LATTICES OF SUBMANIFOLDS IN MANIFOLDS OF ALGEBRAS

A. A. Akataev and D. M. Smirnov

Let m, n be fixed integers satisfying the inequalities $1 \leqslant m \leqslant n$. We denote by $\mathfrak{A}_{m,n}$ the manifold of algebras $A=\langle A, \varphi_1,\ldots,\varphi_n, \omega_1,\ldots,\omega_m\rangle$ of the type $\langle m,\ldots,m, n,\ldots,n\rangle$, defined by the identities

$$\left.\begin{array}{l}\varphi_i(\omega_1(x_1,\ldots,x_n),\ldots,\omega_m(x_1,\ldots,x_n))=x_i\ (i=1,\ldots,n),\\ \omega_j(\varphi_1(x_1,\ldots,x_m),\ldots,\varphi_n(x_1,\ldots,x_m))=x_j\ (j=1,\ldots,m).\end{array}\right\} \quad (1)$$

Swierczkowski [1] established that when $m<n$ the manifold $\mathfrak{A}_{m,n}$ contains only a finite number of isomorphic types of free algebras of finite rank. In the earlier article of Jonsson and Tarski [2] they considered the case $m=1$, $n=2$ and proved that in the manifold $\mathfrak{A}_{1,2}$ all free algebras of finite rank are in general isomorphic. In view of these results A. I. Mal'tsev posed to the authors the problem of studying the lattices $L(\mathfrak{A}_{m,n})$ of submanifolds in the manifolds $\mathfrak{A}_{m,n}$ $(m<n)$. It was expected that for some pairs of numbers $m<n$ these lattices might be foreseeable. In particular, A. I. Mal'tsev observed that the manifolds $\mathfrak{A}_{1,n}$ $(n>1)$ are equationally complete (or minimal).

In the study proposed it is proved that the class of manifolds $\mathfrak{A}_{m,n}$ $(m<n)$ contains no other minimal manifolds and that when $n>m>1$ the lattice $L(\mathfrak{A}_{m,n})$ has the power of the continuum and does not satisfy a single one of the termination conditions for increasing and decreasing chains.

We study also the lattices $L(\mathfrak{A}_{n,n})$ of submanifolds in the manifolds $\mathfrak{A}_{n,n}$ $(n=1,2,\ldots)$. This class of manifolds differs in principle from the class of manifolds $\mathfrak{A}_{m,n}$ $(m<n)$, since each manifold $\mathfrak{A}_{n,n}$ possesses a complete spectrum $\{1,2,3,\ldots\}$ of orders of finite algebras and by the Jonsson-Tarski-Fujziwara theorem (see [2]) $\mathfrak{A}_{n,n}$ –free algebras of different ranks–are not isomorphic. We have established that when $n\geqslant 2$ the manifold $\mathfrak{A}_{n,n}$ has a continuous set of equationally complete submanifolds and so the lattice $L(\mathfrak{A}_{n,n})$ when $n\geqslant 2$ also does not satisfy the termination condition for decreasing chains. The lattice $L(\mathfrak{A}_{1,1})$ is isomorphic to the lattice of positive integers with the relationship of divisibility, completed by the external zero 0 and the external unit I. For arbitrary $n\geqslant 1$ the lattice $L(\mathfrak{A}_{n,n})$ does not satisfy the termination condition for increasing chains.

The main method of the study is the method of modelling identities, which is altogether natural in the study of arbitrary manifolds given by a system of defining identities. The most important part of this method is establishing the rational equivalence (in the sense of A. I. Mal'tsev [3]) of one or another manifold which is being studied to an appropriate manifold of algebras for which the lattice of submanifolds has already to some extent been studied. The following have proved to be the appropriate manifolds for our purposes; the manifold of totally symmetric quasigroups (briefly, TS-quasigroups) defined by the identities

$$xy\cdot y = x = y\cdot yx,$$

and also the class of all infinite and single-element TS-quasigroups, which can be enriched with new operations and transformed into a manifold of universal algebras. We use as the principal result the theorem of A. D. Bol'bot [4] to the effect that a lattice of submanifolds of a manifold of TS-quasigroups has a continuous set of atoms (or points).

The authors dedicate the present work to the enduring memory of the unforgettable Anatolii Ivanovich Mal'tsev.

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1. CONSTRUCTION OF AUXILIARY MODELS

In what follows we shall repeatedly be using the following Jonsson-Tarski result (see [2], p. 100), which we formulate in the form we need:

LEMMA 1. For an arbitrary integer $r \geqslant 2$ every infinite set $\mathcal{M}$ admits the system of operations

$$\varphi_i(x),\ \omega(x_1,\dots,x_r) \qquad (i=1,2,\dots,r),$$

satisfying on this set the identities

$$\omega(\varphi_1(x),\dots,\varphi_r(x))=x,\ \ \varphi_i(\omega(x_1,\dots,x_r))=x_i \quad (i=1,\dots,r).$$

To make the exposition complete we shall prove this statement. Since the power $m=|\mathcal{M}|$ is infinite, it follows that $m=m^2=\dots=m^r$ (see, for example, [5]). So there exists a one-to-one mapping $\omega(x_1,\dots,x_r): M^r \to M$ of the Cartesian power $\mathcal{M}^r$ of the set $\mathcal{M}$ onto $\mathcal{M}$. We put

$$\varphi_i(\omega(x_1,\dots,x_r))=x_i \quad (i=1,\dots,r).$$

Since each element of $\mathcal{M}$ is uniquely representable in the form

$$\omega(x_1,\dots,x_r)$$

for suitable elements $x_1,\dots,x_r$ of $\mathcal{M}$, it follows that the φ_i $(i=1,\dots,r)$ are operations on the set $\mathcal{M}$.

We note that

$$\varphi_1(u)=\varphi_1(v)\,\&\dots\&\,\varphi_r(u)=\varphi_r(v)\Rightarrow u=_1 v.$$

For let $u=\omega(x_1,\dots,x_r)$, $v=\omega(y_1,\dots,y_r)$, where $x_i,y_i\in\mathcal{M}$. Then $\varphi_i(u)=x_i$, $\varphi_i(v)=y_i$, whence

$$x_1=y_1,\dots,x_r=y_r,$$

and so $u=v$.

Now let $\omega(\varphi_1(x),\dots,\varphi_n(x))=y$. Then $\varphi_i(y)=\varphi_i(x)$ for all $i=1,\dots,r$ and, by what has been proved, $y=x$.

The following obvious proposition also holds:

LEMMA 1'. For an arbitrary integer $m\geqslant 1$ every nonempty set $\mathcal{M}$ admits the operations $\varphi_i(x_1,\dots,x_m)$, $\omega_i(x_1,\dots,x_m)$, $(i=1,\dots,m)$, which satisfy on this set the system of identities (1) when $n=m$.

For as φ_i,ω_i it is sufficient to take the selector (or trivial) operations

$$e_i^{(m)}(x_1,\dots,x_m)=x_i \quad (i=1,\dots,m).$$

We shall assume given the integers m,n, satisfying the inequalities $1\leqslant m\leqslant n$. In the sequel we shall find useful the following.

THEOREM 1. Let R be an associative ring with unit 1 and $\xi_1,\xi_2,\dots,\xi_{m+1}$ fixed elements of that ring of which the element ξ_1 is invertible. If on the set R it is possible to define the system of operations

$$\varphi_i(x_1,\dots,x_m),\ \omega_j(x_1,\dots,x_n) \quad (i=1,\dots,n;\ j=1,\dots,m),$$

connected on R by the identities (1), then putting

$$\bar\varphi_1(x_1,\dots,x_{m+1})=\xi_1x_1+\xi_2x_2+\dots+\xi_{m+1}x_{m+1},$$

$$\bar\varphi_{i+1}(x_1,\dots,x_{m+1})=\varphi_i(x_2,\dots,x_{m+1}) \quad (i=1,\dots,n),$$

$$\bar\omega_1(x_1,\dots,x_{n+1})=\xi_1^{-1}(x_1-\xi_2\omega_1(x_2,\dots,x_{n+1})-\dots-\xi_{m+1}\omega_m(x_2,\dots,x_{n+1})),$$

$$\bar\omega_{j+1}(x_1,\dots,x_{n+1})=\omega_j(x_2,\dots,x_{n+1}) \quad (j=1,\dots,m),$$

we obtain the model $\langle \mathcal{R}, \bar{\varphi}_i, \bar{\omega}_j \rangle$ for the system of identities

$$\bar{\varphi}_i(\bar{\omega}_1(x_1,\dots,x_{n+1}),\dots,\bar{\omega}_{m+1}(x_1,\dots,x_{n+1})) = x_i \quad (i=1,\dots,n+1),$$
$$\bar{\omega}_j(\bar{\varphi}_1(x_1,\dots,x_{m+1}),\dots,\bar{\varphi}_{n+1}(x_1,\dots,x_{m+1})) = x_j \quad (j=1,\dots,m+1).$$

Analogously, if $A = \langle A, \varphi_i, \omega_j \rangle$ is an algebra of the manifold $\mathfrak{A}_{m,n}$, then the algebra defined by the set A^* and the system of operations

$$\bar{\varphi}_1(x_1,\dots,x_{m+1}) = x_1,$$
$$\bar{\varphi}_{i+1}(x_1,\dots,x_{m+1}) = \varphi_i(x_2,\dots,x_{m+1}) \quad (i=1,\dots,n),$$
$$\bar{\omega}_1(x_1,\dots,x_{n+1}) = x_1,$$
$$\bar{\omega}_{j+1}(x_1,\dots,x_{n+1}) = \omega_j(x_2,\dots,x_{n+1}) \quad (j=1,\dots,m),$$

belongs to the manifold $\mathfrak{A}_{m+1,n+1}$.

Proof. Let $x_1, x_2, \dots$ be arbitrary elements of the ring R. Direct calculation gives

$$\bar{\varphi}_1(\bar{\omega}_1(x_1,\dots,x_{n+1}),\dots,\bar{\omega}_{m+1}(x_1,\dots,x_{n+1})) = \xi_1\bar{\omega}_1(x_1,\dots,x_{n+1})+\dots$$
$$\dots+\xi_{m+1}\bar{\omega}_{m+1}(x_1,\dots,x_{n+1}) = x_1-\xi_2\omega_1(x_2,\dots,x_{n+1})-\dots-\xi_{m+1}\omega_m(x_2,\dots,x_{n+1})$$
$$+\xi_2\omega_1(x_2,\dots,x_{n+1})+\dots+\xi_{m+1}\omega_m(x_2,\dots,x_{n+1}) = x_1,$$
$$\bar{\omega}_1(\bar{\varphi}_1(x_1,\dots,x_{m+1}),\dots,\bar{\varphi}_{n+1}(x_1,\dots,x_{m+1})) = \xi_1^{-1}\bar{\varphi}_1(x_1,\dots,x_{m+1})-\dots$$
$$\dots-\xi_1^{-1}\xi_{m+1}\omega_m(\bar{\varphi}_2(x_1,\dots,x_{m+1}),\dots,\bar{\varphi}_{n+1}(x_1,\dots,x_{m+1})) = x_1+\xi_1^{-1}\xi_2x_2+\dots$$
$$\dots+\xi_1^{-1}\xi_{m+1}x_{m+1}-\xi_1^{-1}\xi_2\omega_1(\varphi_1(x_2,\dots,x_{m+1}),\dots,\varphi_n(x_2,\dots,x_{m+1}))-\dots$$
$$\dots-\xi_1^{-1}\xi_{m+1}\omega_m(\varphi_1(x_2,\dots,x_{m+1}),\dots,\varphi_n(x_2,\dots,x_{m+1})) = x_1.$$

The remaining identities are obvious.

The second part of Theorem 1 may also be verified by direct calculation.

§ 2. THE SERIES OF SUBMANIFOLDS $\mathfrak{A}_{m,n}^{(s)}$ OF THE MANIFOLD $\mathfrak{A}_{m,n}$ $(1<m<n)$

Let $m>1$. We form the terms

$$\Phi(x,y) = \varphi_1(x,y,y,\dots,y),$$
$$\Phi_{m,n}^{(s)}(x) = \Phi(x,\Phi(x,\dots,\Phi(x,x))\dots),$$

where the letter Φ has s entries $(s=1,2,\dots)$. The identity

$$\Phi_{m,n}^{(s)}(x) = x \tag{3}$$

defines within the manifold $\mathfrak{A}_{m,n}$ $(1<m<n)$ a submanifold which we shall denote by $\mathfrak{A}_{m,n}^{(s)}$.

THEOREM 2. If $2 \le m < n$ and $1 \le s < t$ then the manifolds $\mathfrak{A}_{m,n}^{(s)}$ and $\mathfrak{A}_{m,n}^{(t)}$ are distinct and the inclusion $\mathfrak{A}_{m,n}^{(s)} \subset \mathfrak{A}_{m,n}^{(t)}$ holds if and only if s/t.

COROLLARY. If $2 \le m < n$ the manifold $\mathfrak{A}_{m,n}$ possesses an infinite, strictly increasing chain of submanifolds, for example

$$\mathfrak{A}^{(s)}_{m,n} < \mathfrak{A}^{(2s)}_{m,n} < \mathfrak{A}^{(2^2 s)}_{m,n} < \dots,$$

where s is an arbitrary positive integer.

Before proceeding to prove Theorem 2, we shall construct a special model for the system of the identities (1), (3), defining the manifold $\mathfrak{A}^{(s)}_{m,n}$ by assuming that $1 < m < n$ and $s \geq 1$.

We put $\tau = n - m + 1$. Since $n > m$, it follows that $\tau > 1$. By Lemma 1 the set of all complex numbers K is a support of some algebra $K_1 = \langle K, \varphi_i, \omega_1 \rangle$ of the type $\langle 1, \dots, 1, \tau \rangle$, belonging to the manifold $\mathfrak{A}_{1,\tau}$.

Let ξ be a fixed complex number. Putting $\xi_1 = 1$, $\xi_2 = \xi$, $\xi_3 = \xi_4 = \dots = 0$ and using Theorem 1, we construct the sequence of algebras $K_1(\xi) = K_1$, $K_2(\xi), \dots, K_m(\xi)$ possessing the following properties:

1) $K_i(\xi) \in \mathfrak{A}_{i,\tau+i-1}$ $(i = 1, \dots, m)$.

2) The principal set (or support) of the algebra $K_i(\xi) (i=1,\dots,m)$ is the set of all complex numbers K.

3) The principal operations of the algebra $K_{i+1}(\xi)$ may be expressed in terms of the ring operations in K and in terms of the principal operations of the algebra

$$K_i(\xi) \quad (i = 1, \dots, \tau - 1)$$

by the formulas (2).

Since $\tau = n - m + 1$, the algebra $K_m(\xi)$ is of the type

$$\langle m, m, \dots, m, n, \dots, n \rangle$$

and belongs to the manifold $\mathfrak{A}_{m,n}$.

<u>LEMMA 2.</u> If $m \geq 2$, the value of the term $\Phi^{(s)}_{m,n}(x)$ in the algebra $K_m(\xi)$ when $x = a \in K$ is

$$a(1 + \xi + \xi^2 + \dots + \xi^s).$$

<u>Proof.</u> By the construction, the value in the algebra $K_m(\xi)$ of the first principal operation $\varphi_1(x_1, \dots, x_m)$ is calculated from the formula

$$\varphi_1(x_1, \dots, x_m) = x_1 + \xi x_2$$

whence we obtain

$$\begin{aligned} \Phi^{(s)}_{m,n}(a) &= a + \xi(a + \dots + \xi(a + \xi a)) \\ &= a\ (1 + \xi + \xi^2 + \dots + \xi^s). \end{aligned}$$

<u>Proof of Theorem 2.</u> Let $1 \leq s < t$ and $2 \leq m < n$. If ε_t is a t-th power primitive root of 1, then by Lemma 2 the value of the term $\Phi^{(s)}_{m,n}(x)$ in the algebra $K_m(\varepsilon_t)$ when $x = 1$ is $1 + \varepsilon_t + \dots + \varepsilon_t^s \neq 1$. So the identity $\Phi^{(s)}_{m,n}(x) = x$ does not hold in the algebra $K_m(\varepsilon_t)$ and so $K_m(\varepsilon_t)$ does not belong to the manifold $\mathfrak{A}^{(s)}_{m,n}$. However, for an arbitrary element $a \in K_m(\varepsilon_t)$ the value of the term $\Phi^{(t)}_{m,n}(x)$ in $K_m(\varepsilon_t)$ when $x = a$ is $a(1 + \varepsilon_t + \dots + \varepsilon_t^t) = a$ and so $K_m(\varepsilon_t) \in \mathfrak{A}^{(t)}_{m,n}$. Thus it has been proved that $\mathfrak{A}^{(s)}_{m,n} \neq \mathfrak{A}^{(t)}_{m,n}$.

If $\mathfrak{A}^{(s)}_{m,n} \subseteq \mathfrak{A}^{(t)}_{m,n}$, $s > 1$, and ε_s is a s-th primitive root of 1, then $K_m(\varepsilon_s) \in \mathfrak{A}^{(t)}_{m,n}$. So the value of the term $\Phi^{(t)}_{m,n}(x)$ in $K_m(\varepsilon_s)$ for an arbitrary element x of it must be equal to x. In particular, when $x = 1$ we obtain, in view of Lemma 2: $1 + \varepsilon_s + \varepsilon_s^2 + \dots + \varepsilon_s^t = 1$, whence $\varepsilon_s^t = 1$ and so s/t.

Conversely, if $1 \leq s < t$ and s/t, then it is clear that

$$\mathfrak{A}^{(s)}_{m,n} \subseteq \mathfrak{A}^{(t)}_{m,n}.$$

This proves Theorem 2.

§ 3. RATIONAL EQUIVALENCE

The class of algebras $\mathcal{L}$ is said to be rationally equivalent to the class of algebras $\mathcal{L}'$ (see [3], [6]) if there exists a finite chain of classes of algebras $\mathcal{L}_1 = \mathcal{L}, \mathcal{L}_2, \ldots, \mathcal{L}_1 = \mathcal{L}'$, which starts with $\mathcal{L}$ and finishes with the class $\mathcal{L}'$ and in which each successive class $\mathcal{L}_{i+1}$ may be obtained from the preceding class $\mathcal{L}_i$ either by joining on one terminal operation from the principal operations of the class $\mathcal{L}_i$ or by omitting one principal operation of $\mathcal{L}_i$ which must in this context be expressible termally in terms of the other principal operations of $\mathcal{L}_i$. This transition from $\mathcal{L}_i$ to $\mathcal{L}_{i+1}$ will be called simple enrichment in the one case and simple contraction in the other.

THEOREM 3. For arbitrary integers

$$1 \leq m \leq n$$

the manifold $\mathfrak{A}_{m+1,n+1}$ contains a proper submanifold $\mathfrak{A}^{(0)}_{m+1,n+1}$ rationally equivalent to the manifold $\mathfrak{A}_{m,n}$.

Proof. We shall denote the principal operations of algebras belonging to the class $\mathfrak{A}_{m,n}$ unequivocally by

$$\varphi_i(x_1, \ldots, x_m), \quad \omega_j(x_1, \ldots, x_n) \quad (i = 1, \ldots, n; j = 1, \ldots, m),$$

and the principal operations of algebras belonging to $\mathfrak{A}_{m+1,n+1}$ by

$$\bar{\varphi}_i(x_1, \ldots, x_{m+1}), \quad \bar{\omega}_j(x_1, \ldots, x_{n+1}) \quad (i = 1, \ldots, n+1; j = 1, \ldots, m+1).$$

Let $\mathfrak{A}^{(0)}_{m+1,n+1}$ be a submanifold of the manifold $\mathfrak{A}_{m+1,n+1}$ defined within $\mathfrak{A}_{m+1,n+1}$ by the identities

$$\left.\begin{aligned} &\bar{\varphi}_1(x_1, \ldots, x_{m+1}) = x_1 = \bar{\omega}_1(x_1, \ldots, x_{n+1}), \\ &\bar{\varphi}_i(x_1, x_2, \ldots, x_{m+1}) = \bar{\varphi}_i(y, x_2, \ldots, x_{m+1}) \quad (i = 2, \ldots, n+1), \\ &\bar{\omega}_j(x_1, x_2, \ldots, x_{n+1}) = \bar{\omega}_j(y, x_2, \ldots, x_{n+1}) \quad (j = 2, \ldots, m+1). \end{aligned}\right\} \tag{4}$$

We shall show that the manifolds $\mathfrak{A}_{m,n}$ and $\mathfrak{A}^{(0)}_{m+1,n+1}$ are rationally equivalent.

In the first place, if the algebra $\mathbf{A} = \langle A, \varphi_i, \omega_j \rangle$ belongs to the manifold $\mathfrak{A}_{m,n}$, then relative to the term operations

$$\bar{\varphi}_1(x_1, \ldots, x_{m+1}) = x_1,$$

$$\bar{\varphi}_{i+1}(x_1, \ldots, x_{m+1}) = \varphi_i(x_2, \ldots, x_{m+1}) \quad (i = 1, \ldots, n),$$

$$\bar{\omega}_1(x_1, \ldots, x_{n+1}) = x_1,$$

$$\bar{\omega}_{j+1}(x_1, \ldots, x_{n+1}) = \omega_j(x_2, \ldots, x_{n+1}) \quad (j = 1, \ldots, m)$$

the algebra $\mathbf{A} = \langle A, \bar{\varphi}_i, \bar{\omega}_j \rangle$ belongs to the manifold $\mathfrak{A}^{(0)}_{m+1,\ \cdot+1}$ by the second part of Theorem 1.

Secondly, the operations φ_i, ω_j of $\mathbf{A}$ are in their turn clearly expressible by means of terms derived from $\bar{\varphi}_i$, $\bar{\omega}_j$.

Thus it is possible by a finite number of simple enrichments and simple contractions to pass from the manifold $\mathfrak{A}_{m,n}$ to a subclass of the class $\mathfrak{A}^{(0)}_{m+1,n+1}$. But in fact this subclass coincides with the manifold $\mathfrak{A}^{(0)}_{m+1,n+1}$.

For let the algebra $\mathbf{A} = \langle A, \bar{\varphi}_i, \bar{\omega}_j \rangle$ belong to the manifold $\mathfrak{A}^{(0)}_{m+1,n+1}$. By identities (4) it is possible to define on the set A the term operations

$$\varphi_i(x_1,\dots,x_m)=\bar\varphi_{i+1}(x_1,x_1,x_2,\dots,x_m)\ (i=1,\dots,n),$$

$$\omega_j(x_1,\dots,x_n)=\bar\omega_{j+1}(x_1,x_1,x_2,\dots,x_n)\ (j=1,\dots,m),$$

relative to which the algebra $\langle A,\varphi_i,\omega_j\rangle$ will belong (as can easily be checked) to the manifold $\mathfrak{A}_{m,n}$. The rational equivalence of the manifolds $\mathfrak{A}_{m,n}$ and $\mathfrak{A}^{(0)}_{m+1,\,n+1}$ has thus been proved.

Since the identity

$$\bar\varphi_1(x,x,\dots,x)=x$$

is true in the manifold $\mathfrak{A}^{(0)}_{m+1,\,n+1}$, it follows that when $n>m$ the inclusion

$$\mathfrak{A}^{(0)}_{m+1,n+1}\subseteq\mathfrak{A}^{(1)}_{m+1,n+1}$$

holds and by Theorem 2 of § 1 the submanifold $\mathfrak{A}^{(0)}_{m+1,n+1}$ is distinct from the manifold $\mathfrak{A}_{m+1,n+1}$. But if $n=m$, then on the set of all complex numbers K we prescribe the selector functions $e_i^{(m)}\ (i=1,\dots,m)$ and put

$$\varphi_i(x_1,\dots,x_m)=\omega_i(x_1,\dots,x_m)=e_i^{(m)}(x_1,\dots,x_m)\quad(i=1,\dots,m).$$

Then by formulas (2) we define in the set K the operations

$$\bar\varphi_i,\ \bar\omega_i\ (i=1,\dots,m+1)$$

with $\xi_1=\xi_2=1$, $\xi_3=\xi_4=\dots=0$. By Theorem 1 we obtain the algebra $\langle K,\bar\varphi_i,\bar\omega_i\rangle$ of the manifold $\mathfrak{A}_{m+1,n+1}$ in which the value of the operation $\bar\varphi_1$ can be calculated from the formula $\bar\varphi_1(x_1,\dots,x_{m+1})=x_1+x_2$ and depends in essence on two arguments. Consequently, this algebra does not belong to the manifold $\mathfrak{A}^{(0)}_{m+1,n+1}$. Theorem 3 has thus been completely proved.

Note: In view of the inclusion

$$\mathfrak{A}^{(0)}_{m+1,n+1}\subseteq\mathfrak{A}^{(1)}_{m+1,n+1}$$

for $n>m$, the question naturally arises of whether these manifolds coincide. We shall show however, that the manifold $\mathfrak{A}^{(0)}_{m+1,n+1}$ is distinct from the manifold $\mathfrak{A}^{(1)}_{m+1,n+1}$ for all m,n, $1\le m<n$.

In § 2 we constructed the algebra K_1 on the set of all complex numbers K and belonging to the manifold $\mathfrak{A}_{1,r}$, where $r=n-m+1$. Starting from it and using Theorem 1, we construct the algebras $K_1,K_2,\dots$ with the common support K and such that K_{i+1} belongs to the manifold $\mathfrak{A}_{i+1,\,r+i}$ and the principal operations of K_{i+1} may be expressed in terms of the principal operations of K_i by formulas (2) for $\xi_1=2$, $\xi_2=-1$, $\xi_3=\xi_4=\dots=0$. Then the algebra K_{m+1} belongs to the manifold $\mathfrak{A}^{(1)}_{m+1,n+1}$ since the value of the first principal operation φ_1 in this algebra is given by the formula

$$\varphi_1(x_1,\dots,x_{m+1})=2x_1-x_2,$$

from which it can be seen that the identity $\varphi_1(x,\dots,x)=x$ holds in K_{m+1}. At the same time the algebra K_{m+1} does not belong to the manifold $\mathfrak{A}^{(0)}_{m+1,n+1}$ since in it the operation φ_1 depends essentially on two variables.

We recall that a groupoid with the identity relationships

$$(xy)y=x=y(yx)$$

is said to be a totally symmetric quasigroup (briefly a TS-quasigroup). The class of all TS-quasigroups coincides with the class of all quasigroups in which the three principal operations of multiplication and left and right division coincide.

By Lemma 1 we can, for each integer $r > 1$ in an arbitrary finite quasigroup G, define the system of equations

$$\varphi_1(x), \ldots, \varphi_r(x), \omega(x_1, \ldots, x_r)$$

in such a way that the algebra $\langle G, \varphi_1, \ldots, \varphi_r, \omega \rangle$ belongs to the manifold $\mathfrak{A}_{1,r}$. On the other hand, by the theorem of Swierczkowski and the Jonnson-Tarski-Fujiwara theorem, both mentioned in the introduction, the manifold $\mathfrak{A}_{1,r}$ for arbitrary $r > 1$ contains no finite algebras of order greater than 1. So the class of all infinite one-element TS-quasigroups may be regarded as a manifold of universal algebras of the type $\langle 2, 1, \ldots, 1, r \rangle$ $(r \geq 2)$ with the system of defining identities

$$\left. \begin{array}{c} xy \cdot y = x = y \cdot yx, \\ \omega(\varphi_1(x), \ldots, \varphi_r(x)) = x, \quad \varphi_i(\omega(x_1, \ldots, x_r)) = x_i \quad (i = 1, \ldots, r). \end{array} \right\} \tag{5}$$

We adopt the convention of denoting this manifold by $\mathfrak{A}_{1,r}(TS)$.

<u>THEOREM 4.</u> For arbitrary integers $n > m \geq 2$ the manifold $\mathfrak{A}_{m,n}$ contains a proper submanifold rationally equivalent to the manifold $\mathfrak{A}_{1,r}$ (TS), where $r = n - m + 1$.

<u>Proof.</u> By Theorem 3 each preceding manifold in the sequence of manifolds

$$\mathfrak{A}_{1,r}, \mathfrak{A}_{2,r+1}, \mathfrak{A}_{3,r+2}, \ldots, \mathfrak{A}_{m,n}$$

is rationally equivalent to the following one. Since $m \geq 2$, it is sufficient to prove Theorem 4 for the manifold $\mathfrak{A}_{2,r+1}$.

Let us agree to denote the principal operations of the algebras of the manifold $\mathfrak{A}_{2,r+1}$ by $\varphi_i(x_1, x_2), \omega_j(x_1, \ldots, x_{r+1})$ $(i = 1, \ldots, r+1, j = 1, 2)$. We recall that in $\mathfrak{A}_{2,r+1}$ they are linked by the system of identities (1) for $m = 2$, $n = r + 1$. Let $\mathfrak{T}_{2,r+1}$ be the submanifold of $\mathfrak{A}_{2,r+1}$ defined within $\mathfrak{A}_{2,r+1}$ by the identities

$$\left. \begin{array}{l} \varphi_1(\varphi_1(x, y), y) = x, \quad \varphi_1(x, \varphi_1(x, y)) = y, \\ \varphi_{i+1}(x, y) = \varphi_{i+1}(z, y) \quad (i = 1, \ldots, r), \\ \omega_2(x_1, x_2, \ldots, x_{r+1}) = \omega_2(y, x_2, \ldots, x_{r+1}), \\ \omega_1(x_1, \ldots, x_{r+1}) = \varphi_1(x_1, \omega_2(x_1, \ldots, x_{r+1})). \end{array} \right\} \tag{6}$$

We shall show that the manifold $\mathfrak{T}_{2,r+1}$ is rationally equivalent to $\mathfrak{A}_{1,r}$ (TS).

In the first place, if the algebra $\mathbf{A} = \langle A, \varphi_i, \omega_j \rangle$ belongs to the manifold $\mathfrak{T}_{2,r+1}$ then, putting

$$x \cdot y = \varphi_1(x, y),$$

$$\bar{\varphi}_i(y) = \varphi_{i+1}(x, y) \quad (i = 1, \ldots, r),$$

$$\bar{\omega}(x_1, \ldots, x_r) = \omega_2(x_1, x_1, x_2, \ldots, x_r),$$

we obtain the algebra $\bar{\mathbf{A}} = \langle A, \cdot, \bar{\varphi}_i, \bar{\omega}_j \rangle$ which clearly belongs to the manifold $\mathfrak{A}_{1,r}$ (TS), i.e., it satisfies the system of identities (5).

Secondly, the operations φ_i, ω_j of $\mathbf{A}$ are in their turn expressible termally in terms of the operations $\cdot, \bar{\varphi}_i, \bar{\omega}_j$:

$$\varphi_1(x, y) = x \cdot y, \quad \varphi_{i+1}(x, y) = \varphi_{i+1}(z, y) = \bar{\varphi}_i(y) \quad (i = 1, \ldots, r),$$

$$\omega_2(x_1, x_2, \ldots, x_{r+1}) = \omega_2(y, x_2, \ldots, x_{r+1}) = \bar{\omega}(x_2, \ldots, x_{r+1}),$$

$$\omega_1(x_1, \ldots, x_{r+1}) = x_1 \cdot \bar{\omega}(x_2, \ldots, x_{r+1}).$$

Thirdly, if the algebra $\langle A, \cdot, \bar{\varphi}_i, \bar{\omega} \rangle$ belongs to the manifold $\mathfrak{A}_{1,\tau}$ (TS), then, putting

$$\varphi_1(x,y) = x\cdot y,$$

$$\varphi_{i+1}(x,y) = \bar{\varphi}_i(y) \qquad (i=1,\dots,\tau),$$

$$\omega_1(x_1,\dots,x_{\tau+1}) = x_1\cdot \omega(x_2,\dots,x_{\tau+1}),$$

$$\omega_2(x_1,\dots,x_{\tau+1}) = \bar{\omega}(x_2,\dots,x_{\tau+1}),$$

we obtain in view of identities (5) the algebra $\langle A, \varphi_i, \omega_j \rangle$ of the manifold $\mathfrak{L}_{2,\tau+1}$. For the identity relationships (6) are obvious in $\langle A, \varphi_i, \omega_j \rangle$, and the identity relationships (1) for $m=2$, $n=\tau+1$ can be verified without difficulty.

The rational equivalence of the manifolds $\mathfrak{L}_{2,\tau+1}$, $\mathfrak{A}_{1,\tau}$ (TS) has thus been proved.

We shall show by means of Theorem 1 that $\mathfrak{L}_{2,\tau+1}$ is a proper submanifold of $\mathfrak{A}_{2,\tau+1}$. In view of Theorem 3 it will be proved that $\mathfrak{A}_{1,\tau}$ (TS) is rationally equivalent to a proper submanifold of the manifold $\mathfrak{A}_{m,n}$.

By Theorem 1 there exists in the set of all complex numbers K a model of the system of defining identities (1) of the manifold $\mathfrak{A}_{2,\tau+1}$, in which the value of the principal operation φ_1 may be calculated from the formula $\varphi_1(x,y) = 2x-y$. In this model $\varphi_1(\varphi_1(x,y),y)=4x-3y$ and so it does not belong to the manifold $\mathfrak{L}_{2,\tau+1}$, in which the term $\varphi_1(\varphi_1(x,y),y)$ is identically equal to x . Theorem 4 is thus completely proved.

THEOREM 5. When $m \geq 2$ the manifold $\mathfrak{A}_{m,m}$ contains a proper submanifold which is rationally equivalent to the manifold of all totally symmetric quasigroups.

Proof. By Theorem 3 it is sufficient to prove this statement for $m=2$. Each algebra $A \in \mathfrak{A}_{2,2}$ has four principal operations, let us say φ_1 , φ_2 , ω_1 , ω_2 , which are binary and are connected by the system of identities (1) when $m=n=2$. We distinguish in $\mathfrak{A}_{2,2}$ the submanifold $\mathfrak{L}$ by means of the additional identities

$$\varphi_1(\varphi_1(x,y),y) = x = \varphi_1(x,\varphi_1(x,y)),$$
$$\omega_1(x,y) = \varphi_1(x,y),$$
$$\omega_2(x,y) = y = \varphi_2(x,y).$$

We shall show that the manifold $\mathfrak{L}$ is rationally equivalent to the manifold of TS-quasigroups defined by the identities $xy\cdot y = x = y\cdot yx$.

Let the algebra $\langle A, \varphi_i, \omega_i \rangle$ belong to the manifold $\mathfrak{L}$. Putting

$$xy = \varphi_1(x,y) \qquad (x,y\in A),$$

we clearly obtain a TS-quasigroup. Here the operations φ_i, ω_i in the set A are in their turn termally expressible in terms of a quasigroup multiplication operation. Finally, if the groupoid $\langle A, \cdot \rangle$ is a TS-quasigroup, then, on putting

$$\varphi_1(x,y) = \omega_1(x,y) = x\cdot y, \quad \varphi_2(x,y) = \omega_2(x,y) = y,$$

we clearly obtain the algebra $\langle A, \varphi_1, \varphi_2, \omega_1, \omega_2 \rangle$ of the manifold $\mathfrak{L}$.

It only remains to note that $\mathfrak{L}$ is a proper submanifold of the manifold $\mathfrak{A}_{2,2}$. For by Lemma 1' and Theorem 1 there exists an $\mathfrak{A}_{2,2}$ -algebra $\langle K, \varphi_1, \varphi_2, \omega_1, \omega_2 \rangle$ given on the set of all complex numbers K with the operation $\varphi_1(x,y)=x+y$. This algebra obviously does not belong to the manifold $\mathfrak{L}$.

Theorem 5 is thus proved.

§ 4. EQUATIONALLY COMPLETE MANIFOLDS AND THE POWER OF THE POINT STRATUM OF THE LATTICE $L(\mathfrak{A}_{m,n})$ WHEN $n \geqslant m \geqslant 2$

The system of identities $\mathfrak{Y}$ of the signature Ω is said to be equationally complete if it is compatible (that is its corollaries do not include the identity $x=y$) and for an arbitrary identity τ of the signature Ω either $\mathfrak{Y} \vdash \tau$ or $\{\mathfrak{Y}, \tau\} \vdash x=y$. A manifold of algebras is said to be equationally complete if the set of all its identity relationships is equationally complete.*

Let us denote the lattice of submanifolds of the manifold of algebras $\mathfrak{A}$ by $L(\mathfrak{A})$. In $L(\mathfrak{A})$ atoms or points (cf. [8], p. 24) correspond to equationally complete submanifolds of $\mathfrak{A}$. So we shall call the set of all equationally complete submanifolds of $\mathfrak{A}$ the point stratum of $L(\mathfrak{A})$ and denote it by $EL(\mathfrak{A})$.

THEOREM 6. In the class of manifolds

$$\mathfrak{A}_{m,n} \quad (1 \leqslant m < n)$$

the manifolds $\mathfrak{A}_{1,n}$, and only they, are equationally complete.

Proof. When $m>1$ the manifolds $\mathfrak{A}_{m,n}$ are not equationally complete by Theorem 3. As has already been noted, the equational completeness of the manifolds $\mathfrak{A}_{1,n}$ was discovered by A. I. Mal'tsev. For $n=2$ this fact was given by Mal'tsev as an illustrative example in a course of lectures on the theory of manifolds and quasimanifolds of algebraic systems which he delivered in 1966 at Novosibirsk University. The discussion is the same in the general case. To make our exposition complete we now give that discussion.

Let n be a fixed integer, $n>1$. We shall prove the equational completeness of the following system of identities:

$$\varphi_i(\omega(x_1,\dots,x_n)) = x_i\ ,\ \omega(\varphi_1(x),\dots,\varphi_n(x)) = x \quad (i=1,\dots,n). \tag{7}$$

By Lemma 1 the system of identities (7) is compatible and it is only necessary to show that every identity

$$\Phi(x_1,\dots,x_s) = \Psi(x_1,\dots,x_s) \tag{8}$$

in the functional symbols $\varphi_1,\dots,\varphi_n, \omega$ is either a corollary of the system of identities (7) or is such that the identity $x=y$ can be deduced from (7) and (8).

We shall assume that the terms Φ, Ψ are reduced, that is, do not have subterms of the form $\varphi_i(\omega(x_1,\dots,x_n))$, $\omega(\varphi_1(x),\dots,\varphi_n(x))$, and are graphically distinct. We shall show that with these assumptions the identity $x=y$ can be deduced from (7), (8). We prove this by induction over the number ℓ , which is the sum of the number of entries of ω in Φ and the number of entries of ω in Ψ .

When $\ell=0$ the terms Φ , Ψ do not contain the symbol ω and identity (8) has the form

$$\varphi_{i_1}\dots\varphi_{i_k}(x_{i_{k+1}}) = \varphi_{j_1}\dots\varphi_{j_q}(x_{j_{q+1}}). \tag{8'}$$

Since Φ and Ψ do not coincide graphically, it may be assumed that either $k>q$ or $k=q$, but in this context $i_s \neq j_s$ for some s . When $k>q$, substitutions of the form $x \to \omega(x_1,\dots,x_n)$ and appropriate cancellations by means of the identities (7) can be used to reduce the identity (8') to the form $\varphi_{i_1}\dots\varphi_{i_h}(x) = x$ or $\varphi_{i_1}\dots\varphi_{i_h}(x) = y \ (h \geqslant 1)$. Replacing x in the first of them by $\omega(x_1,\dots,x_n)$ and operating on both sides of the identity so obtained with the operator φ_i , with $i \neq i_h$, we obtain an identity of the form $\varphi_{i_1}\dots\varphi_{i_h}(x) = y$. So when $k>q$ identities (7), (8') imply the identity $x=y$. When $k=q$ the argument is analogous.

Let the statement be proved for identities with the number ℓ less than some value ℓ_0 , where $\ell_0>0$, and let (8) be an identity with $\ell=\ell_0$. Since both sides of (8) are reduced, it must be of the form:

*Other equivalent definitions of an equationally complete manifold are to be found in Tarski's article [7].

$\omega(\Phi_1,\ldots,\Phi_n)=\Psi$ or $\omega(\Phi_1,\ldots,\Phi_n)=\omega(\Psi_1,\ldots,\Psi_n)$, where the term Ψ does not contain the symbol ω , and the terms Φ_i and Ψ_i are graphically distinct for at least one value of i , let us say i_1 . In the second case, operating on both sides with the operators φ_{i_1} , we arrive at the reduced identity $\Phi_1=\Psi_1$ with a smaller value of ℓ. In the first place, operating on both sides with the operators $\varphi_1,\ldots,\varphi_n$, we obtain

$$\Phi_1=\varphi_1(\Psi),\ldots,\Phi_n=\varphi_n(\Psi)$$

If in at least one of these identities the two sides are graphically distinct, then by the induction hypothesis the identity $x=y$ is deducible from (7), (8). It remains to note that the term $\varphi_i(\Psi)$ cannot coincide graphically with the term Φ_i for each $i=1,\ldots,n$, since otherwise the term $\omega(\Phi_1,\ldots,\Phi_n)$ would have been graphically equal to the term

$$\omega(\varphi_1(\Psi),\ldots,\varphi_n(\Psi)),$$

which is not reduced, and we would have obtained a contradiction with the assumption. Theorem 6 is thus proved.

THEOREM 7. When $n \geq m \geq 2$ the point stratum $EL(\mathfrak{A}_{m,n})$ of the lattice of submanifolds $L(\mathfrak{A}_{m,n})$ of the manifold $\mathfrak{A}_{m,n}$ has the power of the continuum.

Proof. Let $n>m$. By Theorem 4 it is sufficient to calculate the power of the point stratum of the lattice $L(\mathfrak{A}_{1,\tau}(TS))$, where $\tau=n-m+1$, and the manifold $\mathfrak{A}_{1,\tau}(TS)$ is of the type $\langle 2,1,\ldots,1,\tau\rangle$, and is defined by the system of identities (5). A. D. Bol'bot [4] has proved that the point stratum $EL(TS)$ of the lattice of submanifolds of the manifold TS of all totally symmetric quasigroups has the power of the continuum. But this power is also possessed by the set

$$M=\{T\in EL(TS):\ A\in T,\ |A|>1\Rightarrow |A|\geq \aleph_0\}$$

of those equationally complete manifolds of TS -quasigroups of which all the nontrivial quasigroups are infinite. By virtue of the observations made in § 3 in the course of defining the manifold $\mathfrak{A}_{1,\tau}(TS)$ each manifold $T\in M$ may be considered as the manifold $\mathfrak{A}_{1,\tau}(T)$ of universal algebras of the type $\langle 2,1,\ldots,1,\tau\rangle$ defined by identities (5) and by the quasigroup identities which give the submanifold T within the manifold TS . Since the different manifolds of quasigroups from M do not intersect, the manifolds $\mathfrak{A}_{1,\tau}(T)$ are also pairwise nonintersecting. It is known that each nontrivial manifold of algebras possesses at least one equationally complete submanifold. Distinguishing in each manifold $\mathfrak{A}_{1,\tau}(T)$ $(T\in M)$ an equationally complete submanifold, we obtain a continuous set of equationally complete submanifolds of the manifold $\mathfrak{A}_{1,\tau}(TS)$. On the other hand, the power of the point stratum $EL(\mathfrak{A}_{1,\tau}(TS))$ obviously does not exceed the power of the continuum. So this power is in fact precisely equal to the power of the continuum.

When $m=n\geq 2$ Theorem 7 immediately follows from Theorem 5 and the theorem of A. D. Bol'bot cited above. This completes the proof of Theorem 7.

COROLLARY 1. If $n\geq m\geq 2$, then

$$|L(\mathfrak{A}_{m,n})| = |EL(\mathfrak{A}_{m,n})| = \aleph .$$

COROLLARY 2. If $n\geq m\geq 2$, then the lattice $L(\mathfrak{A}_{m,n})$ does not satisfy the termination condition for decreasing chains.

For since the set of finitely axiomatizable submanifolds of the manifold $\mathfrak{A}_{m,n}$ is no more than denumerable, it follows that when $n\geq m\geq 2$ there exists a submanifold in $\mathfrak{A}_{m,n}$ which does not have a finite basis of identity relationships. So when $n\geq m\geq 2$ there also exists an infinite strictly decreasing chain of submanifolds of $\mathfrak{A}_{m,n}$.

§ 5. THE LATTICE $L(\mathfrak{A}_{1,1})$

The manifold $\mathfrak{A}_{1,1}$ is of the type $\langle 1,1\rangle$ and is defined by the system of identities (1) when $m=n=1$, that is by the identities

$$\varphi\omega(x)=x=\omega\varphi(x). \tag{9}$$

Let $\mathcal{J}$ be a lattice of positive integers with the relationship of divisibility. Joining zero 0 and unit 1 to it externally (cf. [8]), we obtain a new lattice which we shall denote by $\mathcal{J}^*$.

<u>THEOREM 8.</u> Each submanifold of the manifold $\mathfrak{A}_{1,1}$ may be defined within $\mathfrak{A}_{1,1}$ by a single identity and the isomorphism

$$L(\mathfrak{A}_{1,1}) \cong \mathcal{J}^*$$

holds.

<u>Proof.</u> For the operators φ, ω we adopt operator notation, and we shall first show that every submanifold of $\mathfrak{A}_{1,1}$ can be defined within $\mathfrak{A}_{1,1}$ only by one of the identities

$$x = x, \quad x = y, \quad x\varphi^s = x \quad (s \geq 1). \tag{10}$$

We shall denote a submanifold of $\mathfrak{A}_{1,1}$, which consists of single-element algebras, by $\mathfrak{E}$. It may be defined by the identity $x=y$, and the manifold $\mathfrak{A}_{1,1}$ itself (within itself) by the identity $x=x$. We shall denote by $\mathfrak{A}_{1,1}^{(s)}$ the submanifold defined in $\mathfrak{A}_{1,1}$ by the identity $x\varphi^s = x$ $(s \geq 1)$.

Let F be a $\mathfrak{A}_{1,1}$-free algebra with one free generator x. We shall also consider the absolutely free algebra F_0 of the type $\langle 1,1 \rangle$ over the set $\{x\}$. The algebra F will be a factor algebra of F_0 with respect to the verbal congruence θ corresponding to the system of identities (9). We recall that the terms $\Phi, \Psi \in F_0$ are in the relationship θ if and only if it is possible to pass from one of them to the other by a finite number of transformations of the form $u\varphi\omega \rightleftharpoons u, u\omega\varphi \rightleftharpoons u$. It follows from this that every term $x\alpha_1\alpha_2\ldots\alpha_k$ of φ, ω is comparable in relation to θ with a term of the form $x\varphi^s$ or $x\omega^s$ $(s \geq 0)$.

Since what we are interested in is identity relationships in $\mathfrak{A}_{1,1}$-algebras which are not corollaries of the system (9), we may restrict ourselves to considering formal equations in elements of F, that is equations of the form:

$$x\varphi^s = x\varphi^t, \quad x\omega^s = x\omega^t, \quad x\varphi^s = x\omega^t,$$

$$x\varphi^s = y\varphi^t, \quad x\omega^s = y\omega^t, \quad x\varphi^s = y\omega^t,$$

where $s \geq t \geq 0$. Relative to (9) they are equivalent respectively to the identities

$$x\varphi^{s-t} = x, \quad x\omega^{s-t} = x, \quad x\varphi^{s+t} = x,$$

$$x\varphi^{s-t} = y, \quad x\omega^{s-t} = y, \quad x\varphi^{s+t} = y,$$

which in there turn can be reduced by means of identities (9) to identities of the form

$$x\varphi^s = x, \quad x\varphi^s = y \quad (s \geq 0).$$

Since the identity relationship $x\varphi^s = y$ implies the identity $x=y$, it follows that every submanifold of $\mathfrak{A}_{1,1}$ which is distinct from $\mathfrak{E}$ can be defined within $\mathfrak{A}_{1,1}$ only by a system of identities of the form $x\varphi^s = x$ $(s \geq 0)$.

Let $\mathfrak{E} \neq \mathfrak{B}$ be an arbitrary submanifold of $\mathfrak{A}_{1,1}$. If it does not have identity relationships of the form $x\varphi^s = x$ $(s > 0)$, then by what has been proved $\mathfrak{B} = \mathfrak{A}_{1,1}$.

Let $\mathfrak{B}$ be distinct not only from $\mathfrak{E}$ but also from $\mathfrak{A}_{1,1}$. We shall denote by s_0 the least positive value of the number s for which the identity $x\varphi^s = x$ holds in $\mathfrak{B}$. Then for an arbitrary identity relationship $x\varphi^s = x$ in $\mathfrak{B}$ we shall have s_0 / s. Thus the submanifold $\mathfrak{B}$ may be defined within $\mathfrak{A}_{1,1}$ only by the single identity $x\varphi^{s_0} = x$, that is $\mathfrak{B} = \mathfrak{A}_{1,1}^{(s_0)}$.

We shall show that if $s \neq t$, then $\mathfrak{A}_{1,1}^{(s)} \neq \mathfrak{A}_{1,1}^{(t)}$. Let $t > s$, $A = \{1, 2, \ldots, t\}$, $\varphi = (1, 2, \ldots, t)$ be a cyclic permutation over the set A, $\omega = \varphi^{-1}$. Then the algebra $\mathcal{A} = \langle A, \varphi, \omega \rangle$ belongs to the manifold

$\mathfrak{A}^{(t)}_{1,1}$, since $x\varphi\omega = x\omega\varphi = x$ and $x\varphi^t = x$ for arbitrary $x \in A$. However, $A \notin \mathfrak{A}^{(s)}_{1,1}$ since, for example, $1\varphi^s = s+1 \neq 1$.

Clearly, if s/t , then $\mathfrak{A}^{(s)}_{1,1} \subseteq \mathfrak{A}^{(t)}_{1,1}$. Conversely, let $\mathfrak{A}^{(s)}_{1,1} \subset \mathfrak{A}^{(t)}_{1,1}$. We shall consider the algebra $\langle A, \varphi, \omega \rangle$, where

$$A = \{1, 2, \ldots, s\}, \quad \varphi = (1, 2, \ldots, s)$$

is a cycle of length s, $\omega = \varphi^{-1}$. Since this algebra belongs to the manifold $\mathfrak{A}^{(s)}_{1,1}$ and the inclusion $\mathfrak{A}^{(s)}_{1,1} \subset \mathfrak{A}^{(t)}_{1,1}$ holds, $x\varphi^t = x$ for all $x \in A$, whence s/t , since the element φ in the permutation group over the set A is of order s . Thus the mapping $\mathfrak{A}_{1,1} \to I$, $\mathfrak{A}^{(s)}_{1,1} \to s$, $\mathfrak{E} \to 0$ is an isomorphism of the lattice $L(\mathfrak{A}_{1,1})$ onto the lattice $\mathfrak{J}^*$.

This proves Theorem 8.

COROLLARY 1. The manifold $\mathfrak{A}^{(1)}_{1,1}$ of the type $\langle 1, 1 \rangle$ defined by the identities

$$x\varphi = x = x\omega,$$

and only it among the submanifolds of the manifold $\mathfrak{A}_{1,1}$, is equationally complete.

From Theorems 3,and 7 we obtain

COROLLARY 2. For an arbitrary integer $m \geq 1$ the lattice of submanifolds $L(\mathfrak{A}_{m,m})$ of the manifold $\mathfrak{A}_{m,m}$ does not satisfy the termination condition for increasing chains.

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