EVERY RECURSIVELY ENUMERABLE EXTENSION OF A THEORY OF LINEAR ORDER HAS A CONSTRUCTIVE MODEL

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It is well known that not every consistent theory, even among those which are finitely axiomatizable, has a constructive model. In view of this it is interesting to examine conditions which are sufficient for a theory to have a constructive model. One such condition is:

A. The theory T has a solvable extension.

Another interesting sufficient condition was obtained in [5], and consists essentially in:

B. The theory T is axiomatizable by a recursively enumerable set of Krom formulas without equalities.

The present article will prove that the following condition is likewise sufficient for a theory to have a constructive model:

C. The theory T is a recursively enumerable extension of a theory of linear order with a finite number of supplementary one-place predicates in signature.

Notice that there exists a completely unsolvable recursively enumerable theory, extending a theory of linear order, without supplementary one-place predicates in signature. In other words, there exist theories which have property C, but not properties A or B.

DEFINITION 1 [1]. A partially ordered set M is said to be tight if every sequence $\{x_i\}_{i\in\omega}$ of elements of M contains an increasing (not necessarily strictly increasing) subsequence.

Denote by W_n the set of all words of the alphabet $A=\{1,2,\dots,n\}$ ordered as follows. If $w_1, w_2 \in W_n$, then $w_1 \leq w_2$ if and only if w_1 is obtained from w_2 by striking out certain letters.

It was shown in [1] that W_n is a tight partially ordered set. We shall also use the result obtained in [1] to the effect that the direct product of two, and hence any finite number, of tight partially ordered sets is tight. Thus the partially ordered set $W_n^m = W_n \times W_n \times \dots \times W_n$ (with m factors) is tight.

DEFINITION 2. Let M be a partially ordered set. A subset $F \subseteq M$ will be called a filter if $w_1 \in F$, $w_1 \leq w_2 \Rightarrow w_2 \in F$. We shall refer to the set $B \subseteq M$ as a basis of the filter F if B consists of pairwise incomparable elements and $w \in F \Leftrightarrow \exists v \in B\ (v \leq w)$.

The following lemma is a slight modification of a lemma given in [1] (where the complement of F, and not the set F itself, was considered).

LEMMA 1. Let M be a tight partially ordered set. Then every filter $F \subseteq M$ has a finite basis.

Here and below, a consistent set of propositions of a certain signature, closed with respect to logical amplifications, will be called a theory.

Our aim will be to prove

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THEOREM. Let T be any recursively enumerable theory of signature $\sigma = \langle =, <, R_1^1, R_2^1, \ldots, R_n^1 \rangle$, extending a theory T_0 of linear order of signature $\sigma_0 = \langle =, < \rangle$. Then T has a constructive model.

Proof. Throughout what follows, the theory T is fixed. We shall assume that this theory has no finite models, since otherwise the theorem would be trivial. It may easily be shown (by introducing 2^n new one-place predicates instead of the n original ones) that the theorem can in general be reduced to the particular case when T contains the statement

$$\left[\bigwedge_{i \neq j} \forall x \left(R_i(x) \longrightarrow \neg R_j(x)\right)\right] \& \left[\forall x \left(R_1(x) \vee R_2(x) \vee \ldots \vee R_n(x)\right)\right].$$

Hence we shall assume that this statement belongs to the theory T.

Let T_{pNF} be the subset of T consisting of formulas specified in the prenex normal form. Obviously, T_{pNF} is a recursively enumerable system of axioms of the theory T. For every formula $\varphi \in T_{\text{pNF}}$ we denote by φ^* the Erbranov form of the statement φ [4]. Let $f_0^{m_0}, f_1^{m_1}, \ldots, f_s^{m_s}, \ldots$ be the list of all functional symbols for all such possible φ^*. We shall assume that, with respect to a formula $\varphi \in T_{\text{pNF}}$ and with respect to the occurrence of the quantor $\exists$ in this formula, an s can effectively be found such that $f_s^{m_s}$ serves for replacing precisely this quantor. In particular, the function $\lambda : \lambda(s) = m_s$ is general recursive. Denote

$$\sigma^* = \langle =, <, R_1^1, R_2^1, \ldots, R_n^1, f_0^{m_0}, f_1^{m_1}, \ldots, f_s^{m_s}, \ldots \rangle.$$

Denote by T^* the theory of signature σ^*, the set of axioms of which may be $\{\varphi^* \mid \varphi \in T_{\text{пнф}}\}$. Obviously, T^* is a recursively enumerable theory, and the restriction of T^* up to the signature σ is precisely T.

We shall construct a model $\mathfrak{M}^* = \langle M, \sigma^* \rangle$ of the theory T^*. The fundamental set of the model $\mathfrak{M}^*$ will be $M = \{a_0, a_1, \ldots, a_s, \ldots ;\ s < \omega\}$.

The equality relation of M is natural: $a_i = a_j \iff i = j$. As the numeration $\nu : N \longrightarrow M$ we shall take the mapping defined as follows: $\nu(s) = a_s$.

Predicates and functions of the signature σ^* will be defined step-wise on M. After step t, the predicates $\langle, R_1, R_2, \ldots, R_n \rangle$ will be defined on the finite set $M_t = \{a_0, a_1, \ldots, a_{\tau(t)}\}$. At this instant the predicate $<$ will specify a linear order on M_t, while the predictates $R_1, R_2, \ldots, R_n$ will define a division of the set M_t into n disjoint subsets. The following conditions will then be satisfied:

1. $M_0 \subseteq M_1 \subseteq M_2 \subseteq \ldots \subseteq M_t \subseteq M_{t+1} \subseteq \ldots$

2. $|M_t| > t$.

The predicates on M_{t+1} will be continuations of the corresponding predicates on M_t.

As distinct from the predicates, the Skolem functions $f_0^{m_0}, f_1^{m_1}, \ldots f_s^{m_s} \ldots$ will be trial and error functions. This means that the value of the function $f_s(a_\alpha, a_\beta, \ldots, a_\gamma)$ is not defined once and for all at a given instant, but is variable from step to step, while always remaining defined. For instance, at the instant $t = 0$ the value of every function will be a_0. This value may be changed several times later, but there always exists an instant t_0, dependent on $s, \alpha, \beta, \ldots, \gamma$ such that, starting from this instant the value of $f_s(a_\alpha, a_\beta, \ldots, a_\gamma)$ will remain the same. This last value is in fact, by definition, the value of the function in the model $\mathfrak{M}^*$. In general, therefore, the Skolem functions are not recursive in the numeration ν.

The process of construction is such that the following condition is satisfied.

3. At every instant t the values of all the functions belong to the set M_t.

Before turning directly to the process by which the model is constructed, let us describe the terms and concepts to be employed.

a) Level s functions. By definition, these include all expressions of the form $f_i(a_\alpha, a_\beta, \ldots, a_\gamma)$, for which $\max\{i, \alpha, \beta, \ldots, \gamma\} = s$.

b) Terms of levels $0 \div S$. (The expression $0 \div S$ is to be read as "from 0 to S.") By definition, these include the following terms of signature σ^* : the variables $x_0, x_1, \ldots, x_S$, and also the terms obtained by a single substitution of these variables in the symbols of the functions $f_0, f_1, \ldots, f_S$. Terms of the levels $0 \div S$ will be denoted by $\tau_0^S, \tau_1^S, \ldots, \tau_{K(S)}^S$.

c) The formula describing the arrangement of the terms of levels $0 \div S$. We define this as the explicit formula $\Phi(x_0, x_1, \ldots, x_S)$ which is a conjunction of formulas of the type $\tau_\alpha^S = \tau_\beta^S$, $\tau_\alpha^S < \tau_\beta^S$ and describes a linear order on the set of all terms of levels $0 \div S$. All we require here is satisfaction of the condition: the terms $x_0, x_1, \ldots, x_S$ are arranged in different classes of equal elements in this order.

d) The formulas describing the arrangement of the terms of levels $0 \div S$ in the model at the instant t. (Under the condition $S \leq t$.) We define this as the formula of the type above described, which is true at this instant in the model under the interpretation x_i in a_i, $i = 0, 1, \ldots, S$.

e) The Φ-representatives. (Let Φ be a formula of the type Φ). We define the Φ-representatives as the system $\xi_1, \xi_2, \ldots, \xi_m$ of terms of levels $0 \div S$, which contains just one term from every class of equal elements in the linear order which describes Φ. Here, $\xi_1 < \xi_2 < \ldots < \xi_m$ and m depends on Φ.

f) The filter of contradictions for Φ. Let $\Phi(x_0, x_1, \ldots, x_S)$ be any formula which describes the arrangement of the terms of levels $0 \div S$, and let $\xi_1, \xi_2, \ldots, \xi_m$ - Φ be Φ-representatives. We define the filter of contradictions for Φ as the set $F(\Phi) \subseteq W_n^{m+1}$ (where n is the number of one-place predicates), which is defined as follows.

Let $w = \alpha_1 \alpha_2 \ldots \alpha_c$ be an arbitrary word of the alphabet $A = \{1, 2, \ldots, n\}$. With $i = 0, 1, \ldots, m$ we denote

$$\varphi_i(w) = \exists y_1 y_2 \ldots y_c [\xi_i < y_1 < y_2 < \ldots < y_c < \xi_{i+1}] \,\&\, [\bigwedge_{K=1}^{c} R_{\alpha_K}(y_K)];$$

we do not write ξ_i in this formula if $i = 0$, while we do not write ξ_{i+1} if $i = m$.

Let w_i, $i = 0, 1, \ldots, m$, be any $m+1$-member sequence of words of the alphabet A; then, by definition

$$(w_0, w_1, \ldots, w_m) \in F(\Phi) \Leftrightarrow \neg \exists x_0 x_1 \ldots x_S [\Phi(x_0, x_1, \ldots, x_S) \,\&\, \bigwedge_{i=0}^{m} \varphi_i(w_i)] \in T^*.$$

It can easily be seen that the set $F(\Phi)$ thus defined is in fact a filter of the partially ordered set W_n^{m+1}.

g) The level S contradiction. Consider an instant t in the construction of the model such that $S \leq t$. Let $\Phi(x_0, x_1, \ldots, x_S)$ be the formula which describes the arrangement of the terms of levels $0 \div S$ in the model at the instant t. Let $\xi_1, \xi_2, \ldots, \xi_m$ be Φ-representatives, and $\bar{\xi}_1, \bar{\xi}_2, \ldots, \bar{\xi}_m$ their values under the interpretation x_i in a_i. Denote by Q the set of elements of M_t which are less than $\bar{\xi}_1$, by Q_i, $i = 1, 2, \ldots, m-1$, the set of elements of M_t which are arranged between $\bar{\xi}_{i-1}$ and $\bar{\xi}_i$, and by Q_m the set of elements of M_t which are greater than $\bar{\xi}_m$.

Denote by w_i, $i = 0, 1, \ldots, m$, a word of the alphabet A of length $|Q_i|$, such that α is in the j-th place of it if the predicate Q_i is true on the j-th element of the set R_α. We shall say that there is a contradiction at the level S at this instant if

$$(w_0, w_1, \ldots, w_m) \in F(\Phi). \tag{2}$$

h) The t-contradiction at level S. The concept of level S contradiction discussed in the previous paragraph is not in general effective; i.e., there may be no algorithm which, for the given S and for the given arrangement of terms of levels $0 \div S$ in the set M_t, yields an answer to the question: is there a contradiction at the level S? The concept of t-contradiction at the level S in a sense approximates the concept of level S contradiction, while being at the same time effectively specified. In essence, it amounts to the following.

It is obvious that the sets $F(\Phi)$, with all possible Φ, are uniformly recursively enumerable. Suppose we have an effective process which enumerates the elements of all possible sets $F(\Phi)$. Let $G^t(\Phi)$ be a finite subset of $F(\Phi)$, evaluated at the instant t, and let $B^t(\Phi)$ be the set of all minimal elements of $G^t(\Phi)$. Denote by $F^t(\Phi)$ the filter for which $B^t(\Phi)$ serves as a basis. Obviously,

$$F^0(\Phi) \subseteq F^1(\Phi) \subseteq \ldots \subseteq F^t(\Phi) \subseteq \ldots \subseteq F(\Phi),$$

while there exists an instant t_0, dependent on Φ, such that $t \geq t_0 \Rightarrow F^t(\Phi) = F(\Phi)$. This is precisely the instant at which there appear in $G^t(\Phi)$ all the elements of the basis of filter $F(\Phi)$, which exists and is finite by Lemma 1.

In order to define the concept of t-contradiction at the level s, we require a word-by-word repetition of the previous paragraph, with just one modification: instead of condition (2), we have to require satisfaction of the condition

$$(w_0, w_1, \ldots, w_m) \in F^t(\Phi). \tag{3}$$

Notice that the concept of t-contradiction at the level s depends on the method of enumerating the set $F(\Phi)$. We choose any one such method and settle on this.

We have

LEMMA 2. For every s there exists an instant $t_0 = t(s)$ such that for every $t \geq t_0$, the concepts of contradiction at the level s and of t-contradiction at the level s are identical.

This follows from the fact that there is a finite number of formulas describing the arrangement of the terms of levels $0 \div s$.

i) Admissible σ-model. This is any σ-model which imbeds isomorphically into the appropriate model of the theory T.

With every finite model $\mathfrak{M} = \langle M, \sigma \rangle$ we associate a word w of the alphabet A of length $|M|$, such that j stands in its α-th place if the predicate $\mathfrak{M}$ is true on the j-th element of the model R_α.

LEMMA 3. The set of words $\mathcal{D} \subseteq W_n^1$ corresponding to admissible finite σ-models, is recursive.

Proof. Obviously, $F = W_n^1 \setminus \mathcal{D}$ is a filter. By Lemma 1, F has a finite basis. It follows that F, and hence $\mathcal{D}$, is a recursive set.

We are now in possession of all the necessary concepts, and can proceed directly to describing the process of model construction.

Construction of the Model

Step $t = 0$. $M_0 = \{a_0\}$, we define predicates on M_0 in such a way that an admissible σ-model is obtained, and all functions are identically equal to a_0.

Assume that step $t-1$ has been completed. Denote by $s_0(t)$ the least $s \leq t-1$ such that there is a t-contradiction at the level s. If there is no such s, we assume that $s_0(t)$ is equal to t.

Step $t > 0$. This consists of two substeps. If $s_0(t) = t$ the first substep is omitted.

First substep. We check if it is possible to redefine the functions of level $s_0(t)$ and extend the predicates to the values of these functions going beyond M_{t-1} in such a way that

1) the resulting σ-model is admissible,

2) the t-contradiction at the level $s_0(t)$ disappears,

3) there arise no t-contradictions at levels below $s_0(t)$.

The further procedure depends on whether or not this is possible.

a) It is possible. Let q be the least number of new elements needed for this, and let n be the greatest number of elements of M_{t-1}. We set $M_t' = M_{t-1} \cup \{a_{n+1}, \ldots, a_{n+q}\}$. We continue the predicates from M_{t-1} to M_t', and also redefine the functions of level $s_0(t)$, assigning values from M_t' to them in such a way as to satisfy all three of the conditions above listed. This terminates the first substep.

b) It is impossible. We set $M_t' = M_{t-1}$, and the first substep terminates here.

Second substep. If, as a result of the first substep, $M_t' \neq M_{t-1}$, we set $M_t = M_t'$, and step t terminates here. If $M_t' = M_{t-1}$ we let n be the greatest number of elements of M_t'. We set $M_t = M_t' \cup \{a_{n+1}\}$.

We retain all the values of the functions, while continuing the predicates from V_t' onto M_t in such a way that

1) the resulting σ-model is admissible,

2) $s_0(t+1)$ takes the greatest possible value. At this point the step t terminates.

Having described the process of constructing the model $\mathfrak{M}^*$, our task is now to show that it satisfies all our requirements.

Conditions 1, 2, and 3 are obviously satisfied.

Let us show that every function $f_i(a_\alpha, a_\beta, \ldots, a_\gamma)$ in fact becomes stabilized from a certain instant. Since only functions of the level $s_0(t)$ may be varied at the instant t, we only have to show that, for every s there exists an instant t_s such that

$$t \geq t_s \implies s_0(t) \geq s. \tag{4}$$

As t_0 we can take 0. Assume that a t_s satisfying condition (4) has already been found. We set

$$t_{s+1} = \max\{t_s, t(s)\} + 1,$$

where $t(s)$ is the function of Lemma 2.

Assume that $s_0(t_{s+1} - 1) = s$. This means that, at the instant $t = t_{s+1} - 1$, there are no contradictions at the levels $0, 1, \ldots, s-1$ while there is a contradiction at the level s. Then, during the first substep of the step $t = t_{s+1} - 1$ condition a) will be satisfied, since otherwise the theory T^* would only have finite models of limited power. As a result, $s_0(t_{s+1}) \geq s+1$ there are no contradictions at the levels $0, 1, \ldots, s$. We have thus ensured the possibility of continuing the predicates on to all the newly introduced elements, without obtaining contradictions at the levels $0, 1, \ldots, s$. In that case, at every subsequent instant t, whether as a result of the first substep or as a result of the second substep, the function $s_0(t)$ will not drop below $s+1$.

If $s_0(t_{s+1} - 1) \geq s+1$, there will be no contradictions at the levels $0, 1, \ldots, s$ at the instant $t = t_{s+1} - 1$. In that case, we have $s_0(t) \geq s+1$ at every subsequent instant t.

Since every finite impoverishment of every finite submodel $\mathfrak{M}' \subseteq \mathfrak{M}^*$ does not contradict T^* and this theory is universally axiomatizable, $\mathfrak{M}^*$ must be a model of the theory T^*. If we denote by $\mathfrak{M}$ the σ-impoverishment of model $\mathfrak{M}^*$, $(\mathfrak{M}, \nu)$ will be a constructive model of theory T. QED.

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