PRESERVATION OF ELEMENTARY AND UNIVERSAL EQUIVALENCE UNDER THE WREATH PRODUCT

E. I. Timoshenko

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Two groups G and G' are called elementarily equivalent if the truth of any closed formula of the group signature on one group implies the truth of the formula on the other group. If each closed formula of the group signature not containing existential quantifiers is true on G if and only if it is true on G', then G and G' are called universally equivalent. The question naturally arises as to the preservation of elementary equivalence when group operations are applied. It is for example well known that elementary equivalence is preserved under the direct and cartesian group products. The analogous question for the free group product is still open. The question regarding preservation of elementary equivalence for the group wreath product was stated by M. I. Kargapolov in [1]. We give a negative answer to this question here. More precisely, we construct groups A, B, A', and B' such that A is elementarily equivalent to A', B is elementarily equivalent to B' but the discrete wreath product of A and B is not equivalent to the discrete wreath product of A' and B'.

Let A and B be additive groups, and G their discrete wreath product. This means that for each element $b \in B$ there is an isomorphism $a \to a^{(b)}$ of A onto its isomorph $A^{(b)}$ and $G = \bar{A} \cdot B$, where $\bar{A} = \prod_{b \in B} A^{(b)}$, $b a^{(b_1)} b^{-1} = a^{(b_1+b)}$ for all $a \in A$, $b, b_1 \in B$. By the bearer of an element $f \in \bar{A}$ we mean the set $\sigma(f)$ of $b \in B$ for which the b -th component of f, denoted $f(b)$, is not equal to 0. Let G_p be the discrete wreath product of the infinite cyclic group A and the direct sum B_p of three cyclic groups of simple order p. We take A to be the additive group of integers and denote the elements of B_p by triples (n,m,s) of real numbers taken modulo p.

LEMMA. There exists an element g_0 belonging to the commutant of G_p which cannot be written in the form

$$[f, (n, m, s)] + [g, (n', m', 0)]$$

for any $f, g \in \bar{A}$ and $(n, m, s), (n', m', 0) \in B_p$. Here as usual $[a, b]$ denotes the commutator $aba^{-1}b^{-1}$.

Proof. We define the element g_0 thus:

$$g_0(0,0,0)=6,\ g_0(0,0,1)=-3,\ g_0(0,1,0)=-2,\ g_0(1,0,0)=-1 \text{ and } g(b)=0$$

for the remaining $b \in B$. It is easy to see from Corollary 4.5 of [2] that g_0 belongs to the commutant of G_p.

We first note that g_0 cannot be written as

$$[f, (n, m, p)] + [g, (n', m', 0)].$$

Otherwise we would have

$$\sum_{i,j=0}^{p-1} g_0(i,j,0) = \sum_{i,j=0}^{p-1} \{[f, (n, m, 0)] + [g, (n', m', 0)]\}(i,j,0). \tag{1}$$

The left side is here equal to 3,as in evident from the definition of g_0. It is easy to verify that

$$\sum_{i,j=0}^{p-1} [f, (n, m, 0)](i,j,0) = \sum_{i,j,k=0}^{p-1} [f', (n, m, 0)](i,j,k), \tag{2}$$

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where f' of A is defined by: $f'(i,j,0)=f(i,j,0)$, $f'(i,j,\kappa)=0$ for $g\leqslant i,j\leqslant p-1, 1\leqslant\kappa\leqslant p-1$. The right side of (2) is equal to 0 by Corollary 4.5 of [2]. The right side of (1) is therefore also equal to 0. This means that g_0 cannot be written as specified in the lemma for s = 0.

Now let $s\neq 0$. We introduce the abbreviations

$$f(i_1,j_1,\kappa_1)-f(i_2,j_2,\kappa_2)=F\begin{pmatrix} i_1 & j_1 & \kappa_1 \\ i_2 & j_2 & \kappa_2\end{pmatrix},$$
$$g(i_1,j_1,\kappa_1)-g(i_2,j_2,\kappa_2)=F'\begin{pmatrix} i_1 & j_1 & \kappa_1 \\ i_2 & j_2 & \kappa_2\end{pmatrix}$$

and examine the following sums:

$$\sum_{\ell=0}^{p-1} F\begin{pmatrix} -\kappa n'-\ell n, & -\kappa m'-\ell m, & -\ell s \\ -\kappa n'-(\ell+1)n, & -\kappa m'-(\ell+1)m, & -(\ell+1)s\end{pmatrix} \tag{3}$$

and

$$\sum_{\ell=0}^{p-1} F'\begin{pmatrix} -\kappa n-\ell n', & -\kappa m-\ell m', & -\kappa s \\ -\kappa n-(\ell+1)n', & -\kappa m-(\ell+1)m', & -\kappa s\end{pmatrix}, \quad 0\leqslant\kappa\leqslant p-1. \tag{4}$$

We note that $F\begin{pmatrix} i & j & \kappa \\ i-n & j-m & \kappa-s\end{pmatrix}+F'\begin{pmatrix} i & j & \kappa \\ i-n' & j-m' & \kappa\end{pmatrix}$ is equal to $g_0(i,j,\kappa)$, provided g_0 is written as the sum of the two commutators specified in the lemma. It is evident that (3) and (4) are equal to 0, so

$$\sum_{\kappa,\ell=0}^{p-1} g_0(-\kappa n-\ell n', -\kappa m-\ell m', -\kappa s)=0.$$

But on the other hand this last sum is not less than 1. For the element $(n', m', 0)$ of B_p is not equal to 0; otherwise $\sum_{\kappa=0}^{p-1} F\begin{pmatrix} -\kappa n, & -\kappa m, & -\kappa s \\ -(\kappa+1)n, & -(\kappa+1)m, & -(\kappa+1)s\end{pmatrix}$ would be equal to 0 on the one hand and equal to $\sum_{\kappa=0}^{p-1} g_0(-\kappa n, -\kappa m, -\kappa s)$ on the other, but this last sum is not less than 3, since $s\neq 0$. So $(n', m', 0)\neq 0$ and this means that the system

$$\begin{aligned} -\kappa n-\ell n' &\equiv 0, \\ -\kappa m-\ell m' &\equiv 0, \\ -\kappa s &\equiv 0 \end{aligned}$$

modulo p has the unique trivial solution $\ell\equiv\kappa\equiv 0$. Therefore each of the three systems

$$\begin{aligned} -\kappa n-\ell n' &\equiv \varepsilon_1, \\ -\kappa m-\ell m' &\equiv \varepsilon_2, \\ -\kappa s &\equiv \varepsilon_3, \end{aligned}$$

where $(\varepsilon_1,\varepsilon_2,\varepsilon_3)=(1,0,0),(0,1,0),(0,0,1)$, has not more than one solution. In addition, not more than two of these three systems are simultaneously compatible. From this and the definition of g_0 we find that

$$\sum_{\kappa,\ell=0}^{p-1} g_0(-\kappa n-\ell n', -\kappa m-\ell m', -\kappa s)\geqslant 1.$$

This contradiction proves the lemma.

Let G be the semidirect product of the Abelian groups A and B with a normal subgroup of A. We shall hereafter require the following easily verifiable relations in G. For any $a\in$A and b$\in$B and any integer m there exists an element $a'\in$A such that

$$[a, b^m]=[a', b]. \tag{5}$$

The following hold for any $a_1, a_2 \in A, b_1, b_2 \in B$:

$$[a_1, b_1][a_2, b_2] = [a_1 a_2, b_1][a_2^{b_1}, b_2 b_1^{-1}] , \tag{6}$$

$$[a, b_1, a_2 b_2] = [a_1, b_2][a_2^{-1}, b_1]. \tag{7}$$

Using (5), (6), and (7), we obtain the following

COROLLARY. Let G be the discrete wreath product of the infinite cyclic group A and the direct sum B of n infinite cyclic groups. Then each element of the commutant of G can be written as the sum of more than $[\frac{n+1}{2}]$ commutators.

Proof. According to (7) each commutator of G can be written as the sum of two commutators of the form $[f, b]$, $f \in \bar{A}$, $b \in B$. But it follows from (5) and (6) that $[f, b] = \sum_{i=1}^{n} [f_i, b_i]$, where $f_i \in A$, b_i is the generator of the i-th direct sum of B. The corollary follows from this by easy computation. We note that the corollary remains true and with almost the same proof if the cyclic sums of B are replaced by locally cyclic ones.

THEOREM 1. If groups A and B are elementarily equivalent to A' and B' respectively, it does not follow that the wreath product G of A and B is elementarily equivalent to the wreath product G' of A' and B'.

Proof. Let A, A', and B be infinite cyclic groups and let B' be the direct sum of an infinite cyclic group and two isomorphs of the additive group of the rational numbers. It is easy to see from the criterion for the elementary equivalence of Abelian groups that B and B' are elementarily equivalent. By applying the corollary we find that each element of the commutant of G is a commutator, i.e., the following

$$\forall x_1 x_2 x_3 x_4 \exists x_5 x_6 ([x_1, x_2][x_3, x_4] = [x_5, x_6])$$

is true in G. On the other hand we can find in G' a g_0' from the commutant which is also not a commutator. For let

$$g_0'(0,0,0) = 6, \; g_0'(0,0,1) = -3, \; g_0(0,1,0) = -2, \; g_0(1,0,0) = -1$$

and $g_0'(b') = 0$ for the remaining $b' \in B$. (The elements of B' are triples (r_1, r_2, n), where r_1, r_2 are rational and n is an integer.) By Corollary 4.5 of [2], g_0' belongs to the commutant of G'. We assume that g_0' is a commutator, i.e.,

$$g_0' = [f, (z_1, z_2, n)] + [g, (q_1, q_2, m)],$$

where $f, g \in \bar{A}'$, $(z_1, z_2, n), (q_1, q_2, m) \in B'$. Using (5) and (6) we rewrite g_0' as:

$$g_0' = [f', (z_1', z_2', 0)] + [g', (q_1', q_2', m')]$$

where $f', g' \in \bar{A}'$, $(z_1', z_2', 0)$, $(q_1', q_2', m') \in B'$.

We examine the finite set

$$b(f') \cup b(g') \cup b(g_0) \cup \{(z_1', z_2', 0), (q_1', q_2', m')\}$$

of triples of rational numbers. Let R be the common denominator of these numbers. We consider $\bar{g}_0$ from the wreath product $\bar{G}$ of the infinite cyclic group C and the direct sum $\bar{B}$ of the three infinite cyclic groups specified by

$$\bar{g}_0(0,0,0) = 6, \; \bar{g}_0(0,0,R) = -3, \; \bar{g}_0(0,R,0) = -2, \; \bar{g}_0(R,0,0) = -1$$
$$\bar{g}_0(\bar{b}) = 0 \quad \text{for the remaining } \bar{b} \in \bar{B} .$$

We easily note that $\bar{g}_0$ can be written as

$$\bar{g}_0 = [\bar{f}, (k, l, s)] + [\bar{g}, (k', l', 0)],$$

where $\bar{f}, \bar{g} \in \bar{C}$, $(k, l, s), (k', l', 0) \in \bar{B}$.

We map $\bar{B}$ homomorphically onto the direct sum of three cyclic groups of simple order P, where P satisfies $l \equiv 1 \ (\mathrm{mod}\, P)$. We extend this homomorphism in a natural manner to a homomorphism φ of $\bar{G}$ onto G_p and find that $\bar{g}_0\varphi = g_0 \in G_p$. This contradicts the lemma, and the theorem is proved.

THEOREM 2. If group A is universally equivalent to group A' and group B is universally equivalent to group B', then the discrete wreath product G of A and B is universally equivalent to the discrete wreath product G' of A' and B'.

Proof. We note that two groups are universally equivalent if and only if each finite submodel of the first group has an isomorphic submodel in the other group and conversely.

Let $M = \{g_1, \ldots, g_n\}$ be an arbitrary finite submodel of G'. By using multiplication notation we can write each element $g_i \in M$ as $g_i = a_i b_{ii}$, where $a_i \in \bar{A}$, $b_{ii} \in B$. Further, let

$$\bigcup_{i=1}^{n} \delta(a_i) = \{b_1, \ldots, b_s\}.$$

We can correspond the finite submodel $\{b_1, \ldots, b_s, b_{11}, \ldots, b_{nn}\}$ of G' with the isomorphic submodel $\{b'_1, \ldots, b'_s, b'_{11}, \ldots, b'_{nn}\}$ of G', b_j being placed in correspondence with b'_j and b_{ii} with b'_{ii}. We now consider the finite submodel

$$\{a_i(b_j)\} \quad (i = 1, \ldots, n, \ j = 1, \ldots, s)$$

of A and find the submodel $\{(a_i(b_j))'\}$ in A' isomorphic to it. It is easy to verify that the mapping $\varphi: g_i \to g'_i = a'_i b'_{ii}$ will be an isomorphism of M onto $M' = \{g'_1, \ldots, g'_n\}$ if we set

$$a'_i(b'_j) = \begin{cases} (a_i(b_j))' & \text{for } 1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant s, \\ 1 & \text{otherwise.} \end{cases}$$

This proves the theorem.

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LITERATURE CITED

1. Kourovskii Notebook, Unsolved Problems of Group Theory, Second Edition [in Russian], Novosibirsk (1967).
2. P. M. Neumann, "On the structure of standard wreath products of groups," Math. Z., 84, No. 4, 343-373 (1964).