POSITIVE EQUIVALENCES

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The concept of a positive equivalence (an effective partition of the natural numbers into recursively enumerable sets) is in a sense a "global" concept, corresponding to the "local" concept of a recursively enumerable set. Many concepts connected with recursively enumerable sets, particularly classical ones, can be naturally extended to positive equivalences. On the other hand, certain properties of positive equivalences often imply nontrivial consequences for sets closed under these equivalences. The purpose of this paper is to illustrate the possibility of such fruitful connections.

1. The Algebra of Positive Equivalences. We begin with the basic definition.

An equivalence relation η on the set N of natural numbers is called positive if the set of pairs $\{\langle x,y\rangle \mid x\eta y\}$ is recursively enumerable.

This definition agrees with A. I. Maltsev's definition [5] of a positive enumeration. (An enumeration ν is positive $\Longleftrightarrow$ the enumerated equivalence $\sim_\nu$ is positive.) All notions pertaining to the theory of enumerations may be found in [1].

Henceforth we will identify positive equivalences η with the recursively enumerable sets $\{\langle x,y\rangle \mid x\eta y\}$. We denote the family of all positive equivalences by $\mathcal{E}_p$. The usual inclusion relation $\subseteq$ partially orders $\mathcal{E}_p$. Note that the family $\mathcal{E}$ of all (not just positive) equivalences on N, ordered by $\subseteq$, is a complete lattice, as is well known. It so happens that $\mathcal{E}_p$ is a sublattice of $\mathcal{E}$. In other words, the following assertion holds.

LEMMA 1. The family $\mathcal{E}_p$ has a least element (under $\subseteq$), which we denote $\mathbf{0}$; $\mathcal{E}_p$ contains a greatest element $\mathbf{1}$; if $\eta_0, \eta_1 \in \mathcal{E}_p$ then $\eta_0 \cap \eta_1 \in \mathcal{E}_p$, and we denote $\eta_0 \cap \eta_1$ by $\eta_0 \wedge \eta_1$. If $\eta_0, \eta_1 \in \mathcal{E}_p$ and $\eta_0 \vee \eta_1$ is the least upper bound of η_0 and η_1 in $\mathcal{E}$, then $\eta_0 \vee \eta_1 \in \mathcal{E}_p$.

Note that $\mathbf{0} = \{\langle x,x\rangle \mid x \in N\}$ and $\mathbf{1} = \{\langle x,y\rangle \mid x,y \in N\}$; i.e., $\mathbf{0}$ is the equality relation on N and $\mathbf{1}$ is the trivial equivalence.

Since $\mathcal{E}_p$ is a family of recursively enumerable sets of pairs of natural numbers, we can try to find a principal computable enumeration for it [4].

Let $\mathcal{R}_2$ be the collection of all recursively enumerable sets of pairs of natural numbers. $\mathcal{R}_2$ has a principal computable enumeration π_2. (If one identifies $\mathcal{R}_2$ with the collection of all recursively enumerable subsets of N, using an enumeration of pairs of natural numbers, then π_2 induces a Post numbering π by this identification.) We let $\Pi_2 = (\mathcal{R}_2, \pi_2)$. The following holds.

LEMMA 2. $\mathcal{E}_p$ is an r-subset of Π_2.

Proof. Let $R \in \mathcal{R}_2$ be an arbitrary recursively enumerable set of pairs of natural numbers. Let R^* be the set $\{\langle x,y\rangle \mid x=y$ or there is a finite sequence $x_0, x_1, \ldots, x_n$ such that $x_0 = x$, $x_n = y$, and, for each $i<n$, $\langle x_i, x_{i+1}\rangle \in R$ or $\langle x_{i+1}, x_i\rangle \in R\}$.

It is easily seen that R^* is the smallest equivalence relation containing R. By definition, R^* is constructed effectively from R, so that $R^* \in \mathcal{E}_p$ for any $R \in \mathcal{R}_2$. Consequently, there is a one-place general recursive function g such that for any n, $\pi_2(n)^* = \pi_2 g(n)$. Having observed that for $R \in \mathcal{E}_p$, $R^* = R$, we obtain an enumeration $\varepsilon : N \to \mathcal{E}_p$, where $\varepsilon = \pi_2 g$. Thus $(\mathcal{E}_p, \varepsilon)$ is a retract of Π_2.

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The lemma is proved.

COROLLARY 1. The enumeration ε of ϑ_p constructed in the proof is a principal computable enumeration.

COROLLARY 2. $(\vartheta_p, \varepsilon)$ is a completely enumerated set.

Remark. The enumerated set $(\vartheta_p, \varepsilon)$ is isolated and the relation $\leqslant_\varepsilon$ coincides with the inclusion relation $\subseteq$.

The injection mapping $i: \vartheta_p \to \mathbb{R}_2$ is a morphism from $(\vartheta_p, \varepsilon)$ into Π_2; the mapping $*: \mathbb{R}_2 \to \vartheta_p$ is a morphism from Π_2 to $(\vartheta_p, \varepsilon)$ and $*i = Id$.

The above definitions of $\wedge$ and $\vee$ give on ϑ_p the structure of a (modular) lattice. It will be shown that the enumeration ε agrees well with the operations $\wedge$, $\vee$.

LEMMA 3. The lattice $(\vartheta_p, \wedge, \vee)$ equipped with the enumeration ε is an enumerated algebra; that is, the mappings $\wedge: \vartheta_p \times \vartheta_p \to \vartheta_p$ and $\vee: \vartheta_p \times \vartheta_p \to \vartheta_p$ are morphisms from $(\vartheta_p, \varepsilon) \times (\vartheta_p, \varepsilon)$ to $(\vartheta_p, \varepsilon)$ in that there are two-place general recursive functions $f^\wedge$ and $f^\vee$ such that for any $x, y \in N$

$$\varepsilon(x) \wedge \varepsilon(y) = \varepsilon f^\wedge(x, y), \text{ and } \varepsilon(x) \vee \varepsilon(y) = \varepsilon f^\vee(x, y).$$

Proof. It is well known that the operations $\cap$ and $\cup$ are effective in Π_2; i.e., there are two-place general recursive functions $f^\cap$ and $f^\cup$ such that $\pi_2(x) \cap \pi_2(y) = \pi_2 f^\cap(x, y)$ and $\pi_2(x) \cap \pi_2(y) = \pi_2 f^\cup(x, y)$ for any $x, y \in N$. Note that $\varepsilon(x) \wedge \varepsilon(y) = \varepsilon(x) \cap \varepsilon(y) = \pi_2 g(x) \cap \pi_2 g(y) = \pi_2 f^\cap(g(x), g(y)) = \varepsilon f^\cap(g(x), g(y))$, since for any z such that $\pi_2 z \in \vartheta_p$, $\pi_2 z = \varepsilon z$ and $\pi_2 f^\cap(g(x), g(y)) \in \vartheta_p$. Thus, $f^\wedge(x, y) \rightleftharpoons f^\cap(g(x), g(y))$ satisfies the requirement of the lemma. Furthermore $\varepsilon(x) \vee \varepsilon(y) = (\varepsilon(x) \cup \varepsilon(y))^* = (\pi_2 g(x) \cup \pi_2 g(y))^* = (\pi_2 f^\cup(g(x), g(y)))^* = \varepsilon f^\cup(g(x), g(y))$, since for any x, $(\pi_2 x)^* = \varepsilon x$. Thus $f^\vee(x, y) \rightleftharpoons f^\cup(g(x), g(y))$ will satisfy the requirement of the lemma.

The lemma is proved.

As has been already mentioned, the lattice ϑ of all equivalences on N is complete, i.e., for any collection of elements of ϑ there is a greatest lower and least upper bound. This is of course not true for ϑ_p. However, any enumerated (with respect to ε, or what is the same thing, with respect to π_2) collection of members of ϑ_p has a least upper bound in ϑ_p. More precisely, the following holds.

LEMMA 4. There is a one-place general recursive function $\overset{\vee}{f}$ such that for any n,

$$\varepsilon \overset{\vee}{f}(n) = \sup_{x \in \pi(n)} \{\varepsilon(x)\},$$

where $\sup \varnothing = \mathbf{0}$.

Proof. As in the proof of Lemma 3, it is sufficient to see that there is a one-place general recursive function Π_2 for $\overset{\cup}{f}$ such that

$$\pi_2 \overset{\cup}{f}(n) = \bigcup_{x \in \pi(n)} \pi_2 x$$

and that $\left(\bigcup_{x \in \pi(n)} (\pi_2 x)^*\right)^* = \left(\bigcup_{x \in \pi(n)} \pi_2 x\right)^*$. The last statement is obvious and the first is well known. Then $\overset{\vee}{f} \rightleftharpoons \overset{\cup}{f}$ satisfies the conclusion of the lemma.

Let η be an arbitrary (not necessarily positive) equivalence on N. The set $M \subseteq N$ is called η-closed if it satisfies the condition $\forall x \forall y\, (x \in M \,\&\, \langle x, y \rangle \in \eta \Rightarrow y \in M)$. For an arbitrary set M the η-closure of M is the set

$$[M]_\eta \rightleftharpoons \{x \mid \exists y\, (y \in M \,\&\, \langle x, y \rangle \in \eta)\}.$$

The following elementary properties of η-closed sets and η-closures are obvious:

1) M is η-closed $\Longleftrightarrow M = [M]_\eta$;

2) $[M]_\eta$ is the smallest η-closed set containing M;

3) If $(\xi \in T)$-η are all M_ξ-closed, then $\bigcup\limits_{\xi\in T} M_\xi$ and $\bigcap\limits_{\xi\in T} M_\xi$ are η-closed;

4) If $M = \bigcup\limits_{\xi\in T} M_\xi$, then $[M]_\eta = \bigcup\limits_{\xi\in T}[M_\xi]_\eta$.

Remark. For intersections, the last property does not hold; that is, it is not true in general that $[\bigcap\limits_{\xi\in T} M_\xi]_\tau = \bigcap\limits_{\xi\in T}[M_\xi]_\eta$; but always $[\bigcap\limits_{\xi\in T} M_\xi]_\eta \subseteq \bigcap\limits_{\xi\in T}[M_\xi]_\eta$.

If R is recursively enumerable and η is a positive equivalence, then $[R]_\eta$ is recursively enumerable. More precisely, the following obvious lemma holds.

LEMMA 5. The mapping $\zeta : R\times\eta \rightsquigarrow [R]_\eta$ is a morphism from $\Pi\times(\mathcal{D}_P, \varepsilon)$ to Π_2.

Let R_η, where η is a positive equivalence, denote the collection of all η-closed recursively enumerable sets. Each collection R_η is a distributive lattice with respect to the usual operations of intersection and union. If $\eta_0 \subseteq \eta_1$, any η_1-closed set is η_0-closed. So one has a natural injection $i_{\eta_1\eta_0} : R_{\eta_1} \to R_{\eta_0}$. This injection is an isomorphism of the distributive lattice R_{η_1} into the distributive lattice R_{η_0}. The closure operation $[\ ]_{\eta_1}$ maps R_{η_0} onto R_{η_1}, and we denote this mapping $P_{\eta_0\eta_1}$. Note that this mapping is an upper semilattice homomorphism (see [4]) but in general is not a lattice homomorphism. Thus, if $\eta_0 \subseteq \eta_1$, there are upper semilattice homomorphisms $i_{\eta_1\eta_0} : R_{\eta_1} \to R_{\eta_0}$, $P_{\eta_0\eta_1} : R_{\eta_0} \to R_{\eta_1}$ such that the composition $P_{\eta_0\eta_1} i_{\eta_1\eta_0}$ is the identity mapping of R_{η_1} onto itself. Hence R_{η_1} is a retract of R_{η_0}. If one introduces the principal computable enumeration π_η of the collection R_η (possible, since every R_η is a retract of $R_\emptyset$), then all indicated mappings will be morphisms (by Lemma 5).

LEMMA 6. If $\eta = \eta_0 \vee \eta_1$, then the set M is both η_0-closed and η_1-closed if and only if M is η-closed.

We will consider several examples of positive equivalences.

1. Let R be a recursively enumerable set. Then the set of pairs

$$\eta_R \leftrightharpoons \{\langle x,x\rangle \mid x\in N\} \cup \{\langle x,y\rangle \mid x,y\in R\}$$

is a positive equivalence. Note that

a) $\eta_R = \emptyset \Longleftrightarrow R$ contains at most one element;

b) If R has more than one element, then the only nontrivial equivalence class (i.e., not a unit class) of η_R is R;

c) A set M is η_R-closed if and only if either $M\cap R = \emptyset$ or $M \supseteq R$;

d) $\eta_{R_0} \cap \eta_{R_1} = \eta_{R_0\cap R_1}$;

e) $R_0 \cap R_1 \neq \emptyset \Longrightarrow \eta_{R_0} \vee \eta_{R_1} = \eta_{R_0\cup R_1}$;

f) $R_0 \neq \emptyset \;\&\; R_1 \neq \emptyset \;\&\; \eta_{R_0} \vee \eta_{R_1} = \eta_{R_0\cup R_1} \Longrightarrow R_0 \cap R_1 \neq \emptyset$;

g) $R_0 \subseteq R_1 \Longrightarrow \eta_{R_0} \subseteq \eta_{R_1}$;

h) The mapping $\varphi_0 : R \rightsquigarrow \eta_R$ is a morphism from Π to $(\mathcal{D}_P, \varepsilon)$.

Assertions a)-h) are obvious.

Remark. φ_0 "identifies" only the one-element sets (by a)); φ_0 "nicely" preserves the operations $\cap$, $\cup$ (by d), e), f)); $\varphi_0(\Pi)$ is an n-subobject of $(\mathcal{D}_P, \varepsilon)$. Together all of this shows that the theory of recursively enumerable sets is in a well-defined sense "nicely imbedded" in the theory of positive equivalences.

2. Let R be a recursively enumerable set and let $\gamma_0, \gamma_1, \ldots$ be a strongly computable enumeration of all finite sets [1, p. 122]. We let

$$\eta_R^* \leftrightharpoons \{\langle x, y\rangle \mid \gamma_x \Delta \gamma_y \subseteq R\}$$

(Δ is the symmetric difference). Then

a) η_R^* is a positive equivalence. If we let $S^\omega \leftrightharpoons \{x \mid \gamma_x \subseteq S\}$ and ${}^\omega S \leftrightharpoons \{x \mid \gamma_x \cap S \neq \emptyset\}$, then

b) R^ω is an η_R^*-equivalence class;

c) if $S \cap R = \emptyset$, then ${}^\omega S$ is an η_R^*-closed set.

We will verify these assertions. a) $\gamma_x \Delta \gamma_y \subseteq R$ is equivalent to the equality $\gamma_x \setminus R = \gamma_y \setminus R$, from which it easily follows that η_R^* is an equivalence. The enumerability of η_R^* is clear from the definition.

b) $x \in R^\omega \Longleftrightarrow \gamma_x \subseteq R \Longleftrightarrow \gamma_x \setminus R = \emptyset$ and so, as remarked above, R^ω is the class of elements equivalent to 0 (recall that $\gamma_0 = \emptyset$).

c) $x \in {}^\omega S \;\&\; \langle x, y\rangle \in \eta_R^* \Longleftrightarrow \gamma_x \cap S \neq \emptyset \;\&\; \gamma_x \setminus R = \gamma_y \setminus R$, but since $S \cap R = \emptyset$, $\gamma_x \cap S = (\gamma_x \setminus R) \cap S = (\gamma_y \setminus R) \cap S = \gamma_y \cap R \neq \emptyset$, then $y \in {}^\omega S$;

d) $R_0 \subseteq R_1 \Longrightarrow \eta_{R_0}^* \subseteq \eta_{R_1}^*$;

e) the mapping $\varphi_1 : R \rightsquigarrow \eta_R^*$ is a monomorphism from Π to $(\partial_p, \varepsilon)$. Assertion d) and the fact that φ_1 is a morphism are obvious. The fact that φ_1 is a monomorphism follows from b).

3. Let g be a one-place partial recursive function. Then

$$\eta_g \leftrightharpoons \{\langle x, x\rangle \mid x \in N\} \cup \{\langle x, y\rangle \mid x \in \delta g \;\&\; y \in \delta g \;\&\; g(x) = g(y)\}$$

is a positive equivalence with the following properties:

a) A set M of natural numbers is η_g-closed if and only if $g^{-1}(g(M)) \subseteq M$.

b) If g is general recursive, then M is m-reduced by g to a particular set (any S such that $g(M) \subseteq S \subseteq N \setminus g(N \setminus M)$) if and only if M is η_g-closed.

Property a) is obvious from the definition and b) follows easily from a).

c) $g_0 \subseteq g_1 \Longrightarrow \eta_{g_0} \subseteq \eta_{g_1}$.

d) The mapping $\psi_0 : g \rightsquigarrow \eta_g$ is a morphism from K to $(\partial_p, \varepsilon)$.

Property d) follows easily from the effectiveness of the construction η_g from g .

4. Let g be a one-place partial recursive function. Let $g^0(x) = x$, $g^{z+1}(x) = g(g^z(x))$, and define the set of pairs

$$\eta_g^i \leftrightharpoons \{\langle x, y\rangle \mid \exists z_0 \exists z_1 (g^{z_0}(x) \text{ is defined and } g^{z_1}(y) \text{ is defined and } g^{z_0}(x) = g^{z_1}(y)\}.$$

Then

a) η_g^i is a positive equivalence.

Remark. If we let Γ_g denote the graph of the function g , then $\eta_g^i = (\Gamma_g)^*$ where the operation $*$ was defined in Lemma 2. Thus η_g^i is the smallest equivalence containing the graph of g .

b) $g_0 \subseteq g_1 \Longrightarrow \eta_{g_0}^i \subseteq \eta_{g_1}^i$.

c) The mapping $\psi_1 : g \rightsquigarrow \eta_g^i$ is a morphism from K to $(\partial_p, \varepsilon)$.

It will be shown that ψ_1 is an epimorphism. Indeed, the following holds.

PROPOSITION 1. Let η be an arbitrary positive equivalence. Then there is a one-place partial recursive function g such that $\eta_g^{i} = \eta$.

Proof. Since η is a recursively enumerable set of pairs of natural numbers, there is a strongly computable sequence $\eta^0, \eta^1, \ldots$ of finite sets such that $\eta^0 \subseteq \eta^1 \subseteq \ldots \subseteq \eta^n \subseteq \eta^{n+1} \subseteq \ldots$ and $\eta = \bigcup_{n=0}^{\omega} \eta^n$. Since η is an equivalence the sequence $\eta^0, \eta^1, \ldots$ may be so chosen that

$$1)\ \langle x, y \rangle \in \eta^n \Longrightarrow x, y \leq n;$$

$$2)\ \langle x, x \rangle \in \eta^n \quad \text{for} \quad x \leq n;$$

$$3)\ \langle x, y \rangle \in \eta^n \Longrightarrow \langle y, x \rangle \in \eta^n;$$

$$4)\ \langle x, y \rangle, \langle y, z \rangle \in \eta^n \Longrightarrow \langle x, z \rangle \in \eta^n.$$

Thus one may suppose that η^n is an equivalence relation $\sim_n$ on the set $\{0, 1, \ldots, n\}$.

For any $x \leq n$ let $f(x, n) \rightleftharpoons \mu y (x \sim_n y)$. The function g will be defined as the union of functions with finite domains $g_0 \subseteq g_1 \subseteq \ldots$, where g_0 is the empty function; if g_n has been constructed, then g_{n+1} is defined as follows: for each $x \leq n+1$ with $x \in \delta g_n$, $g_{n+1}(x) \rightleftharpoons g_n(x)$; for $x \notin \delta g_n$, if $f(x, n+1) \neq x$ (note that then $f(x, n+1) < x$), then $g_{n+1}(x) \rightleftharpoons f(x, n+1)$; for other values of x, g_{n+1} is undefined. Clearly, $g \rightleftharpoons \bigcup_{n=0}^{\omega} g_n$ is a partial recursive function.

We note the following property of g_n: g_n is undefined for those and only those x which are either greater than n or which are the least element in an $\sim_n$-equivalence class. This property is obvious by construction.

From this it is easily deduced that g is undefined only on those elements which are minimal in their η-equivalence classes. We shall show that for any x there exists a z with $g^z(x)$ the minimal element in $[\{x\}]_\eta$. In fact, from the construction of g it is clear that for $x \in \delta g$, $g(x) < x$ and $\langle g(x), x \rangle \in \eta$. By considering the strictly decreasing sequence of η-equivalent elements $x > g(x) > g^2(x) > \ldots$, we see that for some $z \geq 0$, $g^z(x)$ is defined and $g(g^z(x))$ is undefined. But since $\langle g^z(x), x \rangle \in \eta$ and g is undefined on $g^z(x)$, $g^z(x)$ is the least element in the class $[\{x\}]_\eta$. Thus, if $\langle x, y \rangle \in \eta$ and z_0 and z_1 are such that $g^{z_0}(x)$ and $g^{z_1}(y)$ are the minimal elements in the classes $[\{x\}]_\eta$ and $[\{y\}]_\eta$ respectively, then $[\{x\}]_\eta = [\{y\}]_\eta$, $g^{z_0}(x) = g^{z_1}(y)$, and $\eta \subseteq \eta_g^{i}$. The opposite inclusion follows from the aforementioned property: if $g^u(x)$ is defined then $\langle x, g^u(x) \rangle \in \eta$.

The proposition is proved.

Now we define some further operations on positive equivalences. Let $\eta_0, \eta_1 \in \mathcal{D}_p$. Then

$$\eta_0 \oplus \eta_1 \rightleftharpoons \{\langle 2x, 2y \rangle \mid \langle x, y \rangle \in \eta_0\} \cup \{\langle 2x+1, 2y+1 \rangle \mid \langle x, y \rangle \in \eta_1\};$$

$$\eta_0 \times \eta_1 \rightleftharpoons \{\langle x, y \rangle \mid \langle e(x), e(y) \rangle \in \eta_0, \langle r(x), r(y) \rangle \in \eta_1\},$$

where r and e are the Cantor enumerating functions [1].

If $\eta \in \mathcal{D}_p$, then

$$\eta_\phi^\omega \rightleftharpoons \{\langle x, y \rangle \mid [\gamma(x)]_\eta = [\gamma(y)]_\eta\};$$

$$\eta^\omega \rightleftharpoons \{\langle x, y \rangle \mid [\gamma(x+1)]_\eta = [\gamma(y+1)]_\eta\}.$$

LEMMA 7. For any $\eta_0, \eta_1, \eta \in \mathcal{D}_p$, $\eta_0 \oplus \eta_1, \eta_0 \times \eta_1, \eta_\phi^\omega$ and $\eta^\omega \in \mathcal{D}_p$. Moreover, the mappings $\oplus : (\eta_0, \eta_1) \rightsquigarrow \eta_0 \oplus \eta_1, \times : (\eta_0, \eta_1) \rightsquigarrow \eta_0 \times \eta_1, \omega_\phi : \eta \rightsquigarrow \eta_\phi^\omega$, and $\omega : \eta \rightarrow \eta^\omega$ are morphisms from, respectively, $(\mathcal{D}_p, \varepsilon) \times (\mathcal{D}_p, \varepsilon)$, $(\mathcal{D}_p, \varepsilon) \times (\mathcal{D}_p, \varepsilon)$, $(\mathcal{D}_p, \varepsilon)$, and $(\mathcal{D}_p, \varepsilon)$ to $(\mathcal{D}_p, \varepsilon)$.

This follows easily by definition.

Remark. If we let $\gamma'(x) \rightleftharpoons \gamma(x+1)$, then γ' is an enumeration of all finite nonempty sets, and, roughly speaking, $\eta^{\omega}_{\emptyset}$ and η^{ω} differ only on the one equivalence class belonging to $\eta^{\omega}_{\emptyset}$ containing only zero.

2. Perfect Equivalences. In this section, we study equivalences of interest when applied to the problem of the existence of m-degrees consisting of a single 1-degree [6].

A positive equivalence $\eta \neq \mathbb{1}$ is called perfect if there does not exist a proper (i.e., other than $\emptyset$ or N) η-closed recursive set.

COROLLARY 1. If $\eta_0 \subseteq \eta_1 \neq \mathbb{1}$ and η_0 is perfect, then so is η_1.

COROLLARY 2. If η is perfect, then any η-equivalence class is infinite.

A proper subset M of N is perfect if M is a η-closed set for some appropriate perfect equivalence η.

PROPOSITION 2. The m-degree of a perfect set consists of a single 1-degree.

Proof. Let η be a perfect equivalence, and let M be a proper η-closed set. In order to prove the lemma it is sufficient to prove two assertions:

$$1)\quad A \leqslant_m M \Longrightarrow A \leqslant_1 M;$$
$$2)\quad M \leqslant_m A \Longrightarrow M \leqslant_1 A.$$

We prove assertion 1). Let f be a general recursive function which m-reduces A to M. From f we will effectively construct a one-one function f_0 such that for any x, $\langle f(x), f_0(x)\rangle \in \eta$. We describe the construction of f_0: $f_0(0) \rightleftharpoons f(0)$. Suppose that $f_0(0), \ldots, f_0(n)$ have already been defined. Compute $f(n+1)$. If $f(n+1) \notin \{f_0(0), \ldots, f_0(n)\}$, then set $f_0(n+1) \rightleftharpoons f(n+1)$. Otherwise compute in sequence the infinite recursively enumerable set $[\{f(n+1)\}]_\eta$ (Corollary 2) and find the first number k (in the order of calculation) in this class which does not belong to $\{f_0(0), \ldots, f_0(n)\}$, and let $f_0(n+1) \rightleftharpoons k$.

We will show that f_0 m-reduces (and, in view of the one-oneness of f, 1-reduces) A to M. If $x \in A$, then $f(x) \in M$. But $\langle f(x), f_0(x)\rangle \in \eta$, and since M is η-closed, $f_0(x) \in M$. Similarly, $x \notin A \Longrightarrow f_0(x) \notin M$. Thus assertion 1) is proved.

We now prove 2). Let g be a general recursive function which m-reduces M to A. Consider the equivalence $\bar{\eta} = \eta \vee \eta_g$. Since $g^{-1}(g(M)) = M$, M is η_g-closed, and by Lemma 6 M is $\bar{\eta}$-closed. Since M is a proper subset, $\bar{\eta} \neq \mathbb{1}$ and by Corollary 1 $\bar{\eta}(\supseteq \eta)$ is perfect. Let K be any proper $\bar{\eta}$-closed set. Since K is η_g-closed, $g^{-1}(g(K)) = K$. Thus $g(K)$ is not recursive (and is, in particular, infinite) since g m-reduces the nonrecursive set K to $g(K)$.

We effectively construct from g a one-one function g_0 such that for any x, $g_0(x) \in g([\{x\}]_{\bar{\eta}})$. The construction is analogous to the construction of f_0 in the proof of 1): $g_0(0) \rightleftharpoons g(0)$. Suppose $g_0(0), \ldots, g_0(n)$, have been defined. Compute $g(n+1)$. If $g(n+1) \notin \{g_0(0), \ldots, g_0(n)\}$, we let $g_0(n+1) \rightleftharpoons g(n+1)$. If $g(n+1) \in \{g_0(0), \ldots, g_0(n)\}$, then compute in sequence the recursively enumerable set $g([\{n+1\}]_{\bar{\eta}})$ and find the first (in the order computed) number $k \in g([\{n+1\}]_{\bar{\eta}})$ in this set which does not belong to $\{g_0(0), \ldots, g_0(n)\}$. Let $g_0(n+1) \rightleftharpoons k$. The construction is complete.

We verify that g_0 m-reduces (in fact 1-reduces) M to A. If $x \in M$, then $g(x) \in g([\{x\}]_{\bar{\eta}}) \subseteq A$, and then $g_0(x) \in A$. If $x \notin M$, then $g(x) \in g([\{x\}]_{\bar{\eta}}) \subseteq N \setminus A$ and $g_0(x) \in g([\{x\}]_{\bar{\eta}}) \subseteq N \setminus A$, so $g_0(x) \notin A$.

The proposition is proved.

Remark. In [9] Young in 1966 gave an example of an undecidable positive equivalence η (of the form η_g^i) $\neq \mathbb{1}$ such that any proper η-closed recursively enumerable set is the union of a finite number of equivalence classes. Thus, this is an example of a perfect equivalence, and consequently its equivalence classes are examples of recursively enumerable sets whose m-degree consists of a single 1-degree.

The following proposition indicates several methods of constructing perfect equivalences.

PROPOSITION 3. Let η be a positive equivalence and let A and B be disjoint η-closed recursively enumerable sets not separable by a recursive η-closed set (i.e., there is no recursive η-closed set R with $A \subseteq R$ and $R \cap B = \emptyset$); then

1) the equivalence $\bar{\eta} = \eta_{\emptyset}^{\omega} \vee \eta_A^{*} \vee \eta\omega_B$ is perfect;

2) the sets A^{ω} and ${}^{\omega}B$ are perfect.

Proof. First notice that 2) follows from 1) and the fact that A^{ω} and ${}^{\omega}B$ are $\bar{\eta}$-closed sets. On account of Lemma 6, we need only verify that A^{ω} and ${}^{\omega}B$ are $\eta_{\emptyset}^{\omega}$-closed, η_A^{*}-closed, and η_{ω_B}-closed. We check this for A^{ω}. Suppose that $x \in A^{\omega}$ and $\langle x, y\rangle \in \eta_A^{*}$. Then $\gamma(x) \subseteq A$ and $[\gamma(x)]_{\eta} = [\gamma(y)]_{\eta}$. But since A is η-closed, $\gamma(y) \subseteq [\gamma(y)]_{\eta} = [\gamma(x)]_{\eta} \subseteq A$. Hence $y \in A^{\omega}$. Property b) of Example 2 shows that A^{ω} is η_A^{*}-closed. Furthermore, suppose that $x \in A^{\omega}$. Then $\gamma(x) \subseteq A$. But since $A \cap B = \emptyset$, $\gamma(x) \cap B = \emptyset$ and $x \notin {}^{\omega}B$. Thus $A^{\omega} \cap {}^{\omega}B = \emptyset$ and from c) of Example 1, it follows that A^{ω} is η_{ω_B}-closed. And so A^{ω} is $\bar{\eta}$-closed. We leave it to the reader to verify in an analogous way that ${}^{\omega}B$ is $\bar{\eta}$-closed. We note further that A^{ω} and ${}^{\omega}B$ are $\bar{\eta}$-equivalence classes.

We now turn to the proof of 1). Suppose that it is false. Then there exists a proper recursive $\bar{\eta}$-closed set R_0. Without loss of generality, we may suppose that ${}^{\omega}B \subseteq R_0$. Since R_0 is a proper subset of N, there is an $x_0 \in N \setminus R_0$. Consider the set

$$R \rightleftharpoons \{x \mid \exists y\,(\gamma(y) = \gamma(x_0) \cup \{x\} \;\&\; y \in R_0\}.$$

Since R_0 is recursive, so is R. We will show that R is η-closed. Let $x \in R$ and $\langle x, z\rangle \in \eta$. Then for $\gamma(y_1) = \gamma(x_0) \cup \{x\}$ and $y_1 \in R_0$, suppose that y_2 is such that $\gamma(y_2) = \gamma(x_0) \cup \{z\}$. Consider the η-closed sets $\gamma(y_1)$ and $\gamma(y_2)$. $[\gamma(y_1)]_{\eta} = [\gamma(x_0) \cup \{x\}]_{\eta} = [\gamma(x_0)]_{\eta} \cup [\{x\}]_{\eta} = [\gamma(x_0)]_{\eta} \cup [\{z\}]_{\eta} = [\gamma(y_2)]_{\eta}$. Hence $\langle y_1, y_2\rangle \in \eta_{\emptyset}^{\omega} \subseteq \bar{\eta}$. Since R_0 is $\bar{\eta}$-closed, $y_2 \in R_0$ and $z \in R$.

Furthermore, if $x \in B$, then for $\gamma(y) = \gamma(x_0) \cup \{x\}$, $\gamma(y) \cap B \neq \emptyset$ and $y \in {}^{\omega}B \subseteq R_0$, so that $x \in R$. And so $B \subseteq R$. If $x \in A$, then for $\gamma(y) = \gamma(x_0) \cup \{x\}$ we have $\gamma(y) \Delta \gamma(x_0) = \{x\} \subseteq A$. Hence $\langle y, x_0\rangle \in \eta_A^{*} \subseteq \bar{\eta}$. Since R_0 is $\bar{\eta}$-closed and $x_0 \notin R_0$, it follows that $y \notin R_0$ and thus $x \notin R$ and $A \cap R = \emptyset$. Hence R is an η-closed recursive set separating A and B. We have obtained a contradiction and proved the proposition.

COROLLARY 1. Under the conditions of the proposition, ${}^{\omega}A$ and B^{ω} are perfect.

This follows since the conditions on A and B are symmetric.

COROLLARY 2. Let A and B be disjoint recursively inseparable recursively enumerable sets. Then ${}^{\omega}A$, A^{ω}, ${}^{\omega}B$, and B^{ω} are perfect sets.

In place of η one may take $\mathbb{O}$.

Recall that for $C, \mathcal{D} \subseteq N$ $C \oplus \mathcal{D}$ denotes the set $\{2x \mid x \in C\} \cup \{2x+1 \mid x \in \mathcal{D}\}$.

COROLLARY 3. If C is a (not necessarily recursively enumerable) perfects set and $\mathcal{D}$ is an arbitrary subset of N, then $(C \oplus \mathcal{D})^{\omega}$ is perfect. If $\mathcal{D}$ is recursively enumerable, then ${}^{\omega}(C \oplus \mathcal{D})$ is also a perfect set.

Suppose that η_0 is a perfect equivalence such that C is η_0-closed and that x_0 and x_1 are numbers such that $x_0 \in C$, $x_1 \notin C$. We let $A_0 \rightleftharpoons [\{x_0\}]_{\eta_0}$, $B_0 \rightleftharpoons [\{x_1\}]_{\eta_0}$, $\eta \rightleftharpoons \eta_0 \oplus \mathbb{O}$, $A \rightleftharpoons A_0 \oplus \emptyset$, $B \rightleftharpoons B_0 \oplus \emptyset$. It is not hard to check that 1) the equivalence η and the sets A and B satisfy the hypotheses of the proposition and that 2) $(C \oplus \mathcal{D})^{\omega}$ will be $\bar{\eta}$-closed, where $\bar{\eta}$ is defined by the formula in the proposition. To prove that ${}^{\omega}(C \oplus \mathcal{D})$ is perfect, one must take the same η and in place of A and B take the sets $B_0 \oplus \emptyset$ and $A_0 \oplus \mathcal{D}$, respectively.

Remark. Corollary 3 was proved earlier in the case where both C and $\mathcal{D}$ are recursively enumerable.

COROLLARY 4. If C_0 is a perfect set and $C_0 \leq_{tt} C_1$, then the tt-degree of C_1 contains a perfect set. If C_0 and C_1 are recursively enumerable, then the corresponding perfect set may be chosen recursively enumerable.

As such a set, one may take $(C_0 \oplus C_1)^{\omega}$.

The following corollary is formulated as a separate proposition.

PROPOSITION 4. Every recursively enumerable Turing degree >0 contains a perfect recursively enumerable set.

Proof. Suppose that A is a recursively enumerable nonrecursive set. Then A may be written as the union of two disjoint recursively enumerable recursively inseparable sets A_0 and A_1 [10].

It is well known that $A_i \leq_T A$, $i=0,1$. By Corollary 2, A_0^{ω} is perfect and, by Corollary 3, $(A_0^{\omega} \oplus A)^{\omega}$ is perfect. Clearly, $A \equiv_T (A_0^{\omega} \oplus A)^{\omega}$.

The proposition is proved.

COROLLARY [6]. Every recursively enumerable Turing degree >0 contains a recursively enumerable m-degree consisting of a single 1-degree.

The results obtained above naturally lead to a series of questions and conjectures:

1. Will any (recursively enumerable) m-degree which consists of a single 1-degree be a perfect set?

In connection with this question, it is natural to check whether any known nontrivial m-degrees are perfect.

For creative sets this will be noted in the next section.

We consider an example given by C. Jockusch [6].

PROPOSITION 5. If S is a simple nonhypersimple set, then S^{ω} is perfect.

Proof. Let $F_0, F_1, \ldots$ be a strongly enumerable listing of finite sets which shows that S is not hypersimple, i.e.,

a) $F_i \cap F_j = \emptyset$ for $i \neq j$;

b) $F_i \cap (N \setminus S) \neq \emptyset$ for all $i \in N$.

Consider the set $B \rightleftharpoons \{x \mid \exists n (F_n \setminus \gamma(x) \subseteq S)\}$. This set is recursively enumerable and consists of those numbers of finite sets which except for the members of S contain some elements of the list. It is not difficult to check from the definition that B is η_S^*-closed. We will show that S^{ω} and B are disjoint and inseparable by a recursive η_S^*-closed set. If $x \in S^{\omega}$, then $\gamma(x) \subseteq S$. Thus if $F_n \setminus \gamma(x) \subseteq S$ then $F_n \subseteq S$, which is impossible by b). Hence $S^{\omega} \cap B = \emptyset$. Suppose that R is a η_S^*-closed recursive set such that $S^{\omega} \cap R = \emptyset$ and $B \subseteq R$. Let $A \rightleftharpoons \{x \mid x \in R \mathbin{\&} \forall y (\gamma(y) \subsetneq \gamma(x) \Rightarrow y \notin R\}$.

The set A is recursive. Note that, for $x \in A$, $\gamma(x) \neq \emptyset$ and $\gamma(x) \cap S = \emptyset$. Indeed, if $\gamma(x) = \emptyset$, $x=0 \in S^{\omega}$. Suppose that $\gamma(x) \cap S \neq \emptyset$. Then, if $z \in \gamma(x) \cap S$, for y such that $\gamma(y) = \gamma(x) \setminus \{z\}$ we have $\langle x, y \rangle \in \eta_S^*$, and since R is η_S^*-closed and $x \in R$, then $y \in R$. But $\gamma(y) \subsetneq \gamma(x)$, and so $x \notin A$. For each n there is $x_n \in A$ such that $\gamma(x_n) \subseteq F_n$. Since $F_i \cap F_j = \emptyset$, $\gamma(x_i) \cap \gamma(x_j) = \emptyset$. It follows that A is infinite and the recursively enumerable set $T \rightleftharpoons \bigcup_{x \in A} \gamma(x)$ is infinite and lies in the complement of S, which is impossible. By applying Proposition 2 to the equivalence η_S^* and the sets S^{ω} and B, we see that $(S^{\omega})^{\omega}$ is perfect. But it is easy to see that $(S^{\omega})^{\omega}$ is recursively isomorphic to S^{ω}. Thus S^{ω} is perfect.

The proposition is proved.

Remark. One may avoid using Proposition 2 and the isomorphism of $(S^{\omega})^{\omega}$ and S^{ω} since it is possible to prove that $\eta_S^{*} \vee \eta_B$ is perfect.

Corollary 4 makes probable the hypothesis that any (recursively enumerable) tt-degree contains a (recursively enumerable) perfect set.

Note that an affirmative answer to this hypothesis in the recursively enumerable case would be implied, in view of the previous results, from an affirmative answer to either of the following two questions:

2. For any recursively enumerable nonrecursive set K, can one find disjoint recursively enumerable recursively inseparable sets S_0 and S_1 such that $S_0 \leqslant_{tt} R$?

3. For any recursively enumerable nonrecursive set R, can one find a simple nonhypersimple set such that $S \leqslant_{tt} R$?

With regard to the latter question, one should recall that if one does not require the absence of hypersimplicity, a theorem of Dekker [10] gives a positive answer. Moreover, instead of tt-reducibility, Dekker's theorem obtains positive reducibility.

To emphasize the difficulty of obtaining a positive answer to Problem 3 we point out that the analog of this problem for positive reducibility has a negative answer.

Remark. After our paper had been completed, negative answers to Questions 2 and 3 were obtained. Indeed, A. N. Degtev proved that no simple nonhypersimple set is tt-reducible to a hypersimple set with a retraceable complement. Later S. D. Denisov proved that no perfect set is tt-reducible to a hypersimple set with a retraceable complement.

PROPOSITION 6. If S is a simple nonhypersimple set and Γ is a hypersimple set, then S is not positively reducible to Γ.

Proof. By reducing formulas to conjunctive normal form, it is easily shown that S is positively reducible to Γ if and only if S is m-reducible to the set $({}^{\omega}\Gamma)^{\omega}$.

Now we prove an auxiliary lemma.

LEMMA 8. If Γ is hypersimple, so is ${}^{\omega}\Gamma$.

Proof. Suppose not. Let $F_0, F_1, \ldots$ be a strongly enumerable list of finite sets such that $F_n \cap (N \setminus {}^{\omega}\Gamma) \neq \emptyset$ for all n. Without loss of generality, we may suppose that $0 \notin F_0$. We will effectively construct a list of finite sets $\Phi_0, \Phi_1, \ldots$ as follows: $\Phi_0 \rightleftharpoons \cup\{\gamma(x) \mid x \in F_0\}$. Suppose that $\Phi_0, \Phi_1, \ldots, \Phi_n$ have been constructed. Let $\psi_n \rightleftharpoons \bigcup_{i=0}^{n} \Phi_i$ and find the least k such that, for any $x \in F_k$, $\gamma(x) \setminus \psi_n \neq \emptyset$ (i.e., $\gamma(x) \nsubseteq \psi_n$). Note that one can find such a k because ψ_n is a finite set, it has a finite number of subsets, and the sets F_n are pairwise disjoint. Let $\Phi_{n+1} \rightleftharpoons [\cup\{\gamma(x) \mid x \in F_k\}] \setminus \psi_n$. Thus $\Phi_0, \Phi_1, \ldots$ is a strong listing of finite sets. We will show that for any n, $\Phi_n \cap (N \setminus \Gamma) \neq \emptyset$. For Φ_0: since $F_0 \cap (N \setminus {}^{\omega}\Gamma) \neq \emptyset$ there is $x \in \Gamma_0$ such that $x \notin {}^{\omega}\Gamma$. Hence $\gamma(x) \cap \Gamma = \emptyset$. But $x \neq 0$, $\gamma(x) \neq \emptyset$, and $\gamma(x) \subseteq \Phi_0$. So $\Phi_0 \cap (N \setminus \Gamma) \supseteq \gamma(x) \neq \emptyset$. For Φ_{n+1}: since $F_k \cap (N \setminus {}^{\omega}\Gamma) \neq \emptyset$ there is $x \in F_k$ such that $x \notin {}^{\omega}\Gamma$, i.e., $\gamma(x) \subseteq N \setminus \Gamma$. But $\gamma(x) \setminus \psi_n \neq \emptyset$, so $\Phi_{n+1} \cap (N \setminus \Gamma) \supseteq \gamma(x) \setminus \psi_n \neq \emptyset$.

The lemma is proved.

Remark. It is easily shown that if Γ is simple, so is ${}^{\omega}\Gamma$. However, it is impossible to show that if Γ is hyperhypersimple, so is ${}^{\omega}\Gamma$. Indeed, suppose that Γ is a hyperhypersimple noncomplete set. Suppose that ${}^{\omega}\Gamma$ is hyperhypersimple. Note that $\Gamma \equiv_{T} {}^{\omega}\Gamma$. Furthermore consider the superset R of ${}^{\omega}\Gamma$ defined by: $R \rightleftharpoons {}^{\omega}\Gamma \cup \{x \mid \overline{\overline{\gamma(x)}} \in K\}$, where K is a creative set and $\overline{\overline{\gamma(x)}}$ denotes the number of elements in $\gamma(x)$. Since $R \supseteq {}^{\omega}\Gamma$ and ${}^{\omega}\Gamma$ is hyperhypersimple, it follows [2] that $R \leqslant_{m}^{\omega} \Gamma$. In particular, $R \leqslant_{T} {}^{\omega}\Gamma$ and $R \leqslant_{T} \Gamma$. The following equivalence holds: $n \in K \Longleftrightarrow$ for the first n elements of the complement of Γ, $x_1, \ldots, x_n$, and x such that $\gamma(x) = \{x_1, \ldots, x_n\}$, $x \in R$. This equivalence shows that K is recursive in Γ, which is impossible.

We now return to the proof of Proposition 6. It is sufficient to show that if S is a simple non-hypersimple set and Γ is hypersimple, then S is not m-reducible to Γ^{ω}. Suppose the contrary, and let f m-reduce S to Γ^{ω}. Let $F_0, F_1, \ldots$ be a strong listing of finite sets such that for any n $F_n \cap (N \setminus S) \neq \phi$. Let $\Phi_n \leftrightharpoons \cup \{\gamma(x) \mid x \in f(F_n)\}$. We note the following:

1) For any n, $\Phi_n \cap (N \setminus \Gamma) \neq \phi$. Indeed since $F_n \cap (N \setminus S) \neq \phi$, then for $x \in F_n \cap (N \setminus S)$, $f(x) \notin \Gamma^{\omega}$. Hence $\gamma f(x) \cap (N \setminus \Gamma) \neq \phi$, but $\gamma f(x) \subseteq \Phi_n$.

2) For any $y \in N \setminus \Gamma$ there is only a finite number of n with $y \in \Phi_n$. Suppose otherwise. By letting

$$Y \leftrightharpoons \{x \mid y \in \gamma(x)\},$$

we have $f^{-1}(Y) \subseteq N \setminus S$, since $Y \subseteq N \setminus \Gamma^{\omega}$. But since, for infinitely many n, $F_n \cap f^{-1}(Y) \neq \phi$, $f^{-1}(Y)$ is infinite. But Y is a recursive set, and consequently $f^{-1}(Y)$ is an infinite recursive set lying in the complement of S, which is impossible.

We now effectively construct a strong listing of finite sets $\psi_0, \psi_1, \ldots$ as follows: $\psi_0 \leftrightharpoons \Phi_0$. Suppose $\psi_0, \psi_1, \ldots, \psi_n$ have been constructed. Compute in turn the elements of Γ, obtaining $\Gamma^0 \subseteq \Gamma^1 \subseteq \ldots$; then find the first pair (ℓ, m) such that

$$(\Phi_\ell \setminus \Gamma^m) \cap (\bigcup_{i=0}^{n} \psi_i) = \phi.$$

One can always find such a pair, since for ℓ sufficiently large no elements of $(\bigcup_{i=0}^{n} \psi_i) \cap (N \setminus \Gamma)$ occur in Φ_ℓ [such an ℓ exists by Property 2)], and for m sufficiently large every number in $\Phi_\ell \cap \Gamma$ is already in Γ^m, so that $(\Phi_\ell \setminus \Gamma^m) \cap (\bigcup_{i=0}^{n} \psi_i) = \phi$. We let $\psi_{n+1} \leftrightharpoons \Phi_\ell \setminus \Gamma^m$. We see at once that $\psi_{n+1} \cap (\bigcup_{i=0}^{n} \psi_n) = \phi$ and $\psi_{n+1} \cap (N \setminus \Gamma) = \Phi_\ell \cap (N \setminus \Gamma) \neq \phi$. Thus $\psi_0, \psi_1, \ldots$ is a strong listing of finite sets such that $\psi_n \cap (N \setminus \Gamma) \neq \phi$ for all n. Hence Γ is not hypersimple — a contradiction.

The proposition is proved.

In the conclusion to this section we explain the behavior of perfect equivalences with respect to the operations introduced in Sec. 1.

LEMMA 9. If η is a perfect equivalence, so is η^{ω}.

Proof. Suppose otherwise, and let R be a proper η^{ω}-closed recursive set. We define R_n for $n = 1, 2, \ldots$ as follows: $R_n \leftrightharpoons \{x \mid x \in R, \overline{\overline{\gamma'(x)}} = n\}$. Then $R = \bigcup_{n=1}^{\omega} R_n$ and $R_i \cap R_j = \phi, i \neq j$. Every R_n is a recursive set. By passing to the complement of R if necessary, we may suppose that $R_1 \neq \phi$. Further, if $N_n \leftrightharpoons \{x \mid \overline{\overline{\gamma'(x)}} = n\}$ let k be the least number such that $R_k \neq N_k$ (since $R \neq N$, such a k exists). Note also that since $R_1 \neq \phi$, $R_n \neq \phi$ for all n.

We consider two cases: $k = 1$ and $k > 1$.

$k = 1$. Let $\bar{R} \leftrightharpoons \{x \mid \exists y (y \in R_1 \,\&\, \gamma'(y) = \{x\})\}$. $\bar{R}$ is a recursive set and, since $R_1 \neq \phi$ and $R_1 \neq N_1$, $\bar{R}$ is a proper subset of N. We show that $\bar{R}$ is η-closed. Let $x \in \bar{R}$ and $\langle x, z \rangle \in \eta$. Then for y_0, y_1 such that $\gamma'(y_0) = \{x\}$, $\gamma'(y_1) = \{z\}$, we have $[\gamma'(y_0)]_\eta = [\{x\}]_\eta = [\{z\}]_\eta = [\gamma'(y_1)]_\eta$, and hence $\langle y_0, y_1 \rangle \in \eta^{\omega}$. But $y_0 \in R_1$, and hence $y_1 \in R_1$, since R is η^{ω}-closed. But then $z \in \bar{R}$. And so $\bar{R}$ is a proper η-closed recursive set — an impossibility.

$k = n+1,\ n > 0$. Then $R_n = N_n$, $R_{n+1} \neq \phi$, and $R_{n+1} \neq N_{n+1}$. Suppose that $y \in N_{n+1} \setminus R_{n+1}$ and $z_0 \in \gamma'(y)$, and let y_0 be such that $\gamma'(y_0) = \gamma'(y) \setminus \{z_0\}$. Since $\overline{\overline{\gamma'(y_0)}} = n$, $y_0 \in R_n = N_n$. Define $\bar{R} \leftrightharpoons \{x \mid \exists u (u \in R \,\&\, \gamma'(u) = \gamma'(y_0) \cup \{x\})\}$. Note that $z_0 \notin \bar{R}$, but $\bar{R} \neq \phi$, since if $x \in \gamma'(y_0)$, then $x \in \bar{R}$. So $\bar{R}$ is a proper recursive set. It is easy to check (as above) that $\bar{R}$ is η-closed — an impossibility.

The lemma is proved.

Remark. The equivalence $\eta_\emptyset^\omega$ has a proper closed class, consisting of only o. Thus $\eta_\emptyset^\omega$ is never perfect.

LEMMA 10. Suppose that η_0 and η_1 are positive equivalences. The product $\eta = \eta_0 \times \eta_1$ is perfect if and only if both η_0 and η_1 are perfect.

Proof. Necessity. Suppose that η is perfect but η_0 is not. Then there is a proper η_0-closed recursive set R. Define $\bar{R} \rightleftharpoons \{x \mid \ell(x) \in R\}$. Since R is a proper recursive set, $\bar{R}$ is also. It remains to show that $\bar{R}$ is η-closed. Let $x \in \bar{R}$ and $\langle x, y \rangle \in \eta$. Then $\ell(x) \in R$ and $\langle \ell(x), \ell(y) \rangle \in \eta_0$. But R is η_0-closed so that $\ell(y) \in R$ and $y \in \bar{R}$. The necessity is proved.

Sufficiency. Suppose that η_0 and η_1 are perfect. Suppose that there is a proper recursive η-closed set R. Let $R_n \rightleftharpoons \{x \mid x \in R,\ \ell(x) = n\}$. Then $R = \bigcup_{n=0}^{\infty} R_n$, $R_i \cap R_j = \emptyset$, $i \neq j$. Each R_n is a recursive set. By passing to the complement of R if necessary, we may suppose that $R_0 \neq \emptyset$. Let k be the least number such that $R_k \neq N_k$ $(\rightleftharpoons \{x \mid \ell(x) = k\})$. If $R_k \neq \emptyset$, then by letting $\bar{R} \rightleftharpoons \{x \mid \exists y (y \in R_k \ \&\ x = r(y))\}$ we obtain a proper recursive set which is η_1-closed and which, as is not difficult to check, is an impossibility. If $R_k = \emptyset$, then $k > 0$. Then let $\bar{R} \rightleftharpoons \{x \mid \exists y \in R (x = \ell(y))\}$. Since $k > 0$, $0 \in \bar{R}$, but $k \notin \bar{R}$. Hence $\bar{R}$ is a proper recursive set. We check that $\bar{R}$ is η_0-closed. Let $x \in \bar{R}$, $\langle x, y \rangle \in \eta_0$. Then there is a z such that $z \in R$ and $x = \ell(z)$. Let $\bar{z}$ be such that $\ell(\bar{z}) = y$ and $r(\bar{z}) = r(z)$, i.e., $\bar{z} = c(y, r(z))$. Then $\langle \ell(z), \ell(\bar{z}) \rangle = \langle x, y \rangle \in \eta_0$, $\langle r(z), r(\bar{z}) \rangle = \langle r(z), r(z) \rangle \in \eta_1$ and hence $\langle z, \bar{z} \rangle \in \eta$, $\bar{z} \in R$, and $y = \ell(\bar{z}) \in \bar{R}$. So $\bar{R}$ is a proper recursive η_0-closed set — an impossibility. This contradiction proves the lemma.

Remark. The sum $\eta_0 \oplus \eta_1$ of any positive equivalences η_0 and η_1 is never perfect. Moreover, in a well-defined sense (the following section) the irreducibility to a direct sum is characteristic of perfect enumerations.

3. The Category $\mathfrak{N}_p$ of Positively Enumerated Sets. An enumerated set $\gamma = (S, \nu)$ is called positively enumerated if the enumerable equivalence $\eta_\nu \rightleftharpoons \{\langle x, y \rangle \mid \nu x = \nu y\}$ is positive. The totality of all positively enumerated sets with the usual defintion of morphism constitutes the category $\mathfrak{N}_p$ of positively enumerated sets, which is a full subcategory of the category $\mathfrak{N}$ of all enumerated sets.

We formulate some basic properties of the category $\mathfrak{N}_p$ in the next proposition.

PROPOSITION 7. 1) The category $\mathfrak{N}_p$ admits (finite) direct sums and products; 2) if $\gamma \in \mathfrak{N}_p$ and (γ_0, μ) is a subobject of γ (in the category $\mathfrak{N}$), then $\gamma_0 \in \mathfrak{N}_p$; 3) the notions of subobject, principal subobject, wn-subobject and e-subobject coincide for positively enumerated sets.

Proof. Assertion 1) is easily verified; one need only notice that the direct sum $\gamma_0 \oplus \gamma_1$ and the direct product $\gamma_0 \times \gamma_1$ for positively enumerated sets γ_0 and γ_1 in the category $\mathfrak{N}$ are positively enumerated.

Let $\gamma = (S, \nu) \in \mathfrak{N}_p$, i.e., $\eta_\nu = \{\langle x, y \rangle \mid \nu x = \nu y\}$ is recursively enumerable. Let (γ_0, μ) be a subobject of γ and let f be a one-place general recursive function such that $\mu \nu_0 = \nu f$. Since μ is a monomorphism then $\nu_0(x) = \nu_0(y) \Leftrightarrow \mu\nu_0(x) = \mu\nu_0(y)$. Hence

$$\langle x, y \rangle \in \eta_{\nu_0} \Leftrightarrow \nu_0(x) = \nu_0(y) \Leftrightarrow \mu\nu_0(x) = \mu\nu_0(y) \Leftrightarrow \nu f(x) = \nu f(y) \Leftrightarrow \langle f(x), f(y) \rangle \in \eta_\nu,$$

so η_{ν_0} is recursively enumerable. Assertion 2) is proved.

Let (γ_0, μ) be a subobject of the positively enumerated set γ; we will show that (γ_0, μ) is an e-subobject. Let f be a one-place general recursive function such that $\mu\nu_0 = \nu f$. Let $R \rightleftharpoons [f(N)]_{\eta_\nu}$, R be recursively enumerable, and $R = \nu^{-1}(\mu(S_0))$. Suppose that $\eta_\nu^0 \subseteq \eta_\nu^1 \subseteq \ldots$ is a strongly computable sequence of finite sets of pairs such that $\bigcup_{n=0}^{\omega} \eta_\nu^n = \eta_\nu$. Let

$$g(x) \rightleftharpoons \ell(\mu t [\langle f\ell(t), x \rangle \in \eta_\nu^{r(t)}]).$$

It is easy to check that $\delta g = R$ and for $x \in R$, $\mu\nu_0 g(x) = \nu(x)$. From Lemma 3, §4 of [1], it follows that (γ_0, μ) is an e-subobject. Since the implications

$$e\text{-subobject} \Longrightarrow wn\text{-subobject} \Longrightarrow \text{principal subobject} \Longrightarrow \text{subobject}$$

always hold, assertion 3 is proved.

The proposition is proved.

We note that any positively enumerated set $\gamma = (S, \nu)$ is isolated and its ν-order coincides with equality. Thus if S is not a single element set, γ cannot be completely enumerated. Nevertheless, precompletely enumerated sets exist in $\mathfrak{N}_p$.

PROPOSITION 8. Suppose g is a one-place universal partial recursive function, i.e., such that for some two-place general recursive function f, $\lambda x \lambda y g(f(x,y))$ is a two-place universal function for the one-place partial recursive functions. Let $S \rightleftharpoons N/\eta_g^i$, $\gamma \rightleftharpoons (S, \nu)$, where $\nu(x) = [\{x\}]_{\eta_g^i}$. Then

1) γ_g^i is a precompletely enumerated set;

2) any positively enumerated set is a subobject of γ.

Proof. The first assertion was proved in Proposition 5, §8 [1]. We will prove the second. But first we establish an auxiliary assertion which is interesting in itself when considered as properties of the partial recursive algebra.

LEMMA 11. If g is a one-place universal partial recursive function, for any one-place partial recursive function g' one can find a one-place one-one general recursive function f such that $\lambda x f g'(x) = \lambda x g f(x)$; i.e., the domains $\delta f g'$, $\delta g f$ of the functions fg' and gf are the same and, for $x \in \delta f g'$, $fg'(x) = gf(x)$.

Proof. From the definition of the universal functions g it is easy to prove that for any one-place partial recursive function h one can effectively find a one-one general recursive function $\bar{f}$ such that $h = g\bar{f}$. Suppose that $\{f_n\}_{n \in N}$ is a computable sequence of one-place one-one general recursive functions such that for any $n \in N$, $\varkappa_n g' = g f_n$, where $\varkappa_n$ - is the one-place partial-recursive function with Kleene number n. By the fixed-point theorem there is an n_0 such that $\varkappa_{n_0} = f_{n_0}$. If we let $f \rightleftharpoons \varkappa_{n_0} = f_{n_0}$, we have $fg' = gf$.

The lemma is proved.

We return to the proof of the proposition. Suppose that η is an arbitrary positive equivalence. Then by Proposition 1 there is a one-place partial recursive function g' such that $\eta = \eta_{g'}^i$. Let f be a one-one one-place general recursive function such that $fg' = gf$. We will verify that

$$\langle x, y \rangle \in \eta \Longleftrightarrow \langle f(x), f(y) \rangle \in \eta_g^i.$$

Since $\eta = \eta_{g'}^i$, it is necessary to check that

$$\langle x, y \rangle \in \eta_{g'}^i \Longleftrightarrow \langle f(x), f(y) \rangle \in \eta_g^i.$$

We show by induction that for any z, $f(g')^z(x) \simeq g^z f(x)$. If $z = 0$, $f(g')^0(x) = f(x) = g^0 f(x)$. If $z = z_0 + 1$ and $f(g')^{z_0}(x) \simeq g^{z_0} f(x)$, then

$$f(g')^{z_0+1}(x) \simeq fg'(g'^{z_0}(x)) \simeq gf(g'^{z_0}(x)) \simeq g(g^{z_0} f(x)) \simeq g^{z_0+1} f(x).$$

Thus, if $\langle x, y \rangle \in \eta_{g'}^i$ and $g'^{z_0}(x) = g'^{z_1}(y)$, then $fg'^{z_0}(x) = g^{z_0} f(x) = fg'^{z_1}(y) = g^{z_1} f(y)$ and $\langle f(x), f(y) \rangle \in \eta_g^i$. Conversely, if $\langle f(x), f(y) \rangle \in \eta_g^i$, and $g^{z_0} f(x) = g^{z_1} f(y)$, then $fg'^{z_0}(x) = fg'^{z_1}(y)$. But since f is one-one, $g'^{z_0}(x) = g'^{z_1}(y)$ and $\langle x, y \rangle \in \eta_{g'}^i$. The equivalence is proved.

If we let $\bar{S} \rightleftharpoons N/\eta$, $\bar{\nu}(x) \rightleftharpoons [\{x\}]_{\eta}$, we see that the mapping μ defined by $\mu([\{x\}]_{\eta}) \rightleftharpoons [\{f(x)\}]_{\eta_g^i}$ is a monomorphism from $\bar{\gamma}=(\bar{S},\bar{\nu})$ to $\gamma=(S,\nu)$. But any positively enumerated set is isomorphic to a set of the form $\bar{\gamma}$, so any positively enumerated set is a subobject of γ.

The proposition is proved.

Now we characterize those positively enumerated sets whose enumerated equivalences are perfect.

LEMMA 12. Let $\gamma = (S,\nu)$ be a nontrivial positively enumerated set. The enumerated equivalence η_ν is perfect if and only if γ is not a direct sum.

Proof. If (γ_0,μ_0) and (γ_1,μ_1) are subobjects of γ and γ is their direct sum, then $\mu_i(S_i)$ is a completely recursive subset of S, $i=0,1$. Hence $\nu^{-1}(\mu_i(S_i))$ are η_ν-closed (proper, clearly) recursive sets. Thus η_ν is not perfect. Conversely, let R be a proper [recursive] η_ν-closed set. Then $S_0 \rightleftharpoons \nu(R)$ and $S_1 \rightleftharpoons \nu(N\setminus R)$ are completely recursive subsets of S. If we provide them with enumerations ν_0 and ν_1 which are principal with respect to ν, then the enumerated sets (S_0,ν_0) and (S_1,ν_1) together with the natural injections $i_0 : S_0 \to S$ and $i_1 : S_1 \to S$ form a pair of subobjects of γ and γ is their direct sum.

Remark. No precompletely enumerated set is a direct sum (Rice's theorem [1]). Furthermore, any equivalence is precomplete and a positively enumerated set is clearly creative. Thus by Proposition 8 and Lemma 12, a creative set is perfect.

4. Description of the Functor $L^0(\gamma)$.

In the preceding paragraph it was noticed that subobjects of positively enumerated sets are also positively enumerated. However, $\mathfrak{N}_p$ is not closed under epimorphic images because $N \in \mathfrak{N}_p$ and each enumerated set may be represented as a factor-object of N. In this section a certain "compactness" of the subcategory $\mathfrak{N}_p$ in $\mathfrak{N}$ will be proved.

This "compactness" will be shown by the fact that by using an appropriate representation of the enumerated set γ as a factor object of a positively enumerated set, one may "compute" the sublattice $L^0(\gamma)$. We recall that $L^0(\gamma)$ (see [1], pp. 35 and 45) is the set of all subobjects of γ modulo equivalence under γ induced by the following partial ordering $\leq$: if (γ_0,μ_0) and (γ_1,μ_1) are subobjects of γ, then $(\gamma_0,\mu_0) \leq (\gamma_1,\mu_1) \Longleftrightarrow$ there is a morphism $\mu: \gamma_0 \to \gamma_1$ such that $\mu_0 = \mu_1\mu$, i.e., the following diagram is commutative:

$$\begin{tikzcd} \gamma_0 \arrow[drr, "\mu_0"] \arrow[d, "\mu"'] & & \\ & & \gamma \\ \gamma_1 \arrow[urr, "\mu_1"'] & & \end{tikzcd}$$

First we describe the "construction" of $L^0(\gamma)$ for a positively enumerated set $\gamma = (S,\nu)$.

LEMMA 13. If $\gamma=(S,\nu)$ is a positively enumerated set and η is the enumerated equivalence, then $L^0(\gamma)$ is isomorphic to the totality R^- of all nonempty η-closed recursively enumerable sets with the relation of inclusion.

Proof. As has already been mentioned in the preceding paragraph, for positively enumerated sets the notions of subobject and e-subobject coincide. Thus, any subobject (γ_0,μ_0) may be uniquely represented by its image $\mu_0(S_0)$, which in turn may be represented by the nonempty η-closed recursively enumerable set $\nu^{-1}(\mu_0(S_0))$. This correspondence gives a mapping of $L^0(\gamma)$ into R, which is the desired isomorphism, as may be verified without difficulty.

The lemma is proved.

If $\mu:\gamma_0 \to \gamma_1$ is a morphism of enumerated sets, there is a mapping $L^0(\mu): L^0(\gamma_0) \to L^0(\gamma_1)$ (see [1], p. 46), denoted for short by μ_*, possessing the following properties:

1) μ_* is an upper semilattice homomorphism;

2) μ_* is a monomorphism if and only if μ is;

3) $\mu_*(L^0(\gamma_0))$ is an ideal in $L^0(\gamma_1)$;

4) $(\mu_0\mu_1)_* = \mu_{0*}\mu_{1*}$;

5) μ_* is an epimorphism if and only if μ is a factorization.

Properties 1)-4) are easily checked using the definitions. We check 5). Suppose that $\mu_* : \mathcal{L}^\circ(\gamma_0) \longrightarrow \mathcal{L}^\circ(\gamma_1)$ is an epimorphism. Since $\mathcal{L}^\circ(\gamma)$ has a greatest element $\mathbb{1}\,(=(\gamma, Id))$, then $\mu_*(\mathbb{1}) = \mathbb{1}$. This means that $\mu(S_0) = S_1$ and the enumeration of S_1 defined by $\mu\nu_0 : N \longrightarrow S_1$ is equivalent to ν_1. Actually this also implies that μ is a factorization. Conversely, if μ is a factorization, then $\mu_*(\mathbb{1}) = \mathbb{1}$ and, by Property 3), μ_* maps $\mathcal{L}^\circ(\gamma_0)$ onto $\mathcal{L}^\circ(\gamma_1)$, i.e., μ_* is an epimorphism.

If $\mu_0 : \gamma_0 \longrightarrow \gamma$ and $\mu_1 : \gamma_1 \longrightarrow \gamma$ are two factorizations we will say that μ_0 is finer than μ_1 $(\mu_0 \succ \mu_1)$ if there is a morphism $\mu : \gamma_1 \longrightarrow \gamma_0$ such that $\mu_1 = \mu_0\mu$. Note that the morphism μ, if it exists, is unique and is a factorization. If $\mu_0 \preccurlyeq \mu_1$ and $\mu_1 \preccurlyeq \mu_0$, then μ_0 and μ_1 are equivalent under γ.

Let γ be an arbitrary enumerated set. Consider the family A of all equivalence classes of factorizations $\mu_0 : \gamma_0 \longrightarrow \gamma$ such that γ_0 is a positively enumerated set. A may be identified with the family of all positive equivalences η_0 such that $\eta_0 \subseteq \eta$, where η is the enumerated equivalence of the enumerated set $\gamma = (S, \nu)$. In fact in each equivalence class of a factorization there is a unique factorization of the form $\tilde{\nu} : ((N/\eta_0), \nu_{\eta_0}) \longrightarrow \gamma$, where N/η_0 is the collection of η_0-equivalence classes, ν_{η_0} is the enumeration defined by $\nu_{\eta_0}(x) \rightleftharpoons [\{x\}]_{\eta_0}$, and $\tilde{\nu}$ is the morphism induced by the enumeration $\nu : N \longrightarrow S$. We note further that the inclusion relation $\supseteq$ on equivalences corresponds to the relation $\succcurlyeq$ on factorizations. Similarly, let $A \rightleftharpoons \{\eta_0 \mid \eta_0$ is a positive equivalence and $\eta_0 \subseteq \eta\}$. On A the relation $\subseteq$ is directed, i.e., for $\eta_0, \eta_1 \in A$ there is a $\eta_2 \in A$ such that $\eta_0 \subseteq \eta_2$ and $\eta_1 \subseteq \eta_2$. This follows from the fact that $\eta_0, \eta_1 \in A \implies \eta_0 \vee \eta_1 \in A$.

Suppose that $\eta_0 \in A$ and $\mu : \gamma_0 \longrightarrow \gamma$ is the corresponding factorization. Then by Lemma 13 $\mathcal{L}^\circ(\gamma_0) \approx \mathbb{R}^-_{\eta_0}$ and the mapping μ_* induces an epimorphism $\varepsilon_{\eta_0} : \mathbb{R}^-_{\eta_0} \longrightarrow \mathcal{L}^\circ(\gamma)$. If $\eta_0 \subseteq \eta_1 \in A$, then the homomorphism $P_{\eta_0\eta_1}$ defined in Sec. 1 of this paper maps $\mathbb{R}^-_{\eta_0}$ homomorphically onto $\mathbb{R}^-_{\eta_1}$. It is easy to see that the diagram

$$\begin{tikzcd}
\mathbb{R}^-_{\eta_0} \arrow[rr, "P_{\eta_0\eta_1}"] \arrow[dr, "\varepsilon_{\eta_0}"'] & & \mathbb{R}^-_{\eta_1} \arrow[dl, "\varepsilon_{\eta_1}"] \\
& \mathcal{L}^\circ(\gamma) &
\end{tikzcd}$$

is commutative.

The system $\{\mathbb{R}^-_{\eta_0}, P_{\eta_0\eta_1} \mid \eta_0, \eta_1 \in A\}$ forms the direct spectrum of the upper semilattice. Let $\mathbb{R}^-_{\eta}$ denote the limit of this spectrum and let P_{η_0} denote the unique homormorphsim from $\mathbb{R}^-_{\eta_0}$ to $\mathbb{R}^-_{\eta}$.

<u>Remark.</u> If γ is a positively enumerated set, then η is a positive equivalence. In this case, the meaning of $\mathbb{R}^-_{\eta}$ is not ambiguous since the two objects denoted are naturally isomorphic.

Since all homomorphisms $P_{\eta_0\eta_1}$ are epimorphisms, P_{η_0} is an epimorphism. Furthermore, the commutativity of the above diagrams implies the existence of a unique homomorphism (indeed, epimorphism) $\varepsilon_\eta : \mathbb{R}^-_{\eta} \longrightarrow \mathcal{L}^\circ(\gamma)$ such that for any $\eta_0 \in A$ the diagram

$$\begin{tikzcd}
\mathbb{R}^-_{\eta_0} \arrow[rr, "P_{\eta_0}"] \arrow[dr, "\varepsilon_{\eta_0}"'] & & \mathbb{R}^-_{\eta} \arrow[dl, "\varepsilon_{\eta}"] \\
& \mathcal{L}^\circ(\gamma) &
\end{tikzcd}$$

commutes.

The fundamental assertion of this section is that ε_η is an isomorphism.

<u>PROPOSITION 9.</u> For any enumerated set γ the homomorphism $\varepsilon_\eta : \mathbb{R}^-_{\eta} \longrightarrow \mathcal{L}^\circ(\gamma)$ constructed above is an isomorphism from $\mathbb{R}^-_{\eta}$ into $\mathcal{L}^\circ(\gamma)$.

Proof. That ε_η is an epimorphism has already been noticed in its definition. Thus it remains to show that ε_η is an monomorphism. Suppose $\eta_0 \subseteq \eta$ and η_0 is a positive equivalence. We will describe the epimorphism $\varepsilon_{\eta_0} : \mathcal{R}^-_{\eta_0} \to L^o(\gamma)$ explicitly. Let $R \in \mathcal{R}^-_{\eta_0}$ and put $S_0 \leftrightharpoons \nu(R)$. Suppose that f is a one-place general recursive function such that $\rho f = R$. Let $\nu_0 \leftrightharpoons \nu f$ and $\gamma_0 \leftrightharpoons (S_0, \nu_0)$. Then (γ_0, Id) is a subobject of γ. This subobject [more precisely, the class of subobjects equivalent under γ to (γ_0, Id)] is $\varepsilon_{\eta_0}(R)$. Now that we know the explicit form of the mapping ε_{η_0}, we will prove the following assertion:

Suppose that $R_0 \in \mathcal{R}^-_{\eta_0}$, $R_1 \in \mathcal{R}^-_{\eta_1}$, and that $\eta_0 (\subseteq \eta)$, $\eta_1 (\subseteq \eta)$ are positive equivalences. If $\varepsilon_\eta \rho_{\eta_0}(R_0) \leqslant \varepsilon_\eta \rho_{\eta_1}(R_1)$, then there is a positive equivalence η_2 such that $\eta_0 \vee \eta_1 \subseteq \eta_2 \subseteq \eta$ and $\rho_{\eta_0\eta_2}(R_0) \subseteq \rho_{\eta_1\eta_2}(R_1)$.

Note that Proposition 9 follows from this assertion since it means that $\varepsilon_\eta \rho_{\eta_0}(R_0) \leqslant \varepsilon_\eta \rho_{\eta_1}(R_1) \Longrightarrow \rho_{\eta_0}(R_0) \leqslant \rho_{\eta_1}(R_1)$; i.e., for any elements $\alpha, \beta \in \mathcal{R}^-_\eta [\varepsilon_\eta(\alpha) \leqslant \varepsilon_\eta(\beta) \Longrightarrow \alpha \leqslant \beta]$ and, in particular, $[\varepsilon_\eta(\alpha) = \varepsilon_\eta(\beta) \Longrightarrow \alpha = \beta]$. Thus ε_η is a monomorphism.

We return now to the proof of the above assertion. Consider the two commutative diagrams:

$$\begin{tikzcd} \mathcal{R}^-_{\eta_0} \arrow[rr, "\rho_{\eta_0}"] \arrow[dr, "\varepsilon_{\eta_0}"'] & & \mathcal{R}^-_\eta \arrow[dl, "\varepsilon_\eta"] \\ & L^c(\gamma) & \end{tikzcd} \qquad \begin{tikzcd} \mathcal{R}^-_{\eta_1} \arrow[rr, "\rho_{\eta_1}"] \arrow[dr, "\varepsilon_{\eta_1}"'] & & \mathcal{R}^-_\eta \arrow[dl, "\varepsilon_\eta"] \\ & L^c(\gamma) & \end{tikzcd}$$

The commutativity of these diagrams and the hypothesis of the assertion show that $\varepsilon_{\eta_0}(R_0) \leqslant \varepsilon_{\eta_1}(R_1)$. Suppose that γ_0 and γ_1 are subobjects of γ (together with the identity injections Id into γ) which define the classes $\varepsilon_{\eta_0}(R_0)$ and $\varepsilon_{\eta_1}(R_1)$ respectively, and let f_0 and f_1 be general recursive functions such that $\nu_0 = \nu f_0$, $\nu_1 = \nu f_1$. Then $\rho f_0 = R_0$ and $\rho f_1 = R_1$. The inequality $\varepsilon_{\eta_0}(R_0) \leqslant \varepsilon_{\eta_1}(R_1)$ implies that there is a morphism $\mu : \gamma_0 \to \gamma_1$ such that the diagram

$$\begin{tikzcd} \gamma_0 \arrow[d, "\mu"'] \arrow[dr, "Id"] & \\ \gamma_1 \arrow[r, "Id"'] & \gamma \end{tikzcd}$$

commutes.

Let g be a one-place general recursive function such that $\mu\nu_0 = \nu_1 g$. But since S_0 and S_1 are subsets of S, it follows that $S_0 \subseteq S_1$ and μ is the identity injection, i.e., $\nu_0 = \nu_1 g$. Let h be a one-place partial recursive function whose domain δh is R_0 such that for any $x \in R_0$, $f_0 h(x) = x$. Let $H \leftrightharpoons f_1 g h$. H is a partial recursive function with domain R_0. We will show that for any $x \in R_0$, $\nu x = \nu H(x)$. Indeed, $\nu x = \nu f_0 h(x) = \nu_0 h(x) = \nu_1 g h(x) = \nu f_1 g h(x) = \nu H(x)$. We will also show that $x \in R_0 \Longrightarrow H(x) \in R_1$. Indeed, $x \in R_0 = \delta H \Longrightarrow H(x) = f_1(gh(x)) \in \rho f_1 = R_1$.

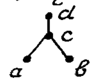
We consider the positive equivalence η^i_H. The implication proved earlier, $x \in R_0 = \delta H \Longrightarrow (\nu x = \nu H(x))$, shows that $\eta^i_H \subseteq \eta$ (η is the enumerated equivalence of the enumeration ν). Since for any $x \in R_0$, $\langle x, H(x) \rangle \in \eta^i_H$ and $H(x) \in R_1$, then $[R_1]_{\eta^i_H} \supseteq R_0$. We let $\eta_2 \leftrightharpoons \eta_0 \vee \eta_1 \vee \eta^i_H$. Then $\eta_2 \subseteq \eta$, $[R_1]_{\eta_2} \supseteq [R_1]_{\eta^i_H} \supseteq R_0$. Recalling the definitions of the mappings $\rho_{\eta_0\eta_2}, \rho_{\eta_1\eta_2}$, we have

$$\rho_{\eta_0\eta_2}(R_0) = [R_0]_{\eta_2} \subseteq [R_1]_{\eta_2} = \rho_{\eta_1\eta_2}(R_1),$$

which we had to prove.

The proposition is proved.

We will consider a typical case. Let M be a proper subset of N. We will consider the enumeration $\nu_M : N \longrightarrow \{0,1\}$, the characteristic function of M. Let $\gamma_M = (\{0,1\}, \nu_M)$. Then $L^0(\gamma_M)$, as is easy to see, coincides with the set of m-degrees less than or equal to the m-degree $d_m(M)$ of M (including the m-degree of the empty set N). The mapping $\varepsilon_\emptyset : R_\emptyset^- \longrightarrow L^0(\gamma_M)$ of the set of all nonempty recursively enumerable sets onto $L^0(\gamma_M)$ is identical to the operator ψ defined by Lachlan in [7, 8].

If η is a positive equivalence such that M is η-closed, then the mapping $\varepsilon_\eta : R_\eta^- \longrightarrow L^0(\gamma_M)$ is the restriction of the operator $\psi (= \varepsilon_\emptyset)$ to the family of η-closed recursively enumerable sets, and the relation $\varepsilon_\eta \rho_{\emptyset\eta} = \varepsilon_0$ corresponds to the equality $\psi(R) = \psi([R]_\eta)$, proved in [8]. Proposition 9 shows that by using appropriate positive equivalences η such that M is η-closed, one can "approximate" $L^0(\gamma_N)$ by the semilattices R_η^-. In certain cases, it is easy to calculate the limit $\varinjlim R_\eta^-$. For example, when M is a maximal set, this limit, clearly, is isomorphic to the semilattice 

(a corresponds to a recursively enumerable set $\subseteq M$; b corresponds to one disjoint from M; c to one which is represented in the form of the union of two nonempty recursively enumerable sets one of which $\subseteq M$ and the other of which is disjoint from M; d corresponds to N).

Similarly, one can prove that if M is η-closed for a positive equivalence such that any η-closed recursively enumerable set consists either of a finite number of equivalence classes or is separable into a finite number of classes from N, then the m-degree of M is minimal (when M is not recursive).

In particular, as we recall, Young gave an example of a positive equivalence with this property and an equivalence η_M for a maximal set M. Moreover, every example known to the author for a recursively enumerable set with a minimal m-degree [3, 6, 8, 9] is η-closed for a positive equivalence η with the above-mentioned property.

The properties considered here and above show that a large number of properties and concepts in the theory of recursively enumerable sets can be relativized to appropriate positive equivalences. We will introduce as examples some definitions and several properties of these definitions.

Let η be a positive equivalence.

A η-closed set R is called η-finite if R consists of a finite number of equivalence classes.

A recursively enumerable nonrecursive η-closed set R is called η-simple if any η-closed recursively enumerable subset of the complement of R is η-finite.

A recursively enumerable nonrecursive η-closed set R is called η-hypersimple if there is no strongly enumerable sequence of finite sets $F_0, F_1, \ldots$ such that $[F_i]_\eta \cap [F_j]_\eta = \emptyset$ for $i \neq j$ and, for all i, $[F_i]_\eta \cap (N \setminus R) \neq \emptyset$.

We state without proof the following analog of Proposition 5.

<u>PROPOSITION 5'</u>. If S is a η-simple non-η-hypersimple set, then S^ω is perfect.

A recursively enumerable nonrecursive η-closed set R is called η-maximal if for any recursively enumerable η-closed subset R', either $R' \setminus R$ or $N \setminus R'$ is η-finite.

The previous remarks show that the m-degree of a η-maximal set R is minimal.

Similarly, one may define the concepts η-immune, η-hyperimmune, η-hyperhypersimple, η-hyperhyperimmune sets, etc. Many of these concepts may (must?) be defined not only for η-closed sets but also for the relation η itself. Thus, for example, one may define the notion of a maximal equivalence η as an undecidable equivalence such that for any η-closed recursively enumerable set R, either R is η-finite or $N \setminus R$ is η-finite.

The importance and necessity of the concepts just introduced evidently needs further justification. However, even now they appear sufficiently natural.

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