ON THE CARDINALITY OF THE UPPER SEMILATTICE OF COMPUTABLE ENUMERATIONS

A. B. Khutoretskii

UDC 517.11:518.5

We shall use the following notations and abbreviations. N is the series of natural numbers; $A \Delta B = (A \setminus B) \cup (B \setminus A)$ is the symmetric difference of sets A and B; $\subset$ denotes strict inclusion for sets; δf is domain of definition while ρf is domain of values of function f; $\bar{\delta} f = \{x / y \leq x \rightarrow y \in \delta f\}$; p.r.f. is a partial recursive function; r.e.s. is a recursively enumerable set; an $\mathfrak{A}$-enumeration is an enumeration of family $\mathfrak{A}$; g.r.f. is a general recursive function.

DEFINITION. The dual sequence of finite sets $\{\tau_n^s / n \in N,\ s \in N\}$ is called an effective sequence of approximation (e.s.a.) for enumeration τ of the family of r.e.s. $\mathfrak{A}$ if the following conditions are met:

1) function $\varphi(n, s)$, equal to the Gödel number of sets τ_n^s, is general recursive;

2) $\bigcup_{s \in N} \tau_n^s = \tau_n$ for all n;

3) $\tau_n^s \subseteq \tau_n^{s+1}$ for all s, n.

One can give an analogous definition for an e.s.a. for an enumeration of a family of p.r.f.

It is clear that an enumeration is computable if and only if it has an e.s.a.

It is known that there exist families of r.e.s. for which the upper semilattice of computable enumerations contains infinitely many elements and has a very rich structure. On the other hand, there are computable families of r.e.s. which have only one (to within equivalence) computable enumeration (cf. [2]). And, finally, a family of r.e.s. may have no computable enumerations at all. The theorem constituting the basic contents of this paper shows that there are no other possibilities.

THEOREM 1. Let the family of r.e.s. $\mathfrak{A}$ have computable enumerations ν and μ, $\nu \not\equiv \mu$. There then exists a computable $\mathfrak{A}$-enumeration τ such that $\mu < \tau$ and $\nu \not\leq \tau$.

Proof. Let $\{\nu_n^s\}$ and $\{\mu_n^s\}$ be e.s.a. for the respective enumerations ν and μ. Let $\varkappa$ be a computable enumeration of the family of all p.r.f., and let $\{\varkappa_n^s\}$ be an e.s.a. for $\varkappa$. We shall construct, by stages, the dual sequence of finite sets $\{\tau_n^s\}$. In the process of construction some numbers will obtain successors. There are two types of successors: $n1$-successors and $n2$-successors (for each n). Each number y can be the successor of several numbers simultaneously. We shall say that x is $n1$-free ($n2$-free) on step s if x was not an $n1$-successor ($n2$-successor) up to step s. Moreover, we will set up a list of certain number pairs (x, y). Each pair on this list will have a mark of the form $\boxed{n1}$ or $\boxed{n2}$ (for some n). In the list at step s there cannot be two pairs with identical marks.

CONSTRUCTION

Step $s = 0$. We set $\tau_n^0 = \varnothing$ for all $n \neq 1$ and $\tau_1^0 = \mu_0^0$.

Step $s > 0$ consists of two half-steps plus finishing touches.

Step $s > 0$. We shall perform the construction under the assumption that the following condition is met: τ_m^{s-1} is defined for all m and u $(\forall m)(\exists x)(\tau_m^{s-1} \subseteq \mu x)$. It is easy to verify that after step s

Translated from Algebra i Logika, Vol. 10, No. 5, pp. 561-569, September-October, 1971. Original article submitted May 8, 1971.

these conditions hold for the sets τ_m^s . Let $(s)_0=n$. If there do not exist numbers $s'<s$, such that $(s')_0 = n$, we then transfer to the finishing touches of step s . If, however, such s' exist, we then let s_1 be the largest of these. If $\hat{\delta}\varkappa_n^{s_1} = \hat{\delta}\varkappa_n^s$ we then go to the finishing touches of step s . Let $\hat{\delta}\varkappa_n^{s_1} \subset \hat{\delta}\varkappa_n^s$.

Half-Step 1. We denote by $2z$ the least even number which does not have a successor.

Case 1. Among the pairs on the list there is none with the mark $\boxed{n1}$. We make the $n1$-successors of the $2z$ first $n1$-free numbers. If there exist pairs x, y , such that $x \in \hat{\delta}\varkappa_n^s$, $y \in \tau_x^{s-1} \Delta \mu_{\varkappa_n^s(x)}^{s-1}$, we then choose from them the pair with the lowest ordinal number and add it to the list of pairs, with the mark $\boxed{n1}$ ascribed to it.

Case 2. In the list there is a pair (x,y) with mark $\boxed{n1}$. If $y \in \tau_x^{s-1} \cap \mu_{\varkappa_n^s(x)}^{s-1}$, we then cross off the list the pair (x,y) with the mark $\boxed{n1}$. (This pair could stay on the list if it bore another mark.) We note immediately that pair (x,y) with these same marks can no longer occur on the list.

Half-Step 2.

Case 1. Among the pairs on the list there are none with the mark $\boxed{n2}$; and there exist numbers $2x$ such that $2x \in \rho\varkappa_n^s$, $2x$ does not have an $n2$-successor, and $2x$ does not have an $m1$-successor with $m \leq n$. We select the largest such number $2x_0$. We choose the first y such that $\mu y \supseteq \tau_{2x_0}^{s-1}$, and we make y $n2$-successors of the number $2x_0$. If $2x_0$ has $m1$-successors with $m>n$, we release all these successors. If there exist pairs (x,y) , such that $x \in \hat{\delta}\varkappa_n^s$, $y \in \nu_x^{s-1} \Delta \tau_{\varkappa_n^s(x)}^{s-1}$, we then choose that pair of them with the least ordinal number and add it to the list, ascribing to it the mark $\boxed{n2}$. We then transfer to the finishing touches on step s .

Case 2. Among the pairs on the list there is one (x,y) with mark $\boxed{n2}$. If $y \in \nu_x^{s-1} \cap \tau_{\varkappa_n^s(x)}^{s-1}$, we then strike the pair (x,y) with mark $\boxed{n2}$ from the list. We then turn to the finishing touches on the step.

If neither case 1 nor case 2 occurs, we transfer to the finishing touches for step s .

Finishing Touches for Step $s>0$. After having performed all the foregoing operations, we do the following.

For all pairs $(2x,y)$ such that y is an $n1$-successor of $2x$, we set $\tau_{2x}^s = \tau_{2x}^{s-1} \cup \nu_y^s$; for all pairs $(2x,y)$ such that y is an $n2$-successor of $2x$, we set $\tau_{2x}^s = \tau_{2x}^{s-1} \cup \mu_y^s$ and we set $\tau_{2x+1}^s = \mu_x^s$ for all $x \leq s$; for all other z we get $\tau_z^s = \tau_z^{s-1}$.

We have described the construction of sequence $\{\tau_n^s\}$. We note that elements were added to τ_m^{s-1} only during the finishing touches for step s , while all preceding actions on step s were necessary after having decided how to supplement each τ_m^{s-1} . It is clear that $\tau_n^s \subseteq \tau_n^{s+1}$ for all n and s . $\tau_n \overset{df}{=} \bigcup_{s>0} \tau_n^s$. τ_n is obviously the computable enumeration of some family of r.e.s. $\mathfrak{A}_1$, $\mathfrak{A}_1 \supseteq \mathfrak{A}$ such that $\tau_{2x+1} = \mu_x$ for all x .

LEMMA 1. Each number $2x$ receives a successor.

Proof. We assume that there exist numbers $2x$ which never obtain successors. Let $2x_0$ be the smallest such number. Let s_0 be a step such that all the numbers $2x<2x_0$ already have successors on step s_0 (we note that a number, once having obtained a successor, thereafter always has a successor since, in losing an $n1$-successor, the number acquires an $n2$-successor).

Let n be a number such that $\varkappa_n$ is a g.r.f. and, on the list of pairs at step s_0 , let there be no pairs with the mark $\boxed{n1}$. (Such an n exists since the list contains a finite number of pairs at step s_0 .) Then, on some step $s>s_0$ there supervenes case 1 of half-step 1 for $(s)_0 = n$. If number $2x_0$ does not have a successor up until step s , it obtains it on step s . Lemma 1 is proven.

LEMMA 2. $(\forall u)(\tau_u \in \mathfrak{A})$.

Proof. $\tau_{2x+1} = \mu_x \in \mathfrak{A}$. Let $u = 2x$. By Lemma 1, each number $2x$ receives a successor y. If this is an $n2$-successor then it is adopted in case 1, half-step 2, of some step s. Then, $\mu y \supseteq \tau_{2x}^{s-1}$. In the future, this successor will not be released. This means that $\tau_{2x}^{s'} = \tau_{2x}^{s'-1} \cup \mu_y^{s'}$ for all $s' \geq s$. Consequently, $\tau_{2x} = \mu_y \in \mathfrak{A}$.

Let y be the $n1$-successor acquired by number $2x$ on step s. Then, $2x$ had no successor prior to step s, which means that $\tau_{2x}^{s-1} = \emptyset$. If y is a constant $n1$-successor of number $2x$, then $\tau_{2x} = \nu_y \in \mathfrak{A}$. If, however, y will be released, then $2x$ will obtain an $n2$-successor z which will not be lost, and $\tau_{2x} = \mu_z \in \mathfrak{A}$. Lemma 2 is proven.

It follows from Lemma 2 that τ is a computable enumeration of family $\mathfrak{A}$.

We denote by A_n^i the set of numbers which acquire ni-successors during the construction process. (Here $i = 1, 2$.)

LEMMA 3. If A_k^2 is a finite set for all $k < n$, then A_n^1 is also a finite set.

Proof. Assume that A_n^1 is infinite. Let $s_0 < s_1 < \ldots < s_i < \ldots$ be an infinite sequence of all steps such that, on step s_i, some number $2x_i$ acquires an $n1$-successor y_i. Then, on each of the steps s_i case 1 of half-step 1 prevails with $(s)_0 = n$. This means that x_n is a g.r.f. and that, at step s_i, the list contains no pairs with mark $\boxed{n1}$. Let z be the largest element of set $\bigcup_{k<n} A_k^2$ which, by hypothesis, is finite. (If $\bigcup_{k<n} A_k^2 = \emptyset$, we then set $z = 0$.) There exists i_0 such that $2x_i > z$ when $i > i_0$ (since each number acquires an $n1$-successor no more than once). Number $2x_i$ can lose its $n1$-successor y_i only in acquiring an $m2$-successor, with $m < n$. This means that the numbers $2x_i$, for $i > i_0$, do not lose successor y_i (since a number greater than z cannot have an $m2$-successor with $m < n$). Then $\tau_{2x_i} = \nu_{y_i}$ for $i > i_0$. But $y_i = i$ since an $n1$-successor is always chosen as the least $n1$-free number. This means that $\tau_{2x_i} = \nu_i$ for $i > i_0$. Let $f(i) = 2x_i$. It is clear that $f(i)$ is a g.r.f. $\nu_i = \tau f(i)$ for $i > i_0$, which means that $\nu \leq \tau$. We now prove that $\tau_x = \mu_{x_n(x)}$ for all x. Assume that $\tau_x \neq \mu_{x_n(x)}$ for some x. There exists y such that $y \in (\tau_x \Delta \mu_{x_n(x)})$. Then, there exists a step s' such that $x \in \hat{\delta} x_n^s$ and $y \in (\tau_x^{s-1} \Delta \mu^{s-1}_{x_n^s(x)})$ for all $s > s'$.

The pairs with ordinal numbers less than the ordinal number of pair (x, y) are of finite number. Each pair falls on the list of pairs having mark $\boxed{n1}$ no more than once. From the fact that, on all steps s_i, there are no pairs on the list with mark $\boxed{n1}$, it follows that pair (x,y), in the final analysis, falls on a list having the mark $\boxed{n1}$. But then, it will never be expunged, since $y \notin \tau_x \cap \mu_{x_n(x)}$, so that the list will henceforth always contain a pair with mark $\boxed{n1}$. We have obtained a contradiction. This means that $\tau_x = \mu_{x_n(x)}$ for all x. Then $\tau \leq \mu$ (x_n is a g.r.f.). Thus, $\nu \leq \tau \leq \mu$, but this contradicts the condition of Theorem 1 that $\nu \not\leq \mu$. This means that A_n^1 is a finite set. Lemma 3 is proven.

LEMMA 4. A_0^1 is a finite set.

Proof. Follows immediately from Lemma 3.

LEMMA 5. If A_k^1 is a finite set for all $k \leq n$, then A_n^2 is also a finite set.

Proof. We assume that A_n^2 is an infinite set. Then, there exists an infinite sequence of steps $s_0 < s_1 < \ldots < s_i < \ldots$, such that, on step s_i, some number $2x_i$ acquires the $n2$-successor y_i. This means that x_n is a g.r.f. Let z be the greatest element of set $\bigcup_{k\leq n} A_k^1$, which is finite by hypothesis. (If $\bigcup_{k\leq n} A_k^1 = \emptyset$ we then set $z = 0$.)

We can find i_0 such that $2x_i > z$ when $i > i_0$. Let z be some even number $z > z$, $z \in \rho x_n$. z cannot have an $m1$-successor with $m \leq n$. From the method of ascription of an $n2$-successor (cf. case 1 of half-step 2) it is clear that such a z necessarily acquires an $n2$-successor. We define function $f(t)$ as follows:

1) $t = 2x+1$. Then, $f(t) = x$.

2) t is even, $t \leq \tau$, $t \in \rho x_n$. There exists a number a_t , such that $\tau_t = \mu a_t$. We set $f(t) = a_t$.

3) t is even, $t > \tau$, $t \in \rho x_n$. Then t , during the construction process, obtains, and does not subsequently lose, $n2$ -successor b_t . We set $f(t) = b_t$. It is clear that $f(t)$ is a p.r.f., $\delta f \supseteq \rho x_n$,and $\tau_t = \mu f(t)$ for $t \in \delta f$.

We now prove that $\nu_x = \tau_{x_n}(x)$ for all x . Assume that $\nu_x \neq \tau_{x_n(x)}$ for some x . Then, there exists a number y such that $y \in (\nu_x \Delta \tau_{x_n(x)})$. We can find a step s' such that $x \in \hat{\delta} x_n^s$ and $y \in (\nu_x^{s-1} \Delta \tau^{s-1}_{x_n^s(x)})$ for all $s > s'$. No pair of numbers can occur more than once on the list of pairs with the mark $\boxed{n2}$. The pairs with ordinal numbers less than the ordinal number of the pair $(x, y)_2$ are of finite number. But, case 1 of half-step 2 for $(s)_0 = n$ occurs infinitely many times, since A_n is infinite. This means that pair (x,y) will, in the final analysis, occur on a list having mark $\boxed{n2}$. But then it will never be thenceforth expunged, since $y \notin \tau_{x_n(x)} \cap \nu_x$. This means that case 1 of half-step 2 no longer holds for $(s)_0 = n$. We have obtained a contradiction. Consequently, $\nu_x = \tau_{x_n(x)}$ for all x . Moreover, we have proven that $\tau_t = \mu f(t)$ for all $t \in \rho x_n$, where $f(t)$ is a p.r.f. Then, $\nu_x = \tau_{x_n(x)} = \mu f x_n(x)$ for all x , with $f x_n$ being a g.r.f. This means that $\nu \leq \mu$, which contradicts Theorem 1. Therefore, A_n^2 is a finite set. Lemma 5 is proven.

<u>LEMMA 6.</u> Set A_n^i is a finite for all n and i .

<u>Proof.</u> Lemma 6 is proven by induction on n . Lemma 4 provides the basis for the induction, while Lemmas 3 and 5 provide the inductive step.

<u>LEMMA 7.</u> $\mu < \tau$.

<u>Proof.</u> It is clear that $\mu \leq \tau$. Since $\tau(2x+1) = \mu(x)$ by construction. It thence remains to prove that $\tau \not\leq \mu$. Let x_n be a g.r.f. We now prove that x_n does not lead enumeration τ to μ . By Lemma 6, the set A_n^1 is finite. This means that, after some step s_0 , no number will obtain an $n1$ -successor. Let $s_1 < s_2 < \ldots < s_i < \ldots$ be all the steps such that $s_i > s_0$, $(s_i)_0 = n$ and, on step s_i , half-step 1 occurs. This sequence is infinite since x_n is a g.r.f. Then, case 1, half-step 1, does not occur on step s_i since no number can obtain an $n1$ -successor on a step $s_i > s_0$. This means that, on each step s_i , on the list of pairs there is a pair (x,y) with mark $\boxed{n1}$. Moreover, on all the steps s_i it is the same pair, since a new pair with the mark $\boxed{n1}$ can occur on the list only when case 1, half-step 1, occurs. But this means that, for all i , $y \in \tau_x^{s_i-1} \Delta \mu^{s_i-1}_{x_n^{s_i}(x)}$. It follows from this that $y \in \tau_x \Delta \mu_{x_n(x)}$, i.e., $\tau_x \neq \mu_{x_n(x)}$. Lemma 7 is proven.

<u>LEMMA 8.</u> $\nu \not\leq \tau$.

<u>Proof.</u> Let x_n be a g.r.f. We then prove that x_n does not reduce ν to τ . If $\rho x_n \cap \{2x / x \in N\}$ is finite then x_n does not reduce ν to τ since $\tau(2x+1) = \mu(x)$, while $\nu \not\leq \mu$. Let $\rho x_n \cap \{2x / x \in N\}$ be infinite. Set A_n^2 is infinite by Lemma 6. This means that, after some step s_0 , no number can acquire an $n2$ -successor (case 1 of half-step 2 does not occur with $(s)_0 = n$). The set $\bigcup_{m \leq n} A_m^1$ is also finite. Therefore, there exists an infinite sequence $s_1 < s_2 < \ldots < s_i < \ldots$ such that the following conditions hold for all i : on step s_i half-step 2 occurs with $(s_i)_0 = n$, $s_i > s_0$ and on step s_i there exists number $2x$ such that $2x \in \rho x_n^{s_i}$, $2x$ does not have an $n2$ -successor, and $2x$ does not have an $m1$ -successor with $m \leq n$. If, with this, case 1 of half-step 2 were to occur, then some number would acquire an $n2$ -successor on step $s_i > s_0$, which is impossible. This means that pair (x, y) with mark $\boxed{n2}$ is on the list at each step s_i . Furthermore, it is always the same pair, since a new pair with mark $\boxed{n2}$ can occur on the list only when case 1 of half-step 2 occurs with $(s)_0 = n$. This means that $y \in (\nu_x^{s_i-1} \Delta \tau^{s_i-1}_{x_n^{s_i}(x)})$ for all i . Then $y \in \nu_x \Delta \tau_{x_n(x)}$, i.e., $\nu_x \neq \tau_{x_n(x)}$. Lemma 8 is proven.

Lemmas 7 and 8 complete the proof of Theorem 1. Theorem 1 is proven.

COROLLARY 1. If family of r.e.s. $\mathfrak{A}$ has two nonequivalent computable enumerations, it then has infinitely many pairwise nonequivalent computable enumerations.

Proof. Let there be two $\mathfrak{A}$-enumerations ν and μ, $\nu \nleq \mu$. By Theorem 1 there exists an $\mathfrak{A}$-enumeration τ such that $\mu < \tau$, $\nu \nleq \tau$. By enumerations ν and τ we can now construct an enumeration τ_1 such that $\tau < \tau_1$, $\nu \nleq \tau_1$. By continuing in this way, we obtain an infinite sequence of enumerations $\mu < \tau < \tau_1 < \ldots < \tau_k < \ldots$ and $\nu \nleq \mu$, $\nu \nleq \tau$, $\nu \nleq \tau_k$ for all k. Corollary 1 is proven.

COROLLARY 2. If family $\mathfrak{A}$ has a principal enumeration ν then, for each nonprincipal computable $\mathfrak{A}$-enumeration μ, there exists an $\mathfrak{A}$-enumeration τ such that $\mu < \tau < \nu$.

Proof. It suffices to apply Theorem 1 to the enumerations ν and μ.

Yu. L. Ershov pointed out that Theorem 1 has

COROLLARY 3 ([1], corollary). For each noncreative r.e. m-th power "a" there exists a noncreative r.e. m-th power "b" such that $a < b$.

Proof. The upper semilattice of r.e. m-th powers L is isomorphic, as is well known, to the upper semilattice $L_1(\mathfrak{A})$ of computable enumerations of the family $\mathfrak{A} = \{\emptyset, \{1\}\}$.

L (and, this means, also $L_1(\mathfrak{A})$) contains a greatest element. Corollary 3 then follows immediately from Corollary 2. The following questions remain open:

1. Do there exist noneffectively discrete [2] families of r.e.s. with unique computable enumerations?

2. If the answer to the first question is affirmative, how is one then to describe the family of r.e.s. having the unique computable enumeration?

In conclusion, the author would like to take this opportunity to thank Yu. L. Ershov, L. A. Lavrov, S. N. Kallibekov, and all the other participants in the seminar "Enumeration Theory" of the Siberian Branch of the Institute of Mathematics, Academy of Sciences of the USSR, for their useful comments on the present work.

LITERATURE CITED

1. S. D. Denisov, "On m-powers of recursively enumerable sets," Algebra i Logika, 9, No. 4, 422-427 (1970).
2. Yu. L. Ershov, "Enumeration of families of general recursive functions," Sibirsk. Matem. Zh., 8, No. 5, 1015-1025 (1969).
3. A. B. Khutoretskii, "On nonprincipal enumerations," Algebra i Logika, 8, No. 6, 726-732 (1969).