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Global Attractors for the Three-Dimensional Navier–Stokes Equations

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In this paper we show that the weak solutions of the Navier–Stokes equations on any bounded, smooth three-dimensional domain have a global attractor for any positive value of the viscosity. The proof of this result, which bypasses the two issues of the possible nonuniqueness of the weak solutions and the possible lack of global regularity of the strong solutions, is based on a new point of view for the construction of the semiflow generated by these equations. We also show that, under added assumptions, this global attractor consists entirely of strong solutions.

KEY WORDS: Attractor; global attractor; Navier–Stokes equations; weak solution.

1. INTRODUCTION

One of the major recent applications of the modern theory of dynamical systems has been the development of dynamical theories for nonlinear partial differential equations which has led to a rigorous basis that the long-time dynamics of these infinite dimensional problems can be characterized by, or approximated by, the dynamics of various finite dimensional systems. What is at the basis of much of this development are the various theories which have shown that the solutions of these equations can be represented in terms of a semiflow on a suitable phase space and that this semiflow has a global attractor in this space (see, e.g., Babin and Vishik, 1983, 1989; Billotti and LaSalle, 1971; Hale, 1988; Ladyzhenskaya, 1972, 1991; Sell and You, 1994; Temam, 1988).

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The very definition of the global attractor is an important technical issue. For the reasons given by Sell and You (1994) and in Section 3, the following concept is appropriate. We shall say that a set $\mathbb{A}$ in a phase space W is a *global attractor* of a semiflow $S(t)$ on W, provided the following four properties are satisfied:

(1) $\mathbb{A}$ is nonempty and compact.

(2) $\mathbb{A}$ is invariant, i.e., one has $S(t)\,\mathbb{A} = \mathbb{A}$, for all $t \geqslant 0$.

(3) There is a bounded neighborhood U of $\mathbb{A}$ in W with the property that $\mathbb{A}$ attracts U, i.e., for every neighborhood V of $\mathbb{A}$, there is a time $T \geqslant 0$ such that $S(t)\,U \subset V$, for all $t > T$.

(4) $\mathbb{A}$ attracts every point in W.

While the theory of global attractors for nonlinear partial differential equations has widespread applicability, there is one very important case that remains open, *viz.*, the Navier–Stokes equations in three dimensions (3D). Even though it has been known for over 20 years that there is a global attractor for these equations in two dimensions (2D) (see Ladyzhenskaya, 1972), the 3D problem is essentially unresolved (see Navier, 1827; Poisson, 1831; Stokes, 1845; Leray, 1933, 1934a, b; Hopf, 1951). Neverheless, one can find some papers which do address this issue.[2] In particular, Foias and Temam (1987) have shown that the 3D problem does admit a *universal attracting set* for the weak solutions. However, this theory does not address the issue of whether Property (3) in the definition of a global attractor is valid. In another direction, for the Navier–Stokes equations on suitable thin 3D domains, it was shown by Raugel and Sell (1993a–c) that the weak solutions do have a global attractor, and that this attractor consists entirely of strong solutions. However, this theory is limited to thin domains. A third example is the related area of inertial forms for the Navier–Stokes equations (see Kwak, 1992; Kwak *et al.*, 1994).

Recall that the Navier–Stokes equations on a suitable smooth bounded domain Ω in $\mathbb{R}^2$, or $\mathbb{R}^3$, have the form

$$u_t - \nu\,\Delta u + (u \cdot \nabla)\,u + \nabla p = f \qquad \text{and} \qquad \nabla \cdot u = 0, \qquad \text{on } \Omega \tag{1.1}$$

where the solution $u = u(t)$ is to satisfy the initial condition

$$u(0) = u_0 \tag{1.2}$$

[2] Comparisons between our theory, as developed herein, and the works of other authors are made in the body of this paper.

for an appropriate divergent-free function u_0. The ordered pair (u_0, f) is referred to as the *data* of the problem, and f is called the *forcing function*. We examine this problem with Dirichlet boundary conditions, where $u = 0$ on $\partial\Omega$, the boundary of Ω. [The theory developed herein extends in a straightforward way to periodic boundary conditions by using the methodology of Temam (1983).] As usual, we let H denote the closure in $L^2(\Omega)$ of the vector fields in $C_0^\infty(\Omega)$ that satisfy $\nabla \cdot u = 0$, and we let V denote the closure of the same set in $H^1(\Omega)$, (see Constantin and Foias, 1988; Temam, 1977, 1983; von Wahl, 1985). In this paper, we assume that $u_0 \in H$ and that the forcing function f satisfies $f \in L^\infty(0, \infty; L^2(\Omega))$.

In the 2D theory, it is known that for given data (u_0, f), there is a unique weak solution $u = u(t)$ of (1.1)–(1.2), that $u(t)$ remains a weak solution for all time $t \geqslant 0$. Moreover, this weak solution instantaneously becomes a strong solution at any time $t = \tau > 0$, *and* it remains a strong solution for all $t \geqslant \tau$. Furthermore, the mapping $S(t)$: $u_0 \to S(t)\, u_0 = u(t)$ defines a semiflow on the phase space H, and that this semiflow is both compact and point dissipative.

In 3D one quickly encounters difficulties in trying to build a similar theory for the weak solutions. While it is the case in 3D that for given data (u_0, f), there is a weak solution $u = u(t)$ of (1.1)–(1.2), that $u(t)$ remains a weak solution for all time $t \geqslant 0$, it is not known whether this solution is uniquely determined by the data. As a result, one cannot conclude that the mapping $S(t)$: $u_0 \to S(t)\, u_0 = u(t)$ defines a semiflow. On the other hand, for *good* data (u_0, f), where $u_0 \in V$, the initial value problem (1.2) does have a unique strong solution on some interval $[0, T)$. However, it is not known whether this strong solution continues to exists for *all* times $t \geqslant 0$, i.e., it is not known whether $T = \infty$. The global regularity problem (GRP) for the 3D Navier–Stokes equations is to show that for all *good* data, the strong solution of the initial value problem (1.2) exists for all time $t \geqslant 0$. This is an open problem in every sense. There is no known 3D Navier–Stokes problem in which GRP has been resolved. No proof. No counterexample.

Thus there are two factors which appear as possible obstructions to the development of a theory of global attractors for the 3D Navier–Stokes equations. In the case of weak solutions, one faces the possibility of non-uniqueness, and in the case of strong solutions, it is the GRP. In this paper we develop a new point of view which bypasses both of these issues. In particular, we show the following:

(1) for every smooth bounded domain Ω in $\mathbb{R}^3$ and for all suitable forcing functions f, the class of weak solutions of the Navier–Stokes equations on Ω generates a semiflow on a suitable phase space; and

(2) under further regularity assumptions on f [for example, f is in $L^2(\Omega)$ and is independent of time], this semiflow has a global attractor.

Since this theory is restricted to weak solutions, we avoid the GRP. The nonuniqueness issue is resolved by replacing H with another phase space W, in which each point is a weak solution. Thus if a given datum $u_0 \in H$ admits several weak solutions, then each of these solutions corresponds to a different point in the space W. [See Sell (1973) for a related construction for ordinary differential equations.]

This paper is organized along the following lines. In Section 2 we present the essentials for the theory of semiflows on a Frechét space. It turns out, as we shall see in Section 3, that, for a suitable forcing function f, the weak solutions of the Navier–Stokes equations can be identified with the restriction of a semiflow on a Frechét space to an appropriate invariant subset W. It is this space W that forms the phase space for the weak solutions of the Navier–Stokes equations. By using the general theory of global attractors for semiflows on metric spaces (see Sell and You, 1994), we describe in Section 3 general sufficient conditions for the semiflow on W to have a global attractor. We also prove the existence of a global attractor when $f \in L^2(\Omega)$. We argue in this case that, for any positive viscosity and any such f, one always has a global attractor in W. If f is time dependent, say that $f \in L^\infty(0, \infty; L^2(\Omega))$, then one needs to incorporate a skew product semiflow into the problem (see Babin and Sell, 1995; Raugel and Sell, 1993a–c; Sacker and Sell, 1977, 1994; Sell, 1967a, b; Vishik, 1992). This time-dependent issue is addressed in Section 4.

Even though we are successful in this paper in presenting a theory of global attractors for the weak solutions of the Navier–Stokes equations, and even though this theory is independent of the resolution of the GRP, it is of interest to do a simple thought experiment and ask how the theory presented herein would be affected by a favorable resolution of the GRP. As an outcome of this analysis, we show in Section 5 that, under additional assumptions, the global attractor consists entirely of strong solutions. For example, it has been shown by Raugel and Sell (1993a–c) that on suitable thin domains, the weak solutions are *ultimately regular*, i.e., there is a time $t_0 > 0$, where t_0 depends on the data, such that the weak solution $u = u(t)$ is regular for all $t \geqslant t_0$. We show in Section 5 that the global attractor arising from the theory present in this paper agrees with the attractor found in the papers of Raugel and Sell cited above. As noted in these papers, this attractor for the thin domain problems consists entirely of strong solutions.

2. THE SPACE OF WEAK SOLUTIONS

We begin with a description of the notation used in this paper. The Navier–Stokes equations are given in (1.1) and the spaces H and V are described above as well. Throughout this paper we assume that the forcing function f satisfies $f \in L^\infty(0, \infty; L^2(\Omega))$. We let H_w denote the Hilbert space H with the weak topology. We use the semicontinuity property of weak convergence, which reads that if a sequence u_n converges to u_0 weakly in H, i.e., $u_n \rightharpoonup u_0$, then one has $\|u_0\| \leqslant \liminf \|u_n\|$. The symbol H always denotes H with the strong topology. Let $\mathbb{P}$ denote the orthogonal projection of $L^2(\Omega)$ onto H. By applying $\mathbb{P}$ to (1.1) we obtain the evolutionary equation

$$u_t + \nu A u + B(u, u) = \mathbb{P} f, \tag{2.1}$$

where $Au = -\mathbb{P}\,\Delta u$ is the *Stokes operator* and the bilinear term B satisfies $B(u, v) = \mathbb{P}((u \cdot \nabla)\, v)$, for suitable functions u and v. It is known that A is a positive, self-adjoint linear operator on H with compact resolvent. Therefore all the fractional powers A^α of A are well defined (see Pazy, 1983). We let $V^2 = \mathscr{D}(A)$ denote the domain of A. One then has $\mathscr{D}(A^{1/2}) = V = V^1$. We let $V^0 = H$ and let V^{-1} denote the dual space to V with respect to the L^2-inner product $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_H$ on H. We also let $\|\cdot\| = \|\cdot\|_H$ denote the L^2-norm on H. One then has three compact imbeddings,

$$V^2 \subsetneq V^1 = V \subsetneq V_0 = H \subsetneq V^{-1}$$

We use the A^α-norms to express the norms on the space $V^{2\alpha}$. Thus one has

$$\|u\|_{V^1} = \|u\|_V = \|A^{1/2} u\| \qquad \text{and} \qquad \|v\|_{V^{-1}} = \|A^{-1/2} v\|$$

See Constantin and Foias (1988) and Temam (1977, 1983) for more details.

We define the trilinear form $b(u, v, w) =^{\text{def}} \langle B(u, v), w \rangle$, where $B(u, v)$ is given above. We use the following inequalities, which are derived by Constantin and Foias (1988), for the form b. There is a constant $C_1 > 0$ such that one has

$$|b(u, v, w)| \leqslant C_1 \|u\|^{1/4} \|A^{1/2} u\|^{3/4} \|v\|^{1/4} \|A^{1/2} v\|^{3/4} \|A^{1/2} w\| \tag{2.2}$$

and

$$\|A^{-1/2} B(u, v)\| \leqslant C_1 \|u\|^{1/4} \|A^{1/2} u\|^{3/4} \|v\|^{1/4} \|A^{1/2} v\|^{3/4} \tag{2.3}$$

for all $u, v, w \in V$.

2.1. Notation. Let X denote any Banach space, with norm $\|\cdot\|_X$. In our applications X denotes one of the three separable Hilbert spaces V, H, or V^{-1}. For $1 \leqslant p < \infty$, we let $L^p_{\text{loc}}(0, \infty; X)$ denote the collection of all functions $\phi: (0, \infty) \to X$ with the property that, for all τ and T with $0 < \tau \leqslant T < \infty$, one has

$$\int_\tau^T \|\phi(s)\|_X^p \, ds < \infty$$

We also let $L^p_{\text{loc}}[0, \infty; X)$ denote the collection of all functions $\phi \in L^p_{\text{loc}}(0, \infty; X)$ with the property that, for all T with $0 < T < \infty$, one has

$$\int_0^T \|\phi(s)\|_X^p \, ds < \infty$$

Similarly for $p = \infty$, we let $L^\infty_{\text{loc}}(0, \infty; X)$ denote the collection of all $\phi: (0, \infty) \to X$ with the property that, for all τ and T with $0 < \tau \leqslant T < \infty$, one has

$$\operatorname*{ess\,sup}_{\tau < s < T} \|\phi(s)\|_X < \infty$$

We also let $L^\infty_{\text{loc}}[0, \infty; X)$ denote the collection of all functions $\phi \in L^\infty_{\text{loc}}(0, \infty; X)$ with the property that, for all T with $0 < T < \infty$, one has

$$\operatorname*{ess\,sup}_{0 < s < T} \|\phi(s)\|_X < \infty$$

For $0 \leqslant a < b \leqslant \infty$ and $1 \leqslant p \leqslant \infty$, the spaces $L^p_{\text{loc}}(a, b; X)$ and $L^p_{\text{loc}}[a, b; X)$ are defined in an analogous manner. See Dunford and Schwartz (1958) or Hille and Phillips (1957) for more details.

For $-\infty \leqslant a < b \leqslant \infty$ we let $L^\infty(a, b; X)$ denote the usual Banach space of functions $\phi: (a, b) \to X$, where

$$\operatorname*{ess\,sup}_{a < s < b} \|\phi(s)\|_X < \infty$$

For a given function $f \in L^\infty(0, \infty; L^2(\Omega))$ we define the norm

$$\|f\|_\infty \stackrel{\text{def}}{=} \operatorname*{ess\,sup}_{0 < s < \infty} \|f(s)\|_{L^2(\Omega)} < \infty$$

We let $C(0, \infty; H_w)$ and $C[0, \infty; H_w)$ denote the spaces of weakly continuous functions with range in H_w and defined, respectively, on the intervals $(0, \infty)$ and $[0, \infty)$.

Some spaces arise quite often in this theory, and it is convenient to adopt some special notation for this situation. In particular, we define

$$Y \stackrel{\text{def}}{=} L^\infty(0, \infty; L^2(\Omega)) \qquad \text{and} \qquad Z \stackrel{\text{def}}{=} L^2_{\text{loc}}[0, \infty; H) \times Y$$

and for any $N_0 \geqslant 0$, we let

$$Y(N_0) \stackrel{\text{def}}{=} \{f \in Y: \|f\|_\infty \leqslant N_0\} \qquad \text{and} \qquad Z(N_0) \stackrel{\text{def}}{=} L^2_{\text{loc}}[0, \infty; H) \times Y(N_0)$$

The space R is the product space

$$R \stackrel{\text{def}}{=} L^2_{\text{loc}}[0, \infty; H) \times L^2_{\text{loc}}[0, \infty; L^2(\Omega))$$

In addition, we make use of other local notation throughout this paper.

The spaces $L^p_{\text{loc}}(0, \infty; X)$ and $L^p_{\text{loc}}[0, \infty; X)$, for $1 \leqslant p < \infty$, are examples of Frechét spaces. This means that they are metrizable, locally convex topological linear spaces, and they are complete (see Kelley and Namioka, 1963; Dunford and Schwartz, 1958). The topology on each of these spaces is generated by a countable family of pseudonorms. In particular, on the space $L^p_{\text{loc}}(0, \infty; X)$ one can use the pseudonorms $N_0 = N_1 = 0$ and

$$N_n(\phi) = \left(\int_{n^{-1}}^{n} \|\phi\|^p_X \, ds\right)^{1/p}, \qquad n = 2, 3, 4, \ldots \tag{2.4}$$

while on the space $L^p_{\text{loc}}[0, \infty; X)$ one can use

$$N_n(\phi) = \left(\int_{n}^{n+1} \|\phi\|^p_X \, ds\right)^{1/p}, \qquad n = 0, 1, 2, \ldots \tag{2.5}$$

An invariant metric on these spaces is then given by $d_p(\phi_1, \phi_2) = d(\phi_1 - \phi_2)$, where

$$d(\phi) = \sum_{n=0}^{\infty} 2^{-n} \min(N_n(\phi), 1) \tag{2.6}$$

Recall that a set B in a linear topological space Z is said to be *bounded* if for every neighbourhood U of the origin in Z there is an $r > 0$ such that $B \subset rU$, where $rU = \{ru: u \in U\}$ (see Kelley and Namioka, 1963). In the case of the Frechét spaces $L^p_{\text{loc}}(0, \infty; X)$ and $L^p_{\text{loc}}[0, \infty; X)$, for $1 \leqslant p < \infty$, it follows that a set B is bounded if and only if one has

$$\sup\{N_n(\phi): \phi \in B\} < \infty, \qquad \text{for each} \quad n = 0, 1, 2, \ldots \tag{2.7}$$

where $N_n(\phi)$ is given above. Recall that in a metric space a set B is compact if and only if it is sequentially compact.

It should be noted that the concept of a bounded set given above is a topological concept. It does not depend on the metric used to construct the topology. In order to illustrate this point, let X be a Banach space with norm $\|\cdot\|_X$. Define N_X by

$$N_X(u) \stackrel{\text{def}}{=} \min\{\|u\|_X, 1\}, \qquad \text{for} \quad u \in X$$

Note that $N_X(u-v)$ is a metric on X, and that N_X and $\|\cdot\|_X$ generate the same topology on X. Furthermore, a set B is bounded in X if and only if

$$\sup\{\|u-v\|_X: u, v \in B\} < \infty$$

A similar characterization of boundedness in terms of N_X is not possible, since $\sup\{N_X(u-v): u, v \in X\} \leqslant 1$. Thus in terms of the invariant metric N_X, every set in X has a finite N_X-diameter, while a set B in X has finite $\|\cdot\|_X$-diameter if and only if it is bounded.

A special situation arises in connection with the topology on the spaces $Y(N_0)$ and $Z(N_0)$, where $N_0 \geqslant 0$ is fixed. Because of the continuous imbedding

$$L^\infty(0, \infty; L^2(\Omega)) \mapsto L^2_{\text{loc}}[0, \infty; L^2(\Omega))$$

we see that $Y(N_0)$ is a subset of $L^2_{\text{loc}} = L^2_{\text{loc}}[0, \infty; L^2(\Omega))$. Moreover, for each $N_0 \geqslant 0$, the space $Y(N_0)$ is a closed, bounded set in L^2_{loc} in the L^2_{loc}-topology. Similarly, for each $N_0 \geqslant 0$, the space $Z(N_0)$ is a closed set in Z in the R-topology.

For a given function f: $(0, \infty) \to X$, where X is a set, we define the *time translate* f_τ by

$$f_\tau(t) = f(\tau + t), \qquad t \geqslant 0$$

where $\tau \geqslant 0$. Note that if $f \in L^\infty(0, \infty; L^2(\Omega))$, then one has

$$\|f_\tau\|^2_\infty \leqslant \|f\|^2_\infty, \qquad \text{for all} \quad \tau \geqslant 0$$

Let $\mathscr{F}$ be a bounded set in Y. Thus $\mathscr{F} \subset Y(N_0)$, for some $N_0 \geqslant 0$. Define $\gamma^+(\mathscr{F}) = \{f_\tau: f \in \mathscr{F} \text{ and } \tau \geqslant 0\}$, and set $L^2_{\text{loc}} = L^2_{\text{loc}}[0, \infty; L^2(\Omega))$. Next define the *hull* as

$$H^+(\mathscr{F}) \stackrel{\text{def}}{=} \text{Cl}_{L^2_{\text{loc}}}(\gamma^+(\mathscr{F}))$$

One then has $H^+(\mathscr{F}) \subset Y(N_0)$ as well, i.e., $H^+(\mathscr{F})$ is a Y-bounded set and it is closed in L^2_{loc}. We are especially interested in the case where the hull is a compact set in L^2_{loc}. Recall that the hull $H^+(\mathscr{F})$ is a compact set in L^2_{loc} if and only if for every T with $0 < T < \infty$, one has

$$\sup \left\{ \int_0^T \|f(s+\tau+h) - f(s+\tau)\|^2 \, ds : f \in \mathscr{F}, \tau \geq 0 \right\} \to 0, \qquad \text{as} \quad h \to 0 \tag{2.8}$$

see Dunford and Schwartz (1958, Part 1, pp. 298–301). Condition (2.8) will be satisfied if $\mathscr{F}$ is Y-bounded, $\mathscr{F} \subset C[0, \infty; L^2(\Omega))$, and $\mathscr{F}$ is a uniformly equicontinuous family of mappings from $[0, \infty)$ into $L^2(\Omega)$. For example, the hull is compact if $\mathscr{F}$ is a bounded set in the Hölder space $C^\alpha[0, \infty; L^2(\Omega))$, for some $\alpha > 0$. For the remainder of this section we assume that the hull $H^+(\mathscr{F})$ is a compact set.

2.2. Bubnov–Galerkin Approximations. We are now prepared to describe the space of weak solutions of the Navier–Stokes equations. In doing this, it will be helpful to recall the role of the Bubnov–Galerkin approximations. The construction of weak solutions is accomplished by taking suitable limits of subsequences of the Bubnov–Galerkin approximations. We do not give all the details of this construction here, although some aspects are incorporated into the Compactness Lemma; see below. The essence of the construction is given in the next paragraph (see Constantin and Foias, 1988; Sell and You, 1995; Temam, 1977, 1983).

Let $\{e_n\}$ be an orthonormal basis in H of eigenvectors of the Stokes operator A, where $Ae_n = \lambda_n e_n$, for $n = 1, 2, 3, \ldots$, and the eigenvalues $\{\lambda_n\}$ satisfy $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots$. Let $P = P_n$ be the orthogonal projection of H onto the finite-dimensional space $PH = \mathrm{Span}\{e_1, \ldots, e_n\}$, for $n = 1, 2, 3, \ldots$. An nth-order Bubnov–Galerkin approximation $p = u_n$ of the initial value problem (1.1)–(1.2) is a solution of the ordinary differential equation

$$p_t + \nu A p + PB(p, p) = P\mathbb{P}f \tag{2.9}$$

with initial condition satisfying $p(0) = P\mathbb{P}u_0$. The solutions of (2.9) exist for all time $t \geq 0$, and the Bubnov–Galerkin approximations satisfy

$$\|p(t)\|^2 \leq e^{-\nu\lambda_1(t - t_0)} \|p(t_0)\|^2 + c_0^2 \|f\|_\infty^2, \qquad \text{for} \quad t \geq t_0 \geq 0$$

where $c_0^2 = (\nu\lambda_1)^{-2}$,

$$\|p(t)\|^2 + 2\nu \int_{t_0}^t \|A^{1/2}p\|^2 \, ds \leq \|p(t_0)\|^2 + 2 \int_{t_0}^t \langle \mathbb{P}f, p \rangle \, ds, \qquad \text{for} \quad t \geq t_0 \geq 0$$

and for all $v \in PH$ one has

$$\langle p(t)-p(t_0), v\rangle + \nu \int_{t_0}^{t} \langle A^{1/2}p(s), A^{1/2}v\rangle\, ds + \int_{t_0}^{t} b(p(s), p(s), v)\, ds$$

$$= \int_{t_0}^{t} \langle \mathbb{P}f(s), v\rangle\, ds, \qquad \text{for} \quad t \geqslant t_0 \geqslant 0 \tag{2.10}$$

As we shall see later, when one takes the limit of a suitable subsequence of the Bubnov–Galerkin approximations, it happens that the first two inequalities hold for almost all t and almost all t_0 with $t > t_0$, while the equality (2.10) holds for all $t \geqslant t_0 \geqslant 0$.

2.3. Weak Solutions. Let $f \in L^\infty(0, \infty; L^2(\Omega))$ be given. We say that a function $\varphi \in L^2_{\text{loc}}[0, \infty; H)$ is a *weak solution of Leray–Hopf class* (*Class LH*), and write $\varphi \in W_{\text{LH}}(f)$, provided that the following four properties hold:

(1) $\varphi \in L^\infty(0, \infty; H) \cap L^2_{\text{loc}}[0, \infty; V)$;

(2) $D_t\varphi \in L^p_{\text{loc}}[0, \infty; V^{-1})$, for $p = \frac{4}{3}$;

(3) for almost all t and almost all t_0 with $t > t_0 > 0$, one has

$$\|\varphi(t)\|^2 \leqslant e^{-\nu\lambda_1(t-t_0)} \|\varphi(t_0)\|^2 + c_0^2 \|f\|_\infty^2 \tag{2.11}$$

and

$$\|\varphi(t)\|^2 + 2\nu \int_{t_0}^{t} \|A^{1/2}\varphi\|^2\, ds \leqslant \|\varphi(t_0)\|^2 + 2 \int_{t_0}^{t} \langle \mathbb{P}f, \varphi\rangle\, ds \tag{2.12}$$

(4) and for all $t \geqslant t_0 \geqslant 0$, one has

$$\langle \varphi(t) - \varphi(t_0), v\rangle + \nu \int_{t_0}^{t} \langle A^{1/2}\varphi, A^{1/2}v\rangle\, ds + \int_{t_0}^{t} b(\varphi, \varphi, v)\, ds = \int_{t_0}^{t} \langle \mathbb{P}f, v\rangle\, ds \tag{2.13}$$

for all $v \in V$.

Notice that since $\varphi \in L^\infty(0, \infty; H)$ one has $\varphi \in L^r_{\text{loc}}[0, \infty; H)$, for every r with $1 \leqslant r \leqslant \infty$. It is the Frechét space $L^2_{\text{loc}}[0, \infty; H)$ that is of special interest in this paper.

The relationship in Item (2) in the definition given above deserves some explanation. What it means is that for every weak solution φ there is a function $\psi \in L^p_{\text{loc}}[0, \infty; V^{-1})$, with $p = \frac{4}{3}$, that satisfies the equation

$$\varphi(t) - \varphi(t_0) = \int_{t_0}^{t} \psi(s)\, ds, \qquad \text{for} \quad 0 \leqslant t_0 \leqslant t < \infty \tag{2.14}$$

where the integral exists in the space V^{-1}. In this case one writes $\psi = D_t \varphi$ [see Constantin and Foias (1988) for more details].

Let $v \in V$ and let φ be a weak solution of Class LH. Because of (2.2), it follows from Property (1) that $b(\varphi, \varphi, v) \in L^1_{\text{loc}}[0, \infty; \mathbb{R})$. Similarly one has $\langle A^{1/2}\varphi, A^{1/2}v\rangle$ and $\langle \mathbb{P}f, v\rangle$ in $L^1_{\text{loc}}[0, \infty; \mathbb{R})$. As a result, it follows from (2.13) that for $t_0 > 0$ one has

$$\lim_{t \to t_0} \langle \varphi(t) - \varphi(t_0), v\rangle = 0, \qquad \text{for all} \quad v \in V$$

and for $t_0 = 0$ one obtains

$$\lim_{t \to 0^+} \langle \varphi(t) - \varphi(0), v\rangle = 0, \qquad \text{for all} \quad v \in V$$

Since V is dense in H, we see that if φ is a weak solution of Class LH, then one has

$$\varphi \in C[0, \infty; H_w) \tag{2.15}$$

Notice that inequality (2.11) is actually valid for *all* $t \geqslant t_0$. This follows from (2.15) and the lower semicontinuity relationship $\|\varphi(t)\|^2 \leqslant \liminf_{t_n \to t} \|\varphi(t_n)\|^2$. By using the Schwarz inequality and the Young inequality, we find that

$$\begin{aligned}\left|2\int_{t_0}^{t} \langle \mathbb{P}f, \varphi\rangle \, ds\right| &\leqslant \|A^{-1/2}\mathbb{P}f\|_\infty \int_{t_0}^{t} \|A^{1/2}\varphi\| \, ds \\ &\leqslant \lambda_1^{-1/2} \|f\|_\infty (t - t_0)^{1/2} \left(\int_{t_0}^{t} \|A^{1/2}\varphi\|^2 \, ds\right)^{1/2} \\ &\leqslant \nu \int_{t_0}^{t} \|A^{1/2}\varphi\|^2 \, ds + \frac{1}{\nu\lambda_1}(t - t_0) \|f\|_\infty^2\end{aligned}$$

By combining this with (2.12) we find that

$$\nu \int_{t_0}^{t} \|A^{1/2}\varphi\|^2 \, ds \leqslant \|\varphi(t_0)\|^2 + (t - t_0)\, c_0 \|f\|_\infty^2, \qquad \text{for almost all} \quad t_0 > 0 \tag{2.16}$$

We do not require inequality (2.11) to hold for $t_0 = 0$. However, since $\varphi \in L^\infty(0, \infty; H)$, it follows from (2.11) that one has

$$\|\varphi(t)\|^2 \leqslant e^{-\nu\lambda_1(t - t_0)} \|\varphi\|_\infty^2 + c_0^2 \|f\|_\infty^2, \qquad \text{for all} \quad t \geqslant t_0 \geqslant 0$$

Indeed, from (2.11) we see that the last inequality is valid on a dense set of $t_0 > 0$. By taking limits in t_0, we see that it holds for all $t_0 \geqslant 0$.

As explained in the next section, it is important for the theory of global attractors that the phase space for the semiflow be a complete metric space. Since the Frechét space $L^2_{\text{loc}} = L^2_{\text{loc}}[0, \infty; H)$ is complete, it would follow that W_{LH} is complete in terms of the invariant metric given by (2.5)–(2.6) on L^2_{loc}, provided that W_{LH} is a closed set in L^2_{loc}. Unfortunately, we are unable to prove that W_{LH} is closed. Instead, we imbed W_{LH} into a larger class W of generalized weak solutions and we will show that W is closed. These new solutions reside in the space $L^2_{\text{loc}}[0, \infty; H)$, but we relax the other conditions somewhat to allow for the possibility of a singularity at $t = 0$.

We say that a function $\varphi \in L^2_{\text{loc}}[0, \infty; H)$ is a *generalized weak solution*, and write $\varphi \in W(f)$, provided that one has

(1) $\varphi \in L^2_{\text{loc}}[0, \infty; H) \cap L^\infty_{\text{loc}}(0, 2; H) \cap L^\infty[1, \infty; H) \cap L^2_{\text{loc}}(0, \infty; V)$;

(2) $D_t \varphi \in L^p_{\text{loc}}(0, \infty; V^{-1})$, for $p = \frac{4}{3}$;

(3) for almost all t and almost all t_0 with $t > t_0$ inequalities (2.11) and (2.12) are valid; and

(4) for all $t \geqslant t_0 > 0$, Eq. (2.13) is valid, for all $v \in V$.

The condition that $\varphi \in L^\infty_{\text{loc}}(0, 2; H) \cap L^\infty[1, \infty; H)$ is one of many ways of saying that $\varphi \in L^\infty_{\text{loc}}(0, \infty; H)$ and $\varphi \in L^\infty[a, \infty; H)$, for every $a > 0$.

It follows from the definitions that one has $W_{\text{LH}}(f) \subset W(f)$, for every $f \in Y$. As argued above, one shows that if φ is a generalized weak solution, then (2.13) implies that

$$\varphi \in C(0, \infty; H_w) \tag{2.17}$$

Also, inequality (2.11) is valid for all $t \geqslant t_0$.

Next we define $\mathscr{W}$ to be the collection of all ordered pairs (φ, f) in Z such that $\varphi \in W(f)$, and $\mathscr{W}_{\text{LH}}$ is defined to be the collection of all such ordered pairs satisfying $\varphi \in W_{\text{LH}}(f)$. Let $\mathscr{W}(N_0) = \mathscr{W} \cap Z(N_0)$ and $\mathscr{W}_{\text{LH}}(N_0) = \mathscr{W}_{\text{LH}} \cap Z(N_0)$. We show below that $\mathscr{W}(N_0)$ is a closed set in $Z(N_0)$. This in turn implies that, for each $f \in Y$, the fiber $W(f)$ is a closed set in $L^2_{\text{loc}}[0, \infty; H)$.

The issue of the long-time dynamics is treated in the next two sections. Here we present some additional properties of the weak solutions in a series of lemmas. First, note that it follows from the definition that a set $B \subset W(f)$ is bounded if and only if

$$\sup \left\{ \int_n^{n+1} \|\varphi\|^2 \, ds \colon \varphi \in B \right\} < \infty, \qquad \text{for each} \quad n = 0, 1, 2, \ldots$$

Consequently, if B is a bounded set in $W(f)$, then

$$\sup\left\{\int_0^1 \|\varphi\|^2\, ds\colon \varphi\in B\right\}<\infty$$

Among other things, the first lemma establishes the converse of the last implication. We define the set

$$B(M_0, N_0) \overset{\text{def}}{=} \left\{(\varphi, f)\in \mathscr{W}\colon \int_0^1 \|\varphi\|^2\, ds \leqslant M_0^2 \text{ and } \|f\|_\infty^2 \leqslant N_0^2\right\} \tag{2.18}$$

for nonnegative numbers M_0 and N_0.

Lemma 1. *The following statements are valid:*

(1) For each $(\varphi, f)\in B(M_0, N_0)$ one has

$$\|\varphi(\tau)\|^2 \leqslant \tau^{-1}M_0^2 + c_0^2 N_0^2, \qquad 0<\tau\leqslant 1 \tag{2.19}$$

and

$$\|\varphi(\tau)\|^2 \leqslant e^{-\nu\lambda_1(\tau-1)}(M_0^2 + c_0^2N_0^2) + c_0^2N_0^2, \qquad \tau\geqslant 1 \tag{2.20}$$

(2) For each $(\varphi, f)\in B(M_0, N_0)$ one has

$$\int_T^{T+1} \|\varphi\|^2\, ds \leqslant e^{-\nu\lambda_1(T-1)}(M_0^2 + c_0^2N_0^2) + c_0^2N_0^2, \qquad T\geqslant 1 \tag{2.21}$$

(3) The set $B(M_0, N_0)$ is bounded in $\mathscr{W}$.

(4) For each τ and T with $0<\tau\leqslant T<\infty$, there exist positive constants $M_i = M_i(\tau, T)$, for $i=1, 2$, such that the following two inequalities are valid:

$$\sup\left\{\int_\tau^T \|A^{1/2}\varphi\|^2\, ds\colon (\varphi, f)\in B(M_0, N_0)\right\} \leqslant M_1(\tau, T)^2 \tag{2.22}$$

and

$$\sup\left\{\int_\tau^T \|A^{-1/2}D_t\varphi\|^p\, ds\colon (\varphi, f)\in B(M_0, N_0)\right\} \leqslant M_2^p, \qquad \textit{for}\quad p=\tfrac{4}{3} \tag{2.23}$$

(5) If in addition, one has $\varphi\in L^2_{\text{loc}}[0, \infty; V)\cap L^\infty(0, \infty; H)$, then the constants M_1 and M_2 in Property (4) can be chosen to be independent of τ, for $0<\tau\leqslant 1$, and the limits $M_i(0, T) = \lim_{\tau\to 0_+} M_i(\tau, T)$ exist, for $i=1, 2$.

Proof. Let $(\varphi, f) \in B(M_0, N_0)$. For $0 < \tau \leqslant 1$, one has $\int_0^\tau \|\varphi\|^2 \, ds \leqslant \int_0^1 \|\varphi\|^2 \, ds \leqslant M_0^2$. Consequently the set

$$\{t \in (0, \tau) \colon \|\varphi(t)\|^2 \leqslant \tau^{-1} M_0^2\}$$

must have positive measure. By using inequality (2.11) one then finds that there is a time t_φ, with $0 < t_\varphi < \tau$ such that $\|\varphi(t_\varphi)\|^2 \leqslant \tau^{-1} M_0^2$ and

$$\|\varphi(t)\|^2 \leqslant e^{-\nu\lambda_1(t - t_\varphi)} \|\varphi(t_\varphi)\|^2 + c_0^2 N_0^2, \qquad \text{for} \quad t \geqslant t_\varphi \tag{2.24}$$

As a result we obtain (2.19) and (2.20). Also, inequality (2.21) is a direct consequence of (2.20). The fact that $B(M_0, N_0)$ is a bounded set in $\mathscr{W}$ now follows from (2.21).

In order to derive (2.22) and (2.23), there is no loss in generality in restricting to the case, where $0 < \tau \leqslant 1 \leqslant T < \infty$. For inequality (2.22) with $t = T$, we let $t_0 = t_\varphi$ be chosen so that one has $0 < t_0 < \tau$ and $\|\varphi(t_0)\|^2 \leqslant \tau^{-1} M_0^2$. Then (2.16) implies that

$$\nu \int_\tau^T \|A^{1/2}\varphi\|^2 \, ds \leqslant \tau^{-1} M_0^2 + (T - \tau)\, c_0 N_0^2 \tag{2.25}$$

which implies (2.22).

In order to prove (2.23), we argue that the equality

$$D_t \varphi = -\nu A\varphi - B(\varphi, \varphi) + \mathbb{P}f$$

is valid in the space $L^p_{\text{loc}}(0, \infty; V^{-1})$, for $p = \frac{4}{3}$. The bound in (2.23) follows from the Minkowski inequality, which implies that

$$\left(\int_\tau^T \|A^{-1/2} D_t \varphi\|^p \, ds\right)^{1/p}$$

is bounded by

$$\nu\left(\int_\tau^T \|A^{1/2}\varphi\|^p \, ds\right)^{1/p} + \left(\int_\tau^T \|A^{-1/2} B(\varphi, \varphi)\|^p \, ds\right)^{1/p} + (T - \tau)^{1/p} N_0 \tag{2.26}$$

Now the Hölder inequality implies that for $p = \frac{4}{3}$ one has

$$\int_\tau^T \|A^{1/2}\varphi\|^p \, ds \leqslant (T - \tau)^{1/3} \left(\int_\tau^T \|A^{1/2}\varphi\|^2 \, ds\right)^{2/3} \tag{2.27}$$

Next we note that (2.3) implies that

$$\int_\tau^T \|A^{-1/2} B(\varphi, \varphi)\|^p \, ds \leqslant C_1^p \operatorname*{ess\,sup}_{\tau \leqslant s \leqslant T} \|\varphi(s)\|^{2/3} \int_\tau^T \|A^{1/2}\varphi\|^2 \, ds \tag{2.28}$$

By combining the last inequality with (2.24), (2.25), (2.26), and (2.27), we obtain (2.23).

Property (5) follows from the argument of the last paragraph along with the two observations: (i) that inequality (2.27) remains valid with $\tau = 0$ and (ii) that one can replace the term ess $\sup_{\tau \leqslant s \leqslant T} \|\varphi(s)\|^{2/3}$ in (2.28) with $\|\varphi\|_{\infty}^{2/3}$. □

As a corollary to the last lemma, we show that if B is a bounded set in $\mathscr{W}$, then B is bounded in other spaces as well.

Lemma 2. *Let B be a bounded set in $\mathscr{W}(N_0)$, for some $N_0 \geqslant 0$. Then the following hold:*

(1) B is a bounded set in $L^2_{\text{loc}}(0, \infty; V) \times Y(N_0)$.

(2) The set $\{(D_t\varphi, f): (\varphi, f) \in B\}$ is bounded in $L^p_{\text{loc}}(0, \infty; V^{-1}) \times Y(N_0)$, for $p = \frac{4}{3}$.

Proof. Since B is bounded, it is contained in $B(M_0, N_0)$ for some nonnegative number M_0; see (2.18). Lemma 2 then follows from Lemma 1 and the definition of boundedness. □

The next lemma gives a sufficient condition for a generalized weak solution to be a solution of Class LH.

Lemma 3. *Let $(\varphi, f) \in \mathscr{W}$ and assume that $\varphi \in L^2_{\text{loc}}[0, \infty; V) \cap L^\infty(0, \infty; H)$. Then one has $(\varphi, f) \in \mathscr{W}_{\text{LH}}$.*

Proof. In reference to the definitions, we need only to verify that if (φ, f) satisfies the hypotheses of this lemma, then one has

(1) Eq. (2.13) is valid at $t_0 = 0$, and

(2) $D_t\varphi \in L^p_{\text{loc}}[0, \infty; V^{-1})$, for $p = \frac{4}{3}$.

Since one has $\varphi \in L^2_{\text{loc}}[0, \infty; V) \cap L^\infty(0, \infty; H)$, it follows from (2.2) that $b(\varphi, \varphi, v)$ and $\langle A^{1/2}\varphi, A^{1/2}v \rangle$ lie in $L^1_{\text{loc}}[0, \infty; \mathbb{R})$, for every $v \in V$, which establishes Item (1). Item (2) follows from Lemma 1, Property (5). □

The next result, which is a compactness lemma, is a key step in our theory. Since one has $\|u\|^2 \leqslant \lambda_1^{-2} \|A^{1/2}u\|^2$, for all $u \in V$, we see that if $\varphi \in L^2_{\text{loc}}(0, \infty; V)$, then $\varphi \in L^2_{\text{loc}}(0, \infty; H)$. Since the forcing function f plays no direct role in this lemma, we make no assumptions on f.

Lemma 4 *(Compactness Lemma). Let φ^n be a sequence in $L^2_{\text{loc}}(0, \infty; V)$, with the properties that (1) φ^n is bounded in $L^2_{\text{loc}}(0, \infty; V)$,* *and (2) the sequence $D_t\varphi^n$ is bounded in $L^p_{\text{loc}}(0, \infty; V^{-1})$, for some p with $1 < p \leqslant \infty$. Then there exists a subsequence of φ^n, which we relabel as φ^n, and a function $\varphi \in L^2_{\text{loc}}(0, \infty; V)$, $D_t\varphi \in L^p_{\text{loc}}(0, \infty; V^{-1})$, and the following properties hold.*

(1) One has $\varphi^n \rightharpoonup \varphi$ weakly in $L^2_{\text{loc}}(0, \infty; V)$.

(2) One has $D_t\varphi^n \rightharpoonup D_t\varphi$ weakly in $L^p_{\text{loc}}(0, \infty; V^{-1})$.

(3) One has $\varphi^n \to \varphi$ strongly in $L^2_{\text{loc}}(0, \infty; H)$ and in $L^2_{\text{loc}}(0, \infty; V^{-1})$.

(4) For each $t \in (0, \infty)$ one has $\varphi^n(t) \to \varphi(t)$ strongly in V^{-1}.

(5) There is a set E in $(0, \infty)$ of Lebesgue measure zero with the property that $\varphi^n(t) \to \varphi(t)$ strongly in H, for each $t \in (0, \infty)\backslash E$.

Proof. For all practical purposes, this compactness lemma is known (see, e.g., Constantin and Foias, 1988, pp. 66–71; Lions, 1969). However, since this lemma is not usually formulated in the manner given above, we give an outline of the proof here.

Let τ and T be given, where $0 < \tau < T < \infty$. Then it follows from the hypotheses that the sequence φ^n is bounded in $L^2(\tau, T; V)$ and the sequence $D_t\varphi^n$ is bounded in $L^p(\tau, T; V^{-1})$. As a result, the seven conclusions stated above follow from the argument of Constantin and Foias (1988) or Lions (1969) subject to the following changes.

(1) The spaces $L^r_{\text{loc}}(0, \infty; X)$ are replaced by $L^r(\tau, T; X)$, for the appropriate choices of r and X.

(2) The convergence statement in Property (5) is restricted to $t \in (\tau, T)$, and E is a set of measure zero in (τ, T).

Next for $m = 1, 2, 3,...$, we construct subsequences φ^n_m and functions φ_m with the following properties.

(1) One has $\varphi_1 = 0$ and $\varphi^n_1 = \varphi^n$, for all n, where φ^n is the sequence described in the hypotheses of this lemma.

(2) For $m = 1, 2, 3,...$, each sequence φ^n_{m+1} is a subsequence of φ^n_m and is chosen so that this subsequence and the function φ_{m+1} satisfy the conclusions of the last paragraph on the interval $I_m = ((m+1)^{-1}, m+1)$.

(3) One has $\varphi^n_{m+1} = \varphi^n_m$, for $n = 1, ..., m$ and $m = 1, 2, 3,....$

Finally, the diagonal subsequence φ^n_n and the function φ defined by $\varphi(t) = \lim_{m \to \infty} \varphi_m(t)$, for $t \in I_m$, satisfy the conclusions of this lemma. □

Lemma 5. *Let (φ^n, f^n) be a convergent sequence in Z with limit (φ_0, f_0) in Z. Assume that $\varphi^n \in W(f^n)$ and that $f^n \in Y(N_0)$, for some $N_0 \geqslant 0$*

and for all n. Then there is a function $\varphi \in W(f_0)$ *such that* $\varphi \overset{a.e.}{=} \varphi_0$. *In particular, for each* $N_0 \geqslant 0$, *the space* $\mathscr{W}(N_0)$ *generated by the weak solutions of the Navier–Stokes equations is a closed set in* Z.

Proof. Let (φ^n, f^n) and (φ_0, f_0) be given as in the hypotheses, where $\varphi^n \in W(f^n)$ and $\|f^n\|_\infty \leqslant N_0$, for all n. Note that, since $\varphi^n \to \varphi_0$ strongly in $L^2_{\text{loc}}[0, \infty; H)$, the function φ_0 satisfies

$$\varphi_0 \overset{\text{a.e.}}{=} \lim_{n \to \infty} \varphi^n, \qquad \text{on } (0, \infty) \tag{2.29}$$

Since (φ^n, f^n) is a convergent sequence, it is bounded in Z. Therefore, there exists a nonnegative number M_0 such that $(\varphi^n, f^n) \in B(M_0, N_0)$, for all n, see (2.18). It then follows from Lemma 2 that the hypotheses of the compactness lemma 4 are satisfied. After a relabeling, we let φ_n denote the subsequence and we let $\varphi \in L^2_{\text{loc}}(0, \infty; V)$ denote the limiting function given by Lemma 4. It then follows from Property (5) of Lemma 4 and (2.29) that $\varphi \overset{\text{a.e.}}{=} \varphi_0$. We now show that $\varphi \in W(f_0)$.

Since one has $\varphi_0 \in L^2_{\text{loc}}[0, \infty; H)$, it follows that $\varphi \in L^2_{\text{loc}}[0, \infty; H)$. By using Property (5) of Lemma 4 and the inequalities (2.19) and (2.20) [as applied to the sequence (φ^n, f^n)], we see that $\varphi \in L^\infty_{\text{loc}}(0, 2; H) \cap L^\infty[1, \infty; H)$. Also, one has $D_t \varphi \in L^p_{\text{loc}}(0, \infty; V^{-1})$ by Property 2 of Lemma 4. Thus the first two conditions in the definition of a generalized weak solution are satisfied. It remains to verify (2.11), (2.12), and (2.13).

Because of Properties (1) and (5) of Lemma 4, we claim that Eq. (2.13), which is valid for each φ^n, is valid (almost everywhere) in the limit as $n \to \infty$. Indeed, one has

$$\begin{aligned}&\int_{t_0}^{t} \langle (A^{1/2}\varphi^n - A^{1/2}\varphi), A^{1/2}v\rangle \, ds \\ &\quad = \left\langle \int_{t_0}^{t} (A^{1/2}\varphi^n - A^{1/2}\varphi)\, ds, A^{1/2}v \right\rangle \to 0, \qquad \text{as } n \to \infty\end{aligned}$$

by Property (1) of Lemma 4. In order to show that

$$\int_{t_0}^{t} (b(\varphi^n, \varphi^n, v) - b(\varphi, \varphi, v))\, ds \to 0, \qquad \text{as } n \to \infty \tag{2.30}$$

we note that the trilinearity of b implies that

$$b(\varphi^n, \varphi^n, v) - b(\varphi, \varphi, v) = b(\varphi^n - \varphi, \varphi^n, v) + b(\varphi, \varphi^n - \varphi, v)$$

Since the solutions φ^n satisfy (2.19) and (2.20), it follows from Property (5) of Lemma 4 that φ satisfies (2.19) and (2.20) almost everywhere. By using this fact, inequality (2.2), and Property (3) of Lemma 4, we see that both

$$\int_{t_0}^{t} b(\varphi^n - \varphi, \varphi^n, v)\, ds \qquad \text{and} \qquad \int_{t_0}^{t} b(\varphi, \varphi^n - \varphi, v)\, ds$$

converge to 0, as $n \to \infty$, which implies (2.30). In order to show that

$$\int_{t_0}^{t} (\langle \mathbb{P} f^n, \varphi^n \rangle - \langle \mathbb{P} f_0, \varphi \rangle)\, ds \to 0, \qquad \text{as} \quad n \to \infty \tag{2.31}$$

we note that

$$\langle \mathbb{P} f^n, \varphi^n \rangle - \langle \mathbb{P} f_0, \varphi \rangle = \langle \mathbb{P}(f^n - f_0), \varphi^n \rangle + \langle \mathbb{P} f_0, \varphi^n - \varphi \rangle$$

As a result, (2.31) follows from the convergence of the sequence f^n, the Schwarz inequality and Property (3) of Lemma 4. By restricting t and t_0 to be in the set $(0, \infty)\backslash E$ and using Property (5) of Lemma 4, we see that (2.13) holds almost everywhere in the limit as $n \to \infty$. By changing $\varphi(t)$ and $\varphi(t_0)$ on a set of measure zero, if necessary, we see that (2.13) is valid for all $t \geqslant t_0 > 0$.

Inequality (2.11) follows from Property (5) of Lemma 4. Also, inequality (2.12) is valid for the sequence φ^n. As $n \to \infty$, the right side of (2.12) has the limit

$$\|\varphi(t_0)\|^2 + 2 \int_{t_0}^{t} \langle \mathbb{P} f_0, \varphi \rangle\, ds, \qquad \text{for} \quad t_0 \in (0, \infty)\backslash E$$

From Property (1) of Lemma 4 and the lower semicontinuity property for weak convergence, one obtains

$$\int_{t_0}^{t} \|A^{1/2}\varphi\|^2\, ds \leqslant \liminf_{n \to \infty} \int_{t_0}^{t} \|A^{1/2}\varphi^n\|^2\, ds$$

Therefore, for $t, t_0 \in (0, \infty)\backslash E$, we see that inequality (2.12) is valid for φ. Consequently, we have $\varphi \in W(f_0)$. □

3. THE SEMIFLOW ON W

In this section we describe the global attractor for the 3D Navier–Stokes equations in the case where the forcing function f is time-independent and $f \in L^2(\Omega)$. The time-dependent case is given in Section 4. Before we describe the dynamical properties of this mapping, it is helpful to review the basic theory of semiflows and global attractors (see Conley, 1978; Hale, 1988; Sell and You, 1994).

3.1. Semiflows and Global Attractors. Let W be a metric space. A *semiflow* on W is defined to be a mapping $\sigma(t, w) = S(t)\, w$, where $\sigma\colon [0, \infty) \times W \to W$ satisfies the following three properties:

(1) $S(0)\, w = w$, for all $w \in W$.

(2) The restricted mapping $\sigma\colon (0, \infty) \times W \to W$ is continuous.

(3) The semigroup property

$$S(s)\, S(t)\, w = S(s+t)\, w, \qquad \text{for} \quad w \in W, \quad s, t \in [0, \infty) \tag{3.1}$$

is valid.

In order to describe the general theory of global attractors, as it will apply in our case, we assume that W is a closed subset of a Frechét space. As a result, W is a complete metric space.

One says that the semiflow $S(t)$ is *compact for* $t > 0$, provided that for every bounded set $B \subset W$ and every $t > 0$, the set $S(t)\, B$ lies in a compact set in W. Also, $S(t)$ is said to be *point dissipative* if there is a bounded set U in W with the property that for every $w \in W$, there is a time $T = T(w)$ such that $S(t)\, w \in U$, for all $t > T$. In this case, the set U is referred to as an *absorbing set* for the semiflow $S(t)$. The proof of the following result on global attractors is given by Billotti and LaSalle (1971). [Also, see Hale (1988), Sell and You (1994), and Temam (1988).]

Theorem A. *Let $S(t)$ be a point dissipative, compact semiflow on a complete metric space. Then $S(t)$ has a global attractor $\mathbb{A}$ in W. Furthermore, $\mathbb{A}$ attracts all bounded sets in W.*

It turns out that under the assumptions of Theorem A, the global attractor is the omega-limit set of the absorbing set U. As shown by Sell and You (1994), the global attractor $\mathbb{A}$ has additional properties, including the following:

(1) $\mathbb{A}$ is maximal in the sense that every compact invariant set in W lies in $\mathbb{A}$.

(2) $\mathbb{A}$ is minimal in the sense that if B is any closed set in W that attracts each compact set in W, then one has $\mathbb{A} \subset B$.

(3) For each bounded set B in W, the omega-limit set $\omega(B)$ satisfies $\omega(B) \subset \mathbb{A}$.

(4) $\mathbb{A}$ is a connected set.

(5) $\mathbb{A}$ is Lyapunov stable, i.e., for every neighborhood V of $\mathbb{A}$ and every $\tau > 0$, there is a neighborhood U of $\mathbb{A}$ with the property that $S(t)\, U \subset V$, for all $t \geqslant \tau$.

3.2. Navier–Stokes Dynamics. Let p satisfy $1 \leqslant p \leqslant \infty$. Then for each $\phi \in L^p_{\mathrm{loc}}(0, \infty; X)$, the time translate ϕ_τ satisfies $\phi_\tau \in L^p_{\mathrm{loc}}(0, \infty; X)$, for all $\tau \geqslant 0$. As an application of Lemma 3, we now prove the following.

Lemma 6. *Let $f \in Y$ and $\varphi \in W(f)$. Then for every $\tau > 0$, the time translate φ_τ is in the space $W_{\mathrm{LH}}(f_\tau)$.*

Proof. Let $\varphi \in W(f)$, and fix $\tau > 0$. Note that the time translate φ_τ is a solution of (1.1), where f is replaced by f_τ, i.e., one has $\varphi_\tau \in W(f_\tau)$. It then follows from (2.19) and (2.20) that the time translate φ_τ is in $L^\infty(0, \infty; H)$. Also, (2.22) implies that $\varphi_\tau \in L^2_{\mathrm{loc}}[0, \infty; V)$. The result now follows from Lemma 3. □

For the remainder of this section we assume that $f \in L^2(\Omega)$. As a result, f does not depend on time, and one has $W =^{\mathrm{def}} W(f) = W(f_\tau)$ and $W_{\mathrm{LH}} =^{\mathrm{def}} W_{\mathrm{LH}}(f) = W_{\mathrm{LH}}(f_\tau)$, for all $\tau \geqslant 0$. We let $S(t)$ be the mapping given by

$$S\colon (\tau, \varphi) \to S(\tau)\, \varphi = \varphi_\tau \tag{3.2}$$

As shown in Lemma 5, W is a closed set in $L^2_{\mathrm{loc}}[0, \infty; H)$. Hence W is a complete metric space in the metric given by (2.6). In order to apply Theorem A to this situation, we need to verify the following properties.

(1) The mapping $S(t)$ given by (3.2) is a semiflow on $L^2_{\mathrm{loc}} = L^2_{\mathrm{loc}}[0, \infty; L^2(\Omega))$ and that $L^2_{\mathrm{loc}}[0, \infty; H)$ and W are positively invariant subsets of L^2_{loc}.

(2) The restriction of the semiflow $S(t)$ to W is compact for $t > 0$.

(3) The restriction of the semiflow $S(t)$ to W is point dissipative.

In addition to these properties, which will then imply that there is a global attractor $\mathbb{A}$ in W, we will prove the following properties.

(4) The global attractor $\mathbb{A}$ lies in W_{LH}.

(5) For every bounded set B in W and for every $t > 0$, the compact set $\mathrm{Cl}_W\, S(t)\, B$ lies in W_{LH}.

Lemma 7. *Let $f \in L^2(\Omega)$ and set $W = W(f)$ and $W_{\mathrm{LH}} = W_{\mathrm{LH}}(f)$. Then the mapping $S(t)$ given by (3.2) is a semiflow on $L^2_{\mathrm{loc}} = L^2_{\mathrm{loc}}[0, \infty; L^2(\Omega))$. Furthermore, the sets $L^2_{\mathrm{loc}}[0, \infty; H)$, W, and W_{LH} are positively invariant sets in L^2_{loc}.*

Proof. In order to show that (3.2) defines a semiflow on L^2_{loc}, the main issue then is to verify the continuity of the mapping $(\tau, \varphi) \to S(\tau)\, \varphi = \varphi_\tau$, for $(\tau, \varphi) \in (0, \infty) \times L^2_{\mathrm{loc}}$. Let τ_n and φ^n be convergent sequences where $\varphi^n \to \varphi$ in L^2_{loc} and $\tau_n \to \tau$ in $(0, \infty)$. Since φ^n is convergent in L^2_{loc}, it is bounded. Since $\tau > 0$, there is no loss in generality in assuming that $0 < \frac{1}{2}\tau \leqslant \tau_n \leqslant 2\tau$. Let I_n denote the closed interval between τ_n and τ. Thus the length of I_n is $|\tau_n - \tau|$.

We now show that

$$d(\varphi^n_{\tau_n} - \varphi_{\tau_n}) \to 0, \qquad \text{as} \quad n \to \infty \tag{3.3}$$

where d is the invariant metric given by (2.6). Let a and b be given where $0 \leqslant a < b < \infty$. It will suffice to show that

$$\int_a^b \|\varphi^n_{\tau_n} - \varphi_{\tau_n}\|^2 \, ds \to 0, \qquad \text{as} \quad n \to \infty$$

Now for $\frac{1}{2}\tau \leqslant \sigma \leqslant \tau$ one has

$$\int_a^b \|\varphi^n_\sigma - \varphi_\sigma\|^2 \, ds = \int_{a+\sigma}^{b+\sigma} \|\varphi^n - \varphi\|^2 \, ds \leqslant \int_{a+(1/2)\tau}^{b+2\tau} \|\varphi^n - \varphi\|^2 \, ds$$

Since $\varphi^n \to \varphi$ in L^2_{loc}, it follows that $\varphi^n_\sigma \to \varphi_\sigma$ in L^2_{loc}, uniformly for $\frac{1}{2}\tau \leqslant \sigma \leqslant 2\tau$. This implies (3.3).

Next we show that

$$d(\varphi_{\tau_n} - \varphi_\tau) \to 0, \qquad \text{as} \quad n \to \infty \tag{3.4}$$

Let $\varepsilon > 0$ be given. Since the C^1 functions of time are dense in $L^2(a + \tau/2, b + 2\tau; H)$, there is a function

$$\psi \in L^2(a + \tau/2, b + 2\tau; H) \cap C^1([a + \tau/2, b + 2\tau]; H)$$

with the property that

$$\int_a^b \|\varphi_\sigma - \psi_\sigma\|^2 \leqslant \varepsilon, \qquad \text{for all} \quad \sigma \in [\tau/2, 2\tau] \tag{3.5}$$

Since ψ is a C^1 function, then for $a \leqslant t \leqslant b$ one has

$$\|\psi(\tau_n + t) - \psi(\tau + t)\| \leqslant \int_{I_n} \|D_t \psi(s + t)\| \, ds \leqslant K\,|\tau_n - \tau|$$

where K is chosen to satisfy $\|D_t \psi(s)\| \leqslant K$, for $a + \frac{1}{2}\tau \leqslant s \leqslant b + 2\tau$. It follows from the last inequality that

$$\int_a^b \|\psi_{\tau_n} - \psi_\tau\|^2 \, ds \leqslant K^2(b - a)\,|\tau_n - \tau|^2$$

Therefore, there is a N such that one has

$$\int_a^b \|\psi_{\tau_n} - \psi_\tau\|^2 \, ds \leqslant \varepsilon, \qquad \text{for all} \quad n \geqslant N \tag{3.6}$$

It then follows from (3.5) and (2.65) that

$$\int_a^b \|\varphi_{\tau_n} - \varphi_\tau\|^2 \, ds \leqslant 9\varepsilon, \qquad \text{for all} \quad n \geqslant N$$

Since ε, a, and b are arbitrary, this implies (3.4). It then follows from (3.3) and (3.4) that

$$\varphi^n_{\tau_n} \to \varphi_\tau \text{ in } L^2_{\text{loc}}, \qquad \text{as} \quad n \to \infty$$

which completes the proof of continuity. Finally, the positive invariance of the space $L^2_{\text{loc}}[0, \infty; H)$ follows from the continuous imbedding $H \mapsto L^2(\Omega)$, and the positive invariance of W and W_{LH} follows from Lemma 6. □

Lemma 8. Let $f \in L^2(\Omega)$ *and set* $W = W(f)$ *and* $W_{\text{LH}} = W_{\text{LH}}(f)$. *Then the restriction of the semiflow* $S(t)$ *to* W *is compact for* $t > 0$, *i.e. for each bounded set* B *in* W *and for each* $\tau > 0$, *the set* $S(\tau) B$ *lies in a compact subset in* W. *Moreover, one has* $\text{Cl}_W \, S(\tau) \, B \subset W_{\text{LH}}$.

Proof. If $S(\tau) B$ lies in a compact set in W for some $\tau > 0$, then by the semigroup property (3.1) $S(\tau + t) B$ lies in a compact set in W, for each $t > 0$. Let τ be fixed where $0 < \tau \leqslant 1$. Since W is a metric space, it suffices to verify that $S(\tau) B$ is sequentially compact in W. Let φ^n be a bounded sequence in W. Then one has $(\varphi^n, f) \in B(M_0, \|f\|_\infty)$, for all n, and for some $M_0 > 0$. As a result, it follows from (2.19) and (2.20) that

$$\|S(\tau) \varphi^n\|^2_\infty \leqslant \tau^{-1} M_0^2 + 2c_0^2 \|f\|^2_\infty, \qquad \text{for all } n \tag{3.7}$$

From (2.22) one finds that

$$\int_m^{m+1} \|A^{1/2} S(\tau) \varphi^n\|^2 \, ds \leqslant M_1(\tau + m, \tau + m + 1)^2, \qquad \text{for all } n \tag{3.8}$$

for $m = 0, 1, 2, \ldots$. From Lemma 2 we see that the hypotheses of the Compactness Lemma 4 are satisfied. After a relabeling, we let φ^n and $\varphi \in L^2_{\text{loc}}(0, \infty; V)$ denote the subsequence and the limit function given by the conclusions of Lemma 4. It then follows from Property (5) of Lemma 4 and (3.7) that

$$\|S(\tau)\,\varphi\|_\infty^2 \leqslant \tau^{-1}M_0^2 + 2c_0^2\,\|f\|_\infty^2 \tag{3.9}$$

Likewise, from Property (1) of Lemma 4 and (3.8) one obtains

$$\int_m^{m+1} \|A^{1/2}S(\tau)\,\varphi\|^2\,ds \leqslant \liminf_{n\to\infty} \int_m^{m+1} \|A^{1/2}S(\tau)\,\varphi^n\|^2\,ds$$
$$\leqslant M_1(\tau+m,\tau+m+1)^2 \tag{3.10}$$

for $m=0, 1, 2,\ldots$. Now (3.9) and (3.10) imply that

$$S(\tau)\,\varphi \in L^\infty(0,\infty; H) \cap L^2_{\text{loc}}[0,\infty; V) \tag{3.11}$$

and from Lemma 6 we conclude that $S(\tau)\,\varphi \in W_{\text{LH}} \subset W$. Hence $S(\tau)\,B$ lies in a compact set in W.

In order to show that the set $\text{Cl}_W\,S(\tau)\,B$ lies in W_{LH}, we let $\varphi_0 \in \text{Cl}_W\,S(\tau)\,B$ be given. Then there is a sequence $\varphi^n \in B$ with the property that $S(\tau)\,\varphi^n \to \varphi_0$ in W. Now each of the functions φ^n satisfies (3.9) and (3.10). By using the argument in the last paragraph, we see that inequalities (3.9) and (3.10) remain valid when one replaces $S(\tau)\,\varphi$ with φ_0. As a result, one has $\varphi_0 \in L^\infty(0,\infty; H) \cap L^2_{\text{loc}}[0,\infty; V)$. It then follows from Lemma 3 that $\varphi_0 \in W_{\text{LH}}$. □

Lemma 9. *Let $f \in L^2(\Omega)$ and set $W = W(f)$ and $W_{\text{LH}} = W_{\text{LH}}(f)$. Then the restriction of the semiflow $S(t)$ to W is point dissipative.*

Proof. In order to verify the point dissipative property, we define U to be the set of all $\varphi \in W$ such that $\int_m^{m+1} \|\varphi\|^2\,ds \leqslant 2c_0^2\,\|f\|^2$, for all $m=0, 1, 2,\ldots$. It follows from (2.7) that U is a bounded set in W. Let $(\varphi, f) \in B(M_0, \|f\|_\infty)$. From inequality (2.20) one finds that

$$\int_m^{m+1} \|S(\tau)\,\varphi\|^2\,ds \leqslant e^{-\nu\lambda_1\tau}K^2 + c_0^2\,\|f\|^2$$

$$\text{for all } \tau \geqslant 1 \text{ and all } m=0, 1, 2,\ldots$$

where $K^2 = e^{\nu\lambda_1}(M_0^2 + c_0^2\,\|f\|^2)$. Let $\tau_0 \geqslant 0$ satisfy $e^{-\nu\lambda_1\tau_0}K^2 \leqslant c_0^2\,\|f\|^2$. One then has

$$\int_m^{m+1} \|S(\tau)\,\varphi\|^2\,ds \leqslant 2c_0^2\,\|f\|_\infty^2, \qquad \text{for all } \tau \geqslant \tau_0 \text{ and all } m=0, 1, 2,\ldots$$

This implies that $S(\tau)\varphi \in U$, for all $\tau \geqslant \tau_0$, i.e., the semiflow $S(\tau)$ is point dissipative. □

Theorem 1 (Main Theorem). Let $f \in L^2(\Omega)$ and set $W = W(f)$ and $W_{\mathrm{LH}} = W_{\mathrm{LH}}(f)$. Then there exists a global attractor $\mathbb{A}$ for the weak solutions of the Navier–Stokes equations on Ω, and one has $\mathbb{A} \subset W_{\mathrm{LH}}$. Furthermore, $\mathbb{A}$ attracts all bounded sets in W.

Proof. The existence of the global attractor $\mathbb{A}$ in W and the fact that it attracts all bounded sets in W now follow from Theorem A and Lemmas 5–9. Since $\mathbb{A}$ is invariant, it follows from Lemma 6 that $\mathbb{A} \subset W_{\mathrm{LH}}$. □

As is usual, the global attractor $\mathbb{A}$ consists entirely of solutions of the Navier–Stokes equations which are defined for all $t \in \mathbb{R}$. It follows from the arguments in Lemma 1 that if $\varphi \in \mathbb{A}$, then there are constants c_1 and c_2 such that one has

(1) $\|\varphi(t)\|^2 \leqslant c_0^2 \|f\|_\infty^2$, for all $t \in \mathbb{R}$;
(2) $\int_t^{t+1} \|A^{1/2}\varphi\|^2\, ds \leqslant \nu^{-1} c_0 \|f\|_\infty$, for all $t \in \mathbb{R}$; and
(3) $\int_t^{t+1} \|A^{-1/2} D_t \varphi\|^p\, ds \leqslant (c_1 \|f\|_\infty + c_2 \|f\|_\infty^{5/2})^p$, for all $t \in \mathbb{R}$, where $p = \frac{4}{3}$.

Let us now compare our results with a related theory developed by Foias and Temam (1987). They show that there is a *universal attracting set* in H, that is, there is a set Γ in H with the following properties.

(1) The set Γ is defined as the collection of all $u_0 \in H$ for which there exists a globally defined weak solution $\varphi \in L^\infty(-\infty, \infty; H)$ with $\varphi(0) = u_0$. It is shown that Γ is nonempty and bounded in H.
(2) Every weak solution $\varphi \in W_{\mathrm{LH}}$ satisfies $\varphi(t) \rightharpoonup \Gamma$ in H_{w}, as $t \to \infty$.
(3) The set Γ is compact in H_{w}.
(4) One has $\Gamma \subset V$ if and only if Γ is a bounded set in V.
(5) The set $\Gamma \cap V$ is weakly dense in Γ, i.e., it is dense in the weak topology H_{w}.
(6) The set Γ contains a set Γ_{reg}, where Γ_{reg} is weakly open and weakly dense in Γ, and for every $u_0 \in \Gamma_{\mathrm{reg}}$ there is an $a > 0$ such that, for any weak solution φ with $\varphi(t) \in \Gamma$, for all $t \in \mathbb{R}$ and $\varphi(0) = u_0$, the restriction $\varphi|_{(-a,a)}$ is uniquely determined and $\varphi(t) \in \Gamma \cap V$, for all $t \in (-a, a)$.

In our notation, we note that if $\varphi \in \mathbb{A}$, then one has $\varphi \in C(-\infty, \infty; H_{\mathrm{w}})$. Furthermore, the set $\{\varphi \in \mathbb{A}\}$ is compact in $C(-\infty, \infty; H_{\mathrm{w}})$ in the topology of uniform convergence on compact sets (see Constantin *et al.*, 1985, Chap. 1). As a result, the evaluation mapping of $\mathbb{A}$ into H_w given by $\varphi \to \varphi(0)$ is continuous, and the set $\Gamma_0 = \{\varphi(0): \varphi \in \mathbb{A}\}$ satisfies the six properties listed in the last paragraph.

We now have the global attractor $\mathbb{A}$ with two topologies. First, one has

$$\mathbb{A} \subset W \subset L^2_{\text{loc}} = L^2_{\text{loc}}[0, \infty; H)$$

with the L^2_{loc}-topology, and second, one has

$$\mathbb{A} \subset C(-\infty, \infty; H_w)$$

with the metrizable topology of uniform convergence on compact sets in $(-\infty, \infty)$. It follows from the compactness property of the last paragraph and our Lemma 8 that the two topologies agree on the space $\mathbb{A}$. By using the fact that Γ_0 is the continuous image of the global attractor $\mathbb{A}$, which is compact, one can show that the two sets Γ and Γ_0 are the same.

Let us conclude this section with a summary of the four principal advantages of the point of view based on dynamical systems and developed in this paper.

(1) This approach includes an overall framework for the study of the weak solutions of the Navier–Stokes equations in the context of dynamical systems, or semiflows.

(2) In this framework we are able to apply Theorem A to the study of weak solutions, and thereby show the existence of a global attractor $\mathbb{A}$.

(3) The global attractor $\mathbb{A}$ attracts all bounded sets of weak solutions.

(4) The global attractor $\mathbb{A}$ satisfies the five properties listed at the end of Section 3.1.

The importance of the fact that the global attractor attracts all bounded sets in W cannot be overemphasized. It is this feature, along with the Lyapunov stability of the attractor, that is the source of various robustness theories of global attractors. One can show, for example, that the global attractor $\mathbb{A} = \mathbb{A}(f)$, which depends on $f \in L^2(\Omega)$, is upper semicontinuous in f (see Sell and You, 1995). As simple examples in the plane $\mathbb{R}^2$ show, this property is not shared by an attracting set that is not an attractor. (Recall that an *attracting* set in a semiflow is a nonempty, compact invariant set that attracts every point in a neighborhood of itself.)

4. THE TIME-DEPENDENT CASE

In studying the time-independent problem in Section 3, we did use the fact that if φ is a weak solution of the Navier–Stokes equations, then for every $\tau \geqslant 0$, the time translate φ_τ is also a weak solution. When the forcing function depends on time, this conclusion is no longer valid. Instead, φ_τ is a solution of the translated problem

$$u_t + \nu A u + B(u, u) = \mathbb{P} f_\tau$$

In order to develop a dynamical theory to handle this situation, we use the traditional approach of skew-product flows (see Raugel and Sell, 1993a–c; Sacker and Sell, 1977, 1994; Sell, 1967a, b, 1973; Vishik, 1992).

Let $\mathscr{F}$ be a set in $Y(N_0)$, for some $N_0 \geqslant 0$, and let $H^+(\mathscr{F})$ be the hull of $\mathscr{F}$. Then $H^+(\mathscr{F}) \subset Y(N_0)$, and it follows from the general theory of semiflows (see Sell and You, 1994) that the hull is a positively invariant set for the semiflow on $L^2_{\text{loc}} = L^2_{\text{loc}}[0, \infty; L^2(\Omega))$ given by (3.2). For the remainder of this section we assume that the hull $H^+(\mathscr{F})$ is a compact set; see (2.8). Since the time-translation mapping $S(\tau)$ given by (3.2) is a semiflow on the hull, and since the hull is compact, it follows from Theorem A that there is a global attractor $\mathbb{A}_2 \subset H^+(\mathscr{F})$.

Define $\mathscr{W}(H^+(\mathscr{F}))$, and $\mathscr{W}_{\text{LH}}(H^+(\mathscr{F}))$, to be the collection of all (φ, f) in $\mathscr{W}$, or in $\mathscr{W}_{\text{LH}}$, respectively, with the property that $f \in H^+(\mathscr{F})$. Define the mapping

$$S(\tau)(\varphi, f) = (\varphi_\tau, f_\tau), \qquad \text{for} \quad \tau \geqslant 0 \tag{4.1}$$

where $(\varphi, f) \in R$. From the comments made above, we see that S maps $[0, \infty) \times R$ into R.[3] From Lemma 6 one has $S(\tau)(\varphi, f) \in \mathscr{W}_{\text{LH}}$, whenever $(\varphi, f) \in \mathscr{W}$ and $\tau > 0$. The argument of Lemma 7 now extends in a straightforward manner to establish the following result.

Lemma 10. *Let $H^+(\mathscr{F})$ be a Y-bounded set that is compact in $L^2_{\text{loc}}[0, \infty; L^2(\Omega))$. Then the mapping S given by (4.1) is a semiflow on R. Furthermore, the sets $\mathscr{W}(H^+(\mathscr{F}))$ and $\mathscr{W}_{\text{LH}}(H^+(\mathscr{F}))$ are positively invariant subsets in this semiflow.*

Likewise the argument of Lemma 8 establishes the following result.

Lemma 11. *Let $H^+(\mathscr{F})$ be a Y-bounded set that is compact in $L^2_{\text{loc}}[0, \infty; L^2(\Omega))$. Then the restriction of the semiflow $S(t)$ to $\mathscr{W}(H^+(\mathscr{F}))$*

[3] This semiflow is a skew-product semiflow (see Sacker and Sell, 1977, 1994; Sell, 1967a, b, 1973). Indeed, the time translate f_τ does not depend on the choice of the solution $\varphi \in W(f)$.

is compact for $t > 0$, i.e., for each bounded set B in $\mathscr{W}(H^+(\mathscr{F}))$ and for each $\tau > 0$, the set $S(\tau) B$ lies in a compact subset in $\mathscr{W}$. Moreover, one has $\mathrm{Cl}_{\mathrm{R}}(S(\tau) B) \subset \mathscr{W}_{\mathrm{LH}}(H^+(\mathscr{F}))$, for $\tau > 0$.

Lastly, the argument of Lemma 9 now establishes the following fact.

Lemma 12. *Let $H^+(\mathscr{F})$ be a Y-bounded set that is compact in $L^2_{\mathrm{loc}}[0, \infty; L^2(\Omega))$. Then the restriction of the semiflow $S(t)$ to $\mathscr{W}(H^+(\mathscr{F}))$ is point dissipative.*

It then follows from these lemmas and Theorem A that we have the following result.

Theorem 2. *Let $H^+(\mathscr{F})$ be a Y-bounded set that is compact in $L^2_{\mathrm{loc}}[0, \infty; L^2(\Omega))$. Let $\mathbb{A}_2$ denote the global attractor generated by the semiflow $(\tau, f) \to f_\tau$ on $H^+(\mathscr{F})$. Then there is a global attractor $\mathbb{A}$ in $\mathscr{W}(H^+(\mathscr{F}))$, and the following hold.*

(1) $\mathbb{A} \subset \mathscr{W}_{\mathrm{LH}}(\mathbb{A}_2) \subset \mathscr{W}_{\mathrm{LH}}(H^+(\mathscr{F}))$.

(2) *$\mathbb{A}$ attracts all bounded sets in $\mathscr{W}(H^+(\mathscr{F}))$.*

(3) *If $(\varphi, f) \in \mathbb{A}$, then one has $f \in \mathbb{A}_2$.*

(4) *For every $f \in \mathbb{A}_2$, the set $\{\varphi \in W(f) \colon (\varphi, f) \in \mathbb{A}\}$ is a nonempty, compact set in $W(f)$.*

5. REGULARITY OF THE GLOBAL ATTRACTOR

Now that we have established the existence of a global attractor for the 3D Navier–Stokes equations, the next issues are to analyze the analytical and dynamical properties of this attractor. On the analytical side, one would like to know, for example, to what extent are the weak solutions in the global attractor actually strong solutions (see Foias and Temam, 1987). The results described in this section represent a preliminary study into the analytical properties of the solutions. We focus here on the case where the weak solutions satisfy an ultimate regularity property, and we show that a number of interesting consequences follow in this case.

Recall that a weak solution φ of Class LH is said to be a *strong solution* on an interval $[0, T)$, where $0 < T \leqslant \infty$, provided that one has $\varphi(0) \in V$ and

$$\varphi \in L^\infty_{\mathrm{loc}}[0, T; V) \cap L^2_{\mathrm{loc}}[0, T; V^2) \tag{5.1}$$

(see Constantin and Foias, 1988; Sell and You, 1995; Temam, 1977, 1983). For all data (u_0, f), where $u_0 \in V$ and $f \in Y$, one always has a maximally defined strong solution on an interval $[0, T)$, where $0 < T \leqslant \infty$, and this solution is uniquely determined on any subinterval of $[0, T)$. As a matter of fact, a strong solution φ of the Navier–Stokes equations has additional properties, including the following.

(1) The derivative $D_t \varphi$ satisfies

$$D_t \varphi \in L^2_{\text{loc}}[0, T; H) \subset L^2_{\text{loc}}[0, T; V^{-1})$$

(2) The function φ satisfies (1.1) almost everywhere on the interval $(0, T)$.

(3) The solution φ satisfies the variation of constants formula,

$$\varphi(t) = e^{-\nu A(t-t_0)}\varphi(t_0) + \int_{t_0}^{t} e^{-\nu A(t-s)}[\mathbb{P}f - B(\varphi, \varphi)]\, ds,$$

$$\text{for} \quad 0 \leqslant t_0 \leqslant t < T$$

where the integral exists in the space H.

(4) The function φ is in the Hölder space $C^{\alpha}_{\text{loc}}[0, T; H)$, for an α satisfying $0 < \alpha < \frac{1}{2}$.

In this section we restrict the Navier–Stokes equations to the case where the forcing function satisfies $f \in L^2(\Omega)$. By using the methodology of Section 4, one can extend many of the features described in this section to the time-dependent problem.

Let $f \in L^2(\Omega)$ be fixed and set $W = W(f)$ and $W_{\text{LH}} = W_{\text{LH}}(f)$. For each $\varphi \in W$ we define

$$L(\varphi) \stackrel{\text{def}}{=} \operatorname*{ess\,sup}_{0 < t < \infty} \|A^{1/2}\varphi(t)\|$$

The function L assumes values in the extended interval $[0, \infty]$. This function is monotone nonincreasing, i.e., one has $L(\varphi_\tau) \leqslant L(\varphi_\sigma) \leqslant L(\varphi)$, whenever $0 \leqslant \sigma \leqslant \tau$. Furthermore, the function L is lower semicontinuous on the space W. As a result, if $\varphi^n \to \varphi$ in W, then one has

$$L(\varphi) \leqslant \liminf_{n \to \infty} L(\varphi^n)$$

To put this another way, for every $\varphi \in W$ and every $\varepsilon > 0$, there is a $\delta = \delta(\varphi, \varepsilon) > 0$, such that one has

$$L(\varphi) \leqslant L(\phi) + \varepsilon, \qquad \text{whenever} \quad d(\varphi - \phi) < \delta$$

where d is the invariant metric on $L^2_{\text{loc}}[0, \infty; H)$ given by (2.6).

We say that a weak solution φ is *ultimately regular* if there is a time $t = t_\varphi > 0$ such that

$$L(\varphi_\tau) < \infty, \qquad \text{for all} \quad \tau \geqslant t_\varphi \tag{5.2}$$

If a given weak solution φ is ultimately regular, then the following properties are valid.

(1) Each point ϕ in the omega-limit set $\omega(\varphi)$ lies in the global attractor $\mathbb{A}$ and one has

$$L(\phi) \leqslant \liminf_{\tau \to \infty} L(\varphi_\tau) < \infty \tag{5.3}$$

(2) The set $w(\varphi)$ defined by

$$w(\varphi) \stackrel{\text{def}}{=} \{\phi(0) : \phi \in \omega(\varphi)\}$$

is a bounded set in V and it consists entirely of strong solutions. In particular, if $u_0 \in w(\varphi)$, then there is one and only one weak solution ϕ with $\phi(0) = u_0$, and this solution is a strong solution for all $t \in \mathbb{R}$. Moreover, one has $\phi \in \mathbb{A}$.

(3) There are constants K_1 and K_2 such that

$$\|A^{1/2}\phi(t)\|^2 \leqslant K_1^2, \qquad \text{for} \quad t \in \mathbb{R} \text{ and } \phi \in \omega(\varphi) \tag{5.4}$$

and

$$\int_t^{t+1} \|A\phi(t)\|^2 \leqslant K_2^2, \qquad \text{for} \quad t \in \mathbb{R} \text{ and } \phi \in \omega(\varphi) \tag{5.5}$$

Recall that the lower semicontinuous function L attains its minimum value on the compact set $\mathbb{A}$ (see Maurin, 1967). Define L_0 by $L_0 =^{\text{def}} \inf_{\phi \in \mathbb{A}} L(\phi)$. We assume that L_0 is finite or, equivalently, that there is at least one weak solution $\varphi \in W$ that satisfies (5.2). For any L_1 with $L_0 \leqslant L_1 < \infty$, we define

$$\mathbb{A}_1 \stackrel{\text{def}}{=} \{\phi \in \mathbb{A} : L(\phi) \leqslant L_1\}$$

Then, in addition to properties (1), (2), and (3) given above, the set $\mathbb{A}_1$ satisfies the following properties.

(4) The set $\mathbb{A}_1$ is a nonempty, closed, positively invariant set in $\mathbb{A}$.

(5) The omega limit set $\omega(\mathbb{A}_1)$ is a nonempty, compact, invariant set with

$$\mathbb{A}_3 \stackrel{\text{def}}{=} \omega(\mathbb{A}_1) \subset \mathbb{A}_1 \subset \mathbb{A}$$

(6) The set $\Gamma_3 =^{\text{def}} \{\phi(0): \phi \in \mathbb{A}_3\}$ is a bounded set in V, and (5.4) is valid for all $\phi \in \mathbb{A}_3$ with $K_1^2 = L_1$.

We say that the weak solutions are *ultimately regular on a set* U in W if for each $\varphi \in U$, the weak solution φ is ultimately regular. Notice that in the case of an infinite set U, the concept of being ultimately regular on U does not imply in any uniformity in (5.2), even when the set U is compact, or when U is the global attractor $\mathbb{A}$, or when U is the entire space W.

Assume now that the weak solutions are ultimately regular on a set U, where U is a neighborhood of the global attractor $\mathbb{A}$ in W. There are two points to be made.

(1) For every point $\varphi \in U$, the omega limit set $\omega(\varphi)$ is nonempty and compact and consists entirely of strong solutions. Furthermore, one has $\omega(\varphi) \subset \mathbb{A}$ by the maximality property; see Section 3.1. (We caution the reader that, in general, the attractor $\mathbb{A}$ is larger than the union of the omega limit sets in it, i.e., there may be a point $\phi \in \mathbb{A}$, where ϕ does not lie in any omega limit set.)

(2) It is our conjecture that under this assumption, the set

$$\Gamma_0 = \{\varphi(0): \varphi \in \mathbb{A}\}$$

is a bounded set in V and it consists entirely of strong solutions. If this were the case, then it would follow that there are constants K_1 and K_2 such that (5.4) and (5.5) are valid for all $\phi \in \mathbb{A}$.

While we are unable to prove this conjecture, there is an interesting case where it is valid. Specifically for the Navier–Stokes equations on thin 3D domains, as studied by Raugel and Sell (1993a–c), it is shown that there is a constant $L_2 > 0$, such that for every weak solution $\varphi \in W$, one has

$$\limsup_{\tau \to \infty} L(\varphi_\tau) \leqslant L_2 \tag{5.6}$$

It then follows from (5.6) that one has

$$L(\varphi) \leqslant L_2, \qquad \text{for every} \quad \varphi \in \mathbb{A}$$

which implies that (5.4) is valid for every $\phi \in \mathbb{A}$ with $K_1^2 = L_2$. A standard argument then establishes (5.5) (see Constantin and Foias, 1988). As a result, the set Γ_0 given above is a bounded set in V, and it consists entirely of strong solutions.

The results described in this section show the importance, from a dynamical systems point of view, of the concept of ultimate regularity. We believe that this concept is worthy of further study.

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865/8/1-3