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The Index of Lyapunov Stable Fixed Points in Two Dimensions

E. N. Dancer[1] **and R. Ortega**[2]

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In this paper, we prove that a stable isolated fixed point of an orientation preserving local homeomorphism on R^2 has fixed point index 1. We also give a number of applications to differential equations. In particular, we deduce that a number of existence methods for producing periodic solutions of differential equations in the plane always produce unstable solutions.

KEY WORDS: stability; fixed point index; periodic solutions.

1. INTRODUCTION

The index of a periodic solution of a system of differential equations (periodic in time) is an integer that is well defined whenever the solution is isolated. It can be constructed by means of degree theory and it is usually employed in the proofs of existence and multiplicity of periodic solutions. In this paper we are interested in the connections between the index and the properties of stability of a solution and the main question is, what is the value of the index of a stable periodic solution? It is well-known that if the solution is asymptotically stable, then the index is always one. However, when the solution is only stable this question is more delicate and the answer seems to depend on the dimension of the system (Krasnoselskii and Zabreiko, 1984, p. 342; Erle, 1993). In this paper we prove that if the system has two dimensions, then every stable and isolated periodic solution has index one.

This result was stated without proof by Krasnoselskii (1968, p. 192). It is possible to give an elementary proof, based on Poincaré–Bendixson theory,

[1] Department of Mathematics and Statistics, University of Sydney, Sydney, NSW 2006, Australia.

[2] Departamento de Matemática Aplicada, Universidad de Granada, 18071 Granada, Spain.

for the special case of a stable equilibrium of an autonomous system (Krasnoselskii *et al.*, 1966, p. 143; Thews, 1989). In the general case it seems that the proof must be based on the use of nonelementary topological properties of the plane. [Note that by Erle (1993) and Krasnoselskii and Zabreiko (1984), the corresponding result is false in more than two dimensions.] We present a complete proof based on Brouwer's lemma on translation arcs. Several proofs of this lemma have been published recently and we shall follow the approach of Brown (1984). In the topology literature Brouwer's lemma is stated for orientation-preserving maps that are homeomorphisms of the plane (in particular, they are onto). In the applications to differential equations the map is not always onto but it is a homeomorphism from the plane to a part of it. For this reason we have employed a modified version of the lemma. We remark that Brouwer's lemma has been applied previously to periodic differential equations to prove the theorem of Massera on second-order systems and the same difficulty appeared. [See Massera (1950) and Pliss (1966, p. 152).]

The main result of this paper can be applied to obtain instability criteria by computing degree. In particular, we show that two classical methods in the theory of periodic solutions of second-order equations (upper and lower solutions, minimization of the action) usually lead to unstable solutions. We present our result in an abstract setting, in terms of Brouwer's index and stability of fixed points of orientation preserving local homeomorphisms of the plane. This presentation is suitable for applications not necessarily related to periodic solutions and we illustrate this fact giving a new proof, based on degree theory, of a classical instability criterion due to Levi-Civita (1901; Siegel and Moser, 1971).

2. MAIN RESULT

Let $U \subset \mathbb{R}^2$ be open and connected and denote by $L(U)$ the class of mappings $f\colon U \to \mathbb{R}^2$ that are continuous, one-to-one and orientation-preserving. [A precise definition of orientation-preserving map can be given using the concept of orientation of a topological manifold (Spanier, 1989, p. 294).]

We remark that each $f \in L(U)$ is a homeomorphism from U onto $f(U)$. In particular, $f(U)$ is open by the invariance of the domain.

Theorem 2.1. *Assume that* $0 \in U$ *and let* $f \in L(U)$ *be such that* $f(0) = 0$. *If* 0 *is stable and isolated as a fixed point of* f, *then*

$$index(I - f, 0) = 1$$

The proof is given at the end of the section. There are many results on how to compute the index of fixed points of planar maps (see Krasnoselskii *et al.*, 1966). These results can be combined with the theorem to obtain instability criteria. We present two examples of such criteria.

Example 2.2 [An Instability Criterion of Levi-Civita (1901)]. Let $f(x, y) = (x_1, y_1)$ be a C^2 map defined in a neighborhood of the origin with the Taylor expansion

$$x_1 = x + a_1 x^2 + 2a_2 xy + a_3 y^2 + \cdots$$
$$y_1 = y + b_1 x^2 + 2b_2 xy + b_3 y^2 + \cdots$$

Define

$$D_1 = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}, \qquad D_2 = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}, \qquad D_3 = \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}$$

and $D = 4D_1 D_2 - D_3^2$. It follows from Theorems 8.1 and 8.4 of Krasnoselskii *et al.* (1966) that if $D \neq 0$, the origin is isolated and the index can take only the value 0, 2, or -2. In consequence 0 is unstable if $D \neq 0$.

Example 2.3 [Instability at the Third Root of Unity (Siegel and Moser, 1971, p. 222)]. We now use complex notation and assume that $f = f(z, \bar{z})$ is a C^2 map with expansion at 0 given by

$$f(z, \bar{z}) = \omega z + az^2 + bz\bar{z} + c\bar{z}^2 + \cdots$$

where ω is a primitive third root of unity, $\omega^2 + \omega + 1 = 0$, and $a, b, c \in \mathbb{C}$. The third iteration f^3 has the expansion

$$f^3(z, \bar{z}) = z + 3c\overline{\omega z}^2 + \cdots$$

If $c \neq 0$, 0 is an isolated 3-periodic point of f and *index*$(I - f^3, 0) = -2$. It follows from the theorem that 0 is unstable with respect to f^3. As a consequence, if $c \neq 0$, the origin is unstable with respect to f.

The following lemmas and definitions are useful for the proof of the theorem.

Lemma 2.4. *Under the assumptions of Theorem 2.1. there exists* $\tilde{f} \in L(\mathbb{R}^2)$ *such that*

(i) $Fix(\tilde{f}) = \{0\}$.

(ii) 0 *is stable with respect to* $\tilde{f}$.

(iii) $Index(I - \tilde{f}, 0) = index(I - f, 0)$.

Proof. In view of the assumptions we can choose two open disks centered at the origin, $D_1 \subset D_2 \subset U$, such that $\overline{D_2} \cap Fix(f) = \{0\}$ and $f^n(\overline{D_1}) \subset D_2$, for each $n \geq 0$. Define $V = \bigcup_{n \geq 0} f^n(D_1)$. Then V is a domain and $f(V) \subset V$. We now fill in the possible holes of V by considering the simply connected domain

$$W = \bigcup \{\hat{\Gamma}: \Gamma \text{ is a Jordan curve in } V\}$$

Here $\hat{\Gamma}$ denotes the bounded component of $\mathbb{R}^2 - \Gamma$. Since f maps homeomorphically D_2 onto $f(D_2)$, it is clear that $f(W) \subset W$. We now consider a homeomorphism Ψ from W onto $\mathbb{R}^2$ and define $\tilde{f} = \Psi \circ f \circ \Psi^{-1}$. (i) and (ii) follow from the construction and (iii) is a consequence of the commutativity theorem for the degree.

Remark. The same technique to fill in the holes is employed by Siegel and Moser (1971, p. 185).

Let $f \in L(\mathbb{R}^2)$. We say that f is locally free if for each $x \in \mathbb{R}^2 - Fix(f)$, there exists an open disk D with $x \in D$ and such that $f^p(D) \cap f^q(D) = \emptyset$ for each $p, q \geq 0, p \neq q$. This definition was introduced by Brown (1985) for the case of homeomorphisms.

Lemma 2.5. Let $f \in L(\mathbb{R}^2)$ be locally free and assume that 0 is an isolated and stable fixed point of f. Then 0 is asymptotically stable.

Proof. [It follows along the lines of the proof of Lemma 3.4 of Brown (1985).] Let U_1 be a bounded positively invariant neighborhood of the origin such that $Fix(f) \cap \overline{U_1} = \{0\}$. We prove $f^n(x) \to 0$ for each $x \in U_1$. Given a small open neighborhood of 0, $U_2 \subset U_1$, by compactness we can find a finite cover of $\overline{U_1} - U_2$ of the form $\{D_i\}_{i=1}^{k}$, where each D_i is a disk such that $f^p(D) \cap f^q(D) = \emptyset$ if $p, q \geq 0, p \neq q$. (We have here used the local freeness of f.) The semiorbit $\{f^n(x): n \geq 0\}$ can intersect D_i at most once and therefore $f^n(x) \in U_2$ for large n.

Let $f \in L(\mathbb{R}^2)$ be given. A translation arc for f is an injective arc $\alpha \subset \mathbb{R}^2$ with extremes p_0, p_1 such that $f(p_0) = p_1$ and $\alpha \cap f(\alpha) = \{p_1\}$. We now state a variant of the classical Brouwer's lemma on translation arcs.

Lemma 2.6. Let $f \in L(\mathbb{R}^2)$ and assume that for each Jordan curve $J \subset \mathbb{R}^2 - Fix(f)$,

$$deg[I - f, \hat{J}] \neq 1$$

Then if α is a translation arc for f, $f^n(\alpha) \cap \alpha = \emptyset$ for each $n \geq 2$.

A proof of a very similar result is given by Brown (1984) for the case where $f(\mathbb{R}^2) = \mathbb{R}^2$. The proof of our result is a simple modification of the proof there.

Proof of Theorem 2.1. In view of Lemma 2.4 it is not restrictive to assume that $f \in L(\mathbb{R}^2)$ and 0 is the unique fixed point. We distinguish two cases.

(1) f is locally free. It follows from Lemma 2.5 that 0 is asymptotically stable, and in such a case it is well-known that $index(I-f, 0) = 1$ (Krasnoselskii and Zabreiko, 1984, p. 235).

(2) f is not locally free. The same proof as in Lemma 4.2 of Brown (1985) shows that there exists a translation arc $\alpha \subset \mathbb{R}^2 - \{0\}$ such that $f^n(\alpha) \cap \alpha \neq \emptyset$ for some $n \geqslant 2$. Lemma 2.6 implies the existence of a domain in $\mathbb{R}^2$ where the degree of $I-f$ is one. Since 0 is the only fixed point of f, this degree must coincide with $index(I-f, 0)$.

3. THE INDEX OF PERIODIC SOLUTIONS

Let us consider the differential equation

$$X' = G(t, X) \tag{3.1}$$

where $G: \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ is continuous, T-periodic in t and such that the solution of the initial value problem is unique. The Poincaré map is defined by $P(x) = X(T, x)$, where $X(t, x)$ is the solution of (3.1) satisfying $X(0, x) = x$. Let U be a connected component of the open set $O = \{x \in \mathbb{R}^2 : X(t, x)$ is defined in $[0, T]\}$, then $P \in L(U)$ since $X(t, \cdot)$ defines an isotopy between I and P.

The index of an isolated T-periodic solution φ is defined by

$$\gamma(\varphi) = index[I - P, \varphi(0)]$$

and Theorem 2.1 says in this context that $\gamma(\varphi) = 1$ if φ is isolated and stable.

We now consider the second-order equation

$$y'' = F(t, y) \tag{3.2}$$

where $F: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies the same kind of assumptions previously imposed to G.

In the first place we analyze the method of upper and lower solutions for this equation. A strict lower solution of (3.2) is a T-periodic function $\alpha \in C^2(\mathbb{R})$ such that

$$\alpha''(t) > F(t, \alpha(t)), \qquad \forall t \in \mathbb{R}$$

The concept of strict upper solution is defined by reversing the inequality.

Proposition 3.1. *Assume that (3.2) has strict upper and lower solutions, β and α, respectively, satisfying $\alpha(t) < \beta(t)$, $\forall t \in \mathbb{R}$. In addition, assume that the number of T-periodic solutions satisfying $\alpha(t) < x(t) < \beta(t)$, $\forall t \in \mathbb{R}$, is finite. Then at least one of them is unstable.*

Proof. The proof of Lemma 3.2 by Ortega (1990) shows the existence of a T-periodic solution of (3.2) with negative index and lying between α and β. In consequence this solution is unstable.

Remark. It is possible to extend the results to equations depending also on y' and satisfying a Nagumo condition. [For example, the proofs of Mawhin (1985) can be adapted].

Equation (3.2) has a variational structure and can be seen as the Euler equation of the action functional

$$A[y] = \int_0^T \{\tfrac{1}{2}(y')^2 + V(t, y)\}\, dt, \qquad y \in H^1(\mathbb{R}/T\mathbb{Z})$$

where $V(t, y) = \int_0^y F(t, z)\, dz$. It is well-known that the critical points of A coincide with the periodic solutions of (3.2).

Proposition 3.2. *Let φ be an isolated T-periodic solution of (3.2) such that A reaches a local minimum at φ. Then φ is unstable.*

Proof. We use the following notation: H is the Hilbert space $H^1(\mathbb{R}/T\mathbb{Z})$ with product $(y, z) = \int_0^T y'z' + yz$. $C^1 = C^1(\mathbb{R}/T\mathbb{Z})$. Given $y \in C(\mathbb{R}/T\mathbb{Z})$, $Ky = u$ is the unique T-periodic solution of the equation

$$-u'' + u = F(t, y(t)) - y(t)$$

It is easily verified that the gradient field $\nabla A\colon H \to H$ is given by $\nabla A(y) = y + K(y)$, $y \in H$. Since K is compact we can apply the results of Amann (1982) to conclude that $index(\nabla A, \varphi) = 1$. The operator $T\colon C^1 \to C^1$ defined by $T(y) = y + K(y)$ has a common core with ∇A. The invariance principle of Krasnoselskii and Zabreiko (1984, p. 141) or the commutativity theorem for the degree implies that $index(T, \varphi) = 1$. The principles of relatedness developed in Chapter 3 of Krasnoselskii and Zabreiko (1984) allow us to conclude that $\gamma(\varphi) = -index(T, \varphi) = -1$ and therefore φ is unstable.

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