4) holds. □

THEOREM 5.3. *Let φ be given by* (5.1). *Then φ satisfies Cauchy's equation for τ_T, where*

(i) *T is strict with generator g, if and only if there exist $k, l > 0$ such that for all x in R^+ and F in Δ^+,*

$$(\varphi(F))(x) = g^{-1}(k \cdot g(F(l \cdot x))).$$

(ii) *T is continuous, non-strict, Archimedean with generator g, if and only if there exists a $k > 0$ such that, for all x in R^+ and all F in Δ^+,*

$$(\varphi(F))(x) = F(k \cdot x).$$

Proof. It is well-known (see, e.g., [1]), that any γ in Aut(R^+) satisfies (5.4) if and only if there is a $k > 0$ such that for all x in R^+, $\gamma(x) = k \cdot x$. Given this, (i) and (ii) follow at once from Theorems 2.2 and 2.4 and the fact that if θ is given by (2.11) then it is an order automorphism if and ony if $k = 1$. □

We now turn to order automorphisms φ of the form (5.2). We begin with the observation that for $0 \leqslant a < \infty$ and $b \neq 0$,

$$\delta^{\vee}_{a,b}(y) = \begin{cases} 0 & \text{for } y = 0, \\ a & \text{for } 0 < y < b, \\ \infty & \text{for } b \leqslant y \leqslant 1, \end{cases} \tag{5.7}$$

and that

$$\varepsilon_\infty^\vee(y) = \begin{cases} 0 & \text{for } y = 0, \\ \infty & \text{for } 0 < y \leqslant 1. \end{cases} \tag{5.8}$$

LEMMA 5.4. *Let* φ *be given by* (5.2). *Then, for all* $\delta_{a,b}$ *in* Δ_δ^+,

$$\varphi(\delta_{a,b}) = \delta_{\beta^{-1}(b),\alpha(a)}. \tag{5.9}$$

Proof. If $\delta_{a,b} \neq \varepsilon_\infty$, then (5.2) and (5.7) yield that

$$(\varphi(\delta_{a,b}))(x) = \alpha(\delta_{a,b}^\vee(\beta(x))) = \begin{cases} \alpha(0) & \text{for } \beta(x) = 0, \\ \alpha(a) & \text{for } 0 < \beta(x) < b, \\ \alpha(\infty) & \text{for } b \leqslant \beta(x) \leqslant 1, \end{cases}$$

$$= \begin{cases} 1 & \text{for } x = \infty, \\ \alpha(a) & \text{for } \beta^{-1}(b) < x < \infty, \\ 0 & 0 \leqslant x \leqslant \beta^{-1}(b), \end{cases}$$

which establishes (5.9) in this case. A similar argument shows that $\varphi(\varepsilon_\infty) = \varepsilon_\infty$. □

LEMMA 5.5. *Let* T *be a continuous t-norm and let* φ *be given by* (5.2). *Then* φ *satisfies Cauchy's equation for* τ_T *if, and only if, for all* x, y *in* I,

$$\alpha(\alpha^{-1}(x) + \alpha^{-1}(y)) = T(x, y), \tag{5.10}$$

and

$$\beta(\beta^{-1}(x) + \beta^{-1}(y)) = T(x, y). \tag{5.11}$$

Proof. As in the proof of Lemma 5.2, it suffices to consider Cauchy's equation on Δ_δ^+. Using Lemma 5.4 we have on the one hand that

$$\varphi(\tau_T(\delta_{a,c}, \delta_{b,d})) = \varphi(\delta_{a+b,T(c,d)})$$
$$= \delta_{\beta^{-1}(T(c,d)),\alpha(a+b)},$$

and on the other hand that

$$\tau_T(\varphi(\delta_{a,c}), \varphi(\delta_{b,d})) = \tau_T(\delta_{\beta^{-1}(c),\alpha(a)}, \delta_{\beta^{-1}(d),\alpha(b)})$$
$$= \delta_{\beta^{-1}(c)+\beta^{-1}(d),T(\alpha(a),\alpha(b))}.$$

Thus φ satisfies Cauchy's equation for τ_T if and only if these two expressions are equal. By Lemma 3.1 this is the case if and only if

$$\alpha(a+b) = T(\alpha(a), \alpha(b)) \quad \text{and} \quad \beta^{-1}(c) + \beta^{-1}(d) = \beta^{-1}(T(c, d)),$$

and these equations are clearly equivalent to (5.10) and (5.11), respectively. □

THEOREM 5.6. *Let φ and τ_T be as in Lemma 5.5. Then φ satisfies Cauchy's equation for τ_T if and only if T is a strict t-norm and α^{-1} and β^{-1} are inner additive generators of T, i.e., if and only if there exists a $k > 0$ such that for all x in R^+ and all F in Δ^+,*

$$(\varphi(F))(x) = (g^{-1}F^{\vee}g^{-1})(k \cdot x), \tag{5.12}$$

where g is any generator of T.

Proof. Using (2.3) and the fact that α and β are strictly decreasing, we have that T has to be strict and that both α^{-1} and β^{-1} generate T. It therefore follows from (2.4) that for some $k > 0$ and all x in R^+,

$$\alpha^{-1}(x) = k \cdot \beta^{-1}(x),$$

which yields (5.12). □

In conclusion we note that, as was to be expected, very few of the order automorphisms of Δ^+ satisfy Cauchy's equation; and these are the ones that essentially act on F by linear scaling on the left and/or the right.

6. Sup-continuous solutions of Cauchy's equation

We now consider the question of finding sup-continuous functions $\varphi: \Delta^+ \to \Delta^+$ that satisfy Cauchy's equation for a given τ_T when T is strict. As is to be expected, this will require the use of powers and roots of elements in the semigroup (Δ^+, τ_T). Since these may not always exist, we will have to impose some restrictions on φ; and this requires some preliminary discussion.

DEFINITION 6.1. Let T be a strict t-norm and g any inner additive generator of T. Then we let

$$\Delta_T^+ = \{F \text{ in } \Delta^+ \mid g \circ F \text{ is convex on } (b_F, \infty)\}, \tag{6.1}$$

where $b_F = \sup_{x \in R^+}\{F(x) = 0\}$.

In view of (2.4), the set Δ_T^+ does not depend on the choice of generator g.

(Note: The set Δ_T^+ is commonly known as the set of T-log-concave elements of Δ^+. The terminology is due to R. A. Moynihan [5] (see also [8, Sec. 7.8]) and stems from the fact that he used multiplicative rather than additive generators to define this set.)

Clearly $b_{\delta_{a,b}} = a$ and, since $\delta_{a,b}$ is constant on (a, ∞), it follows that $g \circ \delta_{a,b}$ is convex, whence

$$\Delta_\delta^+ \subseteq \Delta_T^+. \tag{6.2}$$

The following result is due to B. Schweizer (see [5] and [8, Sec. 7.8]).

THEOREM 6.2. *Let T be a strict t-norm with additive generator g and suppose F is in $\Delta_T^+ \setminus \{\varepsilon_\infty\}$. For any $\mu \geqslant 0$, let F^μ be defined by*

$$F^\mu(x) = g^{-1}(\mu \cdot g(F(x/\mu))) \qquad \text{for } 0 < \mu < \infty, \tag{6.3}$$

$$F^0 = \lim_{\mu \to 0} F^\mu = \varepsilon_0, \tag{6.4}$$

and

$$F^\infty = \lim_{\mu \to \infty} F^\mu = \begin{cases} \varepsilon_\infty & \text{for } F \neq \varepsilon_0 \\ \varepsilon_0 & \text{for } F = \varepsilon_0. \end{cases} \tag{6.5}$$

Then F^μ is in Δ_T^+ and for any $\mu, \nu \geqslant 0$, we have

$$\tau_T(F^\mu, F^\nu) = F^{\mu+\nu}, \tag{6.6}$$

and

$$(F^\mu)^\nu = (F^\nu)^\mu = F^{\mu\nu}. \tag{6.7}$$

Note, however, that, in view of Lemma 3.5, Δ_T^+ is not closed under arbitrary suprema, and hence is not a complete sublattice of Δ^+.

If we let $g^{-1}(\mu g(x)) = \theta(x)$ and $x/\mu = \gamma(x)$ then, for any $\mu > 0$, the mapping $F \to F^\mu$ given by (6.3) is an order automorphism of Δ_T^+ of the form (5.1). This and (5.6) yield that, for $\delta_{a,b} \neq \varepsilon_\infty$,

$$\delta_{a,b}^\mu = \delta_{\mu a, g^{-1}(\mu g(b))}. \tag{6.8}$$

By induction it follows from (6.6) that, for any positive integer n, F^n as defined in (6.3) is precisely the n-th τ_T-power of F as defined in (3.7). The following is also immediate:

COROLLARY 6.3. *Let T be a strict t-norm and G in $\Delta_T^+ \setminus \{\varepsilon_\infty\}$. Then for any $\mu > 0$, there exists a unique H in Δ_T^+, such that*

$$G = H^\mu,$$

namely $H = G^{1/\mu}$.

The root H need not be unique in Δ^+. For, as R. A. Moynihan has shown [5], if H_1 and H_2 have the same log-concave envelope and if, for some integer $n > 0$, H_1^n is in Δ_T^+, then $H_1^n = H_2^n$.

LEMMA 6.4. *Let T be a strict t-norm and φ a sup-continuous solution of Cauchy's equation for τ_T such that, if F is in $\Delta_T^+ \setminus \{\varepsilon_\infty\}$, then for all positive integers n, $\varphi(F^{1/n})$ is in $\Delta_T^+ \setminus \{\varepsilon_\infty\}$. Then, for all $\mu \geqslant 0$,*

$$\varphi(F^\mu) = [\varphi(F)]^\mu. \tag{6.9}$$

Proof. For μ in the set of natural numbers N, this follows from Lemma 4.2(iv). Now let m be in N. Then, by (6.7) and the above, we have $\varphi(F) = \varphi((F^{1/m})^m) = [\varphi(F^{1/m})]^m$. Since $\varphi(F^{1/m})$ is in $\Delta_T^+ \setminus \{\varepsilon_\infty\}$, we can apply Corollary 6.3, which yields $\varphi(F^{1/m}) = [\varphi(F)]^{1/m}$. It follows that (6.9) holds for all rational $\mu \geqslant 0$. Next suppose that $\mu > \nu$. Then, since g and g^{-1} are decreasing and F is nondecreasing, (6.3) implies that $F^\mu \leqslant F^\nu$. Furthermore, since F is left-continuous and g and g^{-1} are continuous, we have

$$\lim_{\nu \downarrow \mu} F^\nu = \sup_{\nu > \mu} F^\nu = F^\mu.$$

Finally, let $\{r_n\}_{n=1}^\infty$ be a decreasing sequence of rational numbers with limit μ. Then

$$F^\mu = \sup_{r_n > \mu} F^{r_n},$$

from which (6.9) follows by the sup-continuity of φ. □

The next lemma is a direct consequence of (3.10) and (6.8) (recall that $\varepsilon_a = \delta_{a,1}$).

LEMMA 6.5. *Let $\delta_{a,b} \neq \varepsilon_\infty$, and let T be a strict t-norm with generator g. Then, for any c in $(0, 1)$, $\delta_{a,b}$ admits the decomposition*

$$\delta_{a,b} = \tau_T(\varepsilon_1^a, \delta_{0,c}^{g(b)/g(c)}). \tag{6.10}$$

We can now prove the central results of this section, namely:

THEOREM 6.6. *Let T be a strict t-norm with generator g and let φ be a sup-continuous solution of Cauchy's equation for τ_T having the property that, for some c in $(0, 1)$ and all positive integers n, $\varphi(\delta_{0,c}^{1/n})$ and $\varphi(\varepsilon_1^{1/n})$ are in $\Delta_T^+ \setminus \{\varepsilon_\infty, \varepsilon_0\}$. Then, for all F in Δ^+,*

$$\varphi(F) = \sup_{t \in R^+} \tau_T([\varphi(\varepsilon_1)]^t, [\varphi(\delta_{0,c})]^{kg(F(t))}), \tag{6.11}$$

where $k = 1/g(c)$.

Proof. By Lemmas 3.5 and 6.5 we have that

$$F = \sup_{t \in R^+} \delta_{t,F(t)} = \sup_{t \in R^+} \tau_T(\varepsilon_1^t, \delta_{0,c}^{kg(F(t))}),$$

whence the sup-continuity of φ, the fact that φ is a solution of Cauchy's equation for τ_T, and (6.9) yield (6.11). □

The converse of Theorem 6.6 is also true; specifically, we have:

THEOREM 6.7. *Let T be a strict t-norm with generator g. Let G and H in $\Delta_T^+ \setminus \{\varepsilon_\infty, \varepsilon_0\}$ and c in $(0, 1)$ be given, and let $\varphi: \Delta^+ \to \Delta^+$ be defined by*

$$\varphi(F) = \sup_{t \in R^+} \tau_T(G^t, H^{kg(F(t))}) \qquad \text{for all } F \text{ in } \Delta^+, \tag{6.12}$$

where $k = 1/g(c)$. Then φ is a sup-continuous solution of Cauchy's equation for τ_T. Moreover, $G = \varphi(\varepsilon_1)$, $H = \varphi(\delta_{0,c})$ and for all positive integers n, $\varphi(\varepsilon_1^{1/n})$ and $\varphi(\delta_{0,c}^{1/n})$ are in $\Delta_T^+ \setminus \{\varepsilon_\infty, \varepsilon_0\}$.

Proof. Suppose that G and H are in $\Delta_T^+ \setminus \{\varepsilon_\infty, \varepsilon_0\}$. Then we have

$$\tau_T(G^t, H^{kg(\delta_{a,b}(t))}) = \begin{cases} \varepsilon_\infty & \text{for } 0 \leqslant t \leqslant a, \\ \tau_T(G^t, H^{kg(b)}) & \text{for } a < t < \infty, \\ \varepsilon_\infty & \text{for } t = \infty. \end{cases}$$

Therefore, letting $F = \delta_{a,b}$ in (6.12), we obtain

$$\varphi(\delta_{a,b}) = \tau_T(G^a, H^{kg(b)}). \tag{6.13}$$

Using (6.8) we have, in particular, that for all positive integers n,

$$\varphi(\delta_{0,c}^{1/n}) = \varphi(\delta_{0,g^{-1}(g(c)/n)}) = \tau_T(G^0, H^{1/n}) = \tau_T(\varepsilon_0, H^{1/n}) = H^{1/n},$$

and, since $g(1) = 0$,

$$\varphi(\varepsilon_1^{1/n}) = \varphi(\delta_{1/n,1}) = \tau_T(G^{1/n}, \varepsilon_0) = G^{1/n}.$$

Hence, using Theorem 6.2, we have that $\varphi(\varepsilon_1^{1/n})$ and $\varphi(\delta_{0,c}^{1/n})$ are in $\Delta_T^+ \setminus \{\varepsilon_\infty, \varepsilon_0\}$.

To show that φ is sup-continuous, we first note that from (6.13), (6.12) and Lemma 3.5 it follows that

$$\sup_{t \in R^+} \varphi(\delta_{t,F(t)}) = \sup_{t \in R^+} \tau_T(G^t, H^{kg(F(t))}) = \varphi(F) = \varphi(\sup_{t \in R^+} \delta_{t,F(t)}).$$

Now let $F = \sup_{\beta \in B} F_\beta$, where F_β is in Δ^+ for all β in some index set B. Since g is continuous and decreasing and H is left-continuous, following the argument in the proof of Lemma 6.4, for any t in R^+, we have

$$\sup_\beta H^{kg(F_\beta(t))} = H^{\inf_\beta kg(F_\beta(t))} = H^{kg(\sup_\beta F_\beta(t))}. \tag{6.14}$$

Next observe that, if L is a complete lattice, if I and J are arbitrary index sets, and if $\{c_{i,j}\}_{i \in I, j \in J}$ is a subset of L, then

$$\sup_i \sup_j c_{i,j} = \sup_j \sup_i c_{i,j}. \tag{6.15}$$

Thus, using (6.12), (6.15), the sup-continuity of τ_T and (6.14), this yields

$$\begin{aligned}\sup_{\beta \in B} \varphi(F_\beta) &= \sup_{\beta \in B} \sup_{t \in R^+} \tau_T(G^t, H^{kg(F_\beta(t))}) \\ &= \sup_{t \in R^+} \sup_{\beta \in B} \tau_T(G^t, H^{kg(F_\beta(t))}) \\ &= \sup_{t \in R^+} \tau_T(G^t, H^{kg(F(t))}) = \varphi(F) = \varphi(\sup_{\beta \in B} F_\beta),\end{aligned}$$

whence φ is sup-continuous.

It remains to show that φ satisfies Cauchy's equation for τ_T. Using (6.13) and the fact that τ_T is commutative and associative, we have that, for all $\delta_{a,c}$ and

$\delta_{b,d}$ in Δ_δ^+,

$$\begin{aligned}\tau_T(\varphi(\delta_{a,c}), \varphi(\delta_{b,d})) &= \tau_T(\tau_T(G^a, H^{kg(c)}), \tau_T(G^b, H^{kg(d)}))\\ &= \tau_T(\tau_T(G^a, G^b), \tau_T(H^{kg(c)}, H^{kg(d)}))\\ &= \tau_T(G^{a+b}, H^{k(g(c)+g(d))}) = \varphi(\delta_{a+b,T(c,d)})\\ &= \varphi(\tau_T(\delta_{a,c}, \delta_{b,d})),\end{aligned}$$

whence the conclusion follows from Lemma 4.4. □

We conclude this section with several remarks:

(i) Since order automorphisms satisfy the hypotheses of Theorem 6.6, they have to be of the form (6.11). Using (5.6), it follows that for order automorphisms of the form (5.1) we have

$$\varphi(\varepsilon_1) = \varepsilon_{\gamma^{-1}(1)} \quad \text{and} \quad \varphi(\delta_{0,c}) = \delta_{0,\theta(c)},$$

and, using (5.9), that for order automorphisms of the form (5.2) we have

$$\varphi(\varepsilon_1) = \delta_{0,\alpha(1)} \quad \text{and} \quad \varphi(\delta_{0,c}) = \varepsilon_{\beta^{-1}(c)},$$

and a simple calculation shows that, with the above, (6.11) yields (5.1) and (5.2), respectively. The converse also holds, i.e., if we let $G = \varepsilon_a$ and $H = \delta_{0,b}$ in (6.12), where $0 < a < \infty$ and $0 < b < 1$, then using (6.13) and Lemma 3.5, the function φ defined by (6.12) is easily seen to be an order automorphism of the form (5.1); similarly, if we let $G = \delta_{0,b}$ and $H = \varepsilon_a$, then φ is an order automorphism of the form (5.2).

(ii) Since Δ^+ contains a copy of R^+, namely $\{\varepsilon_a \mid a \in R^+\}$, equation (6.11) has to include the functions φ that correspond to continuous solutions of Cauchy's equation on R^+. We obtain these by restricting φ in (6.11) to $\{\varepsilon_a \mid a \in R^+\}$ and choosing $\varphi(\varepsilon_1) = \varepsilon_k$ for any $k > 0$. In this case we have $\varphi(\varepsilon_x) = \varepsilon_{kx}$.

(iii) The notion of sup-continuity is a lattice-theoretic one; in probability theory one is usually interested in functions that are weakly continuous, i.e., continuous with respect to the modified Lévy metric (see [8, Sec. 4.2]). In general, neither of these two notions of continuity implies the other. However, as pointed out by R. C. Powers [7], order automorphisms of Δ^+ are weakly continuous as well as sup-continuous. The question whether functions φ defined by (6.12) are weakly continuous is open.

Acknowledgements

The author would like to thank Professors M. F. Janowitz and B. Schweizer for suggesting the problem and for their invaluable suggestions and criticism during the preparation of this manuscript.

This research has been supported by ONR Contract N-00014-87-K-0379.

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