Journal of Dynamics and Differential Equations, Vol. 9, No. 4, 1997

Effects of Periodic Forcing on Delayed Bifurcations

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Received July 11, 1994; revised December 13, 1996

This work is concerned with the rigorous analysis of the effects of small periodic forcing (perturbations) on the dynamical systems which present some interesting phenomena known as delayed bifurcations. We study the dynamical behavior of the system

$$\frac{\partial u}{\partial t} = f(u, I_i + \varepsilon t) + \varepsilon g(u, I_i + \varepsilon t, \varepsilon, t)$$
$$u(t)|_{t=0} = u_0(I_i) + O(\varepsilon) \tag{0.1}$$

where $u_0(I)$ is the solution of $f(u_0(I), I) = 0$ and $I(t) \equiv I_i + \varepsilon t$ is a slowly varying parameter that moves past a critical point I_- of the system so that the linear stability around $u_0(I)$ changes from stable to unstable at I_-. General results are given with respect to the effects of the perturbation $\varepsilon g(u, I(t), \varepsilon, t)$ to several important types of dynamical systems

$$\frac{\partial u}{\partial t} = f(u, I_i + \varepsilon t) \tag{0.2}$$

which present dynamical patterns that there exist persistent unstable solutions in the dynamical systems (delayed bifurcations) in contrast to bifurcations in the classical sense. It is shown that (1) the delayed bifurcations persist if the frequency of $g(\cdot, \cdot, \cdot, t)$ on t is a constant ω which is not a resonant frequency; (2) in case the frequency of $g(\cdot, \cdot, \cdot, t)$ on t is $\omega \equiv \omega(I_i + \varepsilon t)$ that is slowly varying, the resonance frequencies where the delayed bifurcations might be destructed are shifted downward or upward depending on $\omega'(I_-) > 0$ or $\omega'(I_-) < 0$; and (3) delayed pitchfork (simple eigenvalue) bifurcations occur in

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1040-7294/97/1000-0561$12.50/0

a codimension one parameter family of periodic perturbations. (1) is a rigorous analysis of the results in [3], (2) is a new and interesting phenomenon, and (3) is a generalization of the results of Diener [8] and Schecter [19].

KEY WORDS: Delayed bifurcation; resonance; periodic forcing; Hopf bifurcation; simple eigenvalue bifurcation.
AMS Subject Classifications: 34C35, 34D20, 58F14, 92C30.

1. INTRODUCTION

The mathematical analysis of a collection of dynamical behaviors which were known as delayed bifurcations has been developed by many authors [1–4, 7–24]. Typically, these problems involve dynamical systems containing a parameter which is slowly varying with time t or in general, fast–slow systems where the parameter is determined by the slow equations. When the mentioned parameter is kept constant, the dynamical behavior of the systems is well understood. For each value of the parameter, there are "static" solutions (equilibria, or periodic solutions). Further, there is a critical point in the parameter so that the linearized (orbital) stability of the solutions in the classical sense changes as the parameter moves across the critical value. The interesting behavior of delayed bifurcations can be observed when one considers the solutions of initial value problems of these dynamical systems where the parameter is slowly varying with time. In fact, as the parameter slowly passes the critical point, the solutions of the dynamical systems remain close to the unstable "static" solutions mentioned earlier, only bifurcating away from them at some points of the parameter which are above the bifurcation points at distances depending only on the initial parameter values and independent of the slowness of the parameter change. The limits of the solutions as the speed of parameter change goes to zero present the pattern that the solutions of the initial value problems stay close to the "static" solutions (which generally are not the solutions of the systems with a slowly varying parameter) up until certain points above the bifurcation point, then move away. See Fig. 1. These delayed bifurcation phenomena are different from what people understand as bifurcations, and are named as delayed bifurcations. Further, the separations present memory effects in such a way that if the initial parameter is farther away from the critical point, then the separation would be farther above the critical point. These delayed bifurcation phenomena are antiintuitive, and their mathematical structures are rather complicated. The theory, however, offers a good explanation to a number of natural phenomena.

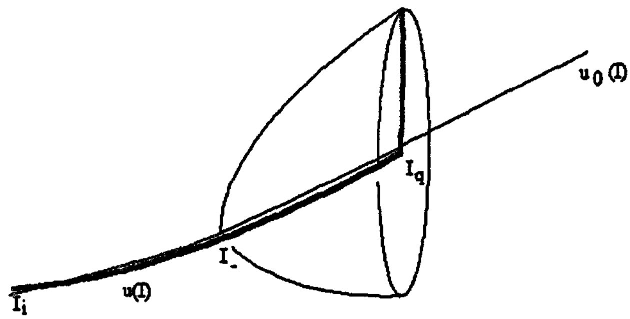

Fig. 1. The behavior of the solutions $u(I)$ present the pattern that $u(I)$ stay close to $u_0(I)$ for $I_i \leqslant I \leqslant I_q$ where $I_q > I_-$ is independent of ε.

We consider the following well-known example to start our mathematical formulation of the problems:

$$\frac{\partial u}{\partial t} = f(u, I_i + \varepsilon t) \tag{1.1}$$

$$u(t)|_{t=0} = u_0(I_i) + O(\varepsilon) \tag{1.2}$$

Here f: $\mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ has analytic extensions for both variables. For each I, the equation $u_t = f(u, I) = 0$ has an equilibrium $u_0(I)$ which is also analytic in I. Assume that there exists $I = I_-$ such that when $I < I_-$, the eigenvalues $\lambda_1(I), \lambda_2(I) = \overline{\lambda_1(I)}$ are in the left-half complex plane, i.e., $Re\, \lambda_j(I) < 0$ for $I < I_-$, and when $I > I_-$, $Re\, \lambda_j(I) > 0$. We may assume that $Im\, \lambda_1(I_-) < 0$ to distinguish the two eigenvalues. Also, $I_i < I_-$. One would agree from the basic stability theory that

$$\|u(t) - u_0(I_i + \varepsilon t)\| = O(\varepsilon) \qquad \text{for} \quad t \in \{t: I(t) \equiv I_i + \varepsilon t \leqslant I_-\} \tag{1.3}$$

Because all the eigenvalues stay in the left half of the complex plane, (1.3) can be achieved by some standard arguments, such as semigroup theory. Standard references can be found, for example, in Refs. 5 and 6. The interesting question remains on the behavior of $u(I(t))$ after $I(t)$ goes across I_-.

It was implied by some quasi-steady-state theory [1, 7, 12] that $u(I(t))$ goes away from $u_0(I(t))$ shortly after I passes I_- since the signs of the eigenvalues change from negative to positive at I_-. Some asymptotic methods had been utilized to determine whether the peeling (separating) point occurs at $I = I_- + O(\varepsilon^{1/2})$ or $I = I_- + O(\varepsilon^{1/3})$ [1, 7, 12].

Surprisingly, in contrast to all these predictions, the experimental results and numerical computations [3, 11, 18] indicated that for some systems, $|u(t) - u_0(I_i + \varepsilon t)| = O(\varepsilon)$ when $I_i \leqslant I(t) \leqslant I_q$, where $I_q = I_q(I_i) > I_-$

is independent of ε as $\varepsilon \to 0^+$. The amount of delay of the bifurcation $I_q - I_-$ is determined by $I_i < I_-$ by a manner of monotone decreasing function which presents a memory effect.

This type of phenomena is of interest for their importance in mathematics and also applications to other fields. We mention below some previous work in the literature to show their connections with physiology and physics. In fact, the theory is applicable to the reversed accommodation phenomena in the experiments of membranes of the giant axon of a squid studied by Jakobsson and Guttman [11]. Essentially, if the membrane is connected with a constant electric current, then there exists a threshold such that the membrane potential can accommodate any current below the threshold, and when the current is above the threshold, the potential starts to oscillate (burst). The discovery of Jakobsson and Guttman in [11] concerns the response of the membrane potential with respect to a continuously increasing current, and it was found by them that if the increasing is slow, the potential will accommodate until the current reaches a point which is substantially higher than the threshold. More significantly, the amount of delay is independent of the slowness of the increasing. Baer *et al.* [3] considered the corresponding mathematical problems by studying the FitzHugh Nagumo equation and made extensive numerical computations for the slow passage problem in the FitzHugh Nagumo equation. In particular, Baer *et al.* [3] indicated that the initial current $I_i < I_-$ and the current I_q where $u(I(t))$ moves away from $u_0(I(t))$ may satisfy the relationship

$$\int_{I_i}^{I_q} Re\, \lambda_1(\tau)\, d\tau = 0 \tag{1.4}$$

which was later confirmed by the author [21, 22]. We also note that other applications of delayed bifurcation in laser and others are available in the literature [9 and references therein].

Rigorous study of delayed bifurcation started at least in the 1970s. Shishkova [20] discovered the delayed bifurcation phenomena in a particular ordinary differential equation system. Later, Neishtadt [14–17] considered the general systems having delayed Hopf bifurcations. Su [21, 22] gave out a rigorous proof of the delayed Hopf bifurcation for the spatially uniform FitzHugh Nagumo equation. The case of a nonspatially uniform FitzHugh Nagumo system was also studied for the delayed Hopf bifurcations [23]. A quite different approach was utilized in order to deal with the parabolic system. Candelpergher *et al.* [4] also attempted to explain the delayed Hopf bifurcation phenomena from the nonstandard analysis point of view. Rigorous analysis for delayed pitchfork bifurcations is also available in the literature. Under the restrictive condition that $u \equiv 0$ is a constant solution of $u_t = f(u, I_i + \varepsilon t)$, Diener and Diener [8] and Schecter [19] independently proved the existence of delayed simple eigenvalue bifurcations. Without the restrictive condition stated above, the problems are rather difficult and open. The reason is that, in general, a nonhomogeneous equation cannot be changed into a homogeneous equation by finite steps of transformations, even in the linear case. Although n-step iterations of some transformations can reduce the nonhomogeneous terms to $O(\varepsilon^n)$, the consequent sequence of the nonhomogeneous terms is divergent as $n \to \infty$ [15]. We extend their results in Section 6. In another direction, there are some results [15, 24] in a more general case where delayed bifurcation occurred to a family of periodic motions whose critical exponents moved across the imaginary axis as the parameter slowly moved past the critical point under some nonresonance conditions.

These mathematical theories have provided a solid theoretical foundation for giving mathematical descriptions of natural phenomena such as the reversed accommodation. A question was seriously imposed by experimentalists on the reason why the reversed accommodation phenomena were observed in some experiments, but not in others, or how sensitive these persistent unstable solutions would be with respect to perturbations. Further, reversed accommodations were found on tissues under laboratory conditons, and it would be interesting to know whether these types of behaviors happen to live tissues where other disturbances exist. Baer *et al.* [3] studied the problems numerically and observed that if (1.1) is added with a small perturbation $g = \delta \sin \omega t$, then unstable solutions persist when $\omega \neq \frac{1}{3}|\omega_0|, \frac{1}{2}|\omega_0|, |\omega_0|, 2|\omega_0|$, where $|\omega_0| = |Im\, \lambda_1(I_-)|$ is the frequency at the Hopf bifurcation point. At those resonance frequencies, the delay amounts $I_q - I_-$ were significantly reduced. Rigorous analysis was not given.

In this work, we resolve these issues in terms of rigorous analysis. We first show the delayed bifurcations of (0.1) by proving the existence of certain persistent unstable solutions under the nonresonance conditions $\omega \neq 2|\omega_0|/n, n \in \mathbb{N}$, and we consider the situations near the resonances. Then we generalize the results into situations where the periodic forcing $\varepsilon g(\cdot, \cdot, \cdot, t)$ is of a slowly varying frequency, and an interesting phenomenon of shifted interference is discovered. Finally, we consider a general version of delayed bifurcations in simple eigenvalue bifurcations related to previous work by Diener [8] and Schecter [19].

We organize this paper in the following way. In Sections 2, 3 and 4, we consider the problems of delayed Hopf bifurcations under the influence of a periodic perturbation. We show that under the nonresonance conditions and some other generic conditions, the delay will persist. In Section 5, we study the problems with a periodic forcing with a slowly varying frequency, and new phenomena of shifted interference are studied. In Section 6, we show that for each scalar ordinary differential equation which has a pitchfork (simple eigenvalue) bifurcation, there exists a codimension one family of periodic perturbations with moderately large norms such that delayed bifurcation phenomena occur under these perturbations.

2. ASSUMPTIONS AND BASIC RESULTS

We begin with the introduction of a basic result of delayed Hopf bifurcations. Let $u(t)$ be solutions of the initial value problems

$$\frac{\partial u}{\partial t} = f(u, I_i + \varepsilon t) \tag{2.1}$$

$$u(t)|_{t=0} = u_0(I_i) + O(\varepsilon) \tag{2.2}$$

where the systems satisfy the following assumptions.

(A1) $f(u, I)\colon \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ has an analytic extension for $|u| < \sigma$, $|I| < r_a$, in the complex plane for some $r_a > 0$.

(A2) For each fixed I, the system

$$v_t = f(v, I) \tag{2.3}$$

has an equilibrium solution $u_0(I)$ which is also analytic in I for $|I| < r_a$.

(A3) The variational system of (2.3) about $u_0(I)$:

$$w_t = f_u(u_0(I), I)\, w \tag{2.4}$$

is a linear system with coefficients depending on the parameter I. Let $A(I) = f_u(u_0(I), I)$. Assume that two eigenvalues of $A(I)$, $\lambda_1(I)$ and $\lambda_2(I)$, are conjugate to each other, i.e., $\lambda_2(I) = \bar{\lambda}_1(I)$ for I on the real axis and $|I| < r_a$. Further, there exists a real number I_- such that $Re\, \lambda_j(I) < 0$ when $I < I_-$, $Re\, \lambda_j(I) > 0$ when $I > I_-$, $Im\, \lambda_1(I_-) < 0$. Also, $I_i < I_-$.

We make some change of variables to simplify the system. We let $I = I_i + \varepsilon t$ be the independent variable, and $y = u(t) - u_0(I_i + \varepsilon t)$. System (2.1) becomes

$$\varepsilon y_I = f_u(u_0(I), I)\, y + f_2(u_0(I), I, y) + \varepsilon G_1(I) \tag{2.5a}$$

$$y|_{I = I_i} = O(\varepsilon) \tag{2.5b}$$

where $f_2 = \sum_{n=2}^{\infty} (1/n!)(\partial^n f/\partial u^n)(u_0(I), I)\, y^n = \sum_{n=2}^{\infty} c_n(u_0(I), I)\, y^n$. The non-homogeneous term $G_1(I) = -(\partial/\partial I)\, u_0(I)$ is a bounded vector function of I.

Proposition 2.1. *Let $y(I, \varepsilon)$ be a family of solutions of (2.5) with initial conditions which satisfy $|y(I_i, \varepsilon)| \leqslant M_1 \varepsilon$. Then there exist $M = M(M_1)$, $I_q = I_q(M_1, M) > I_-$, $\varepsilon_0 = \varepsilon_0(M_1, M)$, such that*

$$|y(I, \varepsilon)| \leqslant M\varepsilon \tag{2.6}$$

whenever $I_i \leqslant I \leqslant I_q$, $\varepsilon \leqslant \varepsilon_0$. Further if I_i is close enough to I_-, then I_q and I_i satisfy the relationship

$$\int_{I_i}^{I_q} Re\, \lambda_1(\tau)\, d\tau = 0 \tag{2.7}$$

The proof of proposition 2.1 can be found in Refs. 14–17 and 21–23. We demonstrate the behavior of $u(I)$ in Fig. 1.

We now consider the delayed bifurcation problems under the periodic forcing in (2.1) which was initially proposed by Baer *et al.* [3]. Before we go into the formal argument, we should point out here the essential effects of resonances. Baer *et al.* [3] had observed from their numerical computations of the FitzHugh Nagumo equation, which is a very typical example of delayed Hopf bifurcation phenomena, that when the forcing frequency ω of the function g was taken to be $2\,|\omega_0|$, $|\omega_0|$, $\frac{1}{2}\,|\omega_0|$, $\frac{1}{3}\,|\omega_0|$, where $|\omega_0| \equiv |Im\, \lambda_1(I_i)|$, the amount of delay is drastically reduced. It is conceivable that due to high sensitivity of the delayed bifurcation phenomena with respect to the roundoff errors in computations, it might be extremely difficult to observe what really happened exactly at those resonant frequencies. We provide an example below by using the following modified Shishkova–Wallet's equation to show that, in fact, when the forcing frequency $\omega = \omega_0$, the delay amount would be $O(\sqrt{\varepsilon}\, |\ln(\varepsilon)|)$ rather than $O(1)$.

Example.

$$\varepsilon u_I = (I + i\omega_0) u + \varepsilon e^{i\omega I/\varepsilon} \tag{2.8a}$$

$$u|_{I = I_i} = O(\varepsilon) \tag{2.8b}$$

for $I_i < I_- \equiv 0$. We define u_+ be the solution of (2.8) with the initial condition $u_+(1) = 0$, and u_- the solution of (2.8) with the initial condition $u_-(-1) = 0$. It can be easily shown through a direct calculation that

$$|u_+(0) - u_-(0)| = \left| \int_{-1}^{1} e^{-(1/2\varepsilon)(s^2 + 2i(\omega_0 - \omega)s)}\, ds \right| \tag{2.9}$$

When $\omega=\omega_0$, the distance $|u_+(0)-u_-(0)|=\int_{-1}^{1} e^{-(1/2\varepsilon)s^2}\,ds \geqslant K\sqrt{\varepsilon}$ for some $K>0$, when $\varepsilon \leqslant \varepsilon_0$ is sufficiently small. Thus, u_- jumps away from u_+ in the sense $|u_+(I)-u_-(I)|=O(1)$ at the point $I_q \equiv \inf\{I>0 \mid \sqrt{\varepsilon}\, e^{(1/(2\varepsilon))I^2}=O(1)\} = O(\sqrt{|\varepsilon \log \varepsilon|})$. In other words, the bifurcation of $u_-(I)$ from $u_0(I)\equiv 0$ occurs near the critical point $I_-=0$ at $I_q = O(\sqrt{|\varepsilon \log(\varepsilon)|})$. Hence the bifurcation has not been delayed substantially. The key element here which prevents the delayed bifurcation is the interference between the frequency at the critical point (of critical exponents) ω_0 and the frequency of the periodic forcing ω. If $\omega \neq \omega_0$, however, then we can derive the estimate $|u_+(0)-u_-(0)| \leqslant e^{-c/\varepsilon}$ for some $c>0$, which implies delayed bifurcations.

This example can be easily modified to

$$\varepsilon u_I = (I+i\omega_0)u + \varepsilon\left(\sum_{n\in\mathbb{Z}} \frac{1}{1+n^2}\, e^{in\omega I/\varepsilon}\right) \tag{2.10a}$$

$$u|_{I=I_i} = O(\varepsilon) \tag{2.10b}$$

where $I_q = O(\sqrt{|\varepsilon \log(\varepsilon)|})$ if $\omega=(1/n)\omega_0$. Thus the resonance effect is a vital factor in our considerations.

We now introduce the setting of our perturbed problem. We study the initial value problems,

$$\frac{\partial u}{\partial t} = F(u, I_i+\varepsilon t, \varepsilon, t) \tag{2.11a}$$

$$u(t)|_{t=0} = u_0(I_i) + O(\varepsilon) \tag{2.11b}$$

under the following hypotheses.

Assumptions. (H1) $F(u, I, \varepsilon, t)\colon \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^2$ has an analytic extension for $|u|<\sigma$, $|I|<r_a$, $t\in(-\infty,\infty)$ in the complex plane when $\varepsilon \leqslant \varepsilon_0$. $F(u, I, 0, t) = f(u, I)$ where f was specified in (A1)–(A3). Namely, $f(u, I)\colon \mathbb{R}^2\times\mathbb{R}\to\mathbb{R}^2$ has an analytic extension for $|u|\leqslant\sigma$, $|I|\leqslant r_a$ in the complex plane. $F(u, I, \varepsilon, t)$ is real analytic in ε for $0\leqslant\varepsilon\leqslant\varepsilon_0$. Consequently,

$$F(u, I, \varepsilon, t) = f(u, I) + \varepsilon f_1(u, I, \varepsilon, t). \tag{2.12}$$

(H2) For each fixed I, and ε set to 0, the reduced system $v_t = f(v, I)$ has an equilibrium solution $u_0(I)$ which is also analytic in I for $|I|<r_a$. The

perturbation $f_1(u, I, \varepsilon, t)$ *is a periodic function of the variable* t *with the period* $2\pi/\omega$. *In particular, we may express*

$$f_1(u, I, \varepsilon, t) = \sum_{n \in \mathbb{Z}} C_n(u, I, \varepsilon)\, e^{in\omega t} \tag{2.13}$$

where C_n *are analytic in all variables and* $\sum_{n \in \mathbb{Z}} |C_n(u, I, \varepsilon)| \leqslant M$ *for* $|u| < \sigma$, $|I| < r_a$ *in the complex plane and* $\varepsilon \leqslant \varepsilon_0$.

(H3) The variational system of (2.3) about $u_0(I)$: $w_t = f_u(u_0(I), I)w$ *is a linear system with coefficients depending on the parameter I. Denote* $A(I) = f_u(u_0(I), I)$. *Assume that two eigenvalues of* $A(I)$, $\lambda_1(I)$ *and* $\lambda_2(I)$, *are conjugate to each other, i.e.,* $\lambda_2(I) = \bar{\lambda}_1(I)$ *for I on the real axis and* $|I| < r_a$. *Further, there exists a real number* I_- *such that* $Re\, \lambda_j(I) < 0$ *when* $I < I_-$, $Re\, \lambda_j(I) > 0$ *when* $I > I_-$, $Im\, \lambda_1(I_-) < 0$, $(d/dI)\, Re\, \lambda_j(I_-) > 0$. *Assume also that there are no branch points of* $\lambda_j(I)$ *in the complex plane so that* $\lambda_j(z)$ *are analytic in the region of consideration, and the matrix* $Q(I)$ *which satisfies* $Q(I)\, A(I)\, Q^{-1} = \begin{pmatrix} \lambda_1(I) & 0 \\ 0 & \lambda_2(I) \end{pmatrix}$ *can also be extended analytically into the complex plane. We assume* $I_i < I_-$.

(H4) (First Nonresonance Condition) Assume that there exists some $0 < r_b < r_a$ *such that for* $|z - I_-| < r_b$ *in the complex plane,* $\lambda_1(z) - in\omega \neq 0$, $\lambda_2(z) - in\omega \neq 0$ *for* $n \in \mathbb{Z}$, *where* $\lambda_j(z)$ *are the analytic extensions of the eigenvalues* $\lambda_j(I)$. *Since* $Re\, \lambda_j(I)|_{I=I_-} = 0$, *(H4) can simply be remembered as* $|\omega_0| \equiv |Im\, \lambda_j(I_-)| \neq n\omega$ *for* $n \in \mathbb{N}$ *if we consider z near* I_-.

(H5) (Second Nonresonance Condition) Assume that there exists some $0 < r_b < r_a$ *such that for* $|z - I_-| < r_b$, *the analytic extensions* $\lambda_j(z)$ *satisfy* $2\lambda_j(z) - i(2n-1)\omega \neq 0$ *for* $n \in \mathbb{Z}$, $j = 1, 2$, *or simply* $2|\omega_0| \equiv 2|Im\, \lambda_1(I_-)| \neq (2n-1)\omega$.

Since we intend to show that $|u(t) - u_0(I_i + \varepsilon t)| = O(\varepsilon)$, we use $y = u(t) - u_0(I_i + \varepsilon t)$ as the new variable and $I = I_i + \varepsilon t$ as the new independent variable. Then (2.11) becomes

$$\varepsilon y_I = f_u(u_0(I), I)\, y + F_0\left(u_0, I, \varepsilon, \frac{I - I_i}{\varepsilon}, y\right) + \varepsilon Q_0\left(u_0, I, \varepsilon, \frac{I - I_i}{\varepsilon}\right) \tag{2.14}$$

where $F_0 = \sum_{n=2}^{\infty} (1/n!)(\partial^n f/\partial u^n)(u_0(I), I)\, y^n + \varepsilon \sum_{n=1}^{\infty} (1/n!)(\partial^n f_1/\partial u^n)(u_0(I), I, \varepsilon, (I - I_i)/\varepsilon)\, y^n$,

$$\varepsilon Q_0\left(u_0, I, \varepsilon, \frac{I - I_i}{\varepsilon}\right) = \varepsilon \sum_{n \in \mathbb{Z}} \tilde{c}_n(I, \varepsilon)\, e^{in\omega(I - I_i)/\varepsilon} \tag{2.15}$$

is of order $O(\varepsilon)$.

We observe here that for fixed I, $F(u, I, \varepsilon, t)$ is periodical on the t-variable, and consequently, $F_0(u_0, I, \varepsilon, t)$ and $Q_0(u_0, I, \varepsilon, t)$ are periodical on the t-variable. Therefore, they can be uniquely expressed as the Fourier series of t whose coefficients depend upon I and ε. Thus, we derive $\varepsilon Q_0(u_0, I, \varepsilon, t) = \varepsilon \sum_{n\in\mathbb{Z}} \tilde{c}_n(I, \varepsilon)\, e^{in\omega t}$. The corresponding series for the case of $I = I_i + \varepsilon t$ naturally leads to (2.15) after the change of variable $t = (I - I_i)/\varepsilon$. Throughout the paper any function which depends on I/ε periodically and also depends on (I, ε) can therefore analogously be decomposed uniquely by following the natural expansion method mentioned above. It should also be noted, however, that the decomposition of $\varepsilon Q_0(u_0, I, \varepsilon, (I - I_i)/\varepsilon)$ into the series with the form of (2.15) is not unique if the natural decomposition method is not followed. The natural decomposition method which we used has the advantage that the coefficients of the series (but not the series itself) have bounded analytic extensions for the z-variable with its total upper bound independent of $\varepsilon \to 0^+$. This property is crucial to the problem in the sense that the desired solution to our problem will be shown to have a similar series expansion.

We may absorb the terms that involve I_i into the coefficients as the terms $e^{in\omega I_i/\varepsilon}$ are bounded by constants. For simplicity, we neglect the dependence on I_i. The initial value problem (2.14) is then equivalent to

$$\varepsilon y_I = A(I)\, y + F_1\left(I, \varepsilon, \frac{I}{\varepsilon}, y\right) + \varepsilon Q_1\left(I, \varepsilon, \frac{I}{\varepsilon}\right) \tag{2.16a}$$

$$y|_{I=I_i} = O(\varepsilon) \tag{2.16b}$$

where F_1 is analytic in y and has the expression

$$F_1\left(I, \varepsilon, \frac{I}{\varepsilon}, y\right) = \sum_{k=1}^{\infty} \sum_{m_1+n_1=k,\, m_1\geq 0,\, n_1\geq 0} B_k^{(m_1, n_1)} y_1^{m_1} y_2^{n_1} \tag{2.17}$$

with $(y_1, y_2)^T = y$. From the properties of F_0 and G_0, the coefficients of F_1 have the form

$$B_1^{(m_1, n_1)}\left(I, \varepsilon, \frac{I}{\varepsilon}\right) = \varepsilon \sum_{n\in\mathbb{Z}} b_{1,n}^{(m_1, n_1)}(I, \varepsilon)\, e^{in\omega I/\varepsilon} \tag{2.18a}$$

$$B_k^{(m_1, n_1)}\left(I, \varepsilon, \frac{I}{\varepsilon}\right) = \sum_{n\in\mathbb{Z}} b_{k,n}^{(m_1, n_1)}(I, \varepsilon)\, e^{in\omega I/\varepsilon} \quad \text{for } k\geq 2 \tag{2.18b}$$

Denote $M_{m\times n}$ the set of $m\times n$ matrices. All terms $b_{k,n}^{(m_1, n_1)}(I, \varepsilon) \in M_{2\times 1}$ are analytic in I. From the fact that $F(u, I, \varepsilon, t)$ of (2.12) is analytic in u for $|u| < \sigma$, and $b_{k,n}^{(m_1, n_1)}(I, \varepsilon)$ correspond to the coefficients of kth power terms in the new system, it is not hard to verify that there exists a constant M_a independent of ε such that for $k \geqslant 1$ and $|I-I_-| < r_b$ in the complex plane,

$$\sum_{n \in \mathbb{Z}} \sum_{m_1 \geqslant 0, n_1 \geqslant 0, m_1+n_1=k} |b_{k,n}^{(m_1,n_1)}(I, \varepsilon)| \leqslant \frac{M_a}{(\sigma/2)^k} \tag{2.19}$$

The nonhomogeneous term εQ_1 has the form

$$Q_1 = \sum_{n \in \mathbb{Z}} q_n(I, \varepsilon)\, e^{in\omega I/\varepsilon} \tag{2.20}$$

where $\sum_{n \in \mathbb{Z}} |q_n(I, \varepsilon)| \leqslant M$ for $|I-I_-| < r_b$ in the complex plane. The method of the decompositions in (2.18) and (2.20) were mentioned earlier in the observation after (2.15). These decompositions are unique based upon the natural method which we used.

We begin with the initial value problems (2.16) for $I_i < I < I_-$.

Lemma 2.2. *Let $y = y(I, \varepsilon)$ be a family of solutions of system (2.16) with initial conditions at I_i satisfying $|y(I, \varepsilon)|\,|_{I=I_i} \leqslant M_1\varepsilon$. Then there exist $r_c = r_c(M)$, $M_2 = M_2(M_1)$ and $\varepsilon_0 = \varepsilon_0(M_1) > 0$ so that when $|I_- - I_i| < r_c$, $0 < \varepsilon < \varepsilon_0$,*

$$|y(I, \varepsilon)| \leqslant M_2 \varepsilon \tag{2.21}$$

for $I_i < I < I_-$.

The proof is somehow standard. If we note the fact that all $e^{in\omega I/\varepsilon}$ terms remain bounded when $I_i < I < I_-$, the proof is not much different from the one of Theorem 1 in Refs. 21 and 22 or Lemma 1 in Ref. 24. We refer the proof to Lemma 1 of Ref. 24.

Lemma 2.3. *Let $I^i > I_-$ be any point above the critical point $I = I_-$, and $y = y(I, \varepsilon)$ be solutions of (2.16) with initial conditions at $I = I^i$ which satisfy $|y(I, \varepsilon)|\,|_{I=I^i} \leqslant M_1\varepsilon$ for any $\varepsilon > 0$. Then there exist $r_d = r_d(M_1)$, $\varepsilon_0 = \varepsilon_0(M_1)$, $M_2 = M_2(M_1)$ so that for $\varepsilon \leqslant \varepsilon_0$,*

$$|y(I, \varepsilon)| \leqslant M_2 \varepsilon \tag{2.22}$$

whenever $I_- \leqslant I \leqslant I^i \leqslant I_- + r_d$.

Proof. If we replace the variable I by $J \equiv 2I_- - I$, then Lemma 2.3 follows immediately by analogous arguments as in Lemma 2.2. □

We also see that the exponential growth property is valid in this case.

Proposition 2.4. Assume that y_A and y_B are two solutions of (2.16) on $I_1 < I < I_2$, and $|y_A(I)| \leqslant M_2\varepsilon$, $|y_B(I)| \leqslant M_2\varepsilon$ whenever $I_1 \leqslant I \leqslant I_2$. Then there exist $M_3 = M_3(M_2)$, $\varepsilon_0 = \varepsilon_0(M_2)$ so that for $\varepsilon \leqslant \varepsilon_0$,

$$\frac{1}{M_3} e^{1/\varepsilon \int_{I_1}^{I_2} Re\,\lambda_1(\tau)\,d\tau} \leqslant \frac{|y_A(I_2) - y_B(I_2)|}{|y_A(I_1) - y_B(I_1)|} \leqslant M_3 e^{1/\varepsilon \int_{I_1}^{I_2} Re\,\lambda_1(\tau)\,d\tau} \qquad (2.23)$$

Proof. See Refs. 21 and 22. We note that there is no restriction on whether I_j are above or below I_-. □

We denote $y(I, s, \varepsilon)$ to be the solution of (2.16) with the initial condition:

$$y(I, s, \varepsilon)|_{I=s} = 0 \qquad (2.24)$$

If we can show that the solution $y(I, I_i, \varepsilon)$ for some $I_i < I_-$ and the solution $y(I, I^i, \varepsilon)$ for $I^i > I_-$ have a distance of magnitude $O(e^{-c/\varepsilon})$ at $I = I_-$ for some $c > 0$, then the delayed bifurcations will follow from Lemmas 2.2 and 2.3 and Proposition 2.4 in the following way: We have $|y(I, I_i, \varepsilon)| \leqslant M_2\varepsilon$ for $I_i < I < I_-$ and $|y(I, I^i, \varepsilon)| \leqslant M_2\varepsilon$ for $I_- < I < I^i$. Adding the fact $|y(I_-, I_i, \varepsilon) - y(I_-, I^i, \varepsilon)| = O(e^{-c/\varepsilon})$ and Proposition 2.4, we get that $|y(I, I_i, \varepsilon)| \leqslant M_2\varepsilon$ when $I_i < I < I_q$ for some $I_q = I_q(c) > I_-$. So it is the key step to obtain the estimate $|y(I_-, I_i, \varepsilon) - y(I_-, I^i, \varepsilon)| = O(e^{-c/\varepsilon})$ in this problem.

To achieve this, we need some additional properties. Let $Q(I)$ be the nonsingular matrix such that $AQ = Q\begin{pmatrix} \lambda_1(I) & 0 \\ 0 & \lambda_2(I) \end{pmatrix}$. If we let $y = QY = Q\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$, then the system (2.16) becomes

$$\varepsilon \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}_I = \begin{pmatrix} \lambda_1(I) & 0 \\ 0 & \lambda_2(I) \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} + Q^{-1} F_1 \left(I, \varepsilon, \frac{I}{\varepsilon}, QY \right)$$
$$+ \varepsilon Q^{-1} Q_1 \left(I, \varepsilon, \frac{I}{\varepsilon} \right) - \varepsilon Q^{-1} Q' Y \qquad (2.25)$$

We rewrite (2.25) as

$$\varepsilon \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}_I = \begin{pmatrix} \lambda_1(I) & 0 \\ 0 & \lambda_2(I) \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} + F_2 \left(I, \varepsilon, \frac{I}{\varepsilon}, Y \right) + \varepsilon Q_2 \left(I, \varepsilon, \frac{I}{\varepsilon} \right) \qquad (2.26)$$

where $F_2 = Q^{-1}F_1 - \varepsilon Q^{-1}(\partial Q/\partial I)\,Y$, $Q_2 = Q^{-1}Q_1$. The nonlinear term F_2 has the expression $F_2 = (F_{2,1}, F_{2,2})^T$, where

$$F_{2,j} = \varepsilon\left(E_{j,1,1}\left(I, \varepsilon, \frac{I}{\varepsilon}\right) Y_1 + E_{j,1,2}\left(I, \varepsilon, \frac{I}{\varepsilon}\right) Y_2\right) + \sum_{2 \leqslant k < \infty,\, 1 \leqslant l \leqslant k+1} E_{j,k,l}\left(I, \varepsilon, \frac{I}{\varepsilon}\right) Y_1^{k-l+1} Y_2^{l-1} \tag{2.27}$$

$$E_{j,k,l}\left(I, \varepsilon, \frac{I}{\varepsilon}\right) = \sum_{n \in \mathbb{Z}} f_{j,k,l,n}(I, \varepsilon)\, e^{in\omega I/\varepsilon} \tag{2.28}$$

The functions $f_{j,k,l,n}(I, \varepsilon)$ have analytic extensions $f_{j,k,l,n}(z, \varepsilon)$, and when $|z - I_-| < r_b$,

$$\sum_{j=1,2} \sum_{0 \leqslant l \leqslant k} \sum_{n \in \mathbb{Z}} |f_{j,k,l,n}(z, \varepsilon)| \leqslant \frac{M_a}{(\sigma/2)^k} \tag{2.29}$$

for $k \geqslant 1$ from (2.19). The coefficients before the linear terms have a factor ε due to the nature of F_2. Similarly, $Q_2 = (Q_{2,1}, Q_{2,2})^T$, where

$$Q_{2,j} = \sum_{n \in \mathbb{Z}} f_{j,0,n}(I, \varepsilon)\, e^{in\omega I/\varepsilon} \tag{2.30}$$

$f_{j,0,n}(I, \varepsilon)$ can be extended analytically to $f_{j,0,n}(z, \varepsilon)$, and

$$\sum_{n \in \mathbb{Z}, j=1,2} |f_{j,0,n}(z, \varepsilon)| \leqslant M \qquad \text{for} \quad |z - I_-| < r_b \tag{2.31}$$

We consider the solutions $Y(I, s, \varepsilon)$ of (2.26) which satisfy the initial conditions

$$Y(I, s, \varepsilon)|_{I=s} = \begin{pmatrix} Y_1(I, s, \varepsilon) \\ Y_2(I, s, \varepsilon) \end{pmatrix}\Bigg|_{I=s} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.32}$$

If we write $(W_1, W_2)^T = (F_{2,1}, F_{2,2})^T + \varepsilon(Q_{2,1}, Q_{2,2})^T$, then (2.26) can be expressed by the integral equations:

$$Y_j(I, s, \varepsilon) = \frac{1}{\varepsilon} \int_s^I e^{1/\varepsilon \int_J^I \lambda_j(\tau)\, d\tau}\, W_j\left(J, \varepsilon, \frac{J}{\varepsilon}, Y_1(J, s, \varepsilon), Y_2(J, s, \varepsilon)\right) dJ, \qquad \text{for} \quad j = 1, 2 \tag{2.33}$$

We intend to find the solutions of (2.33) of the form

$$Y_j(I, s, \varepsilon) = \varepsilon \sum_{n \in \mathbb{Z}} g_{j,n}(I, s, \varepsilon)\, e^{in\omega I/\varepsilon}, \qquad \text{for} \quad j = 1, 2 \tag{2.34}$$

with the property that $\sum_{j=1,2} \sum_{n \in \mathbb{Z}} |g_{j,n}(I, s, \varepsilon)| \leqslant M$.

Let us make some remarks here. In general, for the function $Y_j(I, s, \varepsilon)$, the series representation in the form of (2.34) is not unique. This issue, however, does not bear any effect on our problem. We intend to establish a vector equation for the coefficients $\{g_{j,n}\}$ in such a way that if $\{g_{j,n}\}$ satisfies that vector equation, then the corresponding Y_j will satisfy (2.33). The set of coefficients $\{g_{j,n}\}$ constructed in this way is unique because the above-mentioned vector equation has a unique solution. Further, the key of the problem is to expand $Y_j(I, s, \varepsilon)$ in a series of the form (2.34) whose coefficients $\{g_{j,n}\}$ have totally bounded analytic extensions as $\varepsilon \to 0^+$. The set of $\{g_{j,n}\}$ obtained through the vector equation can exactly do so. Other methods to expand $Y_j(I, s, \varepsilon)$ may lead to a different set of $\{g_{j,n}\}$ which may or may not have a totally bounded analytic extension.

We also note that (2.33) or, equivalently, (2.26) contains a pair of two equations which are conjugate to each other, i.e., $Y_1(I, s, \varepsilon) = \bar{Y}_2(I, s, \varepsilon)$ or $g_{2,n}(I, s, \varepsilon) = \bar{g}_{1,-n}(I, s, \varepsilon)$. Thus, it suffices to solve the equivalent equation

$$Y_1(I, s, \varepsilon) = \frac{1}{\varepsilon} \int_s^I e^{1/\varepsilon \int_J^I \lambda_1(\tau)\, d\tau}\, W_1\left(J, \varepsilon, \frac{J}{\varepsilon}, Y_1, \bar{Y}_1\right) dJ \tag{2.35}$$

where

$$\begin{aligned} W_1 = {} & \varepsilon \left(\sum_{n \in \mathbb{Z}} f_{1,1,1,n}(I, \varepsilon)\, e^{in\omega I/\varepsilon} \right) \left(\varepsilon \sum_{n \in \mathbb{Z}} g_{1,n} e^{in\omega I/\varepsilon} \right) \\ & + \varepsilon \left(\sum_{n \in \mathbb{Z}} f_{1,1,2,n}(I, \varepsilon)\, e^{in\omega I/\varepsilon} \right) \left(\varepsilon \sum_{n \in \mathbb{Z}} \bar{g}_{1,-n} e^{in\omega I/\varepsilon} \right) \\ & + \sum_{k=2}^{\infty} \sum_{1 \leqslant l \leqslant k+1} \left(\sum_{n \in \mathbb{Z}} f_{1,k,l,n}(I, \varepsilon)\, e^{in\omega I/\varepsilon} \right) \varepsilon^k \\ & \times \left(\sum_{n \in \mathbb{Z}} g_{1,n} e^{in\omega I/\varepsilon} \right)^{k-l+1} \left(\sum_{n \in \mathbb{Z}} \bar{g}_{1,-n} e^{in\omega I/\varepsilon} \right)^{l-1} \\ & + \varepsilon \left(\sum_{n \in \mathbb{Z}} f_{1,0,n}(I, \varepsilon)\, e^{in\omega I/\varepsilon} \right) \end{aligned} \tag{2.36}$$

We combine (2.36) into

$$W_1 = \sum_{n \in \mathbb{Z}} h_n(I, \varepsilon, g)\, e^{in\omega I/\varepsilon} + \sum_{n \in \mathbb{Z}} \varepsilon f_{1,0,n}(I, \varepsilon)\, e^{in\omega I/\varepsilon} \tag{2.37}$$

where $g = (g_{1,n}, n \in \mathbb{Z})$,

$$\begin{aligned} h_n(I. \varepsilon, g) &= \varepsilon^2 \sum_{n_1+n_2=n} f_{1,1,1,n_1} g_{1,n_2} + \varepsilon^2 \sum_{n_1+n_2=n} f_{1,1,2,n_1} \bar{g}_{1,-n_2} \\ &+ \sum_{k=2}^{\infty} \varepsilon^k \sum_{1 \leqslant l \leqslant k+1} \Bigg(\sum_{n_1+n_2+\cdots+n_{k+1}} f_{1,k,l,n_1} g_{1,n_2} \cdots \\ &\times g_{1,n_{k-l+2}} \bar{g}_{1,-n_{k-l+3}} \cdots \bar{g}_{1,-n_{k+1}} \Bigg) \end{aligned} \tag{2.38}$$

3. ANALYTIC EXTENSIONS OF SOLUTIONS

Delayed bifurcations are closely related to the analytic extensions of solutions. In fact, Shishkova [20] and Neishtadt[16] indicated delayed bifurcations by showing a bounded analytic extension of the solutions at a neighborhood of I_- in the complex plane where both the neighborhood and the bound are independent of $\varepsilon \to 0^+$. However, with the presence of periodic forcing, there is no such extension. Instead, we shall show that a solution can be expressed by a Fourier series of $e^{(in\omega I + im\omega s)/\varepsilon}$ whose coefficients can be extended analytically and uniformly bounded in ε, but the series itself may be divergent in the complex plane. The idea is the following: we express the right-hand side of (2.35) in a Fourier series of $e^{(in\omega I + im\omega s)/\varepsilon}$, $n \in \mathbb{Z}$, $m \in \mathbb{Z}$, and then Eq. (2.33) will be equivalent to the corresponding vector equation for the coefficients of the Fourier series. Let us introduce several technical lemmas. We denote $cl(D)$ the closure of a set D.

Lemma 3.1. *Let $h(z, \varepsilon)$ and $R(z)$ be analytic in $cl(D)$ for some open and connected region $D \subset \mathbb{C}$. Suppose T is a point in $cl(D)$ where $\mathrm{Re}\, R(z)$ attains its minimum in $cl(D)$. Assume that for any $z \in cl(D)$ there exists a path $\Gamma(z, T) \subset cl(D)$ such that $\Gamma(z, T)$ connects from T to z and $\mathrm{Re}\, R(z)$ is nondecreasing along $\Gamma(z, T)$. Then there exists $\tilde{h}(\cdot, D, \varepsilon)\colon cl(D) \to \mathbb{C}$ such that*

$$\int_{\xi}^{z} h(\tau, \varepsilon)\, e^{R(\tau)/\varepsilon}\, d\tau = \tilde{h}(z, D, \varepsilon)\, e^{R(z)/\varepsilon} - \tilde{h}(\xi, D, \varepsilon)\, e^{R(\xi)/\varepsilon} \tag{3.1}$$

for $z \in cl(D)$ *and* $\xi \in cl(D)$, *and* $\tilde{h}(z, D, \varepsilon)$ *satisfies the property that*

$$\|\tilde{h}(z, D, \varepsilon)\| \leqslant |\Gamma(z, T)| \; \|h(z, \varepsilon)\| \tag{3.2}$$

We denote $\mathbb{T}: (h, R, D) \to \tilde{h}(z, D, \varepsilon)$ *to be the transformation from* h *to* $\tilde{h}$. *Then (3.2) can also be rewritten as* $\|\mathbb{T}\| \leqslant \sup_{z \in cl(D)} |\Gamma(z, T)|$.

Further, if $R'(z) \neq 0$ *in* $cl(D)$, *then*

$$\begin{aligned}\tilde{h}(z, D, \varepsilon) = {} & \frac{\varepsilon}{R'(z)} h(z, \varepsilon) + (-1) \frac{\varepsilon^2}{R'(z)} \left(\frac{h(z, \varepsilon)}{R'(z)} \right)' \\ & + \cdots + (-1)^{m-1} \frac{\varepsilon^m}{R'(z)} h^{[m]}(z, \varepsilon) + O(\varepsilon^m) \\ & - \left[\frac{\varepsilon}{R'(T)} h(T, \varepsilon) + (-1) \frac{\varepsilon^2}{R'(T)} \left(\frac{h(T, \varepsilon)}{R'(T)} \right)' \right. \\ & \left. + \cdots + (-1)^{m-1} \frac{\varepsilon^m}{R'(T)} h^{[m]}(T, \varepsilon) \right] e^{1/\varepsilon (R(T) - R(z))}\end{aligned} \tag{3.3}$$

where $h^{[0]} = h(z, \varepsilon)$, $h^{[m]} = ((1/R'(z))\, h^{[m-1]}(z, \varepsilon))'$.

Proof. We define the function

$$\tilde{h}(z, D, \varepsilon) \equiv e^{-R(z)/\varepsilon} \left(\int_{\tau \in \Gamma(z, T)} h(\tau, \varepsilon)\, e^{R(\tau)/\varepsilon}\, d\tau \right) \tag{3.4}$$

It is easy to verify that $\tilde{h}(z, D, \varepsilon)$ satisfies (3.1). Since $Re\, R(z)$ is non-decreasing along $\Gamma(z, T)$, we obtain from (3.4) that $\|\tilde{h}\| \leqslant |\Gamma(z, T)| \; \|h(z, \varepsilon)\|$.

From integration by parts,

$$\begin{aligned}\int_{\tau \in \Gamma(z, T)} e^{R(\tau)/\varepsilon} h(\tau, \varepsilon)\, d\tau = {} & \tilde{h}^{\{m\}}(z, \varepsilon)\, e^{R(z)/\varepsilon} - \tilde{h}^{\{m\}}(T, \varepsilon)\, e^{R(T)/\varepsilon} \\ & + (-1)^m \varepsilon^m \int_{\tau \in \Gamma(z, T)} h^{[m+1]}(\tau, \varepsilon)\, e^{R(\tau)/\varepsilon}\, d\tau\end{aligned} \tag{3.5}$$

where

$$\begin{aligned}\tilde{h}^{\{m\}}(z, \varepsilon) = {} & \frac{\varepsilon}{R'(z)} h(z, \varepsilon) + (-1) \frac{\varepsilon^2}{R'(z)} \left(\frac{h(z, \varepsilon)}{R'(z)} \right)' \\ & + \cdots + (-1)^{m-1} \frac{\varepsilon^m}{R'(z)} h^{[m]}\end{aligned} \tag{3.6}$$

We denote the last term of (3.5) to be $\tilde{R}^{\{m\}}$. We then obtain

$$|\tilde{R}^{\{m\}}(z, D, \varepsilon)\, e^{-R(z)/\varepsilon}| \leqslant \varepsilon^m \left| \int_{\tau \in \Gamma(z, T)} h^{[m+1]}(\tau, \varepsilon)\, e^{(R(\tau)-R(z))/\varepsilon}\, d\tau \right|$$
$$\leqslant \varepsilon^m \, \|h^{[m+1]}(z, \varepsilon)\| = O(\varepsilon^m)$$

because $Re(R(\tau)-R(z)) \leqslant 0$ for $\tau \in \Gamma(z, T)$. This implies (3.3). □

Remark. The Inequality (3.2) is an estimate of $\tilde{h}$ which does not involve the derivatives of h and is useful for later arguments.

Proposition 3.2. *Let $s_0 < I_-$ be any point sufficiently close to I_-. Assume that ξ is a point in the upper half of the complex z-plane, such that $Re \int_{s_0}^{\xi} \lambda_1(\tau)\, d\tau = 0$, $Re\, \xi \leqslant I_-$, $Im\, \xi \geqslant 0$, i.e., ξ and s_0 are on the same level curve of $\phi_1(z) = Re \int_{I_-}^{z} \lambda_1(\tau)\, d\tau$ in the upper complex plane and to the left of the line $Re\, z = I_-$. Then there exists a nonempty region B_ξ which satisfies the following properties. (1) $\xi \in cl(B_\xi)$. (2) B_ξ is symmetric with respect to the real axis. (3) Let $z = T_n \in cl(B_\xi)$ be a point where $Re\, (\int_{\xi}^{z} \lambda_1(\tau)\, d\tau - in\omega z)$ attains a maximum in $cl(B_\xi)$. For every point $z \in cl(B_\xi)$, $n \in \mathbb{Z}$ there exists $\Gamma_n(z, T_n) \subset cl(B_\xi)$ which connects from T_n to z such that $Re(\int_{\xi}^{y} \lambda_1(\tau)\, d\tau - in\omega y)$ is monotone decreasing as y moves from T_n to z along the path $\Gamma_n(z, T_n)$. (4) $Re \int_{s_0}^{z} \lambda_1(\tau)\, d\tau \leqslant 0$ for $z \in B_\xi$. (5)* $\max_{z \in B_\xi} |\Gamma(z, T_n)|$ *goes to zero as $s_0 \to I_-$.*

Proof. The region B_ξ is constructed by the following considerations. Let Γ_{s_0} be the level curve of $\phi_1(z) = Re \int_{I_-}^{z} \lambda_1(\tau)\, d\tau$ which intersects the real axis at $s_0 < I_-$ and $s_0^* > I_-$. See Fig. 3 or 4. The properties of these level curves with different s_0 are well known from Refs. 16 and 22. Let $B_0 = \{z \mid Re\, z \leqslant I_-,\ z$ s between Γ_{s_0} *and* $\bar{\Gamma}_{s_0}\}$ where $\bar{\Gamma}_{s_0}$ is Γ_{s_0}'s conjugate image. When $s_0 < I_-$ is close enough to I_-, B_0 is a region close to I_- as well. With the fact that for $s < I_-$, $\int_s^z \lambda_1(\tau)\, d\tau - in\omega z = \int_s^z (\lambda_1(\tau) - in\omega)\, d\tau - in\omega s$ and $Re(\int_s^z \lambda_1(\tau)\, d\tau - in\omega z) = Re \int_s^z (\lambda_1(\tau) - in\omega)\, d\tau$, we need to examine the behavior of $\lambda_1(z) - in\omega$ and the corresponding level curves of $\varphi_n = Re \int_{I_-}^{z} (\lambda_1(\tau) - in\omega)\, d\tau$. From the fact that $Re\, \lambda_1(I_-) = 0$, $Im\, \lambda_1(I_-) \neq 0$, we observe that when $z = I$ is on the real axis and near I_-,

$$\lambda_1(I) = a_1(I - I_-) + O((I - I_-)^2) + i(\omega_0 + O(I - I_-)) \tag{3.7}$$

where

$$a_1 = \frac{\partial}{\partial I} Re\, \lambda_1(I)|_{I=I_-} > 0; \qquad \omega_0 = Im\, \lambda_1(I)|_{I=I_-} < 0 \tag{3.8}$$

Thus if we express $z - I_- = x + iy$, then the level curves $Re \int_s^z (\lambda_1 - in\omega)\, d\tau = c$ are essentially in the form of parabolas with high order terms:

$$\tfrac{1}{2} a_1 x^2 - (\omega_0 - n\omega)\, y + O(x^3, y^2) = C \tag{3.9}$$

where $\omega_0 - n\omega \neq 0$ as follows from H4. When $|z - I_-|$ is sufficiently small, all level curves of $\varphi_n(z)$ are classified into two types. For $\omega_0 - n\omega > 0$, the level curves are parabolas opening upward (convex). For $\omega_0 - n\omega < 0$, they are downward (concave). Let $n_0 < 0$ be the unique integer such that $\omega_0 - n\omega < 0$ for $n \geqslant n_0$ and $\omega_0 - n\omega > 0$ for $n \leqslant n_0 - 1$.

We state a sublemma which we later refer as "the rule of the convexity."

Sublemma. *Let assumptions H1–H5 hold. Let $z = z(x)$ be a level curve of $\varphi_n(z) = Re \int_s^z (\lambda_1 - in\omega)\, d\tau$ for $n \in \mathbb{Z}$. There exists $r_0 > 0$ such that for any $0 < r \leqslant r_0$, if $z_1, z_2 \in B_r(I_-)$ and $Re\, z_1 \leqslant Re\, z_2$, and $\varphi_n(z_1) = \varphi_n(z_2)$, then for any x: $Re(z_1 - I_-) \leqslant x \leqslant Re(z_2 - I_-)$, the level curve $z \equiv z(x) = I_- + x + iy(x)$ of $\varphi_n(z)$ which passes through z_i lies inside the ball $B_{2r}(I_-)$ that is uniform for any $n \in \mathbb{Z}$. Furthermore, for any positive number $N > 0$, there exists $r_1 \equiv r_1(N) > 0$ such that for any $0 < r \leqslant \min(r_0, r_1)$, if $\{z_i, i = 1, 2\} \in B_r(I_-)$ belong to the level curve Γ_1 of $\varphi_n(z)$ as well as Γ_2 of $\varphi_m(z)$ for some $m, n \in \mathbb{Z}$, then one of the following three cases occurs. (a) If $0 < \omega_0 - n\omega < \omega_0 - m\omega$, and $|\omega_0 - n\omega| \leqslant N$ for some $N > 0$, then Γ_1 is above Γ_2 for $Re\, z_1 \leqslant Re\, z \leqslant Re\, z_2$. (b) If $\omega_0 - n\omega < \omega_0 - m\omega < 0$, and $|\omega_0 - n\omega| \leqslant N$ for some $N > 0$, then Γ_1 is below Γ_2 for $Re\, z_1 \leqslant Re\, z \leqslant Re\, z_2$. (c) If $\omega_0 - n\omega < 0 < \omega_0 - m\omega$, then Γ_1 is above Γ_2 for $Re\, z_1 \leqslant Re\, z \leqslant Re\, z_2$.*

Proof. Let $z = I_- + x + iy$ be a point of the level curve Γ of $\varphi_n(z)$ which passes z_1 and z_2. Then (x, y) satisfies the relation (3.9). Now since $z_i \in B_r(I_-)$, $i = 1, 2$, $|C| \leqslant |\omega_0 - n\omega|\, r + M_0 r^2$ for some constant $M_0 > 0$.

We assert that there exists r_0 such that for $r \leqslant r_0$, Γ is inside $B_{2r}(I_-)$ for $x \in [x_1, x_2]$, i.e., $|z - I_-| \leqslant 2r$. Otherwise, let x_m be such a point that $|z(x)| < 2r$ for $x_1 \leqslant x < x_m$ and $|z(x_m) - I_-| = 2r$. From (3.9), which is rewritten as

$$\tfrac{1}{2} a_1 x^2 - (\omega_0 - n\omega)\, y + x^3 \Theta_1(x, y) + y^2 \Theta_2(x, y) = C \tag{3.9'}$$

for some Θ_1, Θ_2 where $|\Theta_1| \leqslant M_0$, $|\Theta_2| \leqslant M_0$ when $|z - I_-| \leqslant 2r$, we get that $(|\omega_0 - n\omega| - 2M_0 r)\, |y| \leqslant C + 2a_1 r^2 + M_0 r^3 \leqslant |\omega_0 - n\omega|\, r + (M_0 + 2a_1)\, r^2 + M_0 r^3$. Thus when $r \leqslant r_0 \equiv \min(|\omega_0 - n\omega|/(8M_0 + 16a_1), 1/8, 1/(4M_0)) > 0$, $|y| \leqslant 3r/2$, and consequently $|z - I_-| < 2r$ for $x_1 \leqslant x \leqslant x_m$ which conflicts with the definition of x_m. This implies that the point x_m does not exist and $|z(x)-I_-|<2r$ for $x_1 \leqslant x \leqslant x_2$. Further, by taking the derivative for x once in (3.9′), we derive

$$a_1 x-(\omega_0-n\omega)\, y'+3x^2\Theta_1+x^3\Theta_1'+2yy'\Theta_2+y^2\Theta_2'=0 \qquad (3.9'')$$

where Θ_i' represents the total derivative of the function to the x-variable. Then $|y'| \leqslant C_2 r$ for $r \leqslant r_1$. Finally, by taking one more derivative for x in (3.9″), we get

$$\begin{aligned} a_1-(\omega_0-n\omega)\, y''+6x\Theta_1+6x^2\Theta_1'+x^3\Theta_1'' \\ +2yy''\Theta_2+2y'^2\Theta_2+4y'y\Theta_2'+y^2\Theta_2''=0 \end{aligned}$$

Consequently, $y''=(a_1+O(r))/(\omega_0-n\omega)$. Let us look only at case (a), where $0<\omega_0-n\omega<\omega_0-m\omega$. Other cases are similar. If $z=I_-+x+iy_j(x)\in\Gamma_j$ for $j=1, 2$, then for any sufficient large $M_0>0$ which is uniform for all $r \leqslant r_0$ and $n\in\mathbb{Z}$,

$$y_1''=\frac{a_1+O(r)}{\omega_0-n\omega} \geqslant \frac{a_1-M_0 r}{\omega_0-n\omega}$$

$$y_2''=\frac{a_1+O(r)}{\omega_0-m\omega} \leqslant \frac{a_1+M_0 r}{\omega_0-m\omega}$$

Let $r^0 \equiv a_1/M_0 \inf_{x \geqslant (N+\omega)/N}(x-1)/(x+1)$. Since $|\omega_0-n\omega| \leqslant N$, we find that when

$$r \leqslant r^0 \leqslant \frac{a_1}{M_0}\frac{[(\omega_0-m\omega)/(\omega_0-n\omega)]-1}{[(\omega_0-m\omega)/(\omega_0-n\omega)]+1}=\frac{a_1}{M_0}\frac{(\omega_0-m\omega)-(\omega_0-n\omega)}{(\omega_0-m\omega)+(\omega_0-n\omega)}$$

$y_1''(x) \geqslant y_2''(x)$. Since $y_1(x_j)=y_2(x_j)$ for $j=1, 2$, $y_1(x) \leqslant y_2(x)$ for $x_1 \leqslant x \leqslant x_2$ by the maximum principle of the differential equations. This completes the proof of the sublemma.

Remark. In the remainder of this paper, we take $N=\omega_0$. Since the comparison of two levels curve in the section below (where we refer to the rule of the convexity) will always contain at least one level curves of φ_n with $|\omega_0-n\omega| \leqslant \omega_0$. Thus they satisfy the requirement about the constant N.

We define $\sum_n$ to be the set of all level curves of $\varphi_n(z)=Re\int_s^z(\lambda_1-in\omega)\,d\tau$. The level curves for $Re\int_s^z(\lambda_1(\tau)-in_0\omega)\,d\tau$ which compose $\sum_{n_0}$ are the most concave in the sense that if $\Gamma_1\in\sum_{n_0}$ intersects $\Gamma_2\in\sum_n$, which is the set of all level curves for $Re\int_s^z(\lambda_1(\tau)-in\omega)\,d\tau$, $n\neq n_0$, in two points z_1 and z_2 in the neighborhood of I_-, then for z between z_1 and z_2, Γ_1 is above Γ_2 from the rule of the convexity stated above. See Fig. 2. Similarly, $\sum_{n_0-1}$ are the most convex meaning if $\Gamma_1 \in \sum_{n_0-1}$, $\Gamma_2 \in \sum_n$ for $n \neq n_0 - 1$, and if $\Gamma_1 \cap \Gamma_2 = \{z_1, z_2\}$, Γ_1 is below Γ_2 between (z_1, z_2) also by the sublemma above.

By the nonresonance conditions H4 and H5, $|\omega_0 - n_0\omega| \neq |(\omega_0 - (n_0 - 1)\omega)|$. Thus either (a) $|\omega_0 - n_0\omega| < |\omega_0 - (n_0 - 1)\omega|$ or (b) $|\omega_0 - n_0\omega| > |\omega_0 - (n_0 - 1)\omega|$. Suppose (a) holds. Then when z is close to I_-, the level curves $\sum_{n_0}(z)$ where $z - I_- = x + iy$,

$$\tfrac{1}{2}a_1 x^2 - (\omega_0 - n_0\omega)\, y + O(x^3, y^2) = C \tag{3.10}$$

are the most concave ones in the set of all level curves and their corresponding conjugate images. The conjugate images of $\sum_{n_0}(z)$ written as $\overline{\sum}_{n_0}(z) = \overline{\sum}_{n_0}(x, y)$, which have the expressions

$$\tfrac{1}{2}a_1 x^2 - (\omega_0 - n_0\omega)(-y) + O(x^3, (-y)^2) = C \tag{3.11}$$

are the most convex ones since $\overline{\sum}_{n_0}$ are more convex than $\sum_{n_0-1}$. Assume (b) holds. Then $\sum_{n_0-1}$ are the most convex, and $\overline{\sum}_{n_0-1}$ are the most concave.

By nonresonance conditions H4 and H5,

$$\omega_0 - n\omega \neq -(\omega_0 - m\omega) \tag{3.12}$$

for $m, n \in \mathbb{Z}$. Therefore, we can arrange all $\sum_n$ in an order according to the magnitudes $|\omega_0 - n\omega|$. When z is sufficiently close to I_-, all $\sum_n$ are obeying the rule of convexity and concavity in the order of the magnitudes $|\omega_0 - n\omega|$. We may use the notations $n_1, n_2, ..., n_n, ...$ to express the order of the curves according to the magnitudes, and $\sum^{(1)}, \sum^{(2)}, ..., \sum^{(n)}, ...$ to express the sets of the level curves in such an order. For example, in case (a), $\sum^{(1)} = \sum_{n_0}, ..., \sum^{(k)} = \sum_0$, for some k.

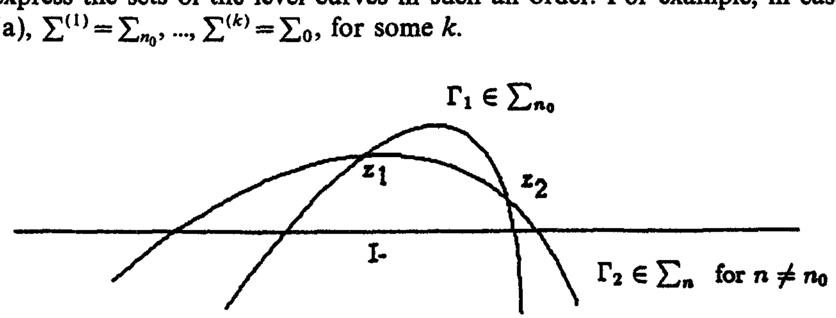

Fig. 2. The convexity argument. If Γ_1 and Γ_2 intersect at z_1 and z_2 near I_-, then Γ_1 is above Γ_2 when z is between z_1 and z_2.

We define that an open set B is $\sum^{(n)}$ accessible if, given $z \in cl(B)$, there exists a path $\Gamma_n(z, T_n) \subset cl(B)$ from T_n to z such that $Re \int_s^y (\lambda_1(\tau) - in\omega)\, d\tau$ is monotone decreasing as y moves from T_n to z along $\Gamma_n(z, T_n)$, where T_n is the maximum point of $Re \int_s^z (\lambda_1(\tau) - in\omega)\, d\tau$ in $cl(B)$.

We also note the very crucial point that if an open and connected region D is not $\sum^{(n)}$ accessible, then there must be a value c such that the set $L(c) = \{z \in D \mid Re(\int_s^z \lambda_1(\tau)\, d\tau - in\omega z) = c\}$ is not connected. The argument is very simple. Assume this assertion is false. Since D is connected, there is a curve Γ which connects any z with T_n. If $\varphi_n(z) = Re(\int_s^z \lambda_1(\tau) - in\omega d\tau)$ is not monotone in Γ, then there exist z_1 and z_2 in Γ such that $\varphi_n(z_1) = \varphi_n(z_2)$ and $\varphi_n(z)$ is monotone decreasing in the segments from T_n to z_1, from z_2 to z, but not between z_1 and z_2. We modify Γ to $\tilde{\Gamma}$ by replacing the segment between z_1 and z_2 with the level curve $L = \{z \in D \mid \varphi_n(z) = \varphi_n(z_1)\}$. Then $\tilde{\Gamma}$ is of the property that $\varphi_n(z)$ is monotone decreasing on $\tilde{\Gamma}$ which contradicts the fact that D is not $\sum^{(n)}$ accessible.

Thus if B_0 is $\sum^{(n)}$ accessible, and we cut off portions of B_0 without disconnecting any of $\Gamma \in \sum^{(n)}$ in the sense that the new set $B_1 \subset B_0$ has the property $(\partial B_1 - \partial B_0) \cap \Gamma$ is either empty or a single point, then B_1 is $\sum^{(n)}$ accessible.

We now construct the region B_ξ in the following way.

The level curve of $\phi_1(z) = Re \int_{I_-}^z \lambda_1(\tau)\, d\tau$ which passes through $z = s_0$: $\Gamma_{s_0}(z)$ and its conjugate image $\bar{\Gamma}_{s_0}$ bound the region B_0 as assumed. We note that $\sum_0 = \sum^{(k)}$ as classified, and there are only finitely many families

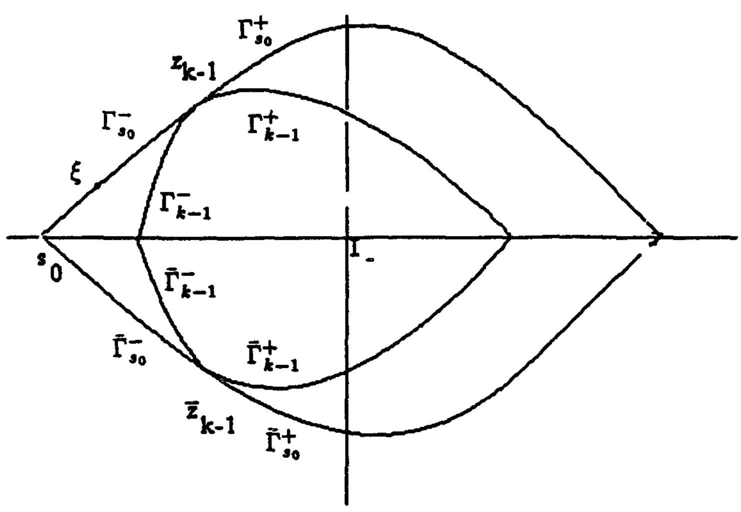

Fig. 3. The construction of B_1. We cut off part of B_0 so that B_1 is $\sum^{(k-1)}$ accessible. When ξ is below z_{k-1}, we take $\Gamma_{s_0}^-$, $\bar{\Gamma}_{s_0}^-$, Γ_{k-1}^+ and $\bar{\Gamma}_{k-1}^+$ as the boundaries of the region B_1.

of $\sum^{(n)}$ which are in front of $\sum_0$ in the order which was described previously. For $n \geq k+1$, B_0 is $\sum^{(n)}$ accessible because of the order in which they are arranged (those level curves are more "flat"). However, B_0 may not be accessible for some $\sum^{(n)}$ for $0 < n \leq k-1$.

We choose the level curve Γ_{k-1} of $Re \int_{I_-}^{z} (\lambda_1(\tau) - in_{k-1}\omega)\, d\tau$ which is tangent to Γ_{s_0} or $\bar{\Gamma}_{s_0}$. Suppose Γ_{s_0} and Γ_{k-1} intersects at $z = z_{k-1}$, we name the portion of Γ_{k-1} left of z_{k-1} as Γ_{k-1}^-, and the portion right of z_{k-1} as Γ_{k-1}^+. Correspondingly, we have $\bar{\Gamma}_{k-1}^-$ and $\bar{\Gamma}_{k-1}^+$. Similarly, we can define $\Gamma_{s_0}^\pm$, $\bar{\Gamma}_{s_0}^\pm$. Define in case (a), when $\xi \in \Gamma_{s_0}$ is to the left of z_{k-1}, B_1 to be the region bounded by $\Gamma_{s_0}^-$, $\bar{\Gamma}_{s_0}^-$, Γ_{k-1}^+ and $\bar{\Gamma}_{k-1}^+$; and in case (b), when $\xi \in \Gamma_{s_0}$ is to the right of z_{k-1}, B_1 to be the region bounded by $\Gamma_{s_0}^+$, $\bar{\Gamma}_{s_0}^+$, Γ_{k-1}^- and $\bar{\Gamma}_{k-1}^-$. See Figs. 3 and 4.

It is obvious that B_1 is accessible for $\sum^{(k-1)}$ because of the way in which we construct B_1 by using several pieces of level curves in $\sum^{(k-1)}$ to form the boundary (namely, Γ_{k-1}^+, $\bar{\Gamma}_{k-1}^+$ or Γ_{k-1}^-, $\bar{\Gamma}_{k-1}^-$). Also, B_1 is accessible for $\sum^{(n)}$, $n \geq k$. This comes from the fact that $\Gamma \cap (\partial B_1 - \partial B_0)$ is empty or a single point for any $\Gamma \in \sum^{(n)}$, $n \geq k$, which follows from the convexity argument. This completes the construction of B_1.

We then follow finitely many steps to find Γ_{k-2}, $\bar{\Gamma}_{k-2}$ which are tangent to ∂B_1, and let $\Gamma_{k-2}^\pm$, $\bar{\Gamma}_{k-2}^\pm$ and ∂B_1 become the new boundaries of B_2 which cut off B_1, and so on. The details of the constructions of $B_2, ..., B_k$ are similar to that of B_1 but lengthy, hence they are not presented. Eventually, we can get B_k which is accessible for all $\sum^{(n)}$, $n \in \mathbb{N}$.

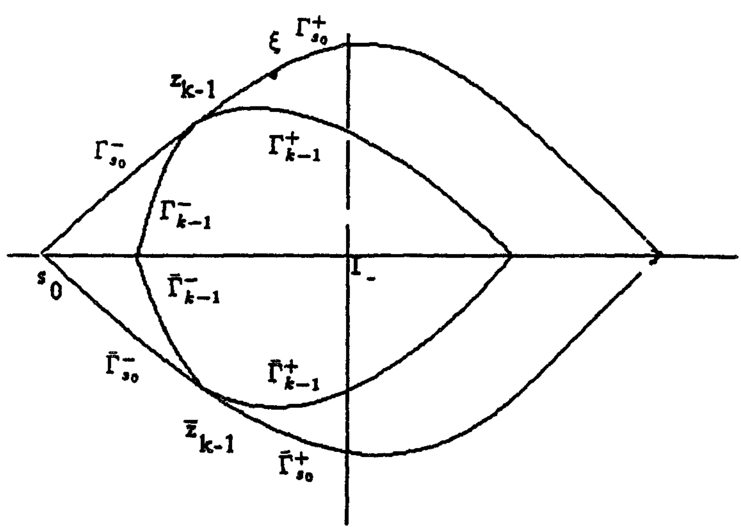

Fig. 4. The construction of B_1. We cut off part of B_0 so that B_1 is $\sum^{(k-1)}$ accessible. When ξ is above z_{k-1}, we take $\Gamma_{s_0}^+$, $\bar{\Gamma}_{s_0}^+$, Γ_{k-1}^- and $\bar{\Gamma}_{k-1}^-$ as the boundaries of the region B_1.

B_k intersects the real axis between two points $z = s_1(\xi) < I_-$ and $z = I_-$. Thus $B_\xi \equiv B_k$ is the desired set which satisfies all the properties. □

Lemma 3.3. *Let the assumptions H1–H5 hold. Then the solution $y(I, s, \varepsilon)$ of (2.16) with the initial condition $y(s, s, \varepsilon) = 0$ has the property that $y(I, s, \varepsilon) = Q(I)\binom{Y_1}{\bar{Y}_1}$ and*

$$Y_1(I, s, \varepsilon) = \varepsilon \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} g_{1,n,m}(I, s, \varepsilon)\, e^{in\omega I/\varepsilon} e^{im\omega s/\varepsilon} \tag{3.13}$$

for some coefficients $\{g_{1,n,m}\}$. There exists a neighborhood N_1 of I_- in the complex plane such that for any $\xi \in N_1$, the functions $g_{1,n,m}(I, s, \varepsilon)$ have analytic extensions $g_{1,n,m}(z, \xi, \varepsilon)$ in z for $z \in B_\xi$, where B_ξ was defined in Proposition 3.2. Further, there exists a constant M independent of $\varepsilon \to 0$ such that for $\xi \in N_1$, $z \in cl(B_\xi) \equiv B_\xi \cup \partial B_\xi$,

$$\sum_{n \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} \sup_{\xi \in N_1,\, z \in B_\xi} |g_{1,n,m}(z, \xi, \varepsilon) \leqslant M \tag{3.14}$$

For fixed $z \in \cap_{\xi \in N_1} B_\xi$, $g_{1,n,m}(z, \xi, \varepsilon)$ are analytic in ξ.

Remark. The coefficients $\{g_{1,n,m}\}$ depend upon variables I, s, ε, and the decomposition of $Y_1(I, s, \varepsilon)$ into series (3.13) is not unique. Lemma 3.3, however, allows us to find a set of coefficients $g_{1,n,m}(z, \xi, \varepsilon)$ which has bounded analytic extensions as $\varepsilon \to 0^+$.

Proof. Let $\mathbb{T}: (h, R, D) \to \tilde{h}$ be as denoted. Denote $\tilde{h}_n(z, B_\xi, \varepsilon) = \mathbb{T}(h_n, \int_z^{I_-} \lambda_1(\tau)\, d\tau + in\omega z, B_\xi)$ and $\tilde{f}_{1,0,n}(z, B_\xi, \varepsilon) = \mathbb{T}(f_{1,0,n}(z, \varepsilon), \int_z^{I_-} \lambda_1(\tau)\, d\tau + in\omega z, B_\xi)$.

From (2.35)–(2.37), using the facts

$$\int_s^I e^{1/\varepsilon \int_J^I \lambda_1(\tau)\, d\tau} h_n(J, \varepsilon)\, e^{in\omega J/\varepsilon}\, dJ$$
$$= \tilde{h}_n(I, B_\xi, \varepsilon)\, e^{in\omega I/\varepsilon} - \tilde{h}_n(s, B_\xi, \varepsilon)\, e^{1/\varepsilon \int_s^I \lambda_1(\tau)\, d\tau} e^{in\omega s/\varepsilon}$$

as well as

$$\int_s^I e^{1/\varepsilon \int_J^I \lambda_1(\tau)\, d\tau} f_{1,0,n}(J, \varepsilon)\, e^{in\omega J/\varepsilon}\, dJ$$
$$= \tilde{f}_{1,0,n}(I, B_\xi, \varepsilon)\, e^{in\omega I/\varepsilon} - \tilde{f}_{1,0,n}(s, B_\xi, \varepsilon)\, e^{1/\varepsilon \int_s^I \lambda_1(\tau)\, d\tau} e^{in\omega s/\varepsilon}$$

we obtain the equivalent equations of (2.35):

$$Y_1 = H_0 + \sum_{n \neq 0} H_n e^{in\omega I/\varepsilon} + R_{1,0} + \sum_{n \neq 0} R_{1,n} e^{in\omega I/\varepsilon} \tag{3.15}$$

where

$$H_0 = \frac{1}{\varepsilon} \left\{ -\sum_{n \in \mathbb{Z}} \tilde{h}_n(s, B_s, \varepsilon)\, e^{1/\varepsilon \int_s^I \lambda_1(\tau)\, d\tau} e^{in\omega s/\varepsilon} + \tilde{h}_0(I, B_s, \varepsilon) \right\} \tag{3.16a}$$

$$H_n = \frac{1}{\varepsilon} \tilde{h}_n(I, B_s, \varepsilon) \qquad \text{for} \quad n \neq 0 \tag{3.16b}$$

$$R_{1,0} = \left\{ -\sum_{n \in \mathbb{Z}} \tilde{f}_{1,0,n}(s, B_s, \varepsilon)\, e^{1/\varepsilon \int_s^I \lambda_1(\tau)\, d\tau} e^{in\omega s/\varepsilon} + \tilde{f}_{1,0,0}(I, B_s, \varepsilon) \right\} \tag{3.16c}$$

$$R_{1,n} = \tilde{f}_{1,0,n}(I, B_s, \varepsilon) \qquad \text{for} \quad n \neq 0 \tag{3.16d}$$

Let $H = (H_n, n \in \mathbb{Z})$, and $R = (R_{1,n}, n \in \mathbb{Z})$. Then a solution of the vector equation

$$g = \frac{1}{\varepsilon}(H(g) + R) \tag{3.17}$$

corresponds to the solution of (2.35), which can be expressed in a Fourier series form as

$$\sum_{n \in \mathbb{Z}} \varepsilon g_{1,n} e^{in\omega I/\varepsilon} = \sum_{n \in \mathbb{Z}} (H_n + R_{1,n})\, e^{in\omega I/\varepsilon}$$

If we further assume that $g_{1,n}$ can be decomposed as

$$g_{1,n} = \sum_{m \in \mathbb{Z}} g_{1,n,m}(I, s, \varepsilon)\, e^{im\omega s/\varepsilon} \tag{3.18}$$

then (3.17) is equivalent to

$$g = \hat{L}(g) \equiv \frac{1}{\varepsilon} \hat{H} + \frac{1}{\varepsilon} \hat{R} \tag{3.19}$$

Here $g = (g_{1,n,m}(I, s, \varepsilon))$, $\hat{H} = (\hat{H}_{n,m}, n \in \mathbb{Z}, m \in \mathbb{Z})$, where $\hat{H}_{n,m} = \theta_m(H_n)$, $\hat{R} = (\hat{R}_{n,m}, n \in \mathbb{Z}, m \in \mathbb{Z})$, where $\hat{R}_{n,m} = \theta_m(R_{1,n})$; and θ_m is defined as the formal projection mapping:

$$\theta_m \colon \sum_{k \in \mathbb{Z}} \alpha_k e^{ik\omega s/\varepsilon} \to \alpha_m \tag{3.20}$$

Note that θ_m is only a formal mapping because the well definedness of θ_m depends on the unique identification of α_k, which is difficult for any general function of s. The mapping θ_m is well defined here only in the sense that once the decomposition of g is assumed, then H_n, $R_{1,n}$ can be expressed uniquely according to the order of $e^{in\omega s/\varepsilon}$ in the natural way. The functions $\theta_m(H_n)$ and $\theta_m(R_{1,n})$ are simply to denote the corresponding coefficients. It can be verified that θ_m is commutative with $\mathbb{T}$. Thus for $n \neq 0$,

$$\hat{H}_{n,m} = \theta_m(H_n) = \frac{1}{\varepsilon}\mathbb{T}\left(\theta_m(h_n), \int_z^{I_-} \lambda_1(\tau)\, d\tau + in\omega z, B_\xi\right)(I, B_s, \varepsilon) \tag{3.21a}$$

$$\hat{R}_{n,m} = \theta_m(R_{1,n}) = \begin{cases} \tilde{f}_{1,0,n}(I, B_s, \varepsilon) & \text{for} \quad m = 0 \\ 0 & \text{for} \quad m \neq 0 \end{cases} \tag{3.21b}$$

$$\begin{aligned} \hat{H}_{0,m} &= \frac{1}{\varepsilon}\theta_m\left\{\tilde{h}_0(I, B_s, \varepsilon) - \sum_{k \in \mathbb{Z}} \tilde{h}_k(s, B_s, \varepsilon)\, e^{1/\varepsilon \int_s^I \lambda_1(\tau)\, d\tau} e^{ik\omega s/\varepsilon}\right\} \\ &= \frac{1}{\varepsilon}\left\{\mathbb{T}\left(\theta_m(h_0), \int_z^{I_-} \lambda_1(\tau)\, d\tau + in\omega z, B_\xi\right)(I, B_s, \varepsilon)\right. \\ &\quad \left. - e^{1/\varepsilon \int_s^I \lambda_1(\tau)\, d\tau} \sum_{k \in \mathbb{Z}} \mathbb{T}\left(\theta_{m-k}(h_k), \int_z^{I_-} \lambda_1(\tau)\, d\tau + in\omega z, B_\xi\right)(s, B_s, \varepsilon)\right\} \end{aligned} \tag{3.21c}$$

$$\begin{aligned} \hat{R}_{0,m} &= \theta_m\left(\tilde{f}_{1,0,0}(I, B_s, \varepsilon) - e^{1/\varepsilon \int_s^I \lambda_1(\tau)\, d\tau} \sum_{n \in \mathbb{Z}} \tilde{f}_{1,0,n}(s, B_s, \varepsilon)\, e^{in\omega s/\varepsilon}\right) \\ &= \begin{cases} (\tilde{f}_{1,0,0}(I, B_s, \varepsilon) - \tilde{f}_{1,0,0}(s, B_s, \varepsilon)\, e^{1/\varepsilon \int_s^I \lambda_1(\tau)\, d\tau}) & \text{for} \quad m = 0 \\ -(\tilde{f}_{1,0,m}(s, B_s, \varepsilon)\, e^{1/\varepsilon \int_s^I \lambda_1(\tau)\, d\tau}) & \text{for} \quad m \neq 0 \end{cases} \end{aligned} \tag{3.21d}$$

For each $\xi \in \mathbb{C}$ which satisfies $Re\,\xi < I_-$ and $Im\,\xi \geqslant 0$, we can find the point $s_0 \equiv s_0(\xi) \leqslant I_-$ in the real line such that $Re \int_{s_0}^{\xi} \lambda_1(\tau)\, d\tau = 0$, i.e, ξ and s_0 are on the same level curve of $\phi_1(z) \equiv Re \int_{I_-}^{z} \lambda_1(\tau)\, d\tau$.

To find the analytic functions $g_{1,n,m}(z, \xi, \varepsilon)$ which extend $g_{1,n,m}(I, s, \varepsilon)$, we assume that $g_{1,n,m}(z, \xi, \varepsilon)$ take the series form of

$$g_{1,n,m}(z, \xi, \varepsilon) = \sum_{l \in \mathbb{Z}} g_{1,n,m,l}(z, \xi, \varepsilon, s_0)\, e^{l/\varepsilon \int_{s_0}^{\xi} \lambda_1(\tau)\, d\tau} \tag{3.22}$$

for each ξ satisfying $Re\,\xi < I_-$, $Im\,\xi > 0$, $|\xi - I_-| \leqslant \min(r_0, r^0)$, and $z \in B_\xi$. Since (2.37) and (3.19) contain $\bar{g}_{1,n,m}$ which also need to be extended, we find the conjugate terms in the form

$$\bar{g}_{1,-n,-m}(\bar{z}, \bar{\xi}, \varepsilon) = \sum_{l \in \mathbb{Z}} \bar{g}_{1,-n,-m,-l}(\bar{z}, \bar{\xi}, \varepsilon, s_0)\, e^{-l/\varepsilon \int_{s_0}^{\xi} \lambda_1(\tau)\, d\tau} \tag{3.23}$$

for $n \in \mathbb{Z}$, $m \in \mathbb{Z}$, $l \in \mathbb{Z}$. The indices $(-l)$ of $\bar{g}_{1,-n,-m,-l}$ are due to the fact $e^{l/\varepsilon \int_{s_0}^{\xi} \lambda_1(\tau)\, d\tau} = e^{-l/\varepsilon \int_{s_0}^{\xi} \lambda_1(\tau)\, d\tau}$ since $\int_{s_0}^{\xi} \lambda_1(\tau)\, d\tau$ is purely imaginary when ξ is on the same level curve as s_0.

We denote the formal projection operator β_l:

$$\beta_l\colon \sum_{k \in \mathbb{Z}} \alpha_k e^{k/\varepsilon \int_{s_0}^{\xi} \lambda_1(\tau)\, d\tau} \to \alpha_l \tag{3.24}$$

Again similar to the case of θ_m, for any function $y(\xi)$ which can be uniquely represented as $\sum_{k \in \mathbb{Z}} \alpha_k(\xi)\, e^{k/\varepsilon \int_{s_0}^{\xi} \lambda_1(\tau)\, d\tau}$, $\beta_l\, y(\xi) \to \alpha_l(\xi)$ is well defined. For the functions concerned below, once (3.22) is assumed, then they can be all expressed uniquely in terms of the series of $e^{k/\varepsilon \int_{s_0}^{\xi} \lambda_1(\tau)\, d\tau}$ in the natural way. Also, β_l are commutative with $\mathbb{T}$.

The analytic extensions of $g_{1,n,m,l}(z, \xi, \varepsilon, s_0)$ in (3.19) will lead to a delay equation which contains $g_{1,n,m,l}(z, \xi, \varepsilon, s_0)$ as well as $g_{1,-n,-m,-l}$ $(\bar{z}, \bar{\xi}, \varepsilon, s_0)$. For our convenience, rather than considering a delay equation for all ξ with $Im\,\xi \geqslant 0$, we study a system of two vector equations:

$$g_{1,n,m,l}(z, \xi, \varepsilon, s_0) = \frac{1}{\varepsilon} \hat{H}_{s_0,n,m,l} + \frac{1}{\varepsilon} \hat{R}_{s_0,n,m,l}, \tag{3.25a}$$

$$g_{1,n,m,l}(z, \bar{\xi}, \varepsilon, s_0) = \frac{1}{\varepsilon} \check{H}_{s_0,n,m,l} + \frac{1}{\varepsilon} \check{R}_{s_0,n,m,l} \tag{3.25b}$$

where $\hat{H}_{s_0,n,m,l}$, $\hat{R}_{s_0,n,m,l}$ and $\check{H}_{s_0,n,m,l}$, $\check{R}_{s_0,n,m,l}$ are defined as follows. For $n \neq 0$,

$$\begin{aligned}\hat{H}_{s_0,n,m,l} &= \frac{1}{\varepsilon} \beta_l \left(\mathbb{T} \left(\theta_m(h_n), \int_z^{I_-} \lambda_1(\tau)\, d\tau + in\omega z, B_\xi \right) (z, B_\xi, \varepsilon) \right) \\ &= \frac{1}{\varepsilon} \mathbb{T} \left(\beta_l \theta_m(h_n), \int_z^{I_-} \lambda_1(\tau)\, d\tau + in\omega z, B_\xi \right) (z, B_\xi, \varepsilon),\end{aligned} \tag{3.26a}$$

$$\hat{R}_{s_0,n,m,l} = \begin{cases} \beta_l(\tilde{f}_{1,0,n}(z, B_\xi, \varepsilon)), & \text{for } m=0 \\ 0, & \text{for } m \neq 0 \end{cases}$$

$$= \begin{cases} (\tilde{f}_{1,0,n}(z, B_\xi, \varepsilon)), & \text{for } m=0, \quad l=0 \\ 0, & \text{otherwise} \end{cases} \tag{3.26b}$$

$$\hat{H}_{s_0,0,m,l} = \frac{1}{\varepsilon}\beta_l\left\{\mathbb{T}\left(\theta_m(h_0), \int_z^{I-}\lambda_1(\tau)\,d\tau + in\omega z, B_\xi\right)(z, B_\xi, \varepsilon)\right.$$

$$- e^{-1/\varepsilon\int_{s_0}^{\xi}\lambda_1(\tau)\,d\tau} e^{1/\varepsilon\int_{s_0}^{z}\lambda_1(\tau)\,d\tau}$$

$$\left.\times\sum_{k\in\mathbb{Z}}\mathbb{T}\left(\theta_{m-k}(h_k), \int_z^{I-}\lambda_1(\tau)\,d\tau + in\omega z, B_\xi\right)(\xi, B_\xi, \varepsilon)\right\}$$

$$= \frac{1}{\varepsilon}\left\{\mathbb{T}\left(\beta_l\theta_m(h_0), \int_z^{I-}\lambda_1(\tau)\,d\tau + in\omega z, B_\xi\right)(z, B_\xi, \varepsilon) - e^{1/\varepsilon\int_{s_0}^{z}\lambda_1(\tau)\,d\tau}\right.$$

$$\left.\times\sum_{k\in\mathbb{Z}}\mathbb{T}\left(\beta_{l+1}\theta_{m-k}(h_k), \int_z^{I-}\lambda_1(\tau)\,d\tau + in\omega z, B_\xi\right)(\xi, B_\xi, \varepsilon)\right\} \tag{3.26c}$$

$$\hat{R}_{s_0,0,m,l} = \begin{cases} \beta_l\{\tilde{f}_{1,0,0}(z, B_\xi, \varepsilon) - \tilde{f}_{1,0,0}(\xi, B_\xi, \varepsilon) \\ \qquad \times e^{1/\varepsilon\int_{s_0}^{z}\lambda_1(\tau)\,d\tau} e^{-1/\varepsilon\int_{s_0}^{\xi}\lambda_1(\tau)\,d\tau}\}, & m=0 \\ \beta_l\{-\tilde{f}_{1,0,m}(\xi, B_\xi, \varepsilon)\, e^{1/\varepsilon\int_{s_0}^{z}\lambda_1(\tau)\,d\tau} e^{-1/\varepsilon\int_{s_0}^{\xi}\lambda_1(\tau)\,d\tau}\}, & m\neq 0 \end{cases}$$

$$= \begin{cases} \tilde{f}_{1,0,0}(z, B_\xi, \varepsilon) & m=0, \quad l=0 \\ -\tilde{f}_{1,0,0}(\xi, B_\xi, \varepsilon)\, e^{1/\varepsilon\int_{s_0}^{z}\lambda_1(\tau)\,d\tau} & m=0, \quad l=-1 \\ 0 & m=0, \quad l\neq 0, \quad l\neq -1 \\ -\tilde{f}_{1,0,m}(\xi, B_\xi, \varepsilon)\, e^{1/\varepsilon\int_{s_0}^{z}\lambda_1(r)\,d\tau} & m\neq 0, \quad l=-1 \\ 0 & m\neq 0, \quad l\neq -1 \end{cases} \tag{3.26d}$$

Equations (3.26a)–(3.26d) are obtained through decomposition from (3.21). We note that Eq. (3.25b) is obtained in a similar way from (3.21) by taking $\bar{\xi}$ (instead of ξ) at the variable s. Thus, we get directly for $n \neq 0$,

$$\check{H}_{s_0,n,m,l}(z, \bar{\xi}, \varepsilon) = \frac{1}{\varepsilon}\mathbb{T}\left(\beta_l\theta_m(h_n), \int_z^{I-}\lambda_1(\tau)\,d\tau + in\omega z, B_\xi\right)(z, B_\xi, \varepsilon) \tag{3.26e}$$

$$\check{R}_{s_0,n,m,l}(z, \bar{\xi}, \varepsilon) = \begin{cases} \tilde{f}_{1,0,n}(z, B_\xi, \varepsilon), & m=0, \quad l=0 \\ 0, & \text{otherwise} \end{cases} \tag{3.26f}$$

$$\check{H}_{s_0,0,m,l}=\frac{1}{\varepsilon}\left\{\mathbb{T}\left(\beta_l\theta_m(h_0),\int_z^{I_-}\lambda_1(\tau)\,d\tau+in\omega z,B_\xi\right)(z,B_\xi.\varepsilon)-e^{1/\varepsilon\int_{s_0}^{\bar{z}}\lambda_1(\tau)\,d\tau}\right.$$

$$\left.\times\sum_{k\in\mathbb{Z}}\mathbb{T}\left(\beta_{l+1}\theta_{m-k}(h_k),\int_z^{I_-}\lambda_1(\tau)\,d\tau+in\omega z,B_\xi\right)(\bar{\xi},B_\xi,\varepsilon)\right\}\tag{3.26g}$$

$$\check{R}_{s_0,0,m,l}=\begin{cases}\tilde{f}_{1,0,0}(z,B_\xi,\varepsilon) & m=0,\ l=0\\ -\tilde{f}_{1,0,0}(\bar{\xi},B_\xi,\varepsilon)\,e^{1/\varepsilon\int_{s_0}^{\bar{z}}\lambda_1(\tau)\,d\tau}, & m=0,\ l=-1\\ 0 & m=0,\ l\neq 0,\ l\neq -1\\ -\tilde{f}_{1,0,m}(\bar{\xi},B_\xi,\varepsilon)\,e^{1/\varepsilon\int_{s_0}^{\bar{z}}\lambda_1(r)\,d\tau}, & m\neq 0,\ l=-1\\ 0 & m\neq 0,\ l\neq -1\end{cases}\tag{3.26d}$$

Let s_0 and ξ be defined above. Consider $z\in B_\xi$ such that Proposition 3.2 is true. In particular, for $z\in B_\xi$, $Re\int_{s_0}^{z}\lambda_1(\tau)\,d\tau\leqslant 0$, therefore $|e^{1/\varepsilon\int_{s_0}^{\bar{z}}\lambda_1(\tau)\,d\tau}|\leqslant 1$. Further, we observe that given any $z\in B_\xi$, there exist $\Gamma_n(z,T_n)\subset cl(B_\xi)$ and $\Gamma_n(\bar{z},T_n)\subset cl(B_\xi)$ such that $\varphi_n(z)$ is decreasing on both curves.

Denote $g=(g_{1,n,m,l}(z,\xi,\varepsilon,s_0),\ g_{1,n,m,l}(z,\bar{\xi},\varepsilon,s_0))$, $(\hat{H}_{s_0},\check{H}_{s_0})=(\hat{H}_{s_0,n,m,l},\check{H}_{s_0,n,m,l})$, $(\hat{R}_{s_0},\check{R}_{s_0})=(\hat{R}_{s_0,n,m,l},\check{R}_{s_0,n,m,l})$,

$$\|g\|=\sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}\sum_{l\in\mathbb{Z}}\sup_{z\in B_\xi}(|g_{1,n,m,l}(z,\xi,\varepsilon,s_0)|+|g_{1,n,m,l}(z,\bar{\xi},\varepsilon,s_0)|)$$

It suffices to prove Lemma 3.3 by showing that (3.25) defines a contraction mapping from $G_\gamma=\{g\colon \|g\|<\gamma\}$ to itself for some positive number γ when $\varepsilon<\varepsilon_0$ and $|s_0-I_-|<r$ for some r independent of ε.

Since both $\hat{H}_{s_0,n,m,l}$ and $\check{H}_{s_0,n,m,l}$ contain $\mathbb{T}(\beta_l\theta_m(h_n),\int_z^{I_-}\lambda_1(\tau)\,d\tau+in\omega z,B_\xi)$ which are evaluated at (z,B_ξ,ε) as well as (ξ,B_ξ,ε) and $|e^{1/\varepsilon\int_{s_0}^{\bar{z}}\lambda_1(\tau)\,d\tau}|\leqslant 1$, we get

$$\frac{1}{\varepsilon}\|(\hat{H}_{s_0}(g),\check{H}_{s_0}(g))\|\leqslant\frac{4}{\varepsilon^2}\sum_{n,m,l\in\mathbb{Z}}\|\mathbb{T}(\theta_m\beta_l(h_n))\|$$

$$\leqslant\frac{4}{\varepsilon^2}\sup_{n\in\mathbb{Z},\,z\in B_\xi}|\Gamma_n(z,T_n)|\sum_{n,m,l\in\mathbb{Z}}\|\theta_m\beta_l(h_n)\|\tag{3.27}$$

The last inequality is from Lemmas 3.1 and Proposition 3.2. Now from (2.29)–(2.31) and (2.37)–(2.38), we get

$$\sum_{n,m,l\in\mathbb{Z}} \|\theta_m\beta_l(h_n)\| \leq \varepsilon^2\left(\sum_{n\in\mathbb{Z}} \|f_{1,1,1,n}\|\right)\|g\| + \varepsilon^2\left(\sum_{n\in\mathbb{Z}} \|f_{1,1,2,n}\|\right)\|g\|$$
$$+\sum_{k=2}^{\infty} \varepsilon^k \sum_{1\leq l\leq k+1}\left(\sum_{n\in\mathbb{Z}} \|f_{1,k,l,n}\|\right)\|g\|^k$$

From (2.29), we obtain, further, that

$$\sum_{n,m,l\in\mathbb{Z}} \|\theta_m\beta_l(h_n)\| \leq \varepsilon^2 M_a\|g\| + \sum_{k=2}^{\infty}\varepsilon^k\frac{M_a\|g\|^k}{(\sigma/2)^k}$$
$$\leq \varepsilon^2 M_a\|g\| + \frac{M_a\varepsilon^2\|g\|^2}{(\sigma/2)^2}\frac{1}{1-2\varepsilon\|g\|/\sigma}$$

Therefore, we derive

$$\frac{1}{\varepsilon}\|(\hat{H}_{s_0}(g), \check{H}_{s_0}(g))\|$$
$$\leq 4(\sup_{n\in\mathbb{Z},\, z\in B_\xi} |\Gamma_n(z, T_n)|)\left(M_a\|g\| + \frac{4M_a\|g\|^2}{\sigma^2}\frac{1}{1-2\varepsilon\|g\|/\sigma}\right) \tag{3.28}$$

Similarly, from (3.26), we have

$$\frac{1}{\varepsilon}\|(\hat{R}_{s_0}, \check{R}_{s_0})\| \leq \frac{4}{\varepsilon}\sum_{n\in\mathbb{Z}}\|\tilde{f}_{1,0,n}\|$$

Then from (3.3), particularly by taking $R(z) = \int_z^{l-} \lambda_1(\tau)\,d\tau + in\omega z$, and using the fact that $|R'(z)| = |\lambda_1(z) - in\omega| > \delta_1$ for $z\in B_\xi$, we derive that there exist $\delta_2 > 0$, $\delta_3 > 0$, and $M_{1,n} = M_{1,n}(f'_{1,0,n})$ which are independent of g such that when $\varepsilon < \varepsilon_0$,

$$\|\tilde{f}_{1,0,n}\| \leq \frac{\varepsilon}{\delta_1}\left(\|f_{1,0,n}\| + \frac{\varepsilon}{\delta_2}\|f'_{1,0,n}\| + \frac{\varepsilon^2}{\delta_3}M_{1,n}(f'_{1,0,n})\right)$$

Consequently, when M is sufficiently large and $\varepsilon \leq \varepsilon_0$,

$$\frac{1}{\varepsilon}\|(\hat{R}_{s_0}, \check{R}_{s_0})\|$$
$$\leq \frac{8}{\delta_1}\left(\sum_{n\in\mathbb{Z}}\|f_{1,0,n}\| + \frac{\varepsilon}{\delta_2}\sum_{n\in\mathbb{Z}}\|f'_{1,0,n}\| + \frac{\varepsilon^2}{\delta_3}\sum_{n\in\mathbb{Z}} M_{1,n}\right) \leq 2M \tag{3.29}$$

If we denote $\hat{L}_{s_0}(g) = 1/\varepsilon(\hat{H}_{s_0}(g), \check{H}_{s_0}(g)) + 1/\varepsilon(\hat{R}_{s_0}, \check{R}_{s_0})$, then when $\varepsilon \leqslant \varepsilon_0$,

$$\begin{aligned}\|\hat{L}_{s_0}(g)\| &\equiv \frac{1}{\varepsilon}\|\hat{H}_{s_0} + \hat{R}_{s_0}\| + \frac{1}{\varepsilon}\|\check{H}_{s_0} + \check{R}_{s_0}\| \\ &\leqslant 4M_a \sup_{z \in B_\xi} (\max_n (|\Gamma_n(z, T_n)|, |\Gamma_n(\bar{z}, T_n)|)) \\ &\quad \times \left(\|g\| + \frac{4\|g\|^2}{\sigma^2}\frac{1}{1 - 2\varepsilon\|g\|/\sigma}\right) + 2M \end{aligned} \tag{3.30}$$

where $|\Gamma_n(z, T_n)|$ express the arc lengths of $\Gamma_n(z, T_n)$. We note that $\sup_{z \in B_\xi} |\Gamma_n(z, T_n)|$ goes to zero as ξ goes to I_-. Thus, when ξ is close enough to I_- in the complex plane, namely, there exists $r_2 = r_2(M) > 0$, whenever $\xi \in N_1 \equiv \{\xi, |\xi - I_-| \leqslant r_2\}$, $\hat{L}_{s_0}(g)$ maps the sphere $\hat{B}_{4M} = \{g: \|g\| \leqslant 4M\}$ to itself. Further, for any $g^{(a)} \in \hat{B}_{4M}$ and $g^{(b)} \in \hat{B}_{4M}$, we show in a similar manner as in (3.28) and (3.29) with $(\hat{R}_{s_0}, \check{R}_{s_0})$ terms canceled that

$$\begin{aligned}&\|\hat{L}_{s_0}(g^{(b)}) - \hat{L}_{s_0}(g^{(a)})\| \\ &\equiv \frac{1}{\varepsilon}\|\hat{H}_{s_0}(g^{(b)}) - \hat{H}_{s_0}(g^{(a)})\| + \frac{1}{\varepsilon}\|\check{H}_{s_0}(g^{(b)}) - \check{H}_{s_0}(g^{(a)})\| \\ &\leqslant \frac{4}{\varepsilon^2} \sup_{z \in B_\xi} (\max_n (|\Gamma_n(z, T_n)|, |\Gamma_n(\bar{z}, T_n)|)) \\ &\quad \times \left\{\varepsilon^2 \left(\sum_{n \in \mathbb{Z}}^{\infty} \|f_{1,1,1,n}\|\right)\|g^{(b)} - g^{(a)}\|\right. \\ &\quad + \varepsilon^2 \left(\sum_{n \in \mathbb{Z}} \|f_{1,1,2,n}\|\right)\|g^{(b)} - g^{(a)}\| \\ &\quad + \sum_{k=2}^{\infty} \varepsilon^k \sum_{1 \leqslant l \leqslant k+1} \left(\sum_{n \in \mathbb{Z}} \|f_{1,k,l,n}\|\right) \\ &\quad \left. \times k(\max(\|g^{(b)}\|, \|g^{(a)}\|))^{k-1}\|g^{(b)} - g^{(a)}\|\right\} \\ &\leqslant 8M_a \sup_{z \in B_\xi} (\max_n (|\Gamma_n(z, T_n)|, |\Gamma_n(\bar{z}, T_n)|)) \\ &\quad \times \left(1 + \frac{16M_a}{\sigma^2}\frac{1}{1 - 8\varepsilon M_a/\sigma}\right)\|g^{(b)} - g^{(a)}\| \\ &\leqslant \frac{1}{2}\|g^{(b)} - g^{(a)}\| \end{aligned} \tag{3.31}$$

when ξ is close enough to I_-, say, $|\xi - I_-| \leqslant r_2$. The fixed point argument provides the existence of the anlytic functions $g(z, \xi, \varepsilon)$ which extend $g(I, s, \varepsilon)$. Note $\bigcap_{\xi \in N_1} B_\xi \neq \phi$. Then for $z \in \bigcap_{\xi \in N_1} B_\xi$, the analyticity of $g(z, \xi, \varepsilon)$ for the ξ variable comes from the analytic dependency of the solutions on their initial time. Standard references in this regard can be found in Ref. 6.

Further, for $|\xi - I_-| \leqslant r_2$, and $z \in B_\xi$, since $Re \int_{s_0}^{\xi} \lambda_1(\tau)\, d\tau = 0$,

$$\begin{aligned}
&\sum_{n,m} \|g_{1,n,m}(z, \xi, \varepsilon)\| + \sum_{n,m} \|g_{1,n,m}(z, \bar{\xi}, \varepsilon)\| \\
&\quad = \sum_{n,m} \left\| \sum_{l \in \mathbb{Z}} g_{1,n,m,l}(z, \xi, \varepsilon, s_0)\, e^{l/\varepsilon \int_{s_0}^{\xi} \lambda_1(\tau)\, d\tau} \right\| \\
&\qquad + \left\| \sum_{l \in \mathbb{Z}} \bar{g}_{1,n,m,-l}(\bar{z}, \xi, \varepsilon, s_0)\, e^{l/\varepsilon \int_{s_0}^{\xi} \lambda_1(\tau)\, d\tau} \right\| \\
&\quad \leqslant \sum_{n,m} \sum_{l} \|g_{1,n,m,l}(z, \xi, \varepsilon, s_0)\| \\
&\qquad + \sum_{n,m} \sum_{l} \|\bar{g}_{1,n,m,-l}(\bar{z}, \xi, \varepsilon, s_0)\| \leqslant 2\, \|g\| \leqslant 8M
\end{aligned}$$

This completes the proof of Lemma 3.3. □

We make some additional remarks here. First, $z = I_-$ belongs to all $cl(B_\xi)$ for $|\xi - I_-| < r_2$. Thus all expressions are valid for $z = I_-$. Second, if the initial parameter $s \geqslant I_-$ is on the other side of the critical point, similar properties are expected. Lemma 3.3 can simply be rewritten for the case of $s \geqslant I_-$ without much change.

We now consider the functions $y_s(I_-, s, \varepsilon)$ which are essential in determining the distance between two solutions of (2.16), $y(I, I^0, \varepsilon)$ and $y(I, I_0, \varepsilon)$ where the solutions $y(I, I^0, \varepsilon)$ and $y(I, I_0, \varepsilon)$ satisfy the initial conditions $y(I, I^0, \varepsilon)|_{I = I^0} = 0$ for $I^0 > I_-$ and $y(I, I_0, \varepsilon)|_{I = I_0} = 0$ for $I_0 < I_-$ respectively.

Lemma 3.4. *Assume H1–H5 hold. Let $y(I, s, \varepsilon)$ be the solutions of (2.16) with the initial conditions $y(I, s, \varepsilon)|_{I=s} = 0$ for $s < I_-$. Then for $s \leqslant I \leqslant I_-$, $(\partial/\partial s)\, y(I, s, \varepsilon)$ can be expressed as $(\partial/\partial s)\, y = ((\partial y_1/\partial s), (\partial y_2/\partial s))^T$, and for $j = 1, 2$,*

$$\begin{aligned}
\frac{\partial}{\partial s} y_j(I, s, \varepsilon) = &\left(\sum_{n,m \in \mathbb{Z}} A_{j,n,m}(I, s, \varepsilon)\, e^{in\omega I/\varepsilon} e^{im\omega s/\varepsilon} \right) e^{1/\varepsilon \int_s^I \lambda_1(\tau)\, d\tau} \\
&+ \left(\sum_{n,m \in \mathbb{Z}} B_{j,n,m}(I, s, \varepsilon)\, e^{in\omega I/\varepsilon} e^{im\omega s/\varepsilon} \right) e^{1/\varepsilon \int_s^I \lambda_2(\tau)\, d\tau} \qquad (3.32)
\end{aligned}$$

where $A_{j,n,m}(I,s,\varepsilon)=\overline{B_{j,-n,-m}(I,s,\varepsilon)}$, *both* $A_{j,n,m}(I,s,\varepsilon)$, $B_{j,n,m}(I,s,\varepsilon)$ *have analytic extensions* $A_{j,n,m}(z,\xi,\varepsilon)$, $B_{j,n,m}(z,\xi,\varepsilon)$ *which are analytic in* ξ *for* $|\xi-I_-|\leqslant r_7$, *and are analytic in* z *for* $z\in B_\xi$. *Furthermore,*

$$\sum_{j,n,m}\|A_{j,n,m}(z,\xi,\varepsilon)\|+\sum_{j,n,m}\|B_{j,n,m}(z,\xi,\varepsilon)\|\leqslant M \tag{3.33}$$

for $|\xi-I_-|\leqslant r_7$, $z\in B_\xi$, *and for any* $\varepsilon\leqslant\varepsilon_0$ *where* $\varepsilon_0=\varepsilon_0(M,\lambda_j)$ *and* $r_7>0$ *is independent of* $\varepsilon\to 0$. *In particular,* $z=I_-$ *belongs to every* $cl(B_\xi)$ *for* $|\xi-I_-|<r_7$, *and therefore*

$$\frac{\partial}{\partial s}y_j(I_-,s,\varepsilon)=2\,Re\left(\left(\sum_{m\in\mathbb{Z}}\hat{A}_{j,m}(s,\varepsilon)\,e^{im\omega s/\varepsilon}\right)e^{1/\varepsilon\int_s^{I_-}\lambda_1(\tau)\,d\tau}\right) \tag{3.34}$$

and $\hat{A}_{j,m}(s,\varepsilon)$ *have analytic extensions* $\hat{A}_{j,m}(\xi,\varepsilon)$ *for* $|\xi-I_-|\leqslant r_7$ *and*

$$\sum_{m\in\mathbb{Z}}\|\hat{A}_{j,m}(\xi,\varepsilon)\|\leqslant M \tag{3.35}$$

Proof. Let $Y=Q^{-1}y$ as defined in Section 2. Then $Y(I,s,\varepsilon)=(Y_1(I,s,\varepsilon),Y_2(I,s,\varepsilon))^T$ satisfies (2.26) and (2.32). From relation (2.32), $Y(s,s,\varepsilon)=0$ for all s, we obtain from (2.17) and (2.26) that

$$\frac{\partial}{\partial s}Y(I,s,\varepsilon)|_{I=s}=-\frac{\partial}{\partial I}Y(I,s,\varepsilon)|_{I=s}=-Q_2\left(s,\varepsilon,\frac{s}{\varepsilon}\right) \tag{3.36}$$

Thus, if we let $\eta=(\partial/\partial s)\,Y(I,s,\varepsilon)$, then

$$\frac{\partial}{\partial I}\eta(I,s,\varepsilon)=\begin{pmatrix}\lambda_1(I) & 0\\ 0 & \lambda_2(I)\end{pmatrix}\eta+\frac{\partial F_2}{\partial Y}\left(I,\varepsilon,\frac{I}{\varepsilon},Y(I,s,\varepsilon)\right)\eta \tag{3.37}$$

$$\eta(I,s,\varepsilon)|_{I=s}=-Q_2\left(s,\varepsilon,\frac{s}{\varepsilon}\right) \tag{3.38}$$

where $Q_2=(Q_{2,1},Q_{2,2})^T$, $Q_{2,j}=\sum_{n\in\mathbb{Z}}f_{j,0,n}(s,\varepsilon)\,e^{in\omega s/\varepsilon}$, $j=1,2$, and $\sum_{n\in\mathbb{Z}}\|f_{j,0,n}(\xi,\varepsilon)\|\leqslant M$ for $|\xi-I_-|<r_2$, and $f_{j,0,n}(\xi,\varepsilon)$ analytically extend $f_{j,0,n}(s,\varepsilon)$. Because $Q_2(s,\varepsilon,s/\varepsilon)$ have the analytic extensions for its coefficients $f_{j,0,n}(I,\varepsilon)$ which satisfy (2.31), it is necessary only to show that the solutions $\eta_j=\eta_j(I,s,\varepsilon)$, $j=1,2$, of

$$\varepsilon \frac{\partial}{\partial I} \eta_j = \begin{pmatrix} \lambda_1(I) & 0 \\ 0 & \lambda_2(I) \end{pmatrix} \eta_j + \frac{\partial F_2}{\partial Y}\left(I, \varepsilon, \frac{I}{\varepsilon}, Y\right) \eta_j \tag{3.39}$$

$$\eta_j(I, s, \varepsilon)|_{I=s} = e_j \tag{3.40}$$

have the desired properties (3.32)–(3.33) because $\eta = -Q_{2,1}(s, \varepsilon, s/\varepsilon)\,\eta_1 - Q_{2,2}(s, \varepsilon, s/\varepsilon)\,\eta_2$. Without loss of generality, we just consider η_1. We express $(\partial F_2/\partial Y)(I, \varepsilon, I/\varepsilon, Y)$ into $(F_{3,1}(I, \varepsilon, I/\varepsilon, Y), F_{3,2}(I, \varepsilon, I/\varepsilon, Y))^T$. Then (3.39)–(3.40) can be written into integral equations

$$\eta_1(I, s, \varepsilon) = \begin{pmatrix} e^{1/\varepsilon \int_s^I \lambda_1(\tau)\, d\tau} \\ 0 \end{pmatrix}$$
$$+ \frac{1}{\varepsilon} \begin{pmatrix} e^{1/\varepsilon \int_s^I \lambda_1(\tau)\, d\tau} \int_s^I e^{-1/\varepsilon \int_s^J \lambda_1(\tau)\, d\tau} F_{3,1} \cdot \eta_1(J, s, \varepsilon)\, dJ \\ e^{1/\varepsilon \int_s^I \lambda_2(\tau)\, d\tau} \int_s^I e^{-1/\varepsilon \int_s^J \lambda_2(\tau)\, d\tau} F_{3,2} \cdot \eta_1(J, s, \varepsilon)\, dJ \end{pmatrix} \tag{3.41}$$

From the fact that $F_{3,j}$ depend upon I/ε periodically and upon $I/\varepsilon, I$ analytically, and $F_{3,j}$ also are analytic functions of Y which themselves are expressed in the Fourier series in Lemma 3.3, $F_{3,j}$ have the property that

$$F_{3,j} = \varepsilon \sum_{n \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} (q_{j,1,n,m}(I, s, \varepsilon), q_{j,2,n,m}(I, s, \varepsilon))\, e^{in\omega I/\varepsilon} e^{im\omega s/\varepsilon} \tag{3.42}$$

where

$$\sum_{j=1,2,\, k=1,2} \sum_{n \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} \|q_{j,k,n,m}(I, s, \varepsilon)\| \leqslant M \tag{3.43}$$

for some constant M. Inequality (3.43) can be easily derived from (2.27)–(2.29) along with (3.13). We look for the solution η_1 of (3.41) of the form

$$\eta_1 = \begin{pmatrix} e^{1/\varepsilon \int_s^I \lambda_1(\tau)\, d\tau} \\ 0 \end{pmatrix} + \sum_{n,m} \begin{pmatrix} \alpha_{1,1,n,m}(I, s, \varepsilon) \\ \alpha_{1,2,n,m}(I, s, \varepsilon) \end{pmatrix} \cdot e^{in\omega I/\varepsilon} e^{im\omega s/\varepsilon} e^{1/\varepsilon \int_s^I \lambda_1(\tau)\, d\tau}$$
$$+ \sum_{n,m} \begin{pmatrix} \beta_{1,1,n,m}(I, s, \varepsilon) \\ \beta_{1,2,n,m}(I, s, \varepsilon) \end{pmatrix} \cdot e^{in\omega I/\varepsilon} e^{im\omega s/\varepsilon} e^{1/\varepsilon \int_s^I \lambda_2(\tau)\, d\tau} \tag{3.44}$$

From (3.41), we obtain that $(\alpha_1, \beta_1) = (\alpha_{1,j,n,m}, \beta_{1,j,n,m} \mid j = 1, 2, n \in \mathbb{Z}, m \in \mathbb{Z})$ must satisfy

$$\sum_{n,m\in\mathbb{Z}} \alpha_{1,j,n,m} e^{in\omega I/\varepsilon} e^{im\omega s/\varepsilon} e^{1/\varepsilon \int_s^I \lambda_1(\tau)\, d\tau}$$

$$+ \sum_{n,m\in\mathbb{Z}} \beta_{1,j,n,m} e^{in\omega I/\varepsilon} e^{im\omega s/\varepsilon} e^{1/\varepsilon \int_s^I \lambda_2(\tau)\, d\tau}$$

$$= \frac{1}{\varepsilon} e^{1/\varepsilon \int_s^I \lambda_j(\tau)\, d\tau} \int_s^I e^{-1/\varepsilon \int_s^J \lambda_j(\tau)\, d\tau} \Bigg[\left(\varepsilon \sum_{n,m} q_{j,1,n,m}(J, s, \varepsilon)\, e^{in\omega J/\varepsilon} e^{im\omega s/\varepsilon} \right)$$

$$\Bigg(e^{1/\varepsilon \int_s^J \lambda_1(\tau)\, d\tau} + \sum_{n,m} (\alpha_{1,1,n,m} e^{1/\varepsilon \int_s^J \lambda_1(\tau)\, d\tau}$$

$$+ \beta_{1,1,n,m} e^{1/\varepsilon \int_s^J \lambda_2(\tau)\, d\tau})\, e^{in\omega J/\varepsilon} e^{in\omega s/\varepsilon} \Bigg)$$

$$+ \left(\varepsilon \sum_{n,m} q_{j,2,n,m}(J, s, \varepsilon)\, e^{in\omega J/\varepsilon} e^{im\omega s/\varepsilon} \right)$$

$$\times \sum_{n,m} (\alpha_{1,2,n,m} e^{1/\varepsilon \int_s^J \lambda_1(\tau)\, d\tau} + \beta_{1,2,n,m} e^{1/\varepsilon \int_s^J \lambda_2(\tau)\, d\tau})\, e^{in\omega J/\varepsilon} e^{in\omega s/\varepsilon} \Bigg] dJ \tag{3.45}$$

We express the right-hand side of Eqs. (3.45) into

$$e^{1/\varepsilon \int_s^I \lambda_1(\tau)\, d\tau} \int_s^I \Bigg[\sum_{n,m} L_{1,1,n,m}(\alpha, J, s, \varepsilon)\, e^{in\omega J/\varepsilon} e^{im\omega s/\varepsilon}$$

$$+ \sum_{n,m} q_{1,1,n,m}(J, s, \varepsilon)\, e^{in\omega J/\varepsilon} e^{im\omega s/\varepsilon}$$

$$+ \sum_{n,m} L_{1,2,n,m}(\beta, J, s, \varepsilon)\, e^{in\omega J/\varepsilon} e^{im\omega s/\varepsilon} e^{1/\varepsilon \int_s^J (\lambda_2 - \lambda_1)(\tau)\, d\tau} \Bigg] dJ \tag{3.46}$$

and

$$e^{1/\varepsilon \int_s^I \lambda_2(\tau)\, d\tau} \int_s^I \sum_{n,m} [L_{2,1,n,m}(\alpha, J, s, \varepsilon)\, e^{in\omega J/\varepsilon} e^{im\omega s/\varepsilon} e^{1/\varepsilon \int_s^J (\lambda_1 - \lambda_2)(\tau)\, d\tau}$$

$$+ L_{2,2,n,m}(\beta, J, s, \varepsilon)\, e^{in\omega J/\varepsilon} e^{im\omega s/\varepsilon}$$

$$+ q_{1,2,n,m}(J, s, \varepsilon)\, e^{in\omega J/\varepsilon} e^{im\omega s/\varepsilon} e^{1/\varepsilon \int_s^J (\lambda_1 - \lambda_2)(\tau)\, d\tau}]\, dJ \tag{3.47}$$

where

$$L_{j,1,n,m} = \sum_{k_1+l_1=n,\, k_2+l_2=m,\, k=1,2} q_{j,k,k_1,k_2}(I, s, \varepsilon) \cdot \alpha_{1,k,l_1,l_2}, \qquad j=1,2, \tag{3.48}$$

$$L_{j,2,n,m} = \sum_{k_1+l_1=n,\, k_2+l_2=m,\, k=1,2} q_{j,k,k_1,k_2}(I, s, \varepsilon) \cdot \beta_{1,k,l_1,l_2}, \qquad j=1,2. \tag{3.49}$$

It can be verified directly that there is a constant $M_b > 0$ such that

$$\begin{aligned}
&\sum_{j,n,m} \|L_{j,1,n,m}\| + \sum_{j,n,m} \|L_{j,2,n,m}\| \\
&\quad \leqslant \left(\sum_{j,n,m} \|q_{j,1,n,m}(I, s, \varepsilon)\| + \sum_{j,n,m} \|q_{j,2,n,m}(I, s, \varepsilon)\| \right) \\
&\quad\quad \times \left(\sum_{j,n,m} \|\alpha_{1,j,n,m}\| + \sum_{j,n,m} \|\beta_{1,j,n,m}\| \right) \leqslant M_b \, \|(\alpha_1, \beta_1)\|
\end{aligned} \tag{3.50}$$

Under hypotheses H4 and H5, when $|\xi - I_-|$ is close enough to 0, it can be easily verified by the Taylor expansions of $in\omega z$ and $\int_{I_-}^{z} (\lambda_2 - \lambda_1 + in\omega)\, d\tau$ that the level curves of $Re(in\omega z), n \in \mathbb{Z}$ and $Re \int_{\xi}^{z} (\lambda_2 - \lambda_1 + in\omega)\, d\tau, n \in \mathbb{Z}$, are less concave than Γ_m and less convex than $\bar{\Gamma}_m$ for $0 \leqslant m \leqslant k$ of which ∂B_ξ consists. Thus there exist $\hat{\Gamma}_{n,1}(z, \xi_n) \subset cl(B_\xi)$ and $\hat{\Gamma}_{n,2}(z, \eta_n^{\pm}) \subset cl(B_\xi)$ such that $Re(in\omega z)$ and $Re \int_{\xi}^{z} (\pm(\lambda_2(\tau) - \lambda_1(\tau)) + in\omega)\, d\tau$ are monotone increasing on the corresponding curves, respectively. The points ξ_n and $\eta_n^{\pm}$ are the minimum points of $Re(in\omega z)$ and $Re \int_{\xi}^{z} (\pm(\lambda_2(\tau) - \lambda_1(\tau)) + in\omega)\, d\tau$ in $cl(B_\xi)$.

Now for $z \in B_\xi$, we can apply Lemma 3.1 with respect to $R(z) = in\omega z$ and $R(z) = \int_{\xi}^{z} (\lambda_2 - \lambda_1 + in\omega)\, d\tau$. Let $\mathbb{T}$ be denoted as in Lemma 3.1. Denote that for $m \in \mathbb{Z}, n \in \mathbb{Z}$,

$$\begin{aligned}
&\tilde{L}_{1,1,n,m}(z, B_\xi, \varepsilon) \equiv \mathbb{T}(L_{1,1,n,m}(\alpha, z, \xi, \varepsilon), in\omega z, B_\xi) \\
&L_{2,2,n,m}(z, B_\xi, \varepsilon) \equiv \mathbb{T}(L_{2,2,n,m}(\beta, z, \xi, \varepsilon), in\omega z, B_\xi) \\
&\tilde{L}_{1,2,n,m}(z, B_\xi, \varepsilon) \equiv \mathbb{T}\left(L_{1,2,n,m}(\beta, z, \xi, \varepsilon), in\omega z + \int_{\xi}^{z} (\lambda_2 - \lambda_1)(\tau)\, d\tau, B_\xi\right) \\
&\tilde{L}_{2,1,n,m}(z, B_\xi, \varepsilon) \equiv \mathbb{T}\left(L_{2,1,n,m}(\alpha, z, \xi, \varepsilon), in\omega z + \int_{\xi}^{z} (\lambda_1 - \lambda_2)(\tau)\, d\tau, B_\xi\right) \\
&\tilde{q}_{1,1,n,m}(z, B_\xi, \varepsilon) \equiv \mathbb{T}(q_{1,1,n,m}(z, \xi, \varepsilon), in\omega z, B_\xi) \\
&\tilde{q}_{1,2,n,m}(z, B_\xi, \varepsilon) \equiv \mathbb{T}\left(q_{1,2,n,m}(z, \xi, \varepsilon), in\omega z + \int_{\xi}^{z} (\lambda_1 - \lambda_2)(\tau)\, d\tau, B_\xi\right)
\end{aligned} \tag{3.51}$$

We now consider the corresponding vector equation of (3.45) after using relation (3.51):

$$\sum_{m,n\in\mathbb{Z}} \alpha_{1,1,n,m} e^{in\omega I/\varepsilon} e^{im\omega s/\varepsilon} e^{1/\varepsilon \int_s^I \lambda_1(\tau)\,d\tau} + \sum_{m,n\in\mathbb{Z}} \beta_{1,1,n,m} e^{in\omega I/\varepsilon} e^{im\omega s/\varepsilon} e^{1/\varepsilon \int_s^I \lambda_2(\tau)\,d\tau}$$
$$= \sum_{m,n} (\tilde{L}_{1,1,n,m}(I, B_s, \varepsilon)\, e^{in\omega I/\varepsilon} - \tilde{L}_{1,1,n,m}(s, B_s, \varepsilon)\, e^{in\omega s/\varepsilon}) \cdot e^{im\omega s/\varepsilon} e^{1/\varepsilon \int_s^I \lambda_1(\tau)\,d\tau}$$
$$+ \sum_{m,n} (\tilde{L}_{1,2,n,m}(I, B_s, \varepsilon)\, e^{in\omega I/\varepsilon} e^{1/\varepsilon \int_s^I \lambda_2(\tau)\,d\tau} - \tilde{L}_{1,2,n,m}(s, B_s, \varepsilon)\, e^{in\omega s/\varepsilon} e^{1/\varepsilon \int_s^I \lambda_1(\tau)\,d\tau})\, e^{im\omega s/\varepsilon}$$
$$+ \sum_{m,n} (\tilde{q}_{1,1,n,m}(I, B_s, \varepsilon)\, e^{in\omega I/\varepsilon} - \tilde{q}_{1,1,n,m}(s, B_s, \varepsilon)\, e^{in\omega s/\varepsilon})\, e^{im\omega s/\varepsilon} e^{1/\varepsilon \int_s^I \lambda_1(\tau)\,d\tau} \tag{3.52a}$$

$$\sum_{m,n\in\mathbb{Z}} \alpha_{1,2,n,m} e^{in\omega I/\varepsilon} e^{im\omega s/\varepsilon} e^{1/\varepsilon \int_s^I \lambda_1(\tau)\,d\tau} + \sum_{m,n\in\mathbb{Z}} \beta_{1,2,n,m} e^{in\omega I/\varepsilon} e^{im\omega s/\varepsilon} e^{1/\varepsilon \int_s^I \lambda_2(\tau)\,d\tau}$$
$$= \sum_{m,n} (\tilde{L}_{2,1,n,m}(I, B_s, \varepsilon)\, e^{in\omega I/\varepsilon} e^{1/\varepsilon \int_s^I \lambda_1(\tau)\,d\tau} - \tilde{L}_{2,1,n,m}(s, B_s, \varepsilon)\, e^{in\omega s/\varepsilon} e^{1/\varepsilon \int_s^I \lambda_2(\tau)\,d\tau})\, e^{im\omega s/\varepsilon}$$
$$+ \sum_{m,n} (\tilde{L}_{2,2,n,m}(I, B_s, \varepsilon)\, e^{in\omega I/\varepsilon} - \tilde{L}_{2,2,n,m}(s, B_s, \varepsilon)\, e^{in\omega s/\varepsilon})\, e^{im\omega s/\varepsilon} e^{1/\varepsilon \int_s^I \lambda_2(\tau)\,d\tau}$$
$$+ \sum_{m,n} (\tilde{q}_{1,2,n,m}(I, B_s, \varepsilon)\, e^{in\omega I/\varepsilon} e^{1/\varepsilon \int_s^I \lambda_1(\tau)\,d\tau} - \tilde{q}_{1,2,n,m}(s, B_s, \varepsilon)\, e^{in\omega s/\varepsilon} e^{1/\varepsilon \int_s^I \lambda_2(\tau)\,d\tau}) \cdot e^{im\omega s/\varepsilon} \tag{3.52b}$$

We compare the coefficients of the series of both sides of the Eq.(3.52) to obtain a vector equation, then we extend it from (I, s) to (z, ξ) for any

$\xi \in B_{r_2}(I_-)$ and $z \in B_\xi$. Lemma 3.3 permits the analytic extensions of the coefficients. The new vector equation becomes

$$\begin{cases} \alpha_{1,1,n,m} = \tilde{L}_{1,1,n,m}(z, B_\xi, \varepsilon) + \tilde{q}_{1,1,n,m}(z, B_\xi, \varepsilon) \quad \text{for} \quad n \neq 0 \\ \alpha_{1,1,0,m} = \tilde{L}_{1,1,0,m}(z, B_\xi, \varepsilon) - \sum_{k+l=m} \tilde{L}_{1,1,k,l}(\xi, B_\xi, \varepsilon) \\ \qquad - \sum_{k+l=m} \tilde{L}_{1,2,k,l}(\xi, B_\xi, \varepsilon) - \sum_{k+l=m} \tilde{q}_{1,1,k,l}(\xi, B_\xi, \varepsilon), \\ \beta_{1,1,n,m} = \tilde{L}_{1,2,n,m}(z, B_\xi, \varepsilon), \\ \alpha_{1,2,n,m} = \tilde{L}_{2,1,n,m}(z, B_\xi, \varepsilon) + \tilde{q}_{1,2,n,m}(z, B_\xi, \varepsilon), \\ \beta_{1,2,n,m} = \tilde{L}_{2,2,n,m}(z, B_\xi, \varepsilon) \quad \text{for} \quad n \neq 0 \\ \beta_{1,2,0,m} = \tilde{L}_{2,2,0,m}(z, B_\xi, \varepsilon) - \sum_{k+l=m} \tilde{L}_{2,1,k,l}(\xi, B_\xi, \varepsilon) \\ \qquad - \sum_{k+l=m} \tilde{L}_{2,2,k,l}(\xi, B_\xi, \varepsilon) - \sum_{k+l=m} \tilde{q}_{1,2,k,l}(\xi, B_\xi, \varepsilon) \end{cases} \tag{3.53}$$

The Banach space of (α_1, β_1) is defined in terms of the norm:

$$\|(\alpha_1, \beta_1)\| = \sum_{j=1,2,\, n \in \mathbb{Z},\, m \in \mathbb{Z}} (\|\alpha_{1,j,n,m}\| + \|\beta_{1,j,m,n}\|) \quad \text{where} \|\cdot\| \equiv \sup_{z \in B_\xi} |\cdot|$$

For $z \in B_\xi$, (3.53) defines a fixed point problem for a vector equation: $(\alpha_1, \beta_1)(z, \xi, \varepsilon) = T((\alpha_1, \beta_1), z, \xi, \varepsilon) + S(z, \xi, \varepsilon)$ where T represents all terms which contain $\tilde{L}_{i,j,n,m}$ that are dependent on (α_1, β_1) and S contains the nonhomogeneous terms involving $\tilde{q}_{1,j,n,m}$. We note that M_b is a constant [referring to (3.50)] which is independent of $\varepsilon \to 0^+$. We can show by using Lemma 3.1 that for $i = 1, 2$, $j = 1, 2$, $n \in \mathbb{Z}$, $m \in \mathbb{Z}$,

$$\|\tilde{L}_{i,j,n,m}(z, B_\xi, \varepsilon)\| \leqslant \sup_{z \in B_\xi} (|\hat{\Gamma}_{n,1}(z, \xi_n)|, |\hat{\Gamma}_{n,2}(z, \eta_n^\pm)|) \, \|L_{i,j,n,m}\| \tag{3.54a}$$

$$\|\tilde{q}_{1,j,n,m}(z, B_\xi, \varepsilon)\| \leqslant \sup_{z \in B_\xi} (|\hat{\Gamma}_{n,1}(z, \xi_n)|, |\hat{\Gamma}_{n,2}(z, \eta_n^\pm)|) \, \|q_{1,j,n,m}\| \tag{3.54b}$$

Thus by combining (3.54) with (3.50), we obtain that

$$\|T((\alpha_1, \beta_1), z, \xi, \varepsilon)\| \leqslant 2M_b \sup_{z \in B_\xi,\, n \in \mathbb{Z}} (|\hat{\Gamma}_{n,1}(z, \xi_n)|, |\hat{\Gamma}_{n,2}(z, \eta_n^\pm)|) \, \|(\alpha_1, \beta_1)\| \tag{3.55a}$$

$$\|S(z, \xi, \varepsilon)\| \leqslant 2M_b \sup_{z \in B_\xi,\, n \in \mathbb{Z}} (|\hat{\Gamma}_{n,1}(z, \xi_n)|, |\hat{\Gamma}_{n,2}(z, \eta_n^\pm)|), \tag{3.55b}$$

$$\|T((\alpha_1^{(a)}, \beta_1^{(a)}), z, \xi, \varepsilon) - T((\alpha_1^{(b)}, \beta_1^{(b)}), z, \xi, \varepsilon)\|$$
$$\leqslant 2M_b \sup_{z \in B_\xi, n \in \mathbb{Z}} (|\hat{\Gamma}_{n,1}(z, \xi_n)|, |\hat{\Gamma}_{n,2}(z, \eta_n^\pm)|) \, \|(\alpha_1^{(a)}, \beta_1^{(a)}) - (\alpha_1^{(b)}, \beta_1^{(b)})\| \tag{3.56}$$

There exists $r_6 = r_6(M_b)$ such that if $|\xi - I_-| \leqslant r_6$, $T((\alpha_1, \beta_1)) + S$ defines a contraction mapping from $B_{4M_b} = \{(\alpha_1, \beta_1) : \|(\alpha_1, \beta_1)\| \leqslant 4M_b\}$ to itself. This can be obtained through controlling $|\xi - I_-| \leqslant r_6$ to get $\sup_{z \in B_\xi, n \in \mathbb{Z}}(|\hat{\Gamma}_{n,1}(z, \xi_n)|, |\hat{\Gamma}_{n,2}(z, \eta_n^\pm)|) \leqslant \min(\frac{1}{4}, 1/(4M_b))$. Thus, the fixed point argument assures the existence of analytic extensions of $(\alpha_1, \beta_1) = (\alpha_{1,j,n,m}, \beta_{1,j,n,m})$. The extensions (α_2, β_2) can be shown in an analogous manner.

For fixed z, $(\alpha_1, \beta_1)(z, \xi, \varepsilon)$, $(\alpha_2, \beta_2)(z, \xi, \varepsilon)$ are analytic in ξ because of the analytic dependence of solutions with respect to their initial parameter. In particular, $z = I_-$ belongs to $cl(B_\xi)$ for any ξ: $|\xi - I_-| \leqslant r_6$. We note that since $\partial y_j/\partial s$ are real, $\lambda_2(I) = \overline{\lambda_1(I)}$, the coefficients $A_{j,n,m}(I, s, \varepsilon)$ and $B_{j,-n,-m}(I, s, \varepsilon)$ are conjugate to each other, i.e., $A_{j,n,m}(I, s, \varepsilon) = \overline{B_{j,-n,-m}(I, s, \varepsilon)}$. Thus we have (3.34)–(3.35) by letting $I = I_-$ in (3.32)–(3.33) and $\hat{A}_{j,m}(\xi, \varepsilon) = \sum_{n \in \mathbb{Z}} A_{j,n,m}(I_-, \xi, \varepsilon) \, e^{in\omega I_-/\varepsilon}$. □

Remark. For the solutions $y(I, s, \varepsilon)$ of (2.16) with the initial condition $y(I, s, \varepsilon)|_{I=s} = 0$ for $s > I_-$, analogous extension results in Lemma 3.4 can also be stated for $y(I, s, \varepsilon)$ when $I_- \leqslant I \leqslant s$. In the regions of ξ for which the extensions from left and right of I_- overlap, we should have the same resulting functions since the functions are the same in the real axis.

4. DEDLAYED HOPF BIFURCATIONS UNDER PERIODIC FORCING

We now derive the delayed bifurcation phenomena. Consider the distance of two solutions of (2.16), one of which is $y(I, I_0, \varepsilon)$ with the initial condition $y(I, I_0, \varepsilon)|_{I=I_0} = 0$ for some $I_0 < I_-$, and the other $y(I, I^0, \varepsilon)$ with the initial condition $y(I, I^0, \varepsilon)|_{I=I^0} = 0$ for $I^0 > I_-$. As we noted previously, if we could obtain an estimate like $|y(I_-, I_0, \varepsilon) - y(I_-, I^0, \varepsilon)| \leqslant Ke^{-c/\varepsilon}$ for some K, $c > 0$, then the results of delayed bifurcations follow from this and Proposition 2.4. In other words, the solution $y(I, I_0, \varepsilon)$ satisfies $|y(I, I_0, \varepsilon)| = O(\varepsilon)$ when $I_0 \leqslant I \leqslant I_q$ for some $I_q > I_-$ as $\varepsilon \to 0^+$. The point $I_q \geqslant \max(I_0^*, I_0^{**})$ (I_0^*, I_0^{**} are defined below) is above the critical point I_-, and is to be determined by the estimates of the quantity c. We introduce the key results here.

Theorem 4.1. *Assume H1–H5 are satisfied. Let* $y(I, I_0, \varepsilon)$ *be the solutions of (2.16) with initial conditions* $y(I, I_0, \varepsilon)|_{I=I_0}=0$ *for any* $I_0<I_-$. *Take* $I^0>I_-$ *such that* $Re\int_{I_0}^{I^0}\lambda_1(\tau)\,d\tau>0$. *Then for any* $M>0$, *there exists* $\varepsilon_0=\varepsilon_0(M)$ *such that for* $|I_0-I_-|\leqslant r_6$ *(*r_6 *is referred to in Lemma 3.4),* $\varepsilon\leqslant\varepsilon_0$, *the solutions* $|y(I, I_0, \varepsilon)-y(I, I^0, \varepsilon)|\leqslant M\varepsilon$ *when and only when* $I_0\leqslant I\leqslant I_q$, *where* I_q *is a point above the critical point. If assumption (4.4) (below) is satisfied, then* I_q *is determined by* $I_q=I_0^*+O(\varepsilon|log(\varepsilon)|)>I_-$, *where* $I_0^*>I_-$ *is the point satisfying the relation* $Re\int_{I_0}^{I_0^*}\lambda_1(\tau)\,d\tau=0$. *Otherwise, without (4.4), we have* $I_q\geqslant I_0^*+O(\varepsilon\,|\log(\varepsilon)|)>I_-$.

Proof. Take I_0 to be any point with $I_--r_6\leqslant I_0<I_-$, and $I^0>I_0^*$ such that $Re\int_{I_0}^{I^0}\lambda_1(\tau)\,d\tau>0$. Using (3.34), we obtain

$$\begin{aligned} D &\equiv y(I_-, I_0, \varepsilon)-y(I_-, I^0, \varepsilon)\\ &=2\,Re\left(\sum_{n\in\mathbb{Z}}\int_{I_0}^{I^0}\hat{A}_n(s,\varepsilon)\,e^{1/\varepsilon\int_s^{I_-}\lambda_1(\tau)\,d\tau+in\omega s/\varepsilon}\,ds\right) \end{aligned}\tag{4.1}$$

where $\hat{A}_n=(\hat{A}_{1,n}, \hat{A}_{2,n})^T$. Under conditions H4–H5, we can apply Lemma 3.1 to treat the integrals. Still denote $\Gamma_n(z, T_n)\in cl(B_\xi)$ to be the curves from T_n to z such that the functions $Re\int_z^{I_-}\lambda_1(\tau)\,d\tau+in\omega z$ are monotone decreasing functions of z on the paths. We construct $\tilde{A}_n(\hat{A}_n, z, \varepsilon)\equiv\mathbb{T}(\hat{A}_n, \int_z^{I_-}\lambda_1(\tau)\,d\tau+in\omega z, B_\xi)$. Then we get

$$\begin{aligned} D=2\,Re\Bigg(&\sum_{n\in\mathbb{Z}}\tilde{A}_n(\hat{A}_n, I^0, \varepsilon)\,e^{1/\varepsilon\int_{I^0}^{I_-}\lambda_1(\tau)\,d\tau+in\omega I^0/\varepsilon}\\ &-\sum_{n\in\mathbb{Z}}\tilde{A}_n(\hat{A}_n, I_0, \varepsilon)\,e^{1/\varepsilon\int_{I_0}^{I_-}\lambda_1(\tau)\,d\tau+in\omega I^0/\varepsilon}\Bigg) \end{aligned}\tag{4.2}$$

where $\|\tilde{A}_n\|\leqslant\|\hat{A}_n(\xi,\varepsilon)\|\max_{z\in B_\xi}|\Gamma_n(z, T_n)|$. Thus we get that

$$\begin{aligned} |D| &\leqslant 4e^{-1/\varepsilon\,Re\int_{I_-}^{I_0}\lambda_1(\tau)\,d\tau}\left(\sum_{n\in\mathbb{Z}}\|\hat{A}_n\|\right)\max_{z\in B_\xi}|\hat{\Gamma}_n(z, T_n)|\\ &\leqslant M_c e^{-1/\varepsilon\,Re\int_{I_-}^{I_0}\lambda_1(\tau)\,d\tau} \end{aligned}\tag{4.3}$$

for a constant $M_c>0$. It was indicated in Ref. 24 that the assumption that for some finite integer $l>0$ and some real number $k_3>0$,

$$\left[\,Re\sum_{n\in\mathbb{Z}}\tilde{A}_n(\hat{A}_n, I_0, \varepsilon)\,e^{l/\varepsilon\,Im\int_{I_0}^{I_-}\lambda_1(\tau)\,d\tau+in\omega I_0/\varepsilon}\right|\geqslant k_3\varepsilon^l\tag{4.4}$$

is quite general in the sense that (4.4) is valid for all but a certain class of systems. In fact, using Lemma 3.1 and particularly (3.3), we can express that for any $l \in \mathbb{N}$,

$$\sum_{n\in\mathbb{Z}} \tilde{A}_n(\hat{A}_n, I_0, \varepsilon)\, e^{i/\varepsilon\, Im \int_{I_0}^{I_-} \lambda_1(\tau)\,d\tau + in\omega I_0/\varepsilon}$$

$$= \sum_{n\in\mathbb{Z}} \Bigg[\sum_{m=1}^{l} (-\varepsilon)^m \frac{\hat{A}_n^{[m]}(I_0, \varepsilon)}{\lambda_1(I_0) - in\omega} + O(\varepsilon^{l+1})$$

$$- \sum_{m=1}^{l} (-\varepsilon)^m \frac{\hat{A}_n^{[m]}(I^0, \varepsilon)}{\lambda_1(I^0) - in\omega}\, e^{1/\varepsilon \int_{I_0}^{I^0} (\lambda_1(\tau) - in\omega)\,d\tau} \Bigg]\, e^{i/\varepsilon\, Im \int_{I_0}^{I_-} \lambda_1(\tau)\,d\tau + in\omega I_0/\varepsilon}$$

$$= \sum_{n\in\mathbb{Z}} \Bigg[\sum_{m=1}^{l} (-1)^m \varepsilon^m \frac{\hat{A}_n^{[m]}(I_0, \varepsilon)}{\lambda_1(I_0) - in\omega} + O(\varepsilon^{l+1}) \Bigg]\, e^{i/\varepsilon\, Im \int_{I_0}^{I_-} \lambda_1(\tau)\,d\tau + in\omega I_0/\varepsilon}$$

From the construction of $\hat{A}_n(I, \varepsilon)$ in Lemma 3.4, we can express $\hat{A}_n(I, \varepsilon) = \sum_{k=0}^{l} \varepsilon^k \hat{A}_{n,k}(I) + O(\varepsilon^{l+1})$ for any finite $l \in \mathbb{N}$. Thus

$$\sum_{n\in\mathbb{Z}} \tilde{A}_n(\hat{A}_n, I_0, \varepsilon)\, e^{i/\varepsilon\, Im \int_{I_0}^{I_-} \lambda_1(\tau)\,d\tau + in\omega I_0/\varepsilon}$$

$$= \sum_{n\in\mathbb{Z}} \sum_{k=0}^{l} \sum_{m=1}^{l} \left[(-1)^m \varepsilon^{m+k} \frac{\hat{A}_{n,k}^{[m]}(I_0)}{\lambda_1(I_0) - in\omega} \right]$$

$$\times e^{i/\varepsilon\, Im \int_{I_0}^{I_-} \lambda_1(\tau)\,d\tau + in\omega I_0/\varepsilon} + O(\varepsilon^{l+1})$$

and its real part is in general greater than $O(e^l)$ for some finite $l \in \mathbb{N}$ unless its coefficients cancel out to make it a smaller amount. In most of the examples which we have calculated, (4.4) is satisfied unless the systems are homogeneous or the nonhomogeneous terms are far smaller than any order of ε to start with (for example, to be exponential small). We, however, cannot produce an easy condition on the equations themselves to determine for any arbitrary system whether (4.4) can be satisfied. Thus (4.4) remains as an assumption.

Since $I_0^* < I^0$, and consequently $\int_{I_0}^{I_-} Re\, \lambda_1(\tau)\,d\tau < \int_{I_0^*}^{I_-} Re\, \lambda_1(\tau)\,d\tau = \int_{I^0}^{I_-} Re\, \lambda_1(\tau)\,d\tau$, the terms in the second group of (4.2) are smaller than those in the first group by a factor of $e^{-1/\varepsilon \int_{I_0^*}^{I^0} Re\, \lambda_1(\tau)\,d\tau}$. Then from (4.2), we can derive that

$$|D| \geq 2 \left| Re \sum_{n\in\mathbb{Z}} \tilde{A}_n(\hat{A}_n, I_0, \varepsilon)\, e^{i/\varepsilon\, Im \int_{I_0}^{I_-} \lambda_1(\tau)\,d\tau + in\omega I_0/\varepsilon} \right| e^{1/\varepsilon \int_{I_0}^{I_-} Re\, \lambda_1(\tau)\,d\tau}$$

$$-2 \left(\sum_{n\in\mathbb{Z}} |\tilde{A}_n(\hat{A}_n, I^0, \varepsilon)| \right) e^{1/\varepsilon \int_{I^0}^{I_-} Re\, \lambda_1(\tau)\,d\tau} \geq k_3 \varepsilon^l e^{-1/\varepsilon\, Re \int_{I_-}^{I_0} \lambda_1(\tau)\,d\tau} \qquad (4.5)$$

when $\varepsilon \leqslant \varepsilon_0$. From Proposition 2.4 and the estimates (4.3) and (4.5), we get that if $|y(I, I_0, \varepsilon) - y(I, I^0, \varepsilon)| \leqslant M\varepsilon$, then

$$M_3 M_c e^{1/\varepsilon \, Re \int_{I_0}^{I} \lambda_1(\tau)\, d\tau} \geqslant |y(I, I_0, \varepsilon) - y(I, I^0, \varepsilon)| = K(I)\, |D|\, e^{1/\varepsilon \, Re \int_{I_-}^{I} \lambda_1(\tau)\, d\tau}$$
$$\geqslant (k_3/M_3)\, \varepsilon^l e^{1/\varepsilon \, Re \int_{I_0}^{I} \lambda_1(\tau)\, d\tau}$$

Thus $|y(I, I_0, \varepsilon) - y(I, I^0, \varepsilon)| \leqslant M\varepsilon$ when and only when $I_0 \leqslant I \leqslant I_q$, where $I_q = I_0^* + O(\varepsilon\, |\log(\varepsilon)|)$ under assumption (4.4). Otherwise, without (4.4), we have $I_q \geqslant I_0^* + O(\varepsilon\, |\log(\varepsilon)|)$. □

Remark. (4.4) is a general condition, and there are exceptional cases. For example, if $Q_1(I, \varepsilon, I/\varepsilon) \equiv 0$ in (2.16), then $I_q > I_0^* + O(\varepsilon\, |\log(\varepsilon)|)$, and $y(I, I_0, \varepsilon) \equiv 0$.

We note that Theorem 4.1 is a general result, but the range of I_0 where the theorem is applicable varies from case to case. In particular, this is so because the conditions require that every $\Gamma_n(z, T_n)$ lies in a close neighborhood of I_-. Thus if $|Im\, \lambda_1(I_-) - in\omega| = \delta \to 0^+$ for some $n \in \mathbb{Z}$, then the range of admissible I_0 can be very small, i.e., $I_0 = I_- - O(\delta)$ since all level curves can be very "tall." Therefore Theorem 4.1 cannot provide much information in the near resonant cases for solutions with initial points left of $I_- - O(\delta)$. We introduce a different result which complements Theorem 4.1 in the sense that it concerns the cases where I_0 is farther left of $I_- - O(\delta)$ in the near-resonant cases as well as other cases.

Theorem 4.2. *Let $y(I, I_0, \varepsilon)$ be the solution of (2.16) with the initial condition $y(I, I_0, \varepsilon)|_{I = I_0} = 0$ for $I_0 < I_-$. Assume H1–H5 are true for the system. Then for any $M > M_2$, there exists $\varepsilon_0 = \varepsilon_0(M) > 0$, such that for $|I_0 - I_-| \leqslant r_6$, $\varepsilon \leqslant \varepsilon_0$,*

$$|y(I, I_0, \varepsilon)| \leqslant M\varepsilon \tag{4.6}$$

when $I_0 \leqslant I \leqslant I_0^{**}$, and $I_0^{**} > I_-$ is expressed below in (4.12).

Proof. Let all notations in Theorem 4.1 stand unless noted otherwise. Then

$$|D| \equiv |y(I_-, I_0, \varepsilon) - y(I_-, I^0, \varepsilon)|$$
$$\leqslant 2 \sum_{n \in \mathbb{Z}} \left| \int_{I_0}^{I^0} \hat{A}_n(s, \varepsilon)\, e^{1/\varepsilon \int_s^{I_-} \lambda_1(\tau)\, d\tau + in\omega s/\varepsilon}\, ds \right| \tag{4.7}$$

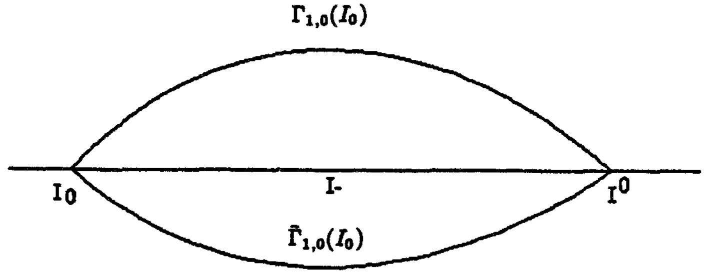

Fig. 5. We choose the integrating paths for terms which involve A_n and $\lambda_1(z)-in\omega$ according to the different values of $Im\,\lambda_1(I_-)-n\omega \equiv \omega_0 - n\omega$. When $\omega_0 - n\omega < 0$, we choose $\Gamma_{1,0}(I_0)$, and when $\omega_0 - n\omega > 0$, we choose $\bar{\Gamma}_{1,0}(I_0)$.

We now take $I^0 = I_0^* > I_-$ to be the point on the positive real axis such that $\phi_1(I^0) = \phi_1(I_0)$ i.e., $Re\int_{I_0}^{I^0} \lambda_1(\tau)\,d\tau = 0$. Such a point is unique from the monotonicity of the harmonic function $\phi_1(I) \equiv Re\int_{I_-}^{I} \lambda_1(\tau)\,d\tau$ for $I > I_-$. The corresponding level curve of $\phi_1(I)$ through I_0 is denoted as $\Gamma_{1,0}(I_0)$.

Let $\chi(I_0) \equiv \min_{z,n} Re\int_{I_-}^{z} (\lambda_1(\tau) - in\omega)\,d\tau$, where for $n \geqslant n_0$ such that $Im\,\lambda_1(I_-) - nw < 0$, z is taken on $\Gamma_{1,0}(I_0) \equiv \{z, Im\,z \geqslant 0,\ \phi_1(z) = Re\int_{I_-}^{z} \lambda_1(\tau)\,d\tau = \phi_1(I_0)\}$, and for $n \leqslant n_0 - 1$ such that $Im\,\lambda_1(I_-) - n\omega > 0$, z is taken on $\bar{\Gamma}_{1,0}(I_0) \equiv \{\bar{z}, Im\,z \geqslant 0,\ \phi_1(z) = Re\int_{I_-}^{z} \lambda_1(\tau)\,d\tau = \phi_1(I_0)\}$. See Fig. 5 for the integrating paths.

It is obvious from the definition that $\chi(I_0) \leqslant \phi_1(I_0)$ because $\phi_1(I_0)$ minimizes only at $n = 0$. We show that because of H4, $Im\,\lambda_1(I_-) - n\omega \neq 0$, for I_0 sufficiently close to the critical point I_-, $\chi(I_0) > 0$. In fact, $Re(\int_{z}^{I_-} \lambda_1(\tau)\,d\tau + in\omega z) = -Re\int_{I_-}^{z} (\lambda_1 - in\omega)\,d\tau$. We see from the discussion of level curves in Section 3 that since $Im\,\lambda_1(I_-) < 0$, the level curve $\Gamma_{1,0}(I_0)\ \subset \mathbb{C}^+$ is in the upper-half complex z-plane. See Fig. 5. Moreover, for n with $Im\,\lambda_1(I_-) - n\omega < Im\,\lambda_1(I_-) < 0$, the level curves $\Gamma_{1,n}(I_0) \equiv \{z \mid Im\,z \geqslant 0, Re\int_{I_-}^{z} (\lambda_1 - in\omega)\,d\tau = Re\int_{I_-}^{I_0} (\lambda_1(\tau) - in\omega)\,d\tau\}$ lie below $\Gamma_{1,0}(I_0)$, and $\Gamma_{1,n}(I_0) \subset \mathbb{C}^+$. In particular, $Re\int_{I_-}^{z} (\lambda_1 - in\omega)\,d\tau \geqslant Re\int_{I_-}^{z} \lambda_1(\tau)\,d\tau$ for $z \in \Gamma_{1,0}$. For n with $0 > Im\,\lambda_1(I_-) - n\omega > Im\,\lambda_1(I_-)$, $\Gamma_{1,n}(I_0)$ stay above $\Gamma_{1,0}(I_0)$. Thus there are only finitely many n's whose level curves need to be considered for the minimizing of $\chi(I_0)$. Also, for n with $Im\,\lambda_1(I_-) - n\omega > 0$, the situations are similar. Thus the minimum is indeed taken as

$$\chi(I_0) = \min\left\{ \min_{z \in \Gamma_{1,0},\ n:\ Im\,\lambda_1(I_-) \leqslant Im\,\lambda_1(I_-) - n\omega < 0} Re\int_{I_-}^{z} (\lambda_1(\tau) - in\omega)\,d\tau, \right.$$
$$\left. \min_{z \in \bar{\Gamma}_{1,0},\ n:\ -Im\,\lambda_1(I_-) > Im\,\lambda_1(I_-) - n\omega > 0} Re\int_{I_-}^{z} (\lambda_1(\tau) - in\omega)\,d\tau \right\} \tag{4.8}$$

where the minimums are taken from finitely many positive terms, and therefore $\chi(I_0) > 0$. The positiveness of each term on the right-hand side of (4.8) can be easily verified from the structure of (3.9) by an implicit function theorem argument. See Ref. 22 (Section 6) for more details of this sort.

To obtain the estimate of D, we use contour integrations to find the right-hand side of (4.7). For n's where $Im\,\lambda_1(I_-) - n\omega < 0$, we take $\Gamma_{1,0}$ as the path,

$$\begin{aligned}
&\left|\int_{I_0}^{I^0} \hat{A}_n(s, \varepsilon)\, e^{1/\varepsilon \int_s^{I_-} \lambda_1(\tau)\, d\tau + in\omega s/\varepsilon}\, ds\right| \\
&\quad = \left|\int_{z \in \Gamma_{1,0}(I_0)} \hat{A}_n(z, \varepsilon)\, e^{1/\varepsilon \int_z^{I_-} \lambda_1(\tau)\, d\tau + in\omega z/\varepsilon}\, dz\right| \\
&\quad \leqslant \int_{z \in \Gamma_{1,0}(I_0)} |\hat{A}_n(z, \varepsilon)|\, e^{1/\varepsilon\, Re(\int_z^{I_-} \lambda_1(\tau)\, d\tau + in\omega z/\varepsilon)}\, |dz| \\
&\quad \leqslant e^{-\chi(I_0)/\varepsilon}\, \|\hat{A}_n\|\, |\Gamma_{1,0}(I_0)| \qquad (4.9)
\end{aligned}$$

where $|\Gamma_{1,0}(I_0)|$ is the length of the level curve $\Gamma_{1,0}(I_0)$. Similarly, for n with $Im\,\lambda_1(I_-) - n\omega > 0$,

$$\begin{aligned}
&\left|\int_{I_0}^{I^0} \hat{A}_n(s, \varepsilon)\, e^{1/\varepsilon \int_s^{I_-} \lambda_1(\tau)\, d\tau + in\omega s/\varepsilon}\, ds\right| \\
&\quad = \left|\int_{z \in \bar{\Gamma}_{1,0}(I_0)} \hat{A}_n(z, \varepsilon)\, e^{1/\varepsilon \int_z^{I_-} \lambda_1(\tau)\, d\tau + in\omega z/\varepsilon}\, dz\right| \\
&\quad \leqslant e^{-\chi(I_0)/\varepsilon}\, \|\hat{A}_n\|\, |\Gamma_{1,0}(I_0)| \qquad (4.10)
\end{aligned}$$

Thus we obtain

$$\begin{aligned}
|D| &= |y(I_-, I_0, \varepsilon) - y(I_-, I^0, \varepsilon)| \\
&\leqslant 2e^{-\chi(I_0)/\varepsilon} \sum_{n \in \mathbb{Z}} \|\hat{A}_n\|\, |\Gamma_{1,0}(I_0)| \qquad (4.11)
\end{aligned}$$

Let $M > M_2$ be a positive number, $M_6 \equiv 2\sum_{n \in \mathbb{Z}} \|\hat{A}_n\|\, |\Gamma_{1,0}(I_0)|$ and $M_0 \equiv (M - M_2)/M_3 M_6 > 0$, where the constants M_2 and M_3 are referred to in Lemma 2.2 and Proposition 2.4.

We define

$$I_0^{**} \equiv \inf \left\{ I \mid I_- < I \leqslant I^0, Re \int_{I_-}^{I} \lambda_1(\tau)\, d\tau = \chi(I_0) - \varepsilon \left| \ln \frac{\varepsilon}{M_0} \right| \right\} \tag{4.12}$$

From Lemma 2.2 and Lemma 2.3, we obtain that $|y(I, I_0, \varepsilon)| \leqslant M_2\varepsilon$ for $I_0 < I < I_-$, and $|y(I, I^0, \varepsilon)| \leqslant M_2\varepsilon$ for $I_- < I < I^0$. We continue the solution $y(I, I_0, \varepsilon)$ for $I > I_-$ to obtain Theorem 4.2. By Proposition 2.4, there exists M_3 such that

$$\begin{aligned} |y(I, I_0, \varepsilon) - y(I, I^0, \varepsilon)| &\leqslant M_3\, |y(I_-, I_0, \varepsilon) - y(I_-, I^0, \varepsilon)|\, e^{1/\varepsilon\, Re \int_{I_0}^{I} \lambda_1(\tau)\, d\tau} \\ &\leqslant M_6 M_3 e^{-(1/\varepsilon)\chi(I_0)} e^{1/\varepsilon\, Re \int_{I_-}^{I} \lambda_1(\tau)\, d\tau}. \end{aligned} \tag{4.13}$$

Thus by the definition of I_0^{**}, when $I_- < I < I_0^{**}$, $|y(I, I_0, \varepsilon) - y(I, I^0, \varepsilon)| \leqslant M_6 M_3 \varepsilon / M_0$. Lemma 2.3 then implies that $|y(I, I_0, \varepsilon)| \leqslant (M_2 + M_6 M_3 / M_0)\varepsilon \leqslant M\varepsilon$ when $I_0 < I < I_0^{**}$ for any $M > M_2$ by the choice of M_0 mentioned above. □

Corollary 4.3. Let H1–H5 hold. Let $y(I, I_i, M_1, \varepsilon)$ be any solution of (2.16) which satisfies the initial conditions $|y(I, I_i, M_1, \varepsilon)|\,|_{I=I_0} \leqslant M_1\varepsilon$, then $|y(I, I_i, M_1, \varepsilon)| \leqslant M\varepsilon$ whenever $I_0 \leqslant I \leqslant I_q$ for any $M > M_2 + M_3(M_1 + M_2) > 0$, $0 < \varepsilon \leqslant \varepsilon_0 = \varepsilon_0(M)$ where $I_q \geqslant \max(I_0^*, I_0^{**})$.

The proof follows from Theorems 4.1 and 4.2 and Proposition 2.4.

Corollary 4.4. Assume H1–H5 hold. Let $I_i < I_0$ be any point left of I_0 which was studied in Theorems 4.1 and 4.2 and Corollary 4.3. If $y(I, I_i, M_1, \varepsilon)$ is a solution of (2.16) with its initial condition $|y(I, I_i, M_1, \varepsilon)|\,|_{I=I_i} \leqslant M_1\varepsilon$, then for any $M > M_2 + 2M_2M_3$, $|y(I, I_i, M_1, \varepsilon)| \leqslant M\varepsilon$ whenever $I_i \leqslant I \leqslant I_q$ for some $I_q \geqslant \max(I_0^*, I_0^{**})$ and $\varepsilon \leqslant \varepsilon_0$.

Proof. Since $|y(I, I_i, M_1, \varepsilon)|\,|_{I=I_i} \leqslant M_1\varepsilon$, $|y(I, I_i, M_1, \varepsilon)|\,|_{I=I_0} \leqslant M_2\varepsilon$ by Lemma 2.2. Corollary 4.3 then implies Corollary 4.4. □

Corollary 4.5 (Memory Effects). Assume H1–H5 hold. Also assume the initial point I_i: $I_0 < I_i < I_-$ is closer to I_- than the point I_0 which was discussed in Theorems 4.1–4.2. Let $y(I, I_i, M_1, \varepsilon)$ be a solution of (2.16) with the initial condition $|y(I, I_i, M_1, \varepsilon)|\,|_{I=I_i} \leqslant M_1\varepsilon$. Let $M_5 \equiv M + M_3(M_1 + M_2)$ (M is referred to in Theorem 4.2). Then $|y(I, I_i, M_1, \varepsilon)| \leqslant M_5\varepsilon$ whenever $I_i < I \leqslant I_q$ where $I_q \geqslant \min(I_0^{**}, I^i)$ and I^i is defined as the unique point $I^i > I_-$ such that $Re \int_{I_i}^{I^i} \lambda_1(\tau)\, d\tau = 0$. Furthermore, there exists r_8 which is independent of ε such that if $|I_i - I_-| \leqslant r_8$, then $[I_i, I^i] \subset (I_0, I_0^{**})$.

If the initial values of $y(I, I_i, M_1, \varepsilon)|_{I=I_i}$ *satisfy* $|y(I, I_i, M_1, \varepsilon) - y(I, I_0, \varepsilon)||_{I=I_i} = M_4\varepsilon$ *for some* $M_4 > 0$, *then for any* $L > 0$, $|y(I, I_i, M_1, \varepsilon) - y(I, I_0, \varepsilon)| < L\varepsilon$ *when and only when* $I_i \leqslant I \leqslant I_q$ *for some* $I_q \in [I^i + \varepsilon \ln[L/(M_3 M_4 Re(\lambda_1(I^i)))] + O(\varepsilon^2), I^i + \varepsilon \ln[LM_3/(M_4 Re(\lambda_1(I^i)))] + O(\varepsilon^2)]$, *and* $|y(I_q, I_i, M_1, \varepsilon) - y(I_q, I_0, \varepsilon)| = L\varepsilon$. *Consequently, if we choose* $L > M$ (M *is referred to in Theorem 4.2*), *then* $|y(I, I_i, M_1, \varepsilon)| \leqslant (L+M)\varepsilon$ *when* $I_i \leqslant I \leqslant I_q$ *and* $|y(I_q, I_i, M_1, \varepsilon)| \geqslant (L-M)\varepsilon$.

Proof. From the initial condition $|y(I, I_i, M_1, \varepsilon)||_{I=I_i} \leqslant M_1\varepsilon$, and $|y(I, I_0, \varepsilon)||_{I=I_i} \leqslant M_2\varepsilon$ from Lemma 2.2, Proposition 2.4 gives

$$\begin{aligned}|y(I, I_i, M_1, \varepsilon) - y(I, I_0, \varepsilon)| &= K(I)|y(I_i, \varepsilon) - y(I_i, I_0, \varepsilon)|\, e^{Re \int_{I_i}^{I} \lambda_1(\tau)\, d\tau} \\ &\leqslant K(I)(M_2 + M_1)\, \varepsilon e^{Re \int_{I_i}^{I} \lambda_1(\tau)\, d\tau} \qquad (4.14)\end{aligned}$$

for some $(1/M_3) \leqslant K(I) \leqslant M_3$. Since $|y(I, I_0, \varepsilon)| \leqslant M\varepsilon$ for $I_0 \leqslant I \leqslant I_0^{**}$ by Theorem 4.2 and $Re \int_{I_-}^{I} \lambda_1(\tau)\, d\tau \leqslant 0$ for $I_i \leqslant I \leqslant I^i$, we derive that when $I_i \leqslant I \leqslant I^i$, $|y(I, I_i, M_1, \varepsilon)| \leqslant M_5\varepsilon$ for $M_5 = M + M_3(M_1 + M_2)$.

Further, if $|I_i - I_-| \leqslant r_8$ such that $[I_i, I^i] \subset (I_0, I_0^{**})$, then $|y(I, I_0, \varepsilon)| \leqslant M\varepsilon$ for $I \in [I_i, I^i]$. From the initial condition $|y(I, I_i, M_1, \varepsilon) - y(I, I_0, \varepsilon)||_{I=I_i} = M_4\varepsilon$, it follows from Proposition 2.4 that

$$\begin{aligned}|y(I, I_i, M_1, \varepsilon) - y(I, I_0, \varepsilon)| &= K(I)\, |y(I_i, \varepsilon) - y(I_i, I_0, \varepsilon)|\, e^{Re \int_{I_i}^{I} \lambda_1(\tau)\, d\tau} \\ &= K(I)\, M_4 \varepsilon e^{Re \int_{I_i}^{I} \lambda_1(\tau)\, d\tau}\end{aligned}$$

for some $(1/M_3) \leqslant K(I) \leqslant M_3$.

Thus for any $L > 0$, when $I_i \leqslant I < I^i + \varepsilon \ln[L/(M_3 M_4\, Re(\lambda_1(I^i)))] + O(\varepsilon^2)$, $Re \int_{I_i}^{I} \lambda_1(\tau)\, d\tau \leqslant \varepsilon \ln(L/(M_3 M_4))$ and $|y(I, I_i, M_1, \varepsilon) - y(I, I_0, \varepsilon)| < L$, and when $I^i + \varepsilon \ln[LM_3/(M_4\, Re(\lambda_1(I^i)))] + O(\varepsilon^2) \leqslant I \leqslant I_0^{**}$, $Re \int_{I_i}^{I} \lambda_1(\tau)\, d\tau \geqslant \varepsilon \ln(LM_3/M_4)$ and $|y(I, I_i, M_1, \varepsilon) - y(I, I_0, \varepsilon)| \geqslant L$. Therefore, $I_q \in [I^i + \varepsilon \ln[L/(M_3 M_4\, Re(\lambda_1(I^i)))] + O(\varepsilon^2),\ I^i + \varepsilon \ln[LM_3/(M_4\, Re(\lambda_1(I^i)))] + O(\varepsilon^2)]$. Let $L > M$. From Theorem 4.2, we get $|y(I, I_0, \varepsilon)| \leqslant M\varepsilon$ for $I \in [I_i, I^i]$. Then

$$\begin{aligned}|y(I, I_i, M_1, \varepsilon)| &\leqslant |y(I, I_0, \varepsilon)| + K(I)\, M_4 \varepsilon e^{Re \int_{I_i}^{I} \lambda_1(\tau)\, d\tau} \\ &< (L+M)\varepsilon\end{aligned}$$

when $I_i \leqslant I < I_q$, and

$$\begin{aligned}|y(I_q, I_i, M_1, \varepsilon)| &\geqslant -|y(I_q, I_0, \varepsilon)| + L\varepsilon \\ &\geqslant (L-M)\varepsilon\end{aligned}$$

when $I_q \leqslant I \leqslant I_0^{**}$. □

We also provide a result showing that the estimates I_0^{**}, the amounts of delays in Theorem 4.2, are indeed sharp for the near resonance frequency cases.

Theorem 4.6. *Assume (H1)–(H5) hold. Also, assume the initial point* I_i *is close enough to* I_- *so that we can find two points* I_0 *and* I^0 *with* $I_0 \leqslant I_i \leqslant I_- \leqslant I^0$ *and* $|I_0 - I^0| \leqslant \sigma_4$ *for some constant* $\sigma_4 >$ *to be specified below. Let* $y(I, I_i, M_1, \varepsilon)$ *be a solution of (2.16) with initial conditions* $|y(I, I_i, M_1, \varepsilon)|\,|_{I=I_i} \leqslant M_1 \varepsilon$. *Then* $|y(I, I_i, M_1, \varepsilon)| \leqslant M\varepsilon$ *when and only when* $I_i \leqslant I \leqslant I_q$ *for some* $I_q = I_q(I_i, \omega) > I_-$ *as stated in Corollary 4.5. Assume the generic condition that* $\hat{A}_m$ *in (3.34) satisfies* $k_1 \varepsilon^l \geqslant |\hat{A}_m(z, \varepsilon)\, e^{im\omega I_-/\varepsilon}| \geqslant |Re\, \hat{A}_m(z, \varepsilon)\, e^{im\omega I_-/\varepsilon}| \geqslant k_2 \varepsilon^l$ *when* $|z - I_-| \leqslant r_7$ *for some* $m \in \mathbb{Z}$, $r_7 > 0$ *and a finite integer* $l > 0$. *Assume in addition that* $Re\, \lambda_1(I)$ *is odd about* I_- *and* $Im\, \lambda_1(I)$ *is even about* I_-. *If the frequency* ω *is near the resonance frequencies so that* $\omega_0 - m\omega = \delta \to 0$, *then the separating point* I_q *satisfies* $c_1 |\delta| \leqslant |I_q - I_-| \leqslant c_2 |\delta|$ *for some constants* $c_j > 0$.

Proof. We let the notations in Theorem 4.2 stand. Thus we get

$$2 \sum_{n \in \mathbb{Z}} \left| \int_{I_0}^{I^0} \hat{A}_n(s, \varepsilon)\, e^{1/\varepsilon \int_s^{I_-} \lambda_1(\tau)\, d\tau + in\omega s/\varepsilon}\, ds \right|$$

$$\geqslant |D| \geqslant \left| 2\, Re \int_{I_0}^{I^0} \hat{A}_m(s, \varepsilon)\, e^{1/\varepsilon \int_s^{I_-} \lambda_1(\tau)\, d\tau + im\omega s/\varepsilon}\, ds \right|$$

$$-2 \sum_{n \neq m} \left| \int_{I_0}^{I^0} \hat{A}_n(s, \varepsilon)\, e^{1/\varepsilon \int_s^{I_-} \lambda_1(\tau)\, d\tau + in\omega s/\varepsilon}\, ds \right| \tag{4.15}$$

Using the assumption that $Re\, \lambda_1(I)$ is odd about I_- and $Im\, \lambda_1(I)$ is even about I_-,

$$\lambda_1(I) - im\omega = a_1(I - I_-) + \sum_{n=1}^{\infty} a_n (I - I_-)^{2n+1}$$

$$+ i \left(\omega_0 - m\omega + \sum_{n=1}^{\infty} b_n (I - I_-)^{2n} \right) \tag{4.16}$$

where $a_1 > 0$ as assumed in H3, a_n and b_n are real numbers. Let $z = I_- + x + iy$. Then we obtain

$$\int_z^{I_-} (\lambda_1(\tau) - im\omega)\, d\tau$$

$$= -\frac{1}{2} a_1 (z - I_-)^2 - \sum_{n=1}^{\infty} \frac{1}{2n+2} a_n (z - I_-)^{2n+2}$$

$$- i\left((\omega_0 - m\omega)(z - I_-) + \sum_{n=1}^{\infty} \frac{1}{2n+1} b_n (z - I_-)^{2n+1}\right)$$

$$= -\frac{1}{2} a_1 (x + iy)^2 - \sum_{n=1}^{\infty} \frac{1}{2n+2} a_n (x + iy)^{2n+2}$$

$$- i\left((\omega_0 - m\omega)(x + iy) + \sum_{n=1}^{\infty} \frac{1}{2n+1} b_n (x + iy)^{2n+1}\right)$$

$$= -\frac{1}{2} a_1 x^2 + \frac{1}{2} a_1 y^2 + \delta y + O(x^4) + O(y^2)$$

$$- i[\delta x + a_1 xy + x(O(x^2) + O(y^2))]. \tag{4.17}$$

To solve the equation

$$Im \int_z^{I_-} (\lambda_1(\tau) - im\omega)\, d\tau = \delta x + a_1 xy + x(O(x^2) + O(y^2))] = 0,$$

we first let $\delta = 0$. There exists $\sigma_0 > 0$ such that if $|x| \leqslant \sigma_0$, then there exists $y = g_0(x)$ which is the solution of $Im \int_z^{I_-} (\lambda_1(\tau) - im\omega)\, d\tau = 0$ at $\delta = 0$. Using the implicit function theorem near $\delta = 0$, we see that there exists $\delta_4 > 0$ such that if $\delta < \delta_4$, we can find a function $y = g(x, \delta) = -(\delta/a_1) + O(\delta^2) + O(|x|^2)$ which is the solution of $Im \int_z^{I_-} (\lambda_1(\tau) - im\omega)\, d\tau = 0$, and $g(x, 0) = g_0(x)$. This implies that there exists a curve l_g in the complex z-plane which has the form $z = I_- + x + iy(x, \delta)$, where $y = g(x, \delta) = -(\delta/a_1) + O(\delta^2) + O(|x|^2)$ such that $Im \int_z^{I_-} (\lambda_1(\tau) - in\omega)\, d\tau = 0$ for $z \in l_g$. We denote $v(I_0) = \{z = I_0 + i\theta g(I_0 - I_-, \delta), 0 \leqslant \theta \leqslant 1\}$, $v(I^0) = \{z = I^0 + i\theta g(I^0 - I_-, \delta), 0 \leqslant \theta \leqslant 1\}$. Thus to replace the integration on the interval (I_0, I^0) for the term related to $\hat{A}_m(z, \varepsilon)$, we construct the contour by following the vertical segment $v(I_0)$ from $z = I_0$ to l_g, then following l_g, and then the vertical segment $v(I^0)$ to $z = I^0$. Namely,

$$\int_{I_0}^{I^0} \hat{A}_m(s, \varepsilon)\, e^{1/\varepsilon \int_s^{I_-} \lambda_1(\tau)\, d\tau + im\omega s/\varepsilon}\, ds$$

$$= \int_{v(I_0) \cup l_g \cup v(I^0)} \hat{A}_m(z, \varepsilon)\, e^{1/\varepsilon \int_z^{I_-} \lambda_1(\tau)\, d\tau + im\omega z/\varepsilon}\, dz$$

For $z \in l_g$, $Im \int_z^{I_-} (\lambda_1(\tau) - im\omega)\, d\tau = 0$ as stated, and from (4.17), we have

$$Re \int_z^{I_-} (\lambda_1(\tau) - im\omega)\, d\tau = -\frac{1}{2} a_1 x^2 + \frac{1}{2} a_1 y^2 + \delta y + O(x^4) + O(y^2)$$

$$= -\frac{1}{2} a_1 x^2 - \frac{\delta^2}{2a_1} + O(x^4) + O(\delta^4)$$

Thus there exist $\sigma_1 > 0$ and $\delta_4' > 0$ such that when $|x| \leqslant \sigma_1$ and $|\delta| \leqslant \delta_4'$,

$$-\frac{1}{2} a_1 x^2 - \frac{\delta^2}{a_1} \leqslant Re \int_z^{I_-} (\lambda_1(\tau) - im\omega)\, d\tau$$

$$\leqslant -\frac{1}{4} a_1 x^2 - \frac{\delta^2}{4a_1}$$

Also, when $z \in l_g$, $dz/dx = 1 + ig'(x, \delta)$, where $g'(x, \delta) = O(x) + O(\delta)$. Under the general assumption that $k_1 \varepsilon^l \geqslant |\hat{A}_m(z, \varepsilon)\, e^{im\omega I_-/\varepsilon}| \geqslant |Re\, \hat{A}_m(z, \varepsilon)\, e^{im\omega I_-/\varepsilon}| \geqslant k_2 \varepsilon^l$ when $|z - I_-| \leqslant r_7$ for a finite integer $l > 0$, we get

$$\left| Re \int_{l_g} \hat{A}_m(z, \varepsilon)\, e^{1/\varepsilon \int_z^{I_-} \lambda_1(\tau)\, d\tau + im\omega z/\varepsilon}\, dz \right|$$

$$= \left| Re \int_{l_g} \hat{A}_m(z, \varepsilon)\, e^{1/\varepsilon \int_z^{I_-} (\lambda_1(\tau) - im\omega)\, d\tau + im\omega I_-/\varepsilon}\, dz \right|$$

$$\leqslant \left| \int_{l_g} \|\hat{A}_m(z, \varepsilon)\|\, e^{1/\varepsilon\, Re \int_z^{I_-} (\lambda_1(\tau) - im\omega)\, d\tau}\, |z'(x)|\, dx \right|$$

$$\leqslant \int_{I_0}^{I^0} k_1 \varepsilon^l e^{-1/(4\varepsilon) a_1 x^2 - \delta^2/(4a_1\varepsilon)} (1 + C_9 |x| + C_9 |\delta|)\, dx$$

$$\leqslant \int_{-\infty}^{\infty} k_1 \varepsilon^l e^{-1/(4\varepsilon) a_1 x^2 - \delta^2/(4a_1\varepsilon)} (1 + C_9 \sigma_4 + C_9 |\delta|)\, dx$$

$$\leqslant c_1 \varepsilon^l \sqrt{\varepsilon}\, e^{-\delta^2/(4a_1\varepsilon)} \tag{4.18a}$$

On the other hand, since $Im\int_z^{I_-}(\lambda_1(\tau)-im\omega)\,d\tau=0$ for $z\in l_g$,

$$
\begin{aligned}
&\left|Re\int_{l_g}\hat{A}_m(z,\varepsilon)\,e^{1/\varepsilon\int_z^{I_-}\lambda_1(\tau)\,d\tau+im\omega z/\varepsilon}\,dz\right| \\
&=\left|Re\int_{l_g}\hat{A}_m(z,\varepsilon)\,e^{1/\varepsilon\int_z^{I_-}(\lambda_1(\tau)-im\omega)\,d\tau+im\omega I_-/\varepsilon}\,dz\right| \\
&=\left|\int_{l_g}[Re\,\hat{A}_m(z,\varepsilon)\,e^{im\omega I_-/\varepsilon}\,z'(x)]\,e^{1/\varepsilon\,Re\int_z^{I_-}(\lambda_1(\tau)-im\omega)\,d\tau}\,dx\right| \\
&=\left|\int_{l_g}[Re\,\hat{A}_m(z,\varepsilon)\,e^{im\omega I_-/\varepsilon}-g'(x,\delta)\,Im\,\hat{A}_m(z,\varepsilon)\,e^{im\omega I_-/\varepsilon}]\right. \\
&\qquad\left.\times e^{1/\varepsilon\,Re\int_z^{I_-}(\lambda_1(\tau)-im\omega)\,d\tau}\,dx\right| \\
&\geqslant\int_{I_0-I_-}^{I^0-I_-}[k_1\varepsilon^l-C_9(|x|+\delta)\sqrt{k_1^2-k_2^2}\,\varepsilon^l]\,e^{-(1/\varepsilon)\,a_1x^2-\delta^2/(a_1\varepsilon)}\,dx \\
&\geqslant\int_{I_0-I_-}^{I^0-I_-}\varepsilon^l e^{-(1/\varepsilon)\,a_1x^2-\delta^2/(a_1\varepsilon)}(k_1-(C_9\sigma_4+C_9\,|\delta_4|)\sqrt{k_1^2-k_2^2})\,dx \\
&\geqslant c_0\varepsilon^l e^{-\delta^2/(a_1\varepsilon)}\int_{I_0-I_-}^{I^0-I_-}e^{-(1/\varepsilon)\,a_1x^2}\,dx \\
&=\frac{c_0\varepsilon^l\sqrt{\varepsilon}}{\sqrt{a_1}}\,e^{-\delta^2/(a_1\varepsilon)}\int_{(I_0-I_-)/\sqrt{\varepsilon}}^{(I^0-I_-)/\sqrt{\varepsilon}}e^{-y^2}\,dy\qquad\text{by letting}\quad y=\frac{\sqrt{a_1}\,x}{\sqrt{\varepsilon}} \\
&\geqslant c_0'\varepsilon^l\sqrt{\varepsilon}\,e^{-\delta^2/(a_1\varepsilon)}
\end{aligned}
\tag{4.18b}
$$

where C_9 is a constant independent of ε and δ, δ_4 and σ_4 are taken to be sufficiently small to allow $k_1-(C_9\delta_4+C_9\sigma_4)\sqrt{k_1^2-k_2^2}>0$.

For $z\in v(I_0)$, we can write $z=I_0+i\theta g(I_0-I_-,\delta)$, then

$$
\begin{aligned}
&Re\int_z^{I_-}(\lambda_1(\tau)-im\omega)\,d\tau \\
&\quad=-\frac{1}{2}a_1x^2+\frac{1}{2}a_1y^2+\delta y+O(x^4)+O(y^2) \\
&\quad=-\frac{1}{2}a_1(I_0-I_-)^2+\frac{a_1\theta^2g^2(I_0-I_-,\delta)}{2} \\
&\qquad+\delta\theta g(I_0-I_-,\delta)+O((I_0-I_-)^4)+O(\theta^4g^4(I_0-I_-,\delta)) \\
&\quad\leqslant-\frac{1}{2}a_1(I_0-I_-)^2+\frac{a_1g^2(I_0-I_-,\delta)}{2} \\
&\qquad+\delta g(I_0-I_-,\delta)+O((I_0-I_-)^4)+O(g^4(I_0-I_-,\delta))
\end{aligned}
$$

since $0 \leqslant \theta \leqslant 1$. Using the property that $g(x, \delta) = -(\delta/a_1) + O(\delta^2) + O(|x|^2)$, we can further derive that there exists $C_{10} > 0$ such that

$$Re \int_z^{I_-} (\lambda_1(\tau) - im\omega)\, d\tau \leqslant -\tfrac{1}{2}a_1(I_0 - I_-)^2 + C_{10}(I_0 - I_-)^4 + C_{10}\delta^2$$
$$\leqslant -\tfrac{1}{4}a_1(I_0 - I_-)^2$$

when $\delta \leqslant \delta_5 \equiv \delta_5(I_0)$. Also, $|z'(\theta)| = |ig(I_0 - I_-, \delta)| = (\delta/a_1) + O(\delta^2) + O((I_- - I_0)^2)$. Similarly, there exists $\delta_6 \equiv \delta_6(I^0)$ such that for $\delta \leqslant \delta_6$, when $z \in v(I^0)$,

$$Re \int_z^{I_-} (\lambda_1(\tau) - im\omega)\, d\tau \leqslant -\tfrac{1}{4}a_1(I^0 - I_-)^2$$

and also, $|z'(\theta)| = |ig(I^0 - I_-, \delta)| = (\delta/a_1) + O(\delta^2) + O((I^0 - I_-)^2)$. Thus we derive the estimate that when $|\omega_0 - m\omega| = |\delta| \to 0^+$, i.e., $|\delta| \leqslant \delta_7 = \min(\delta_5, \delta_6)$,

$$\left| \int_{v(I_0) \cup v(I^0)} \hat{A}_m(z, \varepsilon)\, e^{1/\varepsilon \int_z^{I_-} \lambda_1(\tau)\, d\tau + im\omega z/\varepsilon}\, dz \right|$$
$$\leqslant \left| \int_{v(I_0)} \|\hat{A}_m(z, \varepsilon)\|\, e^{1/\varepsilon\, Re \int_z^{I_-} (\lambda_1(\tau) - im\omega)\, d\tau}\, |z'(\theta)|\, d\theta \right|$$
$$+ \left| \int_{v(I^0)} \|\hat{A}_m(z, \varepsilon)\|\, e^{1/\varepsilon\, Re \int_z^{I_-} (\lambda_1(\tau) - im\omega)\, d\tau}\, |z'(\theta)|\, d\theta \right|$$
$$\leqslant \int_0^1 \|\hat{A}_m(z, \varepsilon)\|\, e^{-(1/4\varepsilon)\, a_1(I_0 - I_-)^2} \delta(1 + C_9\, |I_0 - I_-| + C_9\, |\delta|)\, d\theta$$
$$+ \int_0^1 \|\hat{A}_m(z, \varepsilon)\|\, e^{-(1/4\varepsilon)\, a_1(I^0 - I_-)^2} \delta(1 + C_9\, |I^0 - I_-| + C_9\, |\delta|)\, d\theta$$
$$\leqslant c_2 \varepsilon^l e^{-(1/(4\varepsilon))\, a_1 \min((I_0 - I_-)^2, (I^0 - I_-)^2)} \tag{4.18c}$$

Thus when $\delta \leqslant \delta_7$ and $\varepsilon \leqslant \varepsilon_0$,

$$C_2 e^{K_2\delta^2/(-\varepsilon)} \varepsilon^l \sqrt{\varepsilon} \geqslant \left| Re \int_{I_0}^{I^0} \hat{A}_m(s, \varepsilon)\, e^{1/\varepsilon \int_s^{I_-} \lambda_1(\tau)\, d\tau + im\omega s/\varepsilon}\, ds \right|$$
$$= \left| Re \int_{l_g \cup v(I_0) \cup v(I^0)} \hat{A}_m(z, \varepsilon)\, e^{1/\varepsilon \int_z^{I_-} \lambda_1(\tau)\, d\tau + im\omega z/\varepsilon}\, dz \right|$$
$$\geqslant C_1 e^{-K_1\delta^2/\varepsilon} \sqrt{\varepsilon}\, \varepsilon^l \tag{4.19}$$

for $K_1 \equiv a_1$ and $K_2 \equiv a_1/4$.

The rest of the terms are much smaller since the frequencies are not close to each other in the sense that as $\delta \to 0$, $|\omega_0 - n\omega| > \omega - \delta > \frac{1}{2}\omega$ for $n \neq m$, and therefore for $n \neq m$,

$$\left| \int_{I_0}^{I^0} \hat{A}_n(s, \varepsilon)\, e^{1/\varepsilon \int_s^{I_-} \lambda_1(\tau)\, d\tau + in\omega s/\varepsilon}\, ds \right| \leqslant \|\hat{A}_n\|\, |\Gamma_{1,0}|\, e^{-\chi_1(I_0, \omega)/\varepsilon} \tag{4.20}$$

for some $\chi_1(I_0, \omega) > 0$ which is independent of δ as $\delta \to 0$. We then obtain

$$\sum_{n \neq m} \left| \int_{I_0}^{I^0} \hat{A}_n(s, \varepsilon)\, e^{1/\varepsilon \int_s^{I_-} \lambda_1(\tau)\, d\tau + in\omega s/\varepsilon}\, ds \right| \leqslant M e^{-\chi_1(I_0, \omega)/\varepsilon} \tag{4.21}$$

Thus we have

$$\begin{aligned} &C_2 \varepsilon^l \sqrt{\varepsilon}\, e^{-K_2 \delta^2/\varepsilon} + M e^{-\chi_1(I_0, \omega)/\varepsilon} \\ &\qquad \geqslant |D| \geqslant \varepsilon^l \sqrt{\varepsilon}\, C_1 e^{-K_1 \delta^2/\varepsilon} - M e^{-\chi_1(I_0, \omega)/\varepsilon} \end{aligned} \tag{4.22}$$

When δ is sufficiently small but fixed, we find that for $\varepsilon \leqslant \varepsilon_0$, $|D|$ is bounded from above and from below by $\varepsilon^l \sqrt{\varepsilon}\, e^{-K_j \delta^2/\varepsilon}$. We let $I_{\delta, j} \equiv \inf\{I \mid I > I_-, \int_{I_-}^{I} Re\, \lambda_1(\tau)\, d\tau = K_j \delta^2 - \varepsilon\, |\ln(\varepsilon^l \sqrt{\varepsilon}/M)|\} = I_- + d_j \delta + O(\varepsilon\, |\ln \varepsilon|)$ for $j = 1, 2$. Then following Proposition 2.4, we can show that for $\varepsilon \leqslant \varepsilon_0$, δ is sufficiently small, $|y(I, I_i, \varepsilon)| \leqslant M\varepsilon$ when and only when $I_i \leqslant I \leqslant I_q$ for some $I_{\delta, 1} \leqslant I_q \leqslant I_{\delta, 2}$. Thus the delay amount is a multiple of δ. Finally, we note that this estimate I_δ is consistent with I_0^{**} defined in (4.12) as $|\delta| \to 0^+$ in the sense that they are in the same order, and therefore I_0^{**} is a sharp estimate in the near resonant cases. □

Remark. The assumption on $|Re(\hat{A}_m(z, \varepsilon)\, e^{im\omega I_-/\varepsilon})|$ is a general assumption similar to (4.4). There are exceptional cases. For example, in the sample equation (2.8), if we write it into equations in real variables, we find $\hat{A}_k(z, \varepsilon) \neq 0$ only for $k = 1$ there. In other words, the only resonant frequency there is $\omega = |\omega_0|$. But this is a very special case since the system is linear and the perturbation is of a single term. In general cases, we should expect a full spectrum of all frequencies as in (2.10). Near the resonant frequency region, the delay amounts may decay in a linear fashion as the frequency approaches the resonant frequency.

We also consider the situation where ω is far away from the resonant frequencies such as $\omega > 2\, |\omega_0|$.

Corollary 4.7. *Let all the assumptions and notations be as in Corollary 4.5. If the frequency ω is far away from the resonance region so that*

$2|\omega_0|+\delta \leqslant \omega$ for some $\delta>0$, then the separating point I_q satisfies $|I_q - I_i^*| \leqslant K\varepsilon |\ln \varepsilon|$ for some constant $K>0$, where $I_i^* > I_-$ is the unique point in the real axis that satisfies the relation $Re \int_{I_i}^{I_i^*} \lambda_1(\tau)\, d\tau = 0$.

Proof. We understand this result as saying that when $2|\omega_0|+\delta \leqslant \omega$, the delayed bifurcation phenomena are not much different from the cases of delayed Hopf bifurcations without perturbations. The proof is, in fact, imbedded within the proof of Theorems 4.1 and 4.2. We notice that if $2|\omega_0|+\delta \leqslant \omega$, then B_0 itself is $\sum_n$ accessible for any $n \in \mathbb{Z}$. Therefore $B_\xi = B_0$. Further, for $n \geqslant 0 (n<0$ rep.), when $z \in \Gamma_{1,0}(I_0) (z \in \bar{\Gamma}_{1,0}(I_0)$ resp.), we have

$$\varphi_n(z) \equiv Re\left(\int_{I_-}^{z} \lambda_1(\tau)\, d\tau - in\omega z\right) \geqslant Re \int_{I_-}^{z} \lambda_1(\tau)\, d\tau \qquad (4.23)$$

Thus the terms associated with $n=0$ are the only dominating terms in the integrations of (4.7). Thus $I_q = I_i^* + O(\varepsilon |\ln \varepsilon|)$. □

Remark. It is obvious from Theorem 4.6 and Corollary 4.7 that the delayed Hopf bifurcations can sustain the perturbations of high frequencies much better than lower frequencies. In one way, we say all resonant frequencies are distributed by the formula $\omega = 2|\omega_0|/n$ for $n \in \mathbb{N}$ which are below $2|\omega_0|$. In another way, we observe that in the region $\{\omega \mid \omega < 2|\omega_0|\}$, the delay amount is typically $I_0^{**} \leqslant I_i^*$, where I_0^{**} was specified in (4.12), and in the region $\{\omega \mid \omega > 2|\omega_0|\}$, the delay amount is I_i^*, which is the maximal delay.

5. PERIODIC FORCING WITH SLOWLY VARYING FREQUENCIES AND SHIFTED INTERFERENCES

We now study system (2.1) with a periodic perturbation g which is of a slowly changing frequency $\omega = \omega(I_i + \varepsilon t)$. Such processes are commonly seen in real physical experiments, where various parameters usually vary slowly according to time.

We study the system

$$\frac{\partial u}{\partial t} = F(u, I_i + \varepsilon t, \varepsilon, t) \qquad (5.1a)$$

$$u(t)|_{t=0} = u_0(I_i) + O(\varepsilon) \qquad (5.1b)$$

under hypotheses (H1), (H3) as well as (H2)′, (H4)′, and (H5)′ as follows.

Assumptions. *(H2)′ For each fixed I, and ε set to 0, the reduced system*

$$v_t = f(v, I) \tag{5.2}$$

has an equilibrium solution $u_0(I)$ which is also analytic in I for $|I| < r_a$ for some $r_a > 0$. The perturbation $f_1(u, I, \varepsilon, t)$ is a periodic function of the variable t with the period $2\pi/\omega(I)$, where $\omega(I) > 0$ for $-r_a \leqslant I \leqslant r_a$ and $\omega(I)$ is analytic in I. In particular, we may express

$$f_1(u, I, \varepsilon, t) = \sum_{n \in \mathbb{Z}} C_n(u, I, \varepsilon)\, e^{in\omega(I)t} \tag{5.3}$$

where C_n are analytic in all variables and $\sum_{n \in \mathbb{Z}} |C_n(u, I, \varepsilon)| \leqslant M$ for $|u| < \sigma$, $|I| < r_a$ in the complex plane and $\varepsilon \leqslant \varepsilon_0$.

(H4)′. (First Nonresonance Condition) Assume for $|z - I_-| < r_b$ in the complex plane for $0 < r_b \leqslant r_a$, $\lambda_1(z) - in(\omega(z) + \omega'(z)(z - I_i)) \neq 0$, $\lambda_2(z) - in(\omega(z) + \omega'(z)(z - I_i)) \neq 0$ for $n \in \mathbb{Z}$, where $\lambda_j(z)$ are the analytic extensions of the eigenvalues $\lambda_j(I)$. (H4)′ can also simply be remembered as $|\omega_0| \equiv |Im\, \lambda_j(I_-)| \neq n(\omega(I_-) + \omega'(I_-)(I_- - I_i))$ for $n \in \mathbb{N}$.

(H5)′. (Second Nonresonance Condition) Assume that for $|z - I_-| < r_b$, the analytic extensions $\lambda_j(z)$ satisfy $2\lambda_j(z) - i(2n-1)(\omega(z) + \omega'(z)(z - I_i)) \neq 0$ for $n \in \mathbb{Z}$, $j = 1, 2$, or simply $2\,|\omega_0| \equiv 2\,|Im\, \lambda_1(I_-)| \neq (2n-1)(\omega(I_-) + \omega'(I_-)(I_- - I_i))$.

We observe that the period of the forcing in (5.1) is $2\pi/\omega(I_i + \varepsilon t)$, which is slowly varying with time in the system. Let us also use $y = u(t) - u_0(I_i + \varepsilon t)$ as the new variable and $I = I_i + \varepsilon t$ as the new independent variable. The initial value problem (5.1) is then equivalent to

$$\varepsilon y_I = A(I)\, y + F_1\left(I, \varepsilon, \frac{I - I_i}{\varepsilon}, y\right) + \varepsilon Q_1\left(I, \varepsilon, \frac{I - I_i}{\varepsilon}\right) \tag{5.4a}$$

$$y|_{I = I_i} = O(\varepsilon) \tag{5.4b}$$

where F_1 is analytic in y and has the expression

$$F_1\left(I, \varepsilon, \frac{I - I_i}{\varepsilon}, y\right) = \sum_{k=1}^{\infty} \sum_{m_1 + n_1 = k,\, m_1 \geqslant 0,\, n_1 \geqslant 0} B_k^{(m_1, n_1)} y_1^{m_1} y_2^{n_1} \tag{5.5}$$

with $(y_1, y_2)^T = y$. The term εQ_1 has the form

$$Q_1 = \sum_{n \in \mathbb{Z}} q_n(I, \varepsilon)\, e^{in\omega(I)(I - I_i)/\varepsilon} \tag{5.6}$$

where $\sum_{n \in \mathbb{Z}} |q_n(I, \varepsilon)| \leqslant M$ for $|I - I_-| < r_b$.

Theorem 5.1. *Let $y(I, I_0, \varepsilon)$ be the solution of (5.4) with the initial condition $y(I, I_0, \varepsilon)|_{I=I_0}=0$ for $I_0<I_-$. Assume (H1), (H2)', (H3), (H4)', and (H5)' are true for system (5.4). Then for large enough M, there exist $r_9=r_9(M)>0$, $\varepsilon_0=\varepsilon_0(M)>0$ such that for $|I_0-I_-|\leqslant r_9$, $\varepsilon\leqslant\varepsilon_0$,*

$$|y(I, I_0, \varepsilon)|\leqslant M\varepsilon \tag{5.7}$$

when and only when $I_0\leqslant I\leqslant I_q$, where $I_q=I_0^*+O(\varepsilon\,|\log(\varepsilon)|)>I_-$ is a point above the critical point.

The proof of Theorem 5.1 is essentially similar to the proof of Theorem 4.1. The key point in the proof is also related to the existence of the paths $\Gamma_n(z, T_n)$ along which the harmonic functions $\varphi_n=Re(\int_\xi^z \lambda_1(\tau)\,d\tau-in\omega(z)(z-I_i))$ are monotone decreasing, respectively. The new nonresonance conditions (H4)' and (H5)' assure that $2\omega_0\equiv 2Im\,\lambda_1(I_-)\neq n(\omega(I_-)+\omega'(I_-)(I_- -I_i))$ for any $n\in\mathbb{Z}$. Thus within a small neighborhood of the critical point I_-, the convexity argument remains true. Indeed, from the fact that

$$\begin{aligned}
&\lambda_1(z)-(in\omega(z)(z-I_i))'\\
&\quad=\lambda_1(z)-in\omega'(z)(z-I_i)-in\omega(z)\\
&\quad=a_1(z-I_-)+O((z-I_-)^2)\\
&\qquad+i(\omega_0-n(\omega(I_-)+\omega'(I_-)(I_- -I_i))+O(z-I_-))
\end{aligned} \tag{5.8}$$

where

$$a_1=\frac{\partial}{\partial I}Re\,\lambda_1(I)|_{I=I_-}>0; \qquad \omega_0=Im\,\lambda_1(I)|_{I=I_-}<0, \tag{5.9}$$

we obtain that the equations of the level curve of the function $\varphi_n=Re(\int_\xi^z \lambda_1(\tau)\,d\tau-in\omega(z)(z-I_i))$ can be expressed as $z=I_- +x+iy$, and

$$\tfrac{1}{2}a_1x^2-(\omega_0-n\omega(I_-)-n\omega'(I_-)(I_- -I_i))\,y+O(x^3, y^2)=C \tag{5.10}$$

Under hypotheses (H4)' and (H5)', the level curves would also be parabolas as well. Similar arguments in constructions of the accessible region B_ξ lead to the existence of the analytic extensions of the coefficients of $g_{1,n,m}(z, \xi)$. The rest of the analysis follows through.

Remark 1. Although the delay of bifurcations in this case occurs in analogue to the constant frequency case, the resonant frequencies where the delayed bifurcations vanish are much different from the previous case. In fact, with a Hopf bifurcation whose frequency $|\omega_0|$ is known, in case of periodic perturbations with constant frequency ω, the resonant frequencies are $\omega = 2|\omega_0|/n$ for $n \in \mathbb{N}$ as known from Section 3, while in the slowly varying frequency case, the resonance occurs when $\omega(I_-) + \omega'(I_-)(I_- - I_i) = 2|\omega_0|/n$. Therefore in the latter case, if $\omega'(I) > 0$, then the resonance frequencies which people observe in the experiments should be lower than $2|\omega_0|/n$. For $\omega'(I) < 0$, the result goes to the other direction. Thus the interferences are shifted because of the frequency change. Professor Neishtadt suggested that a new system of the frequency $\Omega = \omega(I) + (\partial\omega/\partial I)(I - I_i)$ is the essential cause of the shift. We are grateful to him for sharing his insight with us.

Remark 2. In Ref. 24, we considered the phenomena of delayed bifurcations from a family of periodic solutions of a constant period. If we consider the related problems of delayed bifurcations from a family of periodic solutions with slowly varying periods, then we expect shifted interferences would occur also in the sense that when the frequency of the periodic solutions and the frequency of the critical exponents form the relation $\omega(I_-) + \omega'(I_-)(I_- - I_i) = 2|\omega_0|/n$, the delay is destructed. These phenomena indeed give us some intuitions on the reasons of the shift that when the solutions of the perturbed system are chasing the static periodic solutions with slowly varying periods, the effective frequencies $\omega(I_-) + \omega'(I_-)(I_- - I_i)$ of solutions are somehow staying behind due to the frequency change. When the "lagged" frequency $\omega(I_-) + \omega'(I_-)(I_- - I_i)$ and the critical exponents ω_0 have an interference at the critical point I_-, the delays are to be destroyed.

Remark 3. We observe the fact that the shift of the interference depends on the initial parameter I_i as a monotonic function. This is another type of memory effect besides the memory effect of the amount of delay in the bifurcation, which also is monotonic function of I_i. The memory effect of the amount of delay provides a method to detect the initial parameter I_i from the parameter point where the separations occur. The phenomena of shifted interferences also provide a second way to detect the initial parameter by looking at the resonance frequency $\omega(I_-)$ and the speed of change $\omega'(I_-)$ to determine the initial parameter I_i under the assumption that ω_0 is a known parameter. This type of result might be of importance in real physical inverse problems.

6. DELAYED SIMPLE EIGENVALUE BIFURCATIONS, A GENERAL CASE

By showing this general result of delayed simple eigenvalue bifurcations, we intend to show the interesting phenomenon that by adding a small but suitable periodic forcing into a system, sometimes delayed bifurcations are created rather than destroyed.

From now on, we modify our method to study the delayed bifurcations of dynamical systems which have a simple eigenvalue slowly moving across zero along the real axis. Such systems have been studied in particular cases by Diener and Diener [8], Schecter [19], Lebovitz and Schaar [13], and Haberman [10]. Our approach to the simple eigenvalue situation is to let the imaginary part of the eigenvalue tend to zero in the previously considered Hopf bifurcation situation.

To be more specific, we consider a one dimensional equation:

$$u_t = f(u, I_i + \varepsilon t) \tag{6.1}$$

$$u(t)|_{t=0} = u_0(I_i) + O(\varepsilon) \tag{6.2}$$

where $f\colon \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ has analytic extensions for both variables. For each I, the equation $u_t = f(u, I) = 0$ has an equilibrium $u_0(I)$ which is also analytic on I. Assume that there exists $I = I_-$ such that when $I < I_-$, $f_u(u_0(I), I) < 0$ and when $I > I_-$, $f_u(u_0(I), I) < 0$. Also, $I_i < I_-$.

Under the assumption that $u_0(I) \equiv 0$, Diener [8] and Schecter [19] independently showed that $|u(I) - u_0(I)| \leqslant M\varepsilon$ for $I_i \leqslant I \equiv I_i + \varepsilon t \leqslant I_q$, where $I_q > I_-$ satisfies $\int_{I_i}^{I_q} f_u(0, I)\, dI = 0$. The methods were essential comparison methods similar to the exponential growth property in Proposition 2.4 in Section 2. There, the assumption $u_0 \equiv 0$ is very crucial in the sense that if the system is perturbed by a constant of magnitude larger than $\varepsilon^{-c/\varepsilon}$ (e.g., ε^n for any $n > 0$) for ε sufficiently small, then the delay vanishes. We consider the following example:

$$\varepsilon u_I = Iu \tag{6.3a}$$

$$u(I_i) = O(\varepsilon) \tag{6.3b}$$

where $I_i < I_- \equiv 0$. Equation (6.3) is a special case of the systems described by Diener [8] and Schecter [19]. The delayed bifurcation phenomena are obvious. The perturbed system

$$\varepsilon u_I = Iu + \sigma, \tag{6.4a}$$

$$u(I_i) = O(\varepsilon) \tag{6.4b}$$

would not present any delay in bifurcation if $\sigma > e^{-c/\varepsilon}$ (e.g., $\sigma = \varepsilon^n$). In fact, $I_q = |\varepsilon \ln \sigma|$ in this case. Lebovitz and Schaar [13] gave some conditions of no delay as well.

Our study of delayed simple eigenvalue bifurcation problems is motivated by the following consideration. The system

$$\varepsilon u_I = Iu + \varepsilon \sum_{n \neq 0} a_n e^{in\omega I/\varepsilon} \tag{6.5a}$$

$$u(I_i) = O(\varepsilon) \tag{6.5b}$$

with $\sum_{n \neq 0} |a_n| \leqslant M$ has a delayed bifurcation pattern. Indeed, if we let u_+ be the solution of (6.5) with the initial condition $u_+(1) = 0$, and u_- be the solution of (6.5) with the initial condition $u_-(-1) = 0$, it can be easily be shown through a direct calculation that

$$|u_+(0) - u_-(0)| = \left| \sum_{n \neq 0} a_n \int_{-1}^{1} e^{-1/(2\varepsilon)(s^2 + 2in\omega s)}\, ds \right| \leqslant M e^{-c(\omega)/\varepsilon} \tag{6.6}$$

by a simple contour integration. Thus we find that by avoiding certain particular resonant frequencies, the delayed bifurcation phenomena persist even with a perturbation of a magnitude $O(\varepsilon)$.

We now consider the generalization of the above situations. We study the dynamical systems

$$u_t = f(u, I_i + \varepsilon t) + \varepsilon \left(A a_0(I_i + \varepsilon t, \varepsilon) + \sum_{n \neq 0} a_n(I_i + \varepsilon t, \varepsilon)\, e^{in\omega t} \right) \tag{6.7a}$$

$$u(t)|_{t=0} = u_0(I_i) + O(\varepsilon) \tag{6.7b}$$

where the system satisfies:

Assumptions. *(C1) $f(u, I)$: $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$ has an analytic extension for $|u| < \sigma$, $|I| < r_a$ in the complex plane.*

(C2) For each fixed I, the system

$$v_t = f(v, I) \tag{6.8}$$

has an equilibrium solution $u_0(I)$ which is also analytic in I for $|I| < r_a$.

(C3) The variational system of (6.8) about $u_0(I)$,

$$w_t = f_u(u_0(I), I) w \tag{6.9}$$

is a linear system with coefficients depending on the parameter I. There exists a real number I_- *such that* $\lambda(I) \equiv f_u(u_0(I), I) < 0$ *when* $I < I_-$; $\lambda(I) \equiv f_u(u_0(I), I) > 0$ *when* $I > I_-$. *Also, assume* $(\partial/\partial I)\,\lambda(I_-) > 0$ *and* $\lambda(I)$ *is odd function at* I_-. *The initial point* $I_i < I_-$.

(C4) The perturbations are real functions in the form of the series $(Aa_0(I_i + \varepsilon t, \varepsilon) + \sum_{n \neq 0} a_n(I_i + \varepsilon t, \varepsilon)\, e^{in\omega t})$, *where* $\omega > 0$ *is a constant, A is a real parameter,* $a_n = \bar{a}_{-n}$. *For* $|z - I_-| \leqslant r_a$ *in the complex plane,* $\varepsilon \leqslant \varepsilon_0$, *the coefficients have analytic extensions* $a_n(z, \varepsilon)$ *which satisfy* $\sum_{n \in \mathbb{Z}} \|a_n(z, \varepsilon)\| \leqslant M$. *Also, assume* $|a_0(I_-, \varepsilon)| \geqslant \delta_{10} > 0$.

Let us also use $y = u(t) - u_0(I_i + \varepsilon t)$ as the new variable and $I = I_i + \varepsilon t$ as the new independent variable. The initial value problem (6.7) is then equivalent to

$$\varepsilon y_I = \lambda(I)\, y + F_1(I, \varepsilon, y) + \varepsilon\left(-u_0'(I) + Aa_0(I, \varepsilon) + \sum_{n \neq 0} a_n(I, \varepsilon)\, e^{in\omega(I - I_i)/\varepsilon}\right) \tag{6.10a}$$

$$y|_{I = I_i} = O(\varepsilon) \tag{6.10}$$

where F_1 is analytic in y and has the expression

$$F_1(I, \varepsilon, y) = \varepsilon B_1(I, \varepsilon)\, y + \sum_{k=1}^{\infty} B_k(I, \varepsilon)\, y^k \tag{6.11}$$

The coefficients $B_k(I, \varepsilon)$ have analytic extensions for I near I_- in the complex plane. From the fact that $F_1(I, \varepsilon, y)$ of (6.10) is an analytic function of y for $|y| < \sigma$, and $B_k(I, \varepsilon)$ correspond to the coefficients of kth power terms in the new system, it is obvious that

$$|B_k(z, \varepsilon)| \leqslant \frac{M_d}{(\sigma/2)^k}, \qquad k \in \mathbb{N} \tag{6.12}$$

for $|z - I_-| < r_b$ in the complex plane, $\varepsilon \leqslant \varepsilon_0$. Under the assumptions C1–C4, we have the general theorem for delayed simple eigenvalue bifurcations as follows.

Theorem 6.1. *Let assumptions C1–C4 hold. Suppose* $y(I, I_i, M_1, \varepsilon)$ *are a family of solutions of (6.10) with the initial conditions satisfying* $|y(I, I_i, M_1, \varepsilon)||_{I = I_i} \leqslant M_1 \varepsilon$ *for* $I_i < I_-$. *Given a set of analytic coefficients* $\{a_n(I, \varepsilon), n \in \mathbb{Z}\}$, $|a_0(I_-, \varepsilon)| \geqslant \delta_{10} > 0$ *for* $\varepsilon \leqslant \varepsilon_0$, *there exists a parameter value* $A_0 \equiv A_0(\{a_n, n \in \mathbb{Z}\}, \varepsilon)$ *such that when* $A = A_0$, *the corresponding system (6.10) presents a delayed bifurcation pattern, i.e., for large enough M,* *there exist* $r_{10}=r_{10}(M)>0, \varepsilon_0=\varepsilon_0(M)>0$, *such that for* $|I_i-I_-|\leqslant r_{10}$, $\varepsilon\leqslant\varepsilon_0$,

$$|y(I, I_i, M_1, \varepsilon)|\leqslant M\sqrt{\varepsilon} \tag{6.13}$$

when and only when $I_i\leqslant I\leqslant I_q$, *where* $I_q=I_i^*+O(\varepsilon)>I_-$ *is a point above the critical point, and* I_i^* *satisfies* $Re\int_{I_i}^{I_i^*}\lambda_1(\tau)\,d\tau=0$.

Remark. We note here that typically $y(I, I_i, M_1, \varepsilon)$ are bounded by $O(\sqrt{\varepsilon})$ rather than $O(\varepsilon)$. This is due to the difference that $\lambda(I_-)=0$ here, which makes the expansions (3.3) invalid in such a way that $|R'(z)|=|\lambda(z)|$ is no longer bounded below near $z=I_-$. The best estimate in Theorem 6.1 is only $O(\sqrt{\varepsilon})$, which is indeed sharp. We are grateful to Professor Neishtadt for pointing this out.

A Sketch of Proof. We start to consider the perturbed systems of (6.10) in the complex variable

$$\varepsilon\frac{dy^\alpha}{dI}=(\lambda(I)+i\alpha)\,y^\alpha+F_1(I, \varepsilon, y^\alpha)$$
$$+\varepsilon\left(-u_0'(I)+Aa_0(I,\varepsilon)+\sum_{n\neq 0}a_n(I,\varepsilon)\,e^{in\omega(I-I_i)/\varepsilon}\right) \tag{6.14a}$$
$$y|_{I=I_i}=O(\varepsilon) \tag{6.14b}$$

which is indeed a two dimensional system in real variables. We prove Theorem 6.1 by the following steps.

(1) For each $\alpha>0$, there exists an A_0^α such that when $A=A_0^\alpha$, the solutions $y_-^\alpha(I)$ and $y_+^\alpha(I)$ of the systems (6.14) satisfy the relation

$$y_-^\alpha(I_-)-y_+^\alpha(I_-)=0, \tag{6.15}$$

where $y_-^\alpha(I)$ and $y_+^\alpha(I)$ are defined as the solutions of (6.14) with the initial conditions $y_-^\alpha|_{I=I_0}=0$ and $y_+^\alpha|_{I=I^0}=0$ for $I_0<I_-$ and $I^0>I_-$, respectively. Here we simply assume that the points $I_0<I_-$ and $I^0>I_-$ satisfy $Re\int_{I_0}^{I^0}\lambda_1(\tau)\,d\tau=0$.

(2) A_0^α are uniformly bounded for $0<\alpha\leqslant\alpha_0$ and $\varepsilon\leqslant\varepsilon_0$.

(3) Let A_0 be any accumulation point of $\{A_0^\alpha\}$ as $\alpha\to 0^+$. Then

$$y_-(I_-)-y_+(I_-)=0 \tag{6.16}$$

from the continuity of the distance depending upon the parameter where $y_-(I)$ and $y_+(I)$ are defined as the solutions of (6.10) with the initial condition $y_-|_{I=I_0}=0$ and $y_+|_{I=I^0}=0$ for $I_0<I_-$ and $I^0>I_-$, respectively.

The delayed bifurcation phenomena follow from relationship (6.16) along with a exponential growth property similar to Proposition 2.4 in Section 2.

We now prove the assertions.

Step 1. We denote $y^\alpha(I, s, \varepsilon)$ to be the solutions of (6.14) with the initial conditions $y^\alpha(I, s, \varepsilon)|_{I=s} = 0$. We adopt Lemmas 2.2 and 3.4 from Sections 2 and 3 with certain modifications. Eventually, we obtain the following.

Lemma 6.2. *Let the assumptions C1–C4 hold. Then the solution $y^\alpha(I, s, \varepsilon)$ of (6.14) with the initial condition: $y^\alpha(I, s, \varepsilon)|_{I=s} = 0$ has the property that for $s < I < I_-$,*

$$y^\alpha(I, s, \varepsilon) = \sqrt{\varepsilon} \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} g^\alpha_{n,m}(I, s, \varepsilon)\, e^{in\omega I/\varepsilon} e^{in\omega s/\varepsilon}. \tag{6.17}$$

There exists a neighborhood N_1 of I_- in the complex plane such that for any $\xi \in N_1$, the functions $g^\alpha_{n,m}(I, s, \varepsilon)$ have analytic extensions $g^\alpha_{n,m}(z, \xi, \varepsilon)$ in z for $z \in B^\alpha_\xi$, where B^α_ξ is a symmetric region left to I_-, $B^\alpha_\xi \cap \{z \mid z \leqslant I_-\} = \{z \mid s_1(\xi, \omega, \alpha) < z < I_-\}$ for some $s_1 < I_-$. Further, for $\xi \in N_1$, $z \in cl(B^\alpha_\xi)$,

$$\sum_{n \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} \sup_{\xi \in N_1, z \in B^\alpha_\xi} |g^\alpha_{n,m}(z, \xi, \varepsilon)| \leqslant M \tag{6.18}$$

for some $M > 0$. For fixed $z \in \bigcap_\xi B^\alpha_\xi$, $g^\alpha_{n,m}(z, \xi, \varepsilon)$ are analytic in ξ.

Remark. The difference between this and Lemma 3.3 is that $R'(z) = \lambda(z) + in\omega + i\alpha$ may vanish near $z = I_-$ as $\alpha \to 0^+$ and $n = 0$. Consequently (3.29) is not valid because $1/R'(z)$ is not bounded when $\alpha \to 0^+$.

Equations (3.1)–(3.2) and the whole scheme for the proof of Lemma 3.3 are still valid with different (but larger) estimates on $\|\tilde{f}_{1,0,n}\|$, where $\tilde{f}_{1,0,n} = \mathbb{T}(f_{1,0,n}, \int_s^z (\lambda(\tau) + i\alpha)\, d\tau + in\omega z, B^\alpha_\xi)$.

Let us first take the case of $\alpha = 0$. We also note that $z = I_-$ is the only point such that $z \in B^\alpha_\xi$ and $R'(z) = \lambda(z) + in\omega = 0$. Thus for z with $|z - I_-| > \sqrt{\varepsilon}$ and $z \in B^\alpha_\xi$, $1/|R'(z)| \geqslant O(1/\sqrt{\varepsilon})$. Thus (3.3) gives $\|\tilde{f}_{1,0,n}(z, B^\alpha_\xi, \varepsilon)\| \leqslant C_4 \sqrt{\varepsilon}$. Then, for the point z with $|z - I_-| \leqslant \sqrt{\varepsilon}$ and $z \in B^\alpha_\xi$, let z_ε be the point with $|z_\varepsilon - I_-| = \sqrt{\varepsilon}$, $Im\, z_\varepsilon = Im\, z$ on the circle $\partial B_{\sqrt{\varepsilon}}(I_-)$, i.e., $z_\varepsilon = I_- + \sqrt{\varepsilon - (Im\,(z_\varepsilon - I_-))^2} + i[Im\, z_\varepsilon]$. From the argument above, since $|z_\varepsilon - I_-| = \sqrt{\varepsilon}$, we find $\|\tilde{f}_{1,0,n}(z_\varepsilon, B^\alpha_\xi, \varepsilon)\| \leqslant C_4 \sqrt{\varepsilon}$, which is going to be used as the initial condition. By differentiating (3.4) with respect to z, we obtain the relation for $\tilde{f}_{1,0,n}(z, B^\alpha_\xi, \varepsilon)$ that

$$\varepsilon \frac{\partial}{\partial z} \tilde{f}_{1,0,n}(z, B^\alpha_\xi, \varepsilon) = (\lambda(z) + i\alpha + in\omega)\, \tilde{f}_{1,0,n}(z, B^\alpha_\xi, \varepsilon) + \varepsilon f_{1,0,n}(z, \varepsilon) \tag{6.19}$$

Now we solve (6.19) along the horizontal line $l_\varepsilon \equiv \{\eta, \eta = z_\varepsilon + \theta(z - z_\varepsilon), 0 \leqslant \theta \leqslant 1\}$ from z_ε to z. Since $Re(\lambda(\eta) + i\alpha + in\omega) = Re\,\lambda(\eta) = O(\sqrt{\varepsilon})$ for $\eta \in l_\varepsilon$, there exists $C_7 > 0$ such that $|Re(\lambda(\eta) + i\alpha + in\omega)| \leqslant C_7\sqrt{\varepsilon}$; we derive

$$
\begin{aligned}
|\tilde{f}_{1,0,n}(\eta, B^\alpha_\xi, \varepsilon)| &= \Big| \tilde{f}_{1,0,n}(z_\varepsilon, B^\alpha_\xi, \varepsilon)\, e^{1/\varepsilon \int_{z_\varepsilon}^{\eta} (\lambda(\tau) + i\alpha + in\omega)\, d\tau} \\
&\quad + \int_{z_\varepsilon}^{\eta} f_{1,0,n}(s, \varepsilon)\, e^{1/\varepsilon \int_s^{\eta} (\lambda(\tau) + i\alpha + in\omega)\, d\tau}\, ds \Big| \\
&\leqslant |\tilde{f}_{1,0,n}(z_\varepsilon, B^\alpha_\xi, \varepsilon)|\, e^{1/\varepsilon\, Re \int_{z_\varepsilon}^{\eta} (\lambda(\tau) + i\alpha + in\omega)\, d\tau} \\
&\quad + \left\| \int_{z_\varepsilon}^{\eta} |f_{1,0,n}(s, \varepsilon)|\, e^{1/\varepsilon\, Re \int_s^{\eta} (\lambda(\tau) + i\alpha + in\omega)\, d\tau}\, ds \right\| \\
&\leqslant |\tilde{f}_{1,0,n}(z_\varepsilon, B^\alpha_\xi, \varepsilon)|\, e^{(1/\varepsilon) C_7 \sqrt{\varepsilon} |\eta - z_\varepsilon|} \\
&\quad + \left\| \int_{z_\varepsilon}^{\eta} |f_{1,0,n}(s, \varepsilon)|\, e^{(1/\varepsilon) C_7 \sqrt{\varepsilon} |\eta - s|}\, ds \right\|
\end{aligned}
$$

Because the initial condition $\|\tilde{f}_{1,0,n}(z_\varepsilon, B^\alpha_\xi, \varepsilon)\| \leqslant C_4\sqrt{\varepsilon}$ and $|s - \eta| \leqslant |z_\varepsilon - \eta| \leqslant |z - z_\varepsilon| \leqslant \sqrt{\varepsilon}$, we get $\|\tilde{f}_{1,0,n}(z, B^\alpha_\xi, \varepsilon)\| = O(\sqrt{\varepsilon})$.

For any general $\alpha \to 0^+$, we find the points $\{O^\alpha_\varepsilon\}$ where $\lambda(z) + i\alpha = 0$ at $z = O^\alpha_\varepsilon$. Since $\lambda(z) + i\alpha$ is analytic, and $\lambda(z) + i\alpha$ is not a constant, $\{O^\alpha_\varepsilon\}$ is a finite set. The estimates $\|\tilde{f}_{1,0,n}(z, B^\alpha_\xi, \varepsilon)\| = O(\sqrt{\varepsilon})$ can be derived following the same procedure as in the $\alpha = 0$ cases. We observe the facts that when z satisfies $|z - O^\alpha_\varepsilon| \geqslant \sqrt{\varepsilon}$ and $z \in B^\alpha_\xi$, $|\lambda(z) + i\alpha| > C_4\sqrt{\varepsilon}$, and when z satisfies $|z - O^\alpha_\varepsilon| \leqslant \sqrt{\varepsilon}$ and $z \in B^\alpha_\xi$, $|Re(\lambda(z) + i\alpha + in\omega)| \leqslant C_7\sqrt{\varepsilon}$. These two facts are critical in the proof of cases where $\alpha = 0$. Thus, we have $\|\tilde{f}_{1,0,n}(z, B^\alpha_\xi, \varepsilon)\| = O(\sqrt{\varepsilon})$, which is independent of α as $\alpha \to 0^+$.

The rest of the derivation of (6.18) can be obtained by an analysis similar to that in Lemma 3.3.

We note that as $\alpha \to 0^+$, $\lim_{\alpha \to 0^+} \cap_{\alpha_1 \leqslant \alpha} B^{\alpha_1}_\xi \neq \phi$. Further, as $\alpha \to 0^+$, the paths $\Gamma^\alpha_n(z, T^\alpha_n)$, upon which the harmonic functions $\varphi^\alpha_n \equiv Re \int_{I_-}^{z} (\lambda(\tau) + i\alpha - in\omega)\, d\tau$ are monotone decreasing, respectively, are nonsingular in the sense that

$$
\sup_{z \in B_\xi} |\Gamma^\alpha_n(z, T^\alpha_n)| \leqslant C(|\xi - I_-|) \tag{6.20}
$$

for some $C > 0$ independent of $\alpha \to 0^+$, when $|\xi - I_-| \leqslant r_{10}$ for some $r_{10} > 0$.

Lemma 6.3. *Assume C1–C4 hold. Let $y^\alpha(I, s, \varepsilon)$ be the solution of (6.14) with the initial condition $y^\alpha(I, s, \varepsilon)|_{I=s} = 0$ for $s < I_-$. Then for $s \leqslant I \leqslant I_-$, $(\partial/\partial s)\, y^\alpha(I, s, \varepsilon)$ can be expressed as*

$$\frac{\partial}{\partial s} y^\alpha(I, s, \varepsilon) = \left(\sum_{n, m \in \mathbb{Z}} A^\alpha_{n,m}(I, s, \varepsilon)\, e^{in\omega I/\varepsilon} e^{im\omega s/\varepsilon} \right) e^{1/\varepsilon \int_s^I (\lambda(\tau) + i\alpha)\, d\tau} \tag{6.21}$$

where $A^\alpha_{n,m}(I, s, \varepsilon)$ have analytic extensions $A^\alpha_{n,m}(z, \xi, \varepsilon)$ which are analytic in ξ for $|\xi - I_-| \leqslant r_{10}$, and are analytic in z for $z \in B^\alpha_\xi$ as defined previously in Lemma 6.2. Furthermore,

$$\sum_{j, n, m} \|A^\alpha_{n,m}(z, \xi, \varepsilon)\| \leqslant M \tag{6.21}$$

for $|\xi - I_-| \leqslant r_{10}$, $z \in cl(B_\xi)$, and for any $\varepsilon \leqslant \varepsilon_0 = \varepsilon_0(M, \lambda_j)$. In particular, $z = I_-$ belongs to $cl(B^\alpha_\xi)$ for any ξ with $|\xi - I_-| < r_{10}$, and therefore

$$\frac{\partial}{\partial s} y^\alpha(I_-, s) = \left(\sum_{m \in \mathbb{Z}} \hat{A}^\alpha_m(s, \varepsilon)\, e^{im\omega s/\varepsilon} \right) e^{1/\varepsilon \int_s^{I_-} (\lambda(\tau) + i\alpha)\, d\tau} \tag{6.22}$$

where $\hat{A}^\alpha_m(s, \varepsilon)$ have analytic extensions $\hat{A}^\alpha_m(\xi, \varepsilon)$ for $|\xi - I_-| \leqslant r_{10}$, and

$$\sum_{m \in \mathbb{Z}} \|\hat{A}^\alpha_m(\xi, \varepsilon)\| \leqslant M \tag{6.23}$$

Lemmas 6.2 and 6.3 are direct applications of the corresponding Lemmas 3.3 and 3.4 to the Eq. (6.10). To continue the sketch of the proof of Theorem 6.1, we apply Lemma 6.3 to derive that

$$y^\alpha_+(I_-) - y^\alpha_-(I_-) = \sum_{n \in \mathbb{Z}} \int_{I_0}^{I^0} \hat{A}^\alpha_n(s, \varepsilon)\, e^{1/\varepsilon \int_s^{I_-} (\lambda(\tau) + i\alpha s)\, d\tau} e^{in\omega s/\varepsilon}\, ds \tag{6.24}$$

where $\hat{A}^\alpha_0 = -(Aa_0(I, \varepsilon) - (du_0/dI) + \varepsilon g(I, \alpha, \varepsilon))$ for some bounded, analytic function $g(I, \alpha, \varepsilon)$.

From the Taylor expansion of $\lambda(I)$, $\lambda(I) = a_1(I - I_-) + O((I - I_-)^2)$, and the fact that $|a_0(I_-, \varepsilon)| \geqslant \delta_{10} > 0$, we obtain the estimate by a method similar to that in (4.19) that, for sufficiently small $\varepsilon \leqslant \varepsilon_0$ and independently $\alpha \leqslant \alpha_0$,

$$C_2 e^{-K_1 \alpha^2/\varepsilon} \sqrt{\varepsilon} \geqslant \left| \int_{I_0}^{I^0} a_0(s, \varepsilon)\, e^{1/\varepsilon \int_s^{I_-} (\lambda(\tau) + i\alpha)\, d\tau}\, ds \right| \geqslant C_1 e^{-K_2 \alpha^2/\varepsilon} \sqrt{\varepsilon} \tag{6.25}$$

for some positive C_1 and k_2 independent of ε and α. It is necessary to require $\alpha \leqslant \alpha_0$ to assure $|a_0(z, \varepsilon)| \geqslant \delta_{10}/2 > 0$ when $|z - I_-| \leqslant K\alpha$. Also, we have

$$\left| \int_{I_0}^{I^0} \left(-\frac{du_0}{dI}(s, \varepsilon) + \varepsilon g(s, \varepsilon) \right) e^{1/\varepsilon \int_s^{I_-} (\lambda(\tau) + i\alpha)\, d\tau}\, ds \right| \leqslant M_2 e^{-K_2\alpha^2/\varepsilon} \sqrt{\varepsilon} \quad (6.26)$$

These estimates are obtained by analyzing the local structure of the function $\int_z^{I_-} (\lambda(\tau) + i\alpha)\, d\tau$ as in (4.25). Details of estimations were carried out in (4.19). By using the analyticity of the functions and the structure of $\lambda(z) + i\alpha + in\omega$ for $n \neq 0$ whose imaginary part satisfies $|Im\, \lambda(z) + \alpha + n\omega| \geqslant \omega/2 > 0$ when $|z - I_-| \leqslant$, $\alpha \leqslant \alpha_0$, we can obtain

$$\int_{I_0}^{I^0} \hat{A}_n^\alpha(s, \varepsilon)\, e^{1/\varepsilon \int_s^{I_-} (\lambda(\tau) + i\alpha)\, d\tau} e^{in\omega s/\varepsilon}\, ds$$

$$\leqslant \|\hat{A}_n^\alpha\|\, |\Gamma_1(I_0)|\, e^{-\chi_2(n, \alpha, \omega)/\varepsilon} \qquad \text{for} \quad n \neq 0 \qquad (6.27)$$

Especially, $\chi_2(n, \alpha, \omega) \geqslant \chi_0 > 0$ when $0 < \alpha \leqslant \alpha_0, n \neq 0$. We choose the path to be $\Gamma_1(I_0)$ for any $n > 0$ where $\Gamma_1(I_0) \equiv \{z \mid Im\, z \geqslant 0, Re \int_{I_-}^{z} (\lambda(\tau) - i\omega)\, d\tau = Re \int_{I_-}^{I_0} (\lambda(\tau) - i\omega)\, d\tau\}$, which is the level curve of the harmonic functions $\psi_1(z) \equiv Re \int_{I_-}^{z} (\lambda(\tau) - i\omega)\, d\tau$. For $n < 0$, the path is then chosen to be $\bar{\Gamma}_1$. The whole procedure is analogous to the similar one in Section 4. Therefore, from (6.21), we obtain

$$\sum_{n \neq 0} \left\| \int_{I_0}^{I^0} \hat{A}_n^\alpha(s, \varepsilon)\, e^{1/\varepsilon \int_s^{I_-} \lambda(\tau)\, d\tau} e^{in\omega s/\varepsilon}\, ds \right\| \leqslant M\, |\Gamma_1(I_0)|\, e^{-\chi_0/\varepsilon}. \qquad (6.28)$$

We also pay attention to the fact that the bounds of the integrations are independent of α as $\alpha \to 0^+$. Since the term in (6.25) dominates the rest of the terms in (6.24), there exist A_0^α which are uniformly bounded with respect to α by $|A_0^\alpha| \leqslant M_2$ such that for $\alpha \leqslant \alpha_0$ and $\varepsilon \leqslant \varepsilon_0$, when $A = A_0^\alpha$,

$$y_-^\alpha(I_-) - y_+^\alpha(I_-) = 0 \qquad (6.29)$$

where $y_-^\alpha(I)$ and $y_+^\alpha(I)$ are two solutions from different sides of the critical point for the equation (6.14) with $A = A_0^\alpha$.

Step 2. Step 2 is implied from the process in which Step 1 was obtained.

Step 3. We take A_0 to be any accumulation point of $\{A_0^\alpha\}$ as $\alpha \to 0^+$. From the continuity dependency of the solutions y_+^α and y_-^α with respect to α, we derive that

$$y_+(I_-) - y_-(I_-) = 0 \tag{6.30}$$

for the solutions y_+ and y_- which are defined in (6.16).

We observe that if the terms $\sum_{n \neq 0} a_n(I, \varepsilon)\, e^{in\omega(I - I_i)/\varepsilon}$ and $a_0(I, \varepsilon)$ are real, then A_0^α can also be chosen to be real. So is A_0. Moreover, if such a condition is met, even though y_+^α and y_-^α might be complex, their limits are real since y_+ and y_- are solutions of real systems with real initial values.

From (6.30), Theorem 6.1 follows by using the fact y_- and y_+ are the same solutions which satisfy $|y(I, I_i, M_1, \varepsilon)| \leqslant M\sqrt{\varepsilon}$ for $I_i = I_0 \leqslant I \leqslant I_q \equiv I^0$. Similarly, we can conclude the delayed bifurcation properties of solutions with initial conditions $|y(I, I_i, M_1, \varepsilon)|\,|_{I = I_i} \leqslant M_1\sqrt{\varepsilon}$ by an argument about exponential growth similar to Proposition 2.4. □

Remarks. (1) If the frequency ω is replaced by $\omega(I_i + \varepsilon t)$, then similar results can be obtained. Analogous results concerning near resonant frequencies can also be given. (2) From Theorem 6.1, the perturbation functions form a codimension one manifold in the periodic function space.

We believe the significance of Theorem 6.1 is to show that delayed simple eigenvalue bifurcations are also a common phenomena in the sense that they occurred in a codimension one family of systems among all systems in which the eigenvalue $\lambda(I)$ is real, and changes sign at $I = I_-$, rather than isolated systems under very restrictive conditions.

ACKNOWLEDGMENTS

The author wishes to thank Professor Hans F. Weinberger for introducing the general subject. Thanks also go to Dr. John Rinzel, who encouraged the author to look at some of the problems presented here. Professor A. I. Neishtadt also made important suggestions, for which the author is grateful. An anonymous referee suggested many improvements, simplifications which helped me a great deal in revising this paper.

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Printed in Belgium