INDEX SETS OF DEGREES OF UNSOLVABILITY

S. Kallibekov

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In the theory of recursive functions recursively enumerable sets are classified by various methods. One of the fundamental classification methods is partitioning these sets into equivalence classes (degrees) according to reducibility type: Turing (T-) reducibility, truth table (tt-) reducibility, many-one (m) reducibility, and one-one (1-) reducibility [3, 4]. The complexity of such classes of recursively enumerable sets is in a certain sense characterized by their index sets (more accurately, by the recursive isomorphism type of their index sets) with respect to the principal computable enumeration (the Post enumeration), which is uniquely defined up to recursive isomorphism. The index sets of the classes of recursively enumerable sets corresponding to a given recursively enumerable degree of unsolvability and recursively enumerable $m(tt)$-degree are studied in [1, 2, 9].

C. Jockusch studied some relationships between these reducibilities in [8] and posed some questions. In particular, he asked if any recursively enumerable degree of unsolvability contains an infinite family of pairwise m-incomparable recursively enumerable m-degrees (an antichain). Previously Yates [2] had proved that in a complete degree there exists an infinite antichain of recursively enumerable m-degrees which are represented by maximal sets. Lerman [7] strengthened this result for recursively enumerable degrees b such that $b' = 0''$

In this note we prove a theorem from which follows characterizations of the recursive isomorphism types of index sets of classes of recursively enumerable sets corresponding to a given recursively enumerable degree of unsolvability [1, 2]. Our theorem differs somewhat from Yates'. In the proof we use an effective method for constructing a recursively enumerable set which is Turing incomparable with the given recursively enumerable, nonrecursive, incomplete set. This method eliminates the application of the recursion theorem to prove the effectiveness of the existence of such a set and gives the recursive function asked about in [5], p. 69. Using the characterizations of the reflexive isomorphism types of recursively enumerable sets corresponding to a recursively enumerable $m(tt)$-degree [9], we deduce from the theorem that any recursively enumerable degree of unsolvability b such that $\Sigma_3(b) \supseteq \Pi_3$ contains an infinite antichain of recursively enumerable $m(tt)$-degrees. From this we obtain a negative answer to one of Rogers' questions ([4], Sec. 9.6): does every recursively enumerable degree of unsolvability contains a recursively enumerable tt-degree which is maximal among all of the recursively enumerable tt-degrees contained in this degree of unsolvability?

Let A be any recursively enumerable set. Consider the classes of recursively enumerable sets $\mathcal{A}_0 = \{R \mid R \equiv_T A\}$, $\mathcal{A}_1 = \{R \mid R <_T A\}$, $\mathcal{A}_2 = \{R \mid A <_T R\}$ and $\mathcal{A}_3 = \{R \mid R \not\leq_T A, A \not\leq_T R\}$. Denote the index sets of these classes by $G(a)$, $G(<a)$, $G(>a)$ and $G(|a)$, respectively; here a is the Turing degree of A. The fundamental definitions and upper bounds for the recursive isomorphism types of the index sets under investigation can be found in [1-4] and [9].

PROPOSITION 1. There exists a recursive function f such that if π_e is a recursively enumerable, nonrecursive, incomplete set, then $\pi_{f(e)}$ is a recursively enumerable set which is Turing incomparable with π_e.

The proof of this proposition is contained in the proof of Theorem 1.

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THEOREM 1. Let A be any recursively enumerable, nonrecursive, incomplete set and let $\mathbf{b}$ be any recursively enumerable degree such that the degree of A is less than or equal to $\mathbf{b}$. Let S be any set in $\Sigma_3(\mathbf{b})$. Then there exists a uniformly recursively enumerable sequence of recursively enumerable sets $\{B_\kappa\}_{\kappa\in N}$ such that for all κ:

$$\kappa\in S \Longrightarrow B_\kappa \text{ has degree } \mathbf{b};$$
$$\kappa\notin S \Longrightarrow B_\kappa \text{ and } A \text{ are incomparable.}$$

Proof. Let $\{L_{\kappa e}\}_{\kappa,e\in N}$ be a uniformly recursively enumerable sequence of recursively enumerable set, uniformly of degree $\leq \mathbf{b}$, such that for a given κ and for every $i\neq j$, $L_{\kappa i}\cap L_{\kappa j}=\phi$, $L_{\kappa i}$ and $L_{\kappa j}$ are recursively separable, and for all κ:

$\kappa\in S \Longrightarrow (\exists e)[\, L_{\kappa e}$ has degree $\mathbf{b}\,\&\,(\forall j<e)\,[\,L_{\kappa j}$ is recursive]];

$\kappa\notin S \Longrightarrow (\forall e)[\, L_{\kappa e}$ is recursive].

(Such a sequence is constructed in [1], Lemma 4.)

Let $L^*_{\kappa e}=\bigcup_{j\leq e} L_{\kappa j}$ for all κ, e. Then we have

$$\kappa\in S \Longrightarrow (\exists e)[(\forall p\geq e)[L^*_{\kappa p} \text{ has degree } \mathbf{b}]\ \&$$
$$\&\ (\forall j<e)[L^*_{\kappa j} \text{ is recursive}]].$$
$$\kappa\notin S \Longrightarrow (\forall e)[L^*_{\kappa e} \text{ is recursive}].$$

Now we transform the sequence $\{L^*_{\kappa e}\}$ into a sequence of embedded recursively enumerable sets,

$$M_{\kappa 0}\supseteq K_0\supseteq M_{\kappa 1}\supseteq K_1\supseteq M_{\kappa 2}\supseteq K_2\supseteq\ldots,$$

where every K_i $(i=0,1,2,\ldots)$ is a creative set and every $M_{\kappa i}$ $(i=0,1,2,\ldots)$ is a recursively enumerable set which is m-equivalent to $L^*_{\kappa i}$. Fix some method of enumeration $\{M_{\kappa i}\}_{i\in N}$ and some general recursive functions g_i, $i=0,1,\ldots$, which enumerate the creative sets K_i without repetition. In general we may assume, although this is not necessary, that every $L_{\kappa e}$, $e=0,1,2,\ldots$, and therefore also $L^*_{\kappa e}$ $(M_{\kappa e})$ can be enumerated without repetition, since $L_{\kappa e}$ is an infinite set for all $e\in N$.

The construction of a set B_κ is carried out stepwise for a given κ. We have two copies of the natural numbers, the A-copy and the B-copy. In the construction we use the signs $\boxed{e}_1$, $\boxed{e}_2$, $e=0,1,\ldots$, ordered as follows:

$$\boxed{0}_1,\ \boxed{0}_2,\ \boxed{1}_1,\ \boxed{1}_2,\ \boxed{2}_1,\ \boxed{2}_2,\ldots \tag{1}$$

The signs $\boxed{e}_1$ are placed at numbers in the A-copy and the signs $\boxed{e}_2$ at numbers in the B-copy. Let $\sigma(t)$ be a recursive function enumerating all pairs of natural numbers, where every pair is enumerated infinitely many times, and let $\ell(t)$ be a recursive function of large range.

Let $B^0_\kappa=\phi$.

Step $2t$. Make $2t$ steps in enumerating every $L^*_{\kappa i}$, $i\leq 2t$, and place the sign $\boxed{t}_1$ at 0 in the A-copy.

Suppose $\sigma(t)=\langle x,y\rangle$, $e=max\{x,y\}$ and the sign $\boxed{e}_1$ is at the number z in the A-copy. Enumerate

$$P_y^{L^{*2t}_{\kappa x}}=\{u\,|\,(\exists v\leq 2t)\,T_1^{L^{*2t}_{\kappa x}}(y,u,v)\},$$
$$P_e^{B^{2t-1}_\kappa}=\{u\,|\,(\exists v\leq 2t)\,T_1^{B^{2t-1}_\kappa}(e,u,v)\}.$$

By a well known property of Kleene's T_1-predicate these sets are finite. If they differ from $N \setminus A^{2t}$ on the segment $\{0,1,\ldots,n\}$, set $B_K^{2t} = B_K^{2t-1}$ and proceed to step $2t+1$. If at least one of the sets coincides with $N \setminus A^{2t}$ on $\{0,1,\ldots,n\}$, set $B_K^{2t} = B_K^{2t-1} \cup M_{Ke}^{2t}$, shift the sign $\boxed{e}_1$ to the number $n+1$ and proceed to step $2t+1$.

Step $2t+1$. Make $2t+1$ steps in enumerating every L_{Ki}^{*}, $i \leq 2t+1$, and place the sign t_2 at 0 in the B-copy. Suppose $\ell(t)=e$ and the sign e_2 is at m in the B-copy. Enumerate

$$P_e^{A^{2t+1}} = \{u \mid (\exists v \leq 2t+1) T_1^{A^{2t+1}}(e,u,v)\}.$$
$$2t$$

If this set is distinct from $N \setminus B_K$ on $\{0,1,\ldots,m\}$, set $B_K^{2t+1} = B_K^{2t}$ and proceed to step $2t+2$. If $P^{A^{2t+1}}$ coincides with $N \setminus B_K^{2t}$ on $\{0,1,\ldots,m\}$, set $\overset{e}{B}{}_K^{2t+1} = B_K^{2t} \cup K_e^{2t}$, shift the sign $\boxed{e}_2$ to $m+1$, and proceed to step $2t+2$.

Let $B_K \rightleftharpoons \bigcup_{t=0}^{\infty} B_K^t$. Then B_K is a recursively enumerable set.

LEMMA 1. If all the L_{Kj}^{*}, $j \leq e_0$ are recursive, then all of the signs $\boxed{j}_1$ and $\boxed{j}_2$ are shifted a finite number of times.

Proof. The proof is by induction on the sequence (1). Let $\boxed{e}_i$, $e < e_0$, be the first sign in (1) which is shifted an infinite number of times. We will consider the cases $i=1$ and $i=2$.

Let $i=1$. Consider the step t_0 up to until which all of the signs preceding $\boxed{e}_1$ in (1) have been stabilized. Then it is clear from the construction that $B_K \supseteq M_{Ke}$ and $B_K \setminus M_{Ke}$ is a finite set. Since all of the $L_{Kj}^{*}, j \leq e$, are recursive, all $P_y^{L_{Kx}^{*}}, x,y \leq e$ are distinct from $\bar{A}$, and therefore there exists a step $s_0(e)$ such that for all $s \geq s_0(e)$,

$$P_y^{L_{Kx}^{*s}} \cap \{0,1,\ldots,n_e\} = P_y^{L_{Kx}^{*}} \cap \{0,1,\ldots,n_e\}$$

for all $x,y \leq e$, where $\{0,1,\ldots,n_e\}$ is an interval on which $\bar{A}$ is distinct from all $P_y^{L_{Kx}^{*}}, x,y \leq e$. Moreover, there exists a step $s_1(e)$, such that for all $s \geq s_1(e)$,

$$(N \setminus A^s) \cap \{0,1,\ldots,n_e\} = \bar{A} \cap \{0,1,\ldots,n_e\}.$$

Let $t_1 \geq max\{s_0(e), s_1(e), t_0\}$. On subsequent steps $2t (t \geq t_1)$ such that $\sigma(t) = \langle x,y \rangle$, $max\{x,y\} = e$ the sign $\boxed{e}_1$ is shifted only when the set $P_e^{B_K^{2t-1}}$ coincides with $N \setminus A^{2t}$ on an initial segment of the natural series, and therefore these sets will coincide on an arbitrarily large segment of the natural series. So $\bar{A} = P_e^{B_K}$. But we observed above that B_K and M_{Ke} are distinct on a finite set. Therefore, B_K is recursive. Consequently, $\bar{A}$ is recursively enumerable in the recursive set B_K and must itself be recursively enumerable, which is impossible.

Let $i=2$. Consider the step t_0 up until which all of the signs preceding $\boxed{e}_2$ in (1) have been stabilized. Then it is clear from the construction that $B_K \supseteq K_e$ and $B_K \setminus K_e$ is finite. Further, the sign $\boxed{e}_2$ is shifted only if $N \setminus B_K^s$ and $P_e^{A^s}$ coincide on an initial segment of the natural series, so $\bar{B}_K = P_e^A$. We noted above that B_K and K_e are distinct on a finite set, and therefore B_K - is creative. At the same time the complement of B_K is recursively enumerable in the incomplete recursively enumerable set A, which is impossible. This proves the lemma.

To complete the proof of the theorem we consider two cases.

Case 1. κ is an arbitrary fixed number, $\kappa \notin S$. Then $(\forall e)$ [$L^*_{\kappa e}$ is recursive], and therefore by Lemma 1 all of the signs $\boxed{e}_i$ $i=1,2$, $e=0,1,\ldots$ are stabilized. Now we claim that $B_\kappa \not\leq_T A$ and $A \not\leq_T B_\kappa$.

Suppose $B_\kappa \leq_T A$ and $\bar{B} = P_e^A$. Let m_e be the final position of the sign $\boxed{e}_2$. There exists a step $s(e)$ such that for all $s \geq s(e)$,

$$(N \setminus B_\kappa^s) \cap \{0,1,\ldots,m_e\} = \bar{B}_\kappa \cap \{0,1,\ldots,m_e\},$$
$$P_e^{A^s} \cap \{0,1,\ldots,m_e\} = P_e^A \cap \{0,1,\ldots,m_e\}.$$

Let $2s+1$ $(s \geq s(e))$ be a step such that $\ell(s)=e$. Then on this step we will have to shift the sign $\boxed{e}_2$. This is a contradiction.

Now suppose $A \leq_T B_\kappa$ and $\bar{A} = P_e^{B_\kappa}$. Let n_e be the final position of the sign $\boxed{e}_1$. There exists a step $s(e)$ such that for all $s \geq s(e)$,

$$(N \setminus A^s) \cap \{0,1,\ldots,n_e\} = \bar{A} \cap \{0,1,\ldots,n_e\},$$
$$P_e^{B_\kappa^s} \cap \{0,1,\ldots,n_e\} = P_e^{B_\kappa} \cap \{0,1,\ldots,n_e\}.$$

Suppose step $2s+1$ $(s \geq s(e))$ is such that $\sigma(s) = \langle x, y \rangle$ and $e = max\{x,y\}$. Then on this step we will have to shift the sign $\boxed{e}_1$. This is a contradiction. Thus $\kappa \notin S \Rightarrow A$ and B_κ are incomparable.

Case 2. κ is an arbitrary fixed number, $\kappa \in S$. Then $(\exists e_0)[(\forall p \geq e_0)$ [$L^*_{\kappa p}$ has degree $\mathbf{f}$] $\&(\forall j < e_0)[L^*_{\kappa j}$ is recursive]]. Then $\bar{A}$ is recursive in every $L^*_{\kappa p}, p \geq e_0$; let $q(p)$ be the smallest q such that $\bar{A} = P_q^{L^*_{\kappa p}}$. Suppose $max\{e_0, q(e_0)\} = \tau_0 \geq e_0$. It is clear from the construction that the sign $\boxed{\tau_0}_1$ is shifted an infinite number of times. Let $\boxed{\tau}_1$ be the first sign in (1) which is shifted an infinite number of times. We will show that B_κ and $M_{\kappa\tau}$ are distinct on a finite set. It is clear from the construction that $B_\kappa \supseteq M_{\kappa\tau}$. Moreover, it is clear from the proof of Lemma 1 (the case $i=2$) that if every sign $\boxed{e}_i$, $e < e_0$, $i=1,2$, and the sign $\boxed{e}_1$ are stabilized, then the sign $\boxed{e_0}_2$ is also stabilized. This means that every sign in (1) up to $\boxed{\tau}_1$ is stabilized, i.e., $B_\kappa \setminus M_{\kappa\tau}$- is a finite set. Thus $\kappa \in S \Rightarrow B_\kappa$ has degree $\mathbf{f}$. This concludes the proof of the theorem.

From the theorem and the corresponding lemmas in [1, 2], we obtain the following corollaries.

COROLLARY 1 [1]. Let $\mathbf{f}$ be any recursively enumerable degree. Then $G(\mathbf{f}) \in \Sigma_3(\mathbf{f})$ and $G(\mathbf{f})$ has the smallest recursive isomorphism type possible for sets in $\Sigma_3(\mathbf{f})$. Therefore, $G(\mathbf{f}) \in \Sigma_3(\mathbf{f}) \setminus \Pi_3(\mathbf{f})$.

COROLLARY 2 [2]. Let $\mathbf{f}$ be any recursively enumerable degree, $\mathbf{f} < 0'$. Then $G(\leq \mathbf{f}) \in \Sigma_3(\mathbf{f})$ and $G(\leq \mathbf{f})$ has the smallest recursive isomorphism type possible for sets in $\Sigma_3(\mathbf{f})$. Therefore, $G(\leq \mathbf{f}) \in \Sigma_3(\mathbf{f}) \setminus \Pi_3(\mathbf{f})$.

COROLLARY 3 [2]. Let $\mathbf{f}$ be any recursively enumerable degree, $\mathbf{f} > 0$. Then $G(\geq \mathbf{f}) \in \Sigma_4$ and $G(\geq \mathbf{f})$ has the smallest recursive isomorphism type possible for sets in Σ_4. Therefore, $G(\geq \mathbf{f}) \in \Sigma_4 \setminus \Pi_4$.

COROLLARY 4 [2]. Let $\mathbf{f}$ be any recursively enumerable degree, $0 < \mathbf{f} < 0'$. Then $G(|\mathbf{f}) \in \Pi_4$ and $G(|\mathbf{f})$ has the smallest recursive isomorphism type possible for sets in Π_4. Therefore, $G(|\mathbf{f}) \in \Pi_4 \setminus \Sigma_4$.

Let $\mathbf{f}$ be any recursively enumerable degree of unsolvability. We wish to find the recursive isomorphism type of the index sets $G(<\mathbf{f}) = G(\leqslant \mathbf{f}) \setminus G(\mathbf{f})$ and $G(>\mathbf{f}) = G(\geqslant \mathbf{f}) \setminus G(\mathbf{f})$. If $\mathbf{f}=\mathbf{0}$, then $G(<\mathbf{0}) = \emptyset$ and $G(>\mathbf{0}) \in \Pi_3$ and $G(>\mathbf{0})$ has the smallest recursive isomorphism type possible for sets in Π_3. If $\mathbf{f}=\mathbf{0}'$, then $G(>\mathbf{0}')=\emptyset$ and $G(<\mathbf{0}') \in \Pi_4$ and $G(<\mathbf{0}')$ has the smallest recursive isomorphism type possible for sets in the class Π_4. Therefore, the case $\mathbf{0} < \mathbf{f} < \mathbf{0}'$ is the interesting one. It can easily be shown that if S is any set in $\Sigma_3(\mathbf{f})$, then $S \not\leqslant_1 G(<\mathbf{f})$ and $\bar{S} \leqslant_1 G(<\mathbf{f})$. Correspondingly, for $G(>\mathbf{f})$, if S is any set in Σ_4, then $\bar{S} \leqslant_1 G(>\mathbf{f})$ and $S \not\leqslant_1 G(>\mathbf{f})$. In order to characterize the recursive isomorphism type of these index sets precisely, we will prove the following propositions.

PROPOSITION 2. Let $\mathbf{f}$ be any recursively enumerable degree of unsolvability, $\mathbf{0} < \mathbf{f} < \mathbf{0}'$. The pair of sets $\langle G(\leqslant \mathbf{f}), G(\mathbf{f})\rangle$ is an m-universal pair for pairs of sets $\langle S_0, S_1\rangle$ such that $S_0 \supseteq S_1$ and $S_0, S_1 \in \Sigma_3(\mathbf{f})$.

Proof. By Theorem 1 of [5], Sec. 6, there exists a recursively enumerable degree $\mathbf{f}_1 < \mathbf{f}$ and $\mathbf{f}_1' = \mathbf{f}'$. Fix such a degree $\mathbf{f}_1$. Now we construct a computable sequence of recursively enumerable sets $\{B_k^1\}$ for S_1 such that for all k,

$k \in S_1 \implies B_k^1$ has degree $\mathbf{f}$;

$k \notin S_1 \implies B_k^1$ has degree $< \mathbf{f}$, but greater than or equal to $\mathbf{f}_1$ (see the proof of Theorem 2 in [1]). Further, for S_0 we construct a computable sequence of recursively enumerable sets $\{B_k^0\}$, S_0 such that for all k,

$k \in S_0$ $\quad B_k^0$ has degree $\mathbf{f}_1$;

$k \notin S_0$ $\quad B_k^0$ has degree incomparable with $\mathbf{f}_1$ and $\mathbf{f}$.

We may assume that B_k^0 and B_ℓ^1 are recursively separable for any k and ℓ. Now define a computable sequence of recursively enumerable sets $\{D_k\}$, setting $D_k = B_k^0 \cup B_k^1$. Then for any k,

$$\begin{aligned} k \in S_1 &\Longrightarrow D_k \text{ has degree } d_k = \mathbf{f}, \\ k \in S_0 \setminus S_1 &\Longrightarrow D_k \text{ has degree } d_k < \mathbf{f}, \\ k \notin S_0 &\Longrightarrow D_k \text{ has degree } d_k \not\leqslant \mathbf{f}. \end{aligned}$$

Now it is obvious that the recursive function f such that $D_k = \pi_{f(k)}$ reduces the pair $\langle S_0, S_1\rangle$ to the pair $\langle G(\leqslant \mathbf{f}), G(\mathbf{f})\rangle$.

COROLLARY. Let $\mathbf{f}$ be a recursively enumerable degree such that $\mathbf{0} < \mathbf{f} < \mathbf{0}'$. Then the set $G(<\mathbf{f})$ is recursively isomorphic to the set $mj_A(G(\mathbf{f}))$ where mj_A is an m-jump with respect to A (see [6], III) and A has degree $\mathbf{f}''$.

Remark. Let $\mathbf{c}$ be any degree of unsolvability such that $\mathbf{c} \geqslant \mathbf{0}'''$ and $\mathbf{c}$ is recursively enumerable in $\mathbf{0}'''$. As noted in [1], Theorem 6, $\mathbf{c}$ can be represented by either of the index set $G(\leqslant \mathbf{f})$ or $G(\mathbf{f})$ for some recursively enumerable degree $\mathbf{f}$; we may also assume that $\mathbf{0} < \mathbf{f} < \mathbf{0}'$. Then the index set $G(<\mathbf{f})$ can also be represented by the degree $\mathbf{c}$, which has another (higher) isomorphism type.

PROPOSITION 3. Let $\mathbf{f}$ be a recursively enumerable degree such that $\mathbf{0} < \mathbf{f} < \mathbf{0}'$. The pair of sets $\langle G(\geqslant \mathbf{f}), G(\mathbf{f})\rangle$ is an m-universal pair for pairs of sets $\langle S_0, S_1\rangle$, such that $S_0 \supseteq S_1$ and $S_0 \in \Sigma_4$, $S_1 \in \Sigma_3(\mathbf{f})$.

The proof is completely analogous to that of Proposition 2.

COROLLARY. If $\mathbf{f}$ is a recursively enumerable degree, $\mathbf{f} < \mathbf{0}'$ and $\mathbf{f}' = \mathbf{0}''$, then the set $G(>\mathbf{f})$ is recursively isomorphic to $mj_A(G(\mathbf{f}))$ where A has degree $\mathbf{f}''$.

As an application of the above results and those obtained in [9], we prove the following theorem.

THEOREM 2. Let $\mathbf{f}$ be a recursively enumerable degree of unsolvability such that $\Sigma_3(\mathbf{f}) \supseteq \Pi_3$. Then if $\mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_n$ are $m(tt)$-incomplete recursively enumerable $m(tt)$-degrees contained in $\mathbf{f}$, there exists a recursively enumerable $m(tt)$-degree $\mathbf{a}_{n+1}$ which is $m(tt)$-incomparable with every $\mathbf{a}_i, i \leq n$.

Proof. We will consider two cases. Let $\mathbf{f}$ be an incomplete recursively enumerable degree of unsolvability as in the hypotheses. Let $A_0, A_1, A_2, \ldots, A_n$ represent the recursively enumerable $m(tt)$-degrees $\mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_n$ respectively. Note that none of the $A_i, i \leq n$ is recursive or $m(tt)$-complete. Consider the set $S = G(\leq \mathbf{a}_0) \cup G(\geq \mathbf{a}_0) \cup G(\leq \mathbf{a}_1) \cup G(\geq \mathbf{a}_1) \cup \ldots \cup G(\leq \mathbf{a}_n) \cup G(\geq \mathbf{a}_n)$. Then, $S \in \Sigma_3$ and $\bar{S} \in \Pi_3$ (see [9]). By Corollary 1, $\bar{S} \leq_1 G(\mathbf{f})$, while $S \leq_1 G(|\mathbf{f})$ (see the proof of Theorem 1). Let f be a recursive function effecting this reduction. By the recursion theorem there exists a number k_0 such that $\pi_{f(k_0)} = \pi_{k_0}$. Now $k_0 \notin S$ since the indices of recursively enumerable sets which are $m(tt)$-comparable with at least one $A_i, i \leq n$ map into indices of sets which are Turing incomparable with every $A_i, i \leq n$. Therefore, $k_0 \in \bar{S}$ and $f(k_0) \in G(\mathbf{f})$. This means that the set $A_{n+1} = \pi_{k_0} = \pi_{f(k_0)}$ has degree of unsolvability $\mathbf{f}$ but has $m(tt)$-degree $\mathbf{a}_{n+1}$, which is $m(tt)$-incomparable with every $\mathbf{a}_i, i \leq n$.

Let $\mathbf{f} = \mathbf{0}'$. Consider the set $S_0 = G(\geq \mathbf{a}_0) \cup G(\geq \mathbf{a}_1) \cup \ldots \cup G(\geq \mathbf{a}_n)$. Then $S_0 \in \Sigma_3$ and $\bar{S}_0 \in \Pi_3$. By Corollary 1, $\bar{S}_0 \leq_1 G(\mathbf{0}')$. Let f be a recursive function effecting this reduction. Again by the recursion theorem there exists a number k_0 such that $\pi_{f(k_0)} = \pi_{k_0}$. Now $k_0 \notin S_0$, since the elements of S_0 map into elements of $G(< \mathbf{0}')$. Therefore, $k_0 \in \bar{S}_0$ and $f(k_0) \in G(\mathbf{0}')$, i.e., the set $D = \pi_{f(k_0)} = \pi_{k_0}$ has degree of unsolvability $\mathbf{0}'$, but the $m(tt)$-degree $\mathbf{d}$ of D is such that $\mathbf{a}_i \not\leq \mathbf{d}$ for all $i \leq n$. Here we cannot assert that $\mathbf{d} \not\leq \mathbf{a}_i, i \leq n$. Therefore we construct a recursively enumerable set C such that $D \oplus C$ is $m(tt)$-incomparable with every $A_i, i \leq n$.

The $m(tt)$-degree of the set $A_{n+1} = D \oplus C$ will also be the required $m(tt)$-degree $\mathbf{a}_{n+1}$. It is clear that A_{n+1} has the Turing degree $\mathbf{0}'$.

Here we will confine ourselves to a short description of the construction of C. Let $D_0 = \{2x \mid x \in D\}$ and $M_0 \supseteq K_0 \supseteq M_1 \supseteq K_1 \supseteq \ldots$ be a computable sequence, where $M_0 = \{2x+1 \mid x \in N\}$, every M_i is an infinite recursive set, and K_i is creative, $i = 0, 1, \ldots; \left\{\varphi_e^t\right\}$ be a computable sequence of graphs of all partial recursive functions in one variable. The construction is carried out stepwise. On the even steps every $A_i, i \leq n$ is $m(tt)$-irreducible to $D_0 \cup C$ by the defined function φ_e on some initial segment $\{0, 1, \ldots, q\}$, if φ_e is completely defined on this segment at the end of this step. For this we use the recursive set M_e. Conversely, on the odd steps $m(tt)$ is not $D_0 \cup C$-reducible to any $A_i, i \leq n$ by the defined function φ_e on some initial segment $\{0, 1, \ldots, r\}$, if φ_e is completely defined on this segment at the end of this step. For this we use the creative set K_e.

COROLLARY 1. If $\mathbf{f}$ is a recursively enumerable degree of unsolvability such that $\Sigma_3(\mathbf{f}) \supseteq \Pi_3$, then this degree contains

a) an infinite antichain of recursively enumerable m-degrees;

b) an infinite antichain of recursively enumerable tt-degrees.

Proof. The proof is by contradiction. Let $\mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_n$ be $m(tt)$-incomparable recursively enumerable $m(tt)$-degrees contained in $\mathbf{f}$. Apply Theorem 2. Then $\mathbf{a}_{n+1}$ is recursively enumerable $m(tt)$-degree contained in $\mathbf{f}$ and $m(tt)$-incomparable with each $\mathbf{a}_i, i \leq n$.

COROLLARY 2. If $\mathbf{f}$ is a recursively enumerable degree of unsolvability such that $\Sigma_3(\mathbf{f}) \supseteq \Pi_3$ and $\mathbf{f} < \mathbf{0}'$, then this degree does not have

a) a recursively enumerable m-degree which is maximal among all recursively enumerable m-degrees contained in $\mathbf{f}$;

b) a recursively enumerable tt-degree which is maximal among all recursively enumerable tt-degrees contained in $\mathbf{f}$.

Proof. The proof is by contradiction. Let $\mathbf{a}_0$ be a maximal recursively enumerable $m(tt)$-degree contained in $\mathbf{f}$. Apply Theorem 2 to find a recursively enumerable $m(tt)$-degree $\mathbf{a}_1$ which is $m(tt)$-incomparable with $\mathbf{a}_0$. Now $\mathbf{a}_0 \oplus \mathbf{a}_1 = \mathbf{a}_2$ is an $m(tt)$-degree which is contained in $\mathbf{f}$ and $\mathbf{a}_0 < \mathbf{a}_2$

Part b) of Corollary 2 answers one of Rogers' questions ([4], Sec. 9.6). Note that the proof of Corollary 2 shows that a recursively enumerable degree of unsolvability $\mathbf{f}$ such that $\Sigma_3(\mathbf{f}) \supseteq \Pi_3$ and $\mathbf{f} < \mathbf{0}'$ has an infinite chain of recursively enumerable $m(tt)$-degrees.

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