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Triangles III: Complex triangle functions

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Summary. This paper is the third in a series of three examining Euclidean triangle geometry via complex cross ratios. In the first two papers, we looked at triangle shapes and triangle coordinates. In this paper, we look at the triangle coordinates of the special points of a triangle, and show that they are functions of its shape. We then show how these functions can be used to prove theorems about triangles, and to gain some insight into what makes a special point of a triangle a centre.

1. Introduction

The geometry of the special points of Euclidean triangles has long fascinated many mathematicians. An extensive inventory of these special points (over a hundred in all) and their properties has recently been compiled by Clark Kimberling [6]. This catalogue is all the more remarkable in that the author also establishes that all pairs of these points determine fewer than 150 lines. These collinearities were discovered experimentally through numerical computer calculations, i.e. without proofs. In principle, of course, the requisite proofs are just mechanical calculations on the trilinear coordinates of these points; in practice, however, the calculations can be very difficult, and best left to a computer algebra program. (See the comments in [3].)

In this paper, the last of a series discussing triangle geometry through complex cross ratios, we build on the notions of shape and triangle coordinate developed in [7] and [8] to discuss these special points from a different perspective. The basic idea, developed in §2, is that the geometric definition of any special point can be encoded into a function of one complex variable. After computing several of these functions, we show in §3 how to use them to discover and prove theorems about

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special points, and illustrate the method by showing that the circumcentre, nine point centre and Fermat points of a triangle must be concyclic. The remaining sections deal with more theoretical considerations: symmetry and centre functions (§4) and Cevian centres (§5).

As a source of information about special points and their properties, the catalogue of [6] is ideal. The requisite mathematical background from [7] and [8] is outlined below.

Identify the Euclidean plane with the complex numbers $\mathbb{C}$, and set $\mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$. The *cycle notation* and its properties are given by

$$z' := \frac{1}{1-z}, \qquad z'' = \frac{z-1}{z} = 1 - \frac{1}{z}, \qquad z''' = z, \qquad zz'z'' = -1$$

for any $z \in \mathbb{C}_\infty$.

The cross ratio of any $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ in $\mathbb{C}_\infty$ with at most two alike is the number

$$[\mathbf{a}, \mathbf{b}; \mathbf{c}, \mathbf{d}] := \frac{(\mathbf{a}-\mathbf{c})(\mathbf{b}-\mathbf{d})}{(\mathbf{a}-\mathbf{d})(\mathbf{b}-\mathbf{c})}$$

("cancel" any terms involving ∞). Cross ratios have the symmetry properties

$$[\mathbf{a}, \mathbf{b}; \mathbf{c}, \mathbf{d}]^{-1} = [\mathbf{a}, \mathbf{b}; \mathbf{d}, \mathbf{c}] = [\mathbf{b}, \mathbf{a}; \mathbf{c}, \mathbf{d}], \qquad [\mathbf{a}, \mathbf{b}; \mathbf{c}, \mathbf{d}]' = [\mathbf{a}, \mathbf{c}; \mathbf{d}, \mathbf{b}].$$

Linear fractional transformations or conjugate linear fractional transformations (which have the form

$$z \to \frac{az+b}{cz+d} \quad \text{or} \quad z \to \frac{a\bar{z}+b}{c\bar{z}+d} \qquad \text{for } ad - bc \neq 0$$

respectively) preserve or conjugate cross ratios. They are similarities or anti-similarities respectively whenever they fix ∞, i.e. whenever $c = 0$. For any distinct $\mathbf{a}$, and $\mathbf{b}$ in $\mathbb{C}$, the mapping $z \to [\infty, \mathbf{a}; \mathbf{b}, z]$ is a similarity.

The *shape* of any (ordered) triangle $\triangle\mathbf{abc}$ is the number $\triangle_{\mathbf{abc}} := [\infty, \mathbf{a}; \mathbf{b}, \mathbf{c}]$. Two triangles are similar whenever they have the same shape and anti-similar whenever they have conjugate shapes. The shapes of other triangles with the same vertices may be found by cycling: triangles $\triangle\mathbf{bca}$ and $\triangle\mathbf{cab}$ have shapes $\triangle_{\mathbf{bca}} = (\triangle_{\mathbf{abc}})'$ and $\triangle_{\mathbf{cab}} = (\triangle_{\mathbf{abc}})''$ respectively. Equilateral triangles have shape $\omega := e^{i\pi/3}$ or $\bar{\omega}$, and isosceles triangles with equal angles at $\mathbf{b}$ and $\mathbf{c}$ have shapes with modulus 1.

To define complex triangle coordinates, first fix an arbitrary, non-degenerate *base triangle* $\triangle\mathbf{abc}$, then for any point $\mathbf{z} \in \mathbb{C}_\infty$, the *triangle coordinate* of $\mathbf{z}$ with respect to the base triangles $\triangle\mathbf{abc}$ is the number

$$\mathbf{z}_\triangle := [\mathbf{z}, \mathbf{a}; \mathbf{b}, \mathbf{c}] \in \mathbb{C}_\infty.$$

The vertices of the base triangle have triangle coordinates $\mathbf{a}_\triangle = 1$, $\mathbf{b}_\triangle = 0$ and $\mathbf{c}_\triangle = \infty$, and $\infty_\triangle = \triangle$. Points inverse in the circumcircle of $\triangle\mathbf{abc}$ have conjugate triangle coordinates. The subscripts $\triangle'$ and $\triangle''$ to refer to the cycled triangles $\triangle\mathbf{bca}$ and $\triangle\mathbf{cab}$ respectively, so

$$\mathbf{z}_{\triangle'} := [\mathbf{z}, \mathbf{b}; \mathbf{c}, \mathbf{a}] = (\mathbf{z}_\triangle)' = \frac{1}{1-\mathbf{z}_\triangle} \quad \text{and} \quad \mathbf{z}_{\triangle''} := [\mathbf{z}, \mathbf{c}; \mathbf{a}, \mathbf{b}] = (\mathbf{z}_\triangle)'' = \frac{\mathbf{z}_\triangle - 1}{\mathbf{z}_\triangle}.$$

Then $\mathbf{z}_\triangle \mathbf{z}_{\triangle'} \mathbf{z}_{\triangle''} = -1$.

We denote the angles of the base triangles by $A := \measuredangle\mathbf{bac}$, $B := \measuredangle\mathbf{cba}$ and $C := \measuredangle\mathbf{acb}$ and its shape by $\triangle := \triangle_{\mathbf{abc}}$. We then have the following useful relations:

$$e^{iA} = \frac{\triangle}{|\triangle|}, \qquad e^{iB} = \frac{\triangle'}{|\triangle'|}, \qquad e^{iC} = \frac{\triangle''}{|\triangle''|}$$

and

$$e^{2iA} = \frac{\triangle}{\bar{\triangle}}, \qquad e^{2iB} = \frac{\triangle'}{\bar{\triangle}'}, \qquad e^{2iC} = \frac{\triangle''}{\bar{\triangle}''}.$$

2. Special points and complex triangle functions

In our discussion of the "special" points of a triangle, we want to include points such as its vertices, the mid-points of its sides, its circumcentre, the feet of its altitudes, its Brocard points, etc., etc. — any of the hundreds of different points defined or discovered over the centuries. These points may be described as follows.

First, each can be expressed in terms of the vertices of the triangle: if $\mathbf{p}$ is a special point of $\triangle\mathbf{abc}$, then $\mathbf{p} = f(\mathbf{a}, \mathbf{b}, \mathbf{c})$ for some function $f: \mathbb{T} \to \mathbb{C}_\infty$ (where $\mathbb{T}$ denotes the set of all non-collinear triples of points in $\mathbb{C}$). For example, f may represent the result of some construction. Second, special points are "similarly situated for similar triangles": if $\triangle\mathbf{abc}$ and $\triangle\hat{\mathbf{a}}\hat{\mathbf{b}}\hat{\mathbf{c}}$ are similar or anti-similar triangles, $\mathbf{p} := f(\mathbf{a}, \mathbf{b}, \mathbf{c})$ and $\hat{\mathbf{p}} := f(\hat{\mathbf{a}}, \hat{\mathbf{b}}, \hat{\mathbf{c}})$, then the similarity or anti-similarity which takes $\triangle\mathbf{abc}$ into $\triangle\hat{\mathbf{a}}\hat{\mathbf{b}}\hat{\mathbf{c}}$ takes $\mathbf{p}$ into $\hat{\mathbf{p}}$.

This latter stipulation places a condition on f. Since any anti-similarity is the composition of a similarity and conjugation, we may express this condition in two parts.

- For any similarity $S\colon \mathbb{C}_\infty \to \mathbb{C}_\infty$, $Sf(\mathbf{a}, \mathbf{b}, \mathbf{c}) = f(S\mathbf{a}, S\mathbf{b}, S\mathbf{c})$.
- $\overline{f(\mathbf{a}, \mathbf{b}, \mathbf{c})} = f(\bar{\mathbf{a}}, \bar{\mathbf{b}}, \bar{\mathbf{c}})$.

The following theorem translates these statements into terms of shapes and triangle coordinates.

TRIANGLE FUNCTION THEOREM. (*a*) *Suppose that* $f\colon \mathbb{T} \to \mathbb{C}_\infty$ *satisfies the conditions*

- $Sf(\mathbf{a}, \mathbf{b}, \mathbf{c}) = f(S\mathbf{a}, S\mathbf{b}, S\mathbf{c})$ *for all similarities* $S\colon \mathbb{C}_\infty \to \mathbb{C}_\infty$,
- $\overline{f(\mathbf{a}, \mathbf{b}, \mathbf{c})} = f(\bar{\mathbf{a}}, \bar{\mathbf{b}}, \bar{\mathbf{c}})$,

for all $(\mathbf{a}, \mathbf{b}, \mathbf{c}) \in \mathbb{T}$. *Define the function* $F\colon (\mathbb{C}\backslash\mathbb{R}) \to \mathbb{C}_\infty$ *by*

$$F(z) := [f(0, 1, z), 0; 1, z].$$

Then with respect to any base triangle $\triangle\mathbf{abc}$ *with shape* $\triangle$, *the point* $\mathbf{p} := f(\mathbf{a}, \mathbf{b}, \mathbf{c})$ *has triangle coordinate*

$$\mathbf{p}_\triangle = F(\triangle).$$

Furthermore, $\overline{F(z)} = F(\bar{z})$ *for all* $z \in \mathbb{C}\backslash\mathbb{R}$.

(*b*) *Suppose the function* $F\colon (\mathbb{C}\backslash\mathbb{R}) \to \mathbb{C}_\infty$ *satisfies* $\overline{F(z)} = F(\bar{z})$ *for all* $z \in \mathbb{C}\backslash\mathbb{R}$. *Define* $f\colon \mathbb{T} \to \mathbb{C}_\infty$ *to be the solution of the relation*

$$[f(\mathbf{a}, \mathbf{b}, \mathbf{c}), \mathbf{a}; \mathbf{b}, \mathbf{c}] = F([\infty, \mathbf{a}; \mathbf{b}, \mathbf{c}]).$$

Then for all $(\mathbf{a}, \mathbf{b}, \mathbf{c}) \in \mathbb{T}$, f *satisfies the assumptions of part* (*a*) *and* F *coincides with the* F *of part* (*a*).

Proof. (a) For any base triangle $\triangle\mathbf{abc}$ and $\mathbf{p} := f(\mathbf{a}, \mathbf{b}, \mathbf{c})$, the similarity $S\colon \mathbb{C}_\infty \to \mathbb{C}_\infty$ given by $Sz := [\infty, \mathbf{a}; \mathbf{b}, z]$ satisfies $\mathbf{S}\mathbf{a} = 0$, $S\mathbf{b} = 1$ and $S\mathbf{c} = \triangle$. Then

$$\begin{aligned}
\mathbf{p}_\triangle &= [\mathbf{p}, \mathbf{a}; \mathbf{b}, \mathbf{c}] \\
&= [f(S\mathbf{a}, S\mathbf{b}, S\mathbf{c}), S\mathbf{a}; S\mathbf{b}, S\mathbf{c}] \\
&= [f(0, 1, \triangle), 0; 1, \triangle] \\
&= F(\triangle).
\end{aligned}$$

Furthermore, for any $z \in \mathbb{C}\backslash\mathbb{R}$,

$$\begin{aligned}\overline{F(z)} &= [\overline{f(0,1,z)}, \bar{0}; \bar{1}, \bar{z}]\\ &= [f(0,1,\bar{z}), 0; 1, \bar{z}]\\ &= F(\bar{z}).\end{aligned}$$

(b) For all $(\mathbf{a}, \mathbf{b}, \mathbf{c}) \in \mathbb{T}$ and any similarity $S: \mathbb{C}_\infty \to \mathbb{C}_\infty$,

$$\begin{aligned}[f(S\mathbf{a}, S\mathbf{b}, S\mathbf{c}), S\mathbf{a}; S\mathbf{b}, S\mathbf{c}] &= F([\infty, S\mathbf{a}; S\mathbf{b}, S\mathbf{c}])\\ &= F([\infty, \mathbf{a}; \mathbf{b}, \mathbf{c}])\\ &= [Sf(\mathbf{a}, \mathbf{b}, \mathbf{c}), S\mathbf{a}; S\mathbf{b}, S\mathbf{c}]\end{aligned}$$

from which $Sf(\mathbf{a}, \mathbf{b}, \mathbf{c}) = f(S\mathbf{a}, S\mathbf{b}, S\mathbf{c})$. Also

$$\begin{aligned}[f(\bar{\mathbf{a}}, \bar{\mathbf{b}}, \bar{\mathbf{c}}), \bar{\mathbf{a}}; \bar{\mathbf{b}}, \bar{\mathbf{c}}] &= F([\infty, \bar{\mathbf{a}}; \bar{\mathbf{b}}, \bar{\mathbf{c}}])\\ &= F(\overline{[\infty, \mathbf{a}; \mathbf{b}, \mathbf{c}]})\\ &= \overline{F([\infty, \mathbf{a}; \mathbf{b}, \mathbf{c}])}\\ &= [\overline{f(\mathbf{a}, \mathbf{b}, \mathbf{c})}, \bar{\mathbf{a}}; \bar{\mathbf{b}}, \bar{\mathbf{c}}],\end{aligned}$$

so $\overline{f(\mathbf{a}, \mathbf{b}, \mathbf{c})} = f(\bar{\mathbf{a}}, \bar{\mathbf{b}}, \bar{\mathbf{c}})$. Since $z = [\infty, 0; 1, z]$, F coincides with the F of part (a). □

The significance of this theorem is that *it translates the geometric definition of any special point into a single function of one complex variable.*

DEFINITION 2.1. A *complex triangle function* is any function $F: (\mathbb{C}\backslash\mathbb{R}) \to \mathbb{C}_\infty$ satisfying $\overline{F(z)} = F(\bar{z})$.

Thus if a complex triangle function F embodies the construction/definition of some special point then, for any particular triangle with shape $\triangle$, the triangle coordinate of its own particular special point $\mathbf{p}$ with respect to that triangle is $\mathbf{p}_\triangle = F(\triangle)$. (We then call $\mathbf{p}$ the "F-point" of the triangle.)

We can use complex triangle functions to study the relations between special points and prove theorems about them. First, however, we must be able to find these functions. [Note: All triangle functions derived in this and the following sections are tabulated at the end.]

If we already have an expression for the special point in terms of the shape of the triangle, the work is done for us.

EXAMPLE 2.1. (*a*) *The vertices of the base triangle have constant functions* 0, 1 *and* ∞ *respectively. The point* ∞ *has triangle function* z.

(*b*) *The mid-points of the sides of the base triangle have functions* $-z$, $2-z$ *and* $z(2z-1)^{-1}$ *respectively.*

(*c*) *The circumcentre of the base triangle has function* $O(z) := \bar{z}$.

(*d*) *The centroid and symmedian points have functions*

$$G(z) := -\frac{z-2}{2z-1}z \quad \textit{and} \quad L(z) := \frac{\bar{z}-2}{2\bar{z}-1}.$$

(*e*) *The Brocard points have functions*

$$\mathrm{BROC}_1(z) := \bar{z}' = \frac{1}{1-\bar{z}} \quad \textit{and} \quad \mathrm{BROC}_2(z) := \bar{z}'' = \frac{\bar{z}-1}{\bar{z}}.$$

Proof. Collect from [8] the corresponding expressions for the triangle coordinates of these points in terms of $\triangle$ and translate them into triangle function form using the relation $\mathbf{p}_\triangle = F(\triangle)$. □

When we know the triangle coordinates of the special points in terms of the angles A, B and C of the base triangle, a little more work is necessary.

EXAMPLE 2.2. (*a*) *The orthocentre has function*

$$H(z) := \frac{z+\bar{z}-2}{2z\bar{z}-z-\bar{z}}z^2.$$

(*b*) *The incentre has function*

$$I(z) := \frac{z|z-1| + |z|(1-z)}{|1-z| - (1-z)}.$$

(*c*) *The nine-point centre has function*

$$N(z) := \frac{z^2-2z+\bar{z}}{-z^2-\bar{z}+2z\bar{z}}z.$$

Proof. From [8], the triangle coordinates for these points are

$$\mathbf{h}_\triangle = \frac{1+e^{-2iB}}{1+e^{2iC}}\triangle, \qquad \mathbf{i}_\triangle = \frac{1-e^{iC}}{1-e^{-iB}}\triangle, \qquad \mathbf{n}_\triangle = \frac{1+e^{-2iB}-e^{2iC}}{1-e^{-2iB}+e^{2iC}}\triangle$$

respectively. By using the relations

$$e^{iB} = \frac{\triangle'}{|\triangle'|}, \qquad e^{iC} = \frac{\triangle''}{|\triangle''|}, \quad \text{and} \quad e^{2iB} = \frac{\triangle'}{\bar{\triangle}'}, \qquad e^{2iC} = \frac{\triangle''}{\bar{\triangle}''},$$

these coordinates may be expressed entirely in terms of $\triangle$; the corresponding functions can then be calculated from the relation $\mathbf{p}_\triangle = F(\triangle)$ as before. □

Some triangle functions can be found directly from the geometry. To simplify the calculations, note that, since $z = [\infty, 0; 1, z] = \triangle_{\mathbf{01z}}$, we may replace any base triangle with shape z by the similar triangle $\triangle 01z$. Thus, if $\mathbf{p} = \mathbf{p}(z)$ is the F-point of $\triangle 01z$, then F is given by $F(z) = [\mathbf{p}(z), 0; 1, z]$.

EXAMPLE 2.3. The foot of the altitude through the third vertex of the base triangle $\triangle 01z$ is $\mathbf{p} = \mathrm{Re}(z) = \frac{1}{2}(z+\bar{z})$. The function for this point is thus

$$[\tfrac{1}{2}(z+\bar{z}), 0; 1, z] = \frac{2-z-\bar{z}}{z-\bar{z}}\, z.$$

EXAMPLE 2.4. Any internal angle bisector of a triangle divides the opposite side in the ratio of the remaining two sides. If the bisector through the first vertex of the base triangle $\triangle 01z$ meets the opposite side at $\mathbf{p}$, then

$$\frac{\mathbf{p}-1}{z-\mathbf{p}} = \frac{1}{|z|},$$

so $\mathbf{p}$ has function

$$[\mathbf{p}, 0; 1, z] = \frac{(\mathbf{p}-1)(0-z)}{(\mathbf{p}-z)(0-1)} = -\frac{z}{|z|}.$$

Similarly, the external bisector at the same vertex meets the opposite side at a point with function $+z/|z|$. □

Many special points of a triangle occur in triples (e.g. the mid-points of the sides, the feet of the altitudes). These points result from "cycling" the same definition or construction around the base triangle.

DEFINITION 2.2. For any triangle function A, the *A-triple* of associated points of the base triangle consists of the points **p**, **q** and **r**, where

$$\mathbf{p}_{\triangle} = A(\triangle), \qquad q_{\triangle'} = A(\triangle'), \qquad \mathbf{r}_{\triangle''} = A(\triangle'').$$

Triangle $\triangle\mathbf{pqr}$ is then called the *associated A-triangle* of the base triangle.

Note that, since $\mathbf{q}_{\triangle} = (\mathbf{q}_{\triangle'})'' = \{A(\triangle')\}''$ and $\mathbf{r}_{\triangle} = (\mathbf{r}_{\triangle''})' = \{A(\triangle'')\}'$, the triangle functions of the A-triple are $A(z)$, $\{A(z')\}''$ and $\{A(z'')\}'$.

EXAMPLE 2.5. From Example 2.4, the internal angle bisectors of the base triangle meet the opposite sides at the points with functions

$$A(z) = -\frac{z}{|z|}, \qquad \{A(z')\}'' = \frac{|1-z| + (1-z)}{|1-z|}, \qquad \{A(z'')\}' = \frac{z|1-z|}{z|1-z| - |z|(1-z)}.$$

EXAMPLE 2.6: ASSOCIATED TRIANGLES. We give the triangle function of the first vertex of each; the functions for the other vertices can be found as in Example 2.5.

(a) the *medial triangle* (with the mid-points of the base triangle as vertices). From Example 2.1, (b) $A(z) = -z$.

(b) the *tangential triangle* (with sides tangent to the circumcircle of the base triangle at its vertices). The tangential triangle is the inverse of the medial triangle in the circumcircle, so since inverse points have conjugate triangle coordinates, $A(z) = -\bar{z}$.

(c) the *anticomplementary triangle* (with the base triangle as medial triangle). If **p** is the vertex of the anticomplementary triangle opposite vertex 0 of the base triangle $\triangle 01z$, then $01\mathbf{p}z$ is a parallelogram, so $\mathbf{p} = 1 + z$. Then $A(z) = [1 + z, 0; 1, z] = z^2$.

(d) the *orthic triangle* (with the feet of the altitudes as vertices). Example 2.3 gives

$$\{A(z'')\}' = \frac{2 - z - \bar{z}}{z - \bar{z}}\, z =: F(z),$$

so (cycle)

$$A(z) \equiv [\{A[(z')'']\}']'' = [F(z')]'' = \frac{z+\bar{z}-2}{2z\bar{z}-z-\bar{z}}\,z.$$

(e) the *excentral triangle* (whose vertices are the excentres). From Example 2.1 of [8], the excentre opposite vertex **a** has triangle coordinate

$$\frac{1+e^{iC}}{1+e^{iB}}\,\triangle.$$

Put $\triangle = z$, $e^{iB} = z/|z'|$ and $e^{iC} = z''/|z''|$; simplify to get

$$A(z) = \frac{z|1-z| - |z|(1-z)}{|1-z|+(1-z)}. \qquad \square$$

Some special points of the base triangle are defined to be other special points of some associated triangle (for example, [6] lists the circumcentre of the tangential triangle, the centroid of the orthic triangle and others as special points). Functions for these "compound" special points may be found using the following theorem.

THEOREM 2.1. *Suppose that the F-point triangle is defined to be the E-point of its associated A-triangle. Then F(z) is the solution of*

$$[F(z), A(z); \{A(z')\}'', \{A(z'')\}'] = E\{[z, A(z); \{A(z')\}'', \{A(z'')\}']\}.$$

Proof. Let **d** be the F-point of the base triangle, i.e. **d** is the E-point of the associated A-triangle $\triangle\mathbf{pqr}$. Then

$$[\mathbf{d}, \mathbf{p}; \mathbf{q}, \mathbf{r}] = E(\triangle_{\mathbf{pqr}}) = E\{[\infty, \mathbf{p}; \mathbf{q}, \mathbf{r}]\},$$

so since the coordinate map $\mathbf{z} \to \mathbf{z}_\triangle$ preserves cross ratios,

$$[\mathbf{d}_\triangle, \mathbf{p}_\triangle; \mathbf{q}_\triangle, \mathbf{r}_\triangle] = E\{[\triangle, \mathbf{p}_\triangle; \mathbf{q}_\triangle, \mathbf{r}_\triangle]\}.$$

In terms of triangle functions (with $\triangle = z$, $\mathbf{d}_\triangle = F(z)$, $\mathbf{p}_\triangle = A(z)$, etc.) this becomes the relation stated. $\square$

EXAMPLE 2.7. The *Spieker centre* of a triangle is defined to be the incentre of its medial triangle [6, point 10]. From Example 2.2, (b),

$$E(z) \equiv I(z) = \frac{z|1-z| + |z|(1-z)}{|1-z| - (1-z)}.$$

From Example 2.1, (a)

$$A(z) = -z, \qquad \{A(z')\}'' = 2 - z, \qquad \{A(z'')\}' = \frac{z}{2-z}.$$

The cross ratio $[z, A(z); \{A(z')\}'', \{A(z'')\}']$ simplifies to z, since the medial triangle is similar to the base triangle, so we must solve

$$\left[\mathrm{SPK}(z), -z; 2-z, \frac{z}{2z-1}\right] = I(z).$$

We obtain

$$\mathrm{SPK}(z) := \frac{|1-z|z - |z| + z(z-2)}{|1-z|z - |z|(2z-1) - z^2} z. \qquad \square$$

Since much of the current work on triangles (e.g. [5] and [6]) is conducted via trilinear coordinates, it is useful to be able to convert trilinears to complex triangle coordinates. The trilinear coordinates of a point with respect to the base triangle are its signed distances from the sides. (A positive distance indicates a point on the same side of the appropriate side of the base triangle as the remaining vertex.) Since only the shape of the triangle is relevant (not its size), trilinears are taken to be *homogeneous* (so for instance, the incentre has trilinears (1. 1, 1), since it is equidistant from the three sides). The following theorem gives the conversion rule.

THEOREM 2.2. *If point* $\mathbf{p}$ *has trilinears* (α, β, γ) *with respect to* $\triangle\mathbf{abc}$, *then*

$$\mathbf{p}_\triangle = \left[\infty, \alpha\frac{|\mathbf{c}-\mathbf{b}|}{\mathbf{c}-\mathbf{b}}; \beta\frac{|\mathbf{a}-\mathbf{c}|}{\mathbf{a}-\mathbf{c}}, \gamma\frac{|\mathbf{b}-\mathbf{a}|}{\mathbf{b}-\mathbf{a}}\right].$$

Proof. We may take α, β and γ to be the exact signed distances of $\mathbf{p}$ from the sides of $\triangle\mathbf{abc}$. If $\triangle\mathbf{stu}$ is the pedal triangle of $\mathbf{p}$ with respect to $\triangle\mathbf{abc}$, then (MTST)

$$\overline{\mathbf{p}_\triangle} = [\infty, \mathbf{s}; \mathbf{t}, \mathbf{u}] = [\infty, -i(\mathbf{p}-\mathbf{s}); -i(\mathbf{p}-\mathbf{t}), -i(\mathbf{p}-\mathbf{u})]. \qquad (*)$$

If $\triangle\mathbf{abc}$ has positive angles, then the angle from $\overrightarrow{\mathbf{bc}}$ to $\overrightarrow{\mathbf{sp}}$ is $\pm\frac{1}{2}\pi$, where the sign is positive for $\mathbf{p}$ on the same side of $\mathbf{bc}$ as $\mathbf{a}$. It follows that

$$\frac{\mathbf{p}-\mathbf{s}}{\mathbf{c}-\mathbf{b}}=\frac{\alpha}{|\mathbf{c}-\mathbf{b}|}i, \qquad \text{i.e. } -i(\mathbf{p}-\mathbf{s})=\alpha\frac{(\mathbf{c}-\mathbf{b})}{|\mathbf{c}-\mathbf{b}|}=\alpha\frac{|\mathbf{c}-\mathbf{b}|}{(\overline{\mathbf{c}-\mathbf{b}})},$$

and similarly,

$$-i(\mathbf{p}-\mathbf{t})=\beta\frac{|\mathbf{a}-\mathbf{c}|}{(\overline{\mathbf{a}-\mathbf{c}})} \quad \text{and} \quad -i(\mathbf{p}-\mathbf{u})=\gamma\frac{|\mathbf{b}-\mathbf{a}|}{(\overline{\mathbf{b}-\mathbf{a}})}.$$

Substitute into (*) and conjugate to get the required formula.

If $\triangle\mathbf{abc}$ has negative angles, then we need merely replace $\mathbf{c}-\mathbf{b}$, $\mathbf{a}-\mathbf{c}$ and $\mathbf{b}-\mathbf{a}$ by their negatives; this has no effect on the formula. □

EXAMPLE 2.8. The *Steiner point* is defined in [2], and, with respect to a triangle $\triangle\mathbf{abc}$ with sides of lengths $\lambda:=|\mathbf{c}-\mathbf{b}|$, $\mu:=|\mathbf{a}-\mathbf{c}|$ and $\nu:=|\mathbf{b}-\mathbf{a}|$, has trilinears [6]

$$(\alpha,\beta,\gamma):=\left(\frac{1}{\lambda(\mu^2-\nu^2)},\frac{1}{\mu(\nu^2-\lambda^2)},\frac{1}{\nu(\lambda^2-\mu^2)}\right).$$

For $z:=\triangle_{\mathbf{abc}}$, $\triangle 01z$ is similar to $\triangle\mathbf{abc}$, so we may assume that $\mathbf{a}=0$, $\mathbf{b}=1$ and $\mathbf{c}=z$. Then

$$\alpha\frac{|\mathbf{c}-\mathbf{b}|}{(\overline{\mathbf{c}-\mathbf{b}})}=\frac{1}{|z-1|\{|z|^2-1\}}\cdot\frac{|z-1|}{\bar{z}-1}=\frac{1}{(z\bar{z}-1)(\bar{z}-1)},$$

and similarly

$$\beta\frac{|\mathbf{a}-\mathbf{c}|}{(\overline{\mathbf{a}-\mathbf{c}})}=\frac{1}{(z\bar{z}-z-\bar{z})\bar{z}} \quad \text{and} \quad \gamma\frac{|\mathbf{b}-\mathbf{a}|}{(\overline{\mathbf{b}-\mathbf{a}})}=\frac{1}{1-z-\bar{z}}.$$

Substitute into the formula of Theorem 2.2 and simplify to get the function

$$\mathrm{STN}(z):=\mathbf{p}_\triangle=\frac{z\bar{z}(z+\bar{z}-z\bar{z})}{z+\bar{z}-1}. \qquad \square$$

If the trilinears are given in terms of the angles A, B and C of the base triangle, then Theorem 2.2 can be reformulated as follows. (Caveat: our angles here are oriented, while those of the trilinear triangle are not. For $\mathrm{Im}(\triangle)<0$, insert minus signs as necessary.)

COROLLARY 2.1. *If* $\mathbf{p}$ *has trilinears* (α, β, γ) *with respect to* $\triangle\mathbf{abc}$, *then*

$$\mathbf{p}_\triangle = [\infty, -\alpha; \beta\, e^{iC}, \gamma\, e^{-iB}].$$

Proof. Multiply each term in the cross ratio of Theorem 2.2 by $-(\mathbf{c}-\mathbf{b})/|\mathbf{c}-\mathbf{b}|$:

$$\begin{aligned}\mathbf{p}_\triangle &= \left[\infty, -\alpha; \beta\frac{(\mathbf{c}-\mathbf{d})}{(\mathbf{c}-\mathbf{a})}\frac{|\mathbf{c}-\mathbf{a}|}{|\mathbf{c}-\mathbf{b}|}, \gamma\frac{(\mathbf{b}-\mathbf{c})}{(\mathbf{b}-\mathbf{a})}\frac{|\mathbf{b}-\mathbf{a}|}{|\mathbf{b}-\mathbf{c}|}\right]\\ &= \left[\infty, -\alpha; \beta\frac{\triangle''}{|\triangle''|}, \gamma\frac{|\triangle'|}{\triangle'}\right]\\ &= [\infty, -\alpha; \beta\, e^{iC}, \gamma\, e^{-iB}].\end{aligned}$$

□

EXAMPLE 2.9. An unidentified point $\mathbf{p}$ (number 36 in the catalogue of [6]) has trilinears

$$\begin{aligned}&(1-2\cos A, 1-2\cos B, 1-2\cos C)\\ &= (1-e^{iA}-e^{-iA}, 1-e^{iB}-e^{-iB}, 1-e^{iC} = -e^{-iC})\\ &= (1+e^{iB}e^{iC}+e^{-iB}e^{-iC}, 1-e^{iB}-e^{-iB}, 1-e^{iC}-e^{-iC})\end{aligned}$$

(since $A = \pi - B - C$). Then

$$\begin{aligned}\mathbf{p}_\triangle &= [\infty, -(1+e^{iB}e^{iC}+e^{-iB}e^{-iC}); (1-e^{iB}-e^{-iB})\,e^{iC}, (1-e^{iC}-e^{-iC})\,e^{-iB}]\\ &= \frac{(1+e^{iB})(e^{iC}-e^{iC}e^{-iB}+e^{iB})}{(1+e^{-iC})(e^{iC}-e^{iC}e^{-iB}+e^{-iB})}\\ &= \frac{(1+e^{iB})}{(1+e^{-iC})}.\end{aligned}$$

From Example 2.1 of [8], this expression is the conjugate of $\mathbf{i}_\triangle$, the triangle coordinate of the incentre of $\triangle\mathbf{abc}$. We thus identify $\mathbf{p}$ as the inverse of the incentre of $\triangle\mathbf{abc}$ in its circumcircle. □

This method works in general: if the trilinears of a special point are given in terms of trigonometric functions of the angles, they can be written in terms of e^{iA}, e^{iB} and e^{iC}. After applying Corollary 2.1, the resulting triangle coordinate may then be converted to a triangle function in z using the relations $e^{iA} = z/|z|$, $e^{2iA} = z/\bar{z}$, etc.

Strictly speaking, the function STN of Example 2.8 is not a triangle function: triangle functions have domain $\mathbb{C}\backslash\mathbb{R}$, but since the Steiner point is not defined for equilateral triangles, STN is undefined for $\omega := e^{i\pi/3}$ and $\bar{\omega}$ (we get "0/0"). Other points share this "deficiency": the Feuerbach point (the point of tangency of the incircle and the nine-point circle) is also undefined for equilateral triangles. Still others may be undefined for isosceles triangles or for some other particular triangle shape. Rather than modifying our definition of triangle function, we simply note that, with caution, much of what we write is also valid for these points. (An exception: Theorem 4.1.)

On the other hand, many of the formulae for triangle functions thus far derived also work for some real numbers (other than 0 and 1), so the domains of these functions may be extended to include these reals. Geometrically, this means extending the definitions of the corresponding special points to certain degenerate triangles.

3. Discovering and proving theorems

Using triangle functions, proofs of theorems about the special points of a triangle can often be reduced to verifications of algebraic identities in a single complex variable z. To illustrate this method, we prove the elementary theorem that the segment joining the mid-points of two sides of a triangle is parallel to the third side and half its length.

Let $\mathbf{p}$ and $\mathbf{q}$ be the mid-points of the sides $\mathbf{bc}$ and $\mathbf{ca}$ of any non-degenerate triangle $\triangle\mathbf{abc}$ and set $\mathscr{R} := (\mathbf{q} - \mathbf{p})/(\mathbf{a} - \mathbf{b})$; then since $\arg \mathscr{R}$ and $|\mathscr{R}|$ give the angle between the vectors $\overrightarrow{\mathbf{pq}}$ and $\overrightarrow{\mathbf{ab}}$ and the ratio of their lengths respectively, we want to prove that $\mathscr{R} \equiv \frac{1}{2}$. Express $\mathscr{R}$ in terms of cross ratios:

$$\mathscr{R} = \frac{[\infty, \mathbf{p}; \mathbf{a}, \mathbf{q}]}{[\infty, \mathbf{a}; \mathbf{p}, \mathbf{b}]}.$$

Apply the coordinate map $\mathbf{z} \to \mathbf{z}_\triangle$ (which preserves cross ratios):

$$\mathscr{R} = \frac{[\triangle, \mathbf{p}_\triangle; \mathbf{a}_\triangle, \mathbf{q}_\triangle]}{[\triangle, \mathbf{a}_\triangle; \mathbf{p}_\triangle, \mathbf{b}_\triangle]}.$$

Since all the points involved are special points, they have triangle functions: for $\triangle = z$, we have $\mathbf{a}_\triangle = 1$, $\mathbf{b}_\triangle = 0$, $\mathbf{p}_\triangle = -z$ and $\mathbf{q}_\triangle = 2 - z$ (Example 2.1, (b)). Substitute into $\mathscr{R}$ and simplify:

$$\mathscr{R} = \frac{[z, -z; 1, 2-z]}{[z, 1; -z, 0]} \equiv \frac{1}{2}.$$

Since $z = \triangle \in \mathbb{C}\backslash\mathbb{R}$ was arbitrary, this proves the statement for all non-degenerate triangles.

This method works for any geometric statement about special points that can be expressed in terms of cross ratios. The following theorem lists some applications of the method to angles, ratios of lengths, collinearity, etc.

THEOREM 3.1. *Let* **p**, **q**, **r** *and* **s** *be the P-, Q-, R- and S-points of any non-degenerate triangle with shape z.*

(a) Set

$$\mathscr{R} := \frac{[z, S(z); P(z), R(z)]}{[z, P(z); S(z), Q(z)]}.$$

If $\mathbf{p} \neq \mathbf{q}$ *and* $\mathbf{r} \neq \mathbf{s}$ *then* $|\mathscr{R}|$ *gives the ratio of the length of* $\overrightarrow{\mathbf{rs}}$ *to that of* $\overrightarrow{\mathbf{pq}}$, *and* $\arg \mathscr{R}$ *gives the angle from* $\overrightarrow{\mathbf{pq}}$ *to* $\overrightarrow{\mathbf{rs}}$. *In particular, the vectors are parallel when* $\mathscr{R}$ *is real and positive, anti-parallel when* $\mathscr{R}$ *is real and negative, and perpendicular when* $\mathscr{R}$ *is imaginary.*

(b) If **p**, **q** *and* **r** *are distinct, then*

$$\measuredangle\, \mathbf{qpr} = \arg[z, P(z); Q(z), R(z)].$$

(c) Points **p**, **q** *and* **r** *are collinear whenever* $[z, P(z); Q(z), R(z)]$ *is real. In this case,* **p** *divides segment* **qr** *the signed ratio* $-[z, P(z); Q(z), R(z)]$, *so* **p** *is between* **q** *and* **r** *whenever* $[z, P(z); Q(z), R(z)]$ *is negative, and is the mid-point of* **qr** *whenever* $[z, P(z); Q(z), R(z)] = -1$.

(d) Points **p**, **q**, **r**, **s** *are concyclic or collinear whenever* $[P(z), Q(z); R(z), S(z)]$ *is real. In this case, the pairs* **p**, **q** *and* **r**, **s** *separate each other whenever* $[P(z), Q(z); R(z), S(z)]$ *is negative, and* **p**, **q** *are harmonic conjugates with respect to* **r**, **s** *whenever* $[P(z), Q(z); R(z), S(z)] = -1$.

Proof. The statements are direct translations into triangle function terms of the corresponding statements about triangle coordinates in Theorem 3.1 of [8]. □

EXAMPLE 3.1: THE EULER LINE. The Euler line of a triangle is the line through its centroid **g** and its circumcentre **o**. Many other special points lie on this line: for example, from part (c) of Theorem 3.1, the identity

$$[z, G(z); N(z), O(z)] \equiv -\frac{2}{3}$$

for

$$G(z) = -\frac{z-2}{2z-1}z, \qquad N(z) = \frac{z^2 - 2z + \bar{z}}{-z^2 - \bar{z} + 2z\bar{z}}z, \quad \text{and} \quad O(z) = \bar{z}$$

(as in Examples 2.1 and 2.2) shows that the nine-point centre **n** is also on this line, and that **g** divides the segment **no** internally in the ratio 2 : 3. Another identity

$$[z, N(z); O(z), H(z)] \equiv -1$$

for

$$H(z) = \frac{z + \bar{z} - 2}{2z\bar{z} - z - \bar{z}}z^2$$

(Example 2.2) shows that the orthocentre **h** also lies on the Euler line, and that **n** is the mid-point of the segment **oh**. □

In principle, we could use Theorem 3.1 as in the above example to verify any of the many collinearities of [6]. Instead, for the remainder of this section, we focus on a theorem about concyclic points. We begin by finding the triangle functions of the isodynamic points and the Fermat points.

The isodynamic points are the intersections of the Apollonian circles of the base triangle [6, points 15 and 16], and are inverses in its circumcircle. From Example 4.2 of [7], they have triangle coordinates $\omega := e^{\pi i/3}$ and $\bar{\omega}$; however $F_1(z) := \omega$ and $F_2(z) := \bar{\omega}$ are *not* triangle functions, since neither satisfies $\overline{F(z)} = F(\bar{z})$. The difficulty lies with the orientation of the base triangle (i.e. the sign of its angles, given by the sign of $\text{Im}(z)$), and may be resolved as follows: any triangle function can be defined on all of $\mathbb{C}\backslash\mathbb{R}$ by defining it first for $\text{Im}(z) > 0$. Then for $\text{Im}(z) < 0$, we have $\text{Im}(\bar{z}) > 0$, so $F(z) = \overline{F(\bar{z})}$.

Thus, for the isodynamic points, the appropriate functions are

$$\text{ISD}_1(z) := \bar{\omega}, \ (\text{Im}(z) > 0) \quad \text{and} \quad \text{ISD}_2(z) := \omega, \ (\text{Im}(z) > 0).$$

(so for $\text{Im}(z) < 0$, $\text{ISD}_1(z) := \omega$ and $\text{ISD}_2(z) := \bar{\omega}$). Furthermore, since

$$[z, \text{ISD}_1(z); O(z), \text{ISD}_2(z)] = \begin{cases} -4\dfrac{\text{Im}(z)\ \text{Im}(z)}{|z-\omega|^2} < 0 & (\text{Im}(z) > 0) \\ +4\dfrac{\text{Im}(z)\ \text{Im}(\omega)}{|z-\omega|^2} < 0 & (\text{Im}(z) < 0) \end{cases},$$

then (Theorem 3.1, (c)) the first isodynamic point is always between the second and the circumcentre, and so is always the "inner" isodynamic point.

A similar ambiguity occurs for the Fermat points. The *first Fermat point* is the point of concurrence of the lines drawn from each vertex of the base triangle to the apex of an equilateral triangle constructed "outward" on the opposite side. The *second Fermat point* is defined similarly, but for "inward" directed equilateral triangles. The triangle functions for the Fermat points may be calculated directly from the complex version of Ceva's theorem (see Example 5.3). Alternately (and more efficiently), since the Fermat points are known to be the isogonal conjugates of the isodynamic points [6], their functions may be calculated directly from the isogonal conjugate formula [8]: the isogonal conjugate $\tilde{\mathbf{p}}$ of any point $\mathbf{p}$ has triangle coordinate

$$\tilde{\mathbf{p}}_\triangle = \frac{\mathrm{Im}[\triangle, \infty; 1, \mathbf{p}_\triangle]}{\mathrm{Im}[\triangle, 0; 1, \mathbf{p}_\triangle]} \triangle \overline{\mathbf{p}_\triangle}.$$

Thus if $\mathbf{p}$ is a special point with function F, its isogonal conjugate has function

$$\tilde{F}(z) := \frac{\mathrm{Im}[z, \infty; 1, F(z)]}{\mathrm{Im}[z, 0; 1, F(z)]} zF(\bar{z}).$$

Apply this formula to the isodynamic points and calculate: the Fermat points have functions

$$\mathrm{FER}_1(z) := \frac{(1-z) + \bar{\omega}(1-\bar{z})}{-\omega\bar{z}(1-z) + \bar{\omega}z(1-\bar{z})} z, \qquad (\mathrm{Im}(z) > 0)$$

and

$$\mathrm{FER}_2(z) := \frac{(1-z) + \omega(1-\bar{z})}{-\bar{\omega}\bar{z}(1-z) + \omega z(1-\bar{z})} z, \qquad (\mathrm{Im}(z) > 0).$$

Denote the isodynamic points by $\mathbf{i}_1$ *and* $\mathbf{i}_2$ and the Fermat points by $\mathbf{f}_1$ and $\mathbf{f}_2$ respectively. It has been discovered [4, 6] that the lines $\mathbf{i}_1\mathbf{f}_1$ and $\mathbf{i}_2\mathbf{f}_2$ are parallel to the Euler line $\mathbf{go}$. (If the triangle is isosceles, the three lines coincide with its axis.) As preparation for the theorem which follows, we prove a somewhat more precise version of this statement.

LEMMA 3.1. *For any non-degenerate triangle, the vectors* $\overrightarrow{\mathbf{i}_1\mathbf{f}_1}$ *and* $\overrightarrow{\mathbf{i}_2\mathbf{f}_2}$ *are respectively anti-parallel and parallel to the vector* $\overrightarrow{\mathbf{go}}$. *If the triangle is not isosceles, then neither of the lines* $\mathbf{i}_1\mathbf{f}_1$, $\mathbf{i}_2\mathbf{f}_2$ *coincides with the line* $\mathbf{go}$.

Proof. Without loss of generality, $\mathrm{Im}(z) > 0$. For $i = 1, 2$, define

$$\mathscr{R}_i(z) = \frac{[z, O(z); \mathrm{ISD}_i(z), G(z)]}{[z, \mathrm{ISD}_i(z); O(z), \mathrm{FER}_i(z)]}.$$

From Theorem 3.1, (a) we must prove that, for all non-real z, $\mathscr{R}_1(z)$ is real and negative and $\mathscr{R}_2(z)$ is real and positive.

The numerator of $\mathscr{R}_1(z)$ works out to

$$-\frac{(z-\bar{\omega})(2z\bar{z} - \bar{z} + z^2 - 2z)}{3z(1-z)(\bar{z}-\bar{\omega})}.$$

For the denominator, we get

$$\mathrm{ISD}_1(z) - \mathrm{FER}_1(z) = \frac{2z\bar{z} - \bar{z} + z^2 - 2z}{-\omega(1-z)\bar{z} + \bar{\omega}(1-\bar{z})z}$$

and, using the relation $\bar{\omega} + 1 = -\omega(\omega - \bar{\omega})$,

$$\bar{z} - \mathrm{FER}_1(z) = -\frac{z(1-z)(\omega-\bar{\omega})(\bar{z}-\omega)}{-\omega(1-z)\bar{z} + \bar{\omega}(1-\bar{z})z},$$

so the denominator of $\mathscr{R}_1(z)$ is

$$\frac{(z-\bar{z}) \cdot (2z\bar{z} - \bar{z} + z^2 - 2z)}{z(1-z)(\omega-\bar{\omega})(\bar{z}-\omega) \cdot (\bar{z}-\bar{\omega})} = \frac{(z-\bar{z})(2z\bar{z} - \bar{z} + z^2 - 2z)(z-\bar{\omega})}{z(1-z)(\omega-\bar{\omega})(\bar{z}-\bar{\omega})|\bar{z}-\omega|^2}.$$

Then

$$\mathscr{R}_1(z) = -\frac{|\bar{z}-\omega|^2(\omega-\bar{\omega})}{3(z-\bar{z})} = -\frac{1}{3}|\bar{z}-\omega|^2 \frac{\mathrm{Im}(\omega)}{\mathrm{Im}(z)},$$

which is real and negative.

Similarly (interchange ω and $\bar{\omega}$),

$$\mathscr{R}_2(z) = -\frac{|\bar{z}-\bar{\omega}|^2(\bar{\omega}-\omega)}{3(z-\bar{z})} = -\frac{1}{3}|\bar{z}-\bar{\omega}|^2 \frac{\mathrm{Im}(\bar{\omega})}{\mathrm{Im}(z)},$$

which is real and positive.

Since **o**, $\mathbf{i}_1$ and $\mathbf{i}_2$ are collinear ($\mathbf{i}_1$ and $\mathbf{i}_2$ are inverses in the circumcircle), if one lies on the Euler line, then both do. In this case, **g**, $\mathbf{i}_1$ and $\mathbf{i}_2$ are collinear, so the cross ratio $[z, G(z); \bar{\omega}, \omega]$ is real. This cross ratio simplifies to E^3 for $E := (z - \bar{\omega})/(z - \omega)$, so $E^3 - \bar{E}^3 = (E - \bar{E})(E + \bar{\omega}\bar{E})(E + \omega\bar{E}) = 0$. The three cases $E - \bar{E} = 0$, $E + \bar{\omega}\bar{E} = 0$ and $E + \omega\bar{E} = 0$ simplify to $|z''|^2 = 1$, $|z'|^2 = 1$ and $|z|^2 = 1$ respectively, so the triangle must be isosceles. Thus if the triangle is not isosceles, neither of the lines $\mathbf{i}_1\mathbf{f}_1$, $\mathbf{i}_2\mathbf{f}_2$ coincides with line **go**. □

Our theorem is unexpected; the known properties of the points involved give no hint of the result.

THEOREM 3.2. *For any scalene triangle, the circumcentre, the first Fermat point, the nine-point centre and the second Fermat point are concyclic, and the first pair always separates the second.*

Proof. From Lemma 3.1, the points cannot be collinear. In light of Theorem 3.1, (d), we must show that, for all non-real z with $|z| \neq 1$, $|z'| \neq 1$ and $|z''| \neq 1$, the cross ratio $\mathscr{R}(z) := [O(z), \mathrm{FER}_1(z); N(z), \mathrm{FER}_2(z)]$ is real and negative. A brute force calculation of this cross ratio is just that: brutal. Hindsight shows that the expression

$$1 - \mathscr{R}(z') = \left[\frac{O(z')}{z'}, \frac{N(z')}{z'}; \frac{\mathrm{FER}_1(z')}{z'}, \frac{\mathrm{FER}_2(z')}{z'}\right]$$

is more amenable to calculation: since z ranges over all non-real numbers whenever z' does, we must now show that $1 - \mathscr{R}(z')$ is real and greater than 1.

Without loss of generality, $\mathrm{Im}(z) > 0$ (and hence $\mathrm{Im}(z') > 0$). We calculate that

$$\frac{O(z')}{z'} = \frac{1 - z}{1 - \bar{z}}, \qquad \frac{N(z')}{z'} = \frac{T_N}{B_N}, \qquad \frac{\mathrm{FER}_i(z')}{z'} = \frac{T_i}{B_i}, \qquad i = 1, 2,$$

for

$$T_N := -z^2 + 2z\bar{z} - \bar{z}, \qquad B_N := z^2 - \bar{z}$$

$$T_1 := -\omega z + \omega z\bar{z} + \bar{\omega}\bar{z} - \bar{\omega}z\bar{z} \qquad B_1 := z + \bar{\omega}\bar{z}$$

$$T_2 := \bar{\omega}z + \bar{\omega}z\bar{z} + \omega\bar{z} - \omega z\bar{z} \qquad B_2 := z + \omega\bar{z}$$

and that

$$1 - \mathscr{R}(z') = \frac{[(1-z)B_1 - (1-\bar{z})T_1][T_N B_2 - T_2 B_N]}{[(1-z)B_2 - (1-\bar{z})T_2][T_N B_1 - T_1 B_N]}.$$

Further calculation gives

$$(1-z)B_1 - (1-\bar{z})T_1 = (1+\omega)z - z^2 - 2\omega z\bar{z} + (\omega - \bar{\omega})\bar{z}^2 z + \bar{\omega}\bar{z}^2$$

and

$$\begin{aligned} T_N B_2 - T_2 B_N &= -z\{(1+\bar{\omega})\bar{z} - \bar{z}^2 - 2\bar{\omega}\bar{z}z + (\bar{\omega} - \omega)z^2\bar{z} + \omega z^2\} \\ &= -z\overline{\{(1-z)B_1(1-\bar{z})T_1\}}, \end{aligned}$$

so the numerator of $1 - \mathscr{R}(z')$ becomes $-z|(1-z)B_1 - (1-\bar{z})T_1|^2$. Similarly (interchange ω and $\bar{\omega}$) the denominator of $1 - \mathscr{R}(z')$ becomes $-z|(1-z)B_2 - (1-\bar{z})T_2|^2$, so

$$1 - \mathscr{R}(z') = \frac{|(1-z)B_1(1-\bar{z})T_1|^2}{|(1-z)B_1 - (1-\bar{z})T_1|^2},$$

which is real.

To show that $1 - \mathscr{R}(z')$ is greater than 1, we show that its numerator is greater than its denominator. A brute force (computer-assisted) calculation shows that

$$|(1-z)B_1 - (1-\bar{z})T_1|^2 - |(1-z)B_2 - (1-\bar{z})T_2|^2 = 4\,\mathrm{Im}(z)\,\mathrm{Im}(\omega)\mathscr{P},$$

where

$$\mathscr{P} := [2(z+\bar{z})^2 + z\bar{z}](1 + z\bar{z}) - (z+\bar{z})[(z+\bar{z})^2 + 5z\bar{z}].$$

Since $\mathrm{Im}(z)$ and $\mathrm{Im}(\omega)$ are positive, we need only show that $\mathscr{P} > 0$. Put $z = r\,e^{i\theta}$ (so $z + \bar{z} = 2r\cos\theta$and $z\bar{z} = r^2$) and rearrange:

$$r^{-2}\mathscr{P} = (8\cos^2\theta + 1)r^2 - 2\cos\theta(4\cos^2\theta + 5)r + (8\cos^2\theta + 1).$$

As a quadratic in r, the right-hand side has discriminant

$$\mathscr{D} := -4\sin^2\theta(4\cos^2\theta - 1)^2 \leq 0.$$

The case $\mathscr{D} = 0$ is easily shown to imply that $|z| = 1$, contrary to hypothesis, so $\mathscr{D} < 0$ for all non-real z. Since the coefficient of r in the quadratic is positive, this implies that $\mathscr{P} > 0$ for all appropriate z, and we are done. □

A few comments: not only was this theorem *proven* by triangle function methods, it was also *discovered* using triangle function methods, with the aid of a computer (a Macintosh IIcx running *Theorist* [S3], with *Geometer's Sketchpad* [S2] for checking and visualization). In brief, the trilinears of the special points listed in [6] were entered, together with the conversion formulae of Theorem 2.2 and Corollary 2.1 and formulae relating angles and side lengths to shapes. Then, for an arbitrary numerical shape, the triangle coordinate of each special point was calculated, and the list scanned for concyclic points (using Theorem 3.1, (d)).

We report some additional discoveries; see [6] for definitions of the points involved.

- The centre of perspective of the orthic triangle and the triangle of alternate interior tangents appears to be concyclic with the symmedian, Feuerbach and crucial points.
- The segment joining the circumcentre and the symmedian point is known to be a diameter of a circle through the Brocard points. Two other point pairs on the same line appear to share this property: the inner/outer isodynamic point and the isogonal conjugate of the second/first Napoleon point.

Unlike the collinearity search of [6], the abovementioned investigation was neither systematic nor exhaustive. The list of [6] may contain other concyclic quadruples — the reader is invited to explore further!

A further comment: complex triangle functions also have potential applications to dynamic computer geometry software such as [S1] or [S2]. If the graphics window is identified with a region of the complex plane, then, given the vertices of any triangle (input by three mouse clicks, say), the command "plot the F-point of the triangle" need only invoke the (previously stored) function F and a few simple calculations, rather than an elaborate geometric construction. Specifically, the F-point of $\triangle\mathbf{abc}$ is the solution $\mathbf{p}$ of $F([\infty, \mathbf{a}; \mathbf{b}, \mathbf{c}]) = [\mathbf{p}, \mathbf{a}; \mathbf{b}, \mathbf{c}]$.

4. Symmetry and centre functions

In general terms, a centre is a special point with symmetry, i.e. one whose definition is independent of the order of the vertices of the base triangle. Suppose that, as in §2, $\mathbf{p} = f(\mathbf{a}, \mathbf{b}, \mathbf{c})$ is such a special point of the base triangle $\triangle\mathbf{abc}$. The symmetry requirement on f may then be split into two parts:

- *cyclic symmetry*: for all $(\mathbf{a}, \mathbf{b}, \mathbf{c}) \in \mathbb{T}, f(\mathbf{a}, \mathbf{b}, \mathbf{c}) = f(\mathbf{c}, \mathbf{a}, \mathbf{b})$,
- *bilateral symmetry* (with respect to side $\mathbf{bc}$): for all $(\mathbf{a}, \mathbf{b}, \mathbf{c}) \in \mathbb{T}$, $f(\mathbf{a}, \mathbf{b}, \mathbf{c}) = f(\mathbf{a}, \mathbf{c}, \mathbf{b})$.

(Thus any f with both types of symmetry is totally symmetric in its arguments.) The following theorem translates these conditions into triangle function properties.

CENTRE FUNCTION THEOREM. *For any triangle function* $F: (\mathbb{C}\backslash\mathbb{R}) \to \mathbb{C}_\infty$, *define the function* $f: \mathbb{T} \to \mathbb{C}_\infty$ *by*

$$[f(\mathbf{a}, \mathbf{b}, \mathbf{c}), \mathbf{a}; \mathbf{b}, \mathbf{c}] = F([\infty, \mathbf{a}; \mathbf{b}, \mathbf{c}]).$$

Then

(*a*) $f(\mathbf{a}, \mathbf{b}, \mathbf{c}) = f(\mathbf{c}, \mathbf{a}, \mathbf{b})$ *for all* $(\mathbf{a}, \mathbf{b}, \mathbf{c}) \in \mathbb{T}$ *if and only if* $F(z') = \{F(z)\}'$ *for all* $z \in \mathbb{C}\backslash\mathbb{R}$,

(*b*) $f(\mathbf{a}, \mathbf{b}, \mathbf{c}) = f(\mathbf{c}, \mathbf{a}, \mathbf{b})$ *for all* $(\mathbf{a}, \mathbf{b}, \mathbf{c}) \in \mathbb{T}$ *if and only if* $F(z^{-1}) = \{F(z)\}^{-1}$ *for all* $z \in \mathbb{C}\backslash\mathbb{R}$.

Proof. We prove only (a); the proof of part (b) is similar. From the triangle function theorem, f must satisfy the calculation rule

$$\lambda f(\mathbf{a}, \mathbf{b}, \mathbf{c}) + \mu = f(\lambda\mathbf{a} + \mu, \lambda\mathbf{b} + \mu, \lambda\mathbf{c} + \mu)$$

for all $(\mathbf{a}, \mathbf{b}, \mathbf{c}) \in \mathbb{T}$ and all similarities $z \to Sz := \lambda z + \mu$, $\lambda, \mu \in \mathbb{C}$, $\lambda \neq 0$. Since

$$F(z) := [f(0, 1, z), 0; 1, z],$$

if f is cyclically symmetric, then for all $z \in \mathbb{C}\backslash\mathbb{R}$,

$$\begin{aligned} F(z') &= [f(0, 1, z'), 0; 1, z'] \\ &= [(z')^{-1}f(z', 0, 1), 0; (z')^{-1}, 1] \\ &= [f(1, 0, 1 - z), 0; 1 - z, 1] \\ &= [-f(1, 0, 1 - z) + 1, 1; z, 0] \\ &= [f(0, 1, z), 0; 1, z]' \\ &= \{F(z)\}'. \end{aligned}$$

Conversely, if $F(z') = \{F(z)\}'$ for all $z \in \mathbb{C}\backslash\mathbb{R}$, then for all $(\mathbf{a}, \mathbf{b}, \mathbf{c}) \in \mathbb{T}$

$$\begin{aligned}[f(\mathbf{a}, \mathbf{b}, \mathbf{c}), \mathbf{a}; \mathbf{b}, \mathbf{c}] &= F([\infty, \mathbf{c}; \mathbf{a}, \mathbf{b}]') \\ &= \{F([\infty, \mathbf{c}; \mathbf{a}, \mathbf{b}])\}' \\ &= [f(\mathbf{c}, \mathbf{a}, \mathbf{b}), \mathbf{c}; \mathbf{a}, \mathbf{b}]' \\ &= [f(\mathbf{c}, \mathbf{a}, \mathbf{b}), \mathbf{a}; \mathbf{b}, \mathbf{c}].\end{aligned}$$

Thus $f(\mathbf{a}, \mathbf{b}, \mathbf{c}) = f(\mathbf{c}, \mathbf{a}, \mathbf{b})$. □

In general terms, this theorem states that centre functions preserve the symmetry properties of cross ratios.

DEFINITION 4.1. A triangle function $F: (\mathbb{C}\backslash\mathbb{R}) \to \mathbb{C}_\infty$ and its associated F-point are

- *cyclically symmetric* whenever $F(z') = \{F(z)\}'$ for all $z \in \mathbb{C}\backslash\mathbb{R}$,
- *bilaterally symmetric* whenever $F(z^{-1}) = \{F(z)\}^{-1}$ for all $z \in \mathbb{C}\backslash\mathbb{R}$.

The function is a *centre function* and its F-point a *centre* whenever they are both cyclically and bilaterally symmetric.

Some examples: the Brocard points are cyclically symmetric but not bilaterally symmetric, the excentre opposite the first vertex of the base triangle is bilaterally symmetric but not cyclically symmetric, and the centroid, circumcentre, incentre and orthocentre are both cyclically and bilaterally symmetric, as are most of the triangle centres listed in [6]. Note that we define bilateral symmetry with respect to the first side only; the conditions for bilateral symmetry with respect to the second and third sides may be found by cycling: $F[(z^{-1})'] = \{[F(z)]^{-1}\}'$ and $F[(z^{-1})''] = \{[F(z)]^{-1}\}''$.

There are clearly many more centre functions than there are identified geometric centres of a triangle. We give some examples, most of which correspond to as yet unidentified centres; we leave the proofs that they actually are centres to the reader. (The complex identities $(z^{-1})' = (z'')^{-1}$ and $(z^{-1})'' = (z')^{-1}$ may be useful.)

EXAMPLE 4.1. Let $g: \mathbb{C} \to \mathbb{C}$ be a non-trivial, odd, multiplicative function which preserves conjugates, i.e. for all z_1 and z_2 in $\mathbb{C}$, $g(-z_1) = -g(z_1)$, $g(z_1 z_2) = g(z_1)g(z_2)$ and $g(\overline{z_1}) = \overline{g(z_1)}$. Define $F: (\mathbb{C}\backslash\mathbb{R}) \to \mathbb{C}_\infty$ by

$$F(z) := \frac{1 + g(1-z)}{g(z) - g(1-z)}\, g(z).$$

For example, if $g(z) = z$, then F is the centroid function (Example 2.1, (d)). □

EXAMPLE 4.2. For any bilaterally symmetric triangle function $A\colon (\mathbb{C}\backslash\mathbb{R}) \to \mathbb{C}_\infty$, define $F\colon (\mathbb{C}\backslash\mathbb{R}) \to \mathbb{C}_\infty$ by

$$F(z) := \overline{[z, A(z); \{A(z')\}'', \{A(z'')\}']}.$$

Note that $A(z)$, $\{A(z')\}''$ and $\{A(z'')\}'$ are the functions of a triple of associated points (Definition 2.2). If these points lie on the sides of the base triangle, then the right-hand side gives the shape of their triangle, so F is the centre function of the Miquel point of this associated triangle. □

EXAMPLE 4.3. Let $B\colon (\mathbb{C}\backslash\mathbb{R}) \to \mathbb{C}_\infty$ be any triangle function not identically 1, and assume that B satisfies $B(z)B(z')B(z'') = 1$ for all $z \in \mathbb{C}\backslash\mathbb{R}$. Define $F\colon (\mathbb{C}\backslash\mathbb{R}) \to \mathbb{C}_\infty$ by

$$F(z) := \{B(z')\}''\{B(z'')\}'.$$

Then F is cyclically symmetric, and is bilaterally symmetric if B is. For $B(z) = z^{-2}$, for example, we get the centroid function once again, while for $B(z) = z^2$, we get the (as yet unidentified) centre function

$$F(z) = -\frac{z-2}{2z-1}z^3. \qquad \square$$

EXAMPLE 4.4. Let $H\colon \mathbb{C} \to \mathbb{C}$ be a homogeneous function which is symmetric in its last two arguments and which preserves conjugates, i.e. for all a, b, c in $\mathbb{C}$,

- $h(\lambda a, \lambda b, \lambda c) = \lambda^n h(a, b, c)$ for so integer n and all complex $\lambda \neq 0$,
- $h(a, b, c) = h(a, c, b)$,
- $h(\bar{a}, \bar{b}, \bar{c}) = \overline{h(a, b, c)}$.

Define $F\colon (\mathbb{C}\backslash\mathbb{R}) \to \mathbb{C}_\infty$ by

$$F(z) := [\infty, h(1-z, z-1); h(z, -1, 1-z), h(-1, 1-z, z)].$$

For example, $h(a, b, c) = \bar{a}$ gives the symmedian function (Example 2.1, (d)). □

EXAMPLE 4.5. For any $\theta \in \mathbb{R}$, define $F_\theta\colon (\mathbb{C}\backslash\mathbb{R}) \to \mathbb{C}_\infty$ by

$$F_\theta(z) := [e^{i\theta}, z; z', z''], \qquad (\mathrm{Im}(z) > 0).$$

For $\theta = 0$, ω and $\bar{\omega}$, we get the point at infinity and the first and second isodynamic points. In general, if $\mathbf{p}_\theta$ is the F_θ-centre of $\triangle 01z$, then

$$[\mathbf{p}_\theta, 0, 1, z] = [e^{i\theta}, z; z', z''], \qquad (\mathrm{Im}(z) > 0).$$

For a fixed z, the left- and right-hand sides are linear fractional transformations of $\mathbf{p}_\theta$ and $e^{i\theta}$ respectively, so $\mathbf{p}_\theta$ is a linear fractional transformation of $e^{i\theta}$. Then, since all $e^{i\theta}$'s lie on a circle, all $\mathbf{p}_\theta$'s lie on the line through the isodynamic points. □

It is intuitively clear that the centres of an isosceles triangle lie on its axis; the algebraic manifestation of this fact is the fact that, for non-real z with $|z| = 1$ and any centre function F, the cross ratio $[z, \bar{z}; 1, F(z)]$ is real (exercise). The centres of an equilateral triangle then lie on all three of its axes, and thus coincide with either the circumcentre or the point at infinity. To distinguish the cases, we have the following theorem.

THEOREM 4.1. *Every cyclically symmetric triangle function F either fixes ω and $\bar{\omega}$ or interchanges them. If it interchanges them, then the F-point of any equilateral triangle coincides with its circumcentre. If it fixes them, then the F-point of any equilateral triangle is the point at infinity.*

Proof. The numbers ω and $\bar{\omega}$ are the only solutions of the relation $z = z' = z''$. Since F preserves this relation and preserves conjugates, it either fixes ω and $\bar{\omega}$ or interchanges them. Suppose it interchanges them; then the F-point $\mathbf{p}$ of any equilateral triangle ($\triangle = \omega$ or $\triangle = \bar{\omega}$) satisfies $\mathbf{p}_\triangle = F(\triangle) = \bar{\triangle} = \mathbf{o}_\triangle$, so $\mathbf{p} = \mathbf{o}$. If F fixes ω and $\bar{\omega}$, then $\mathbf{p}_\triangle = \triangle = \infty_\triangle$, so $\mathbf{p} = \infty$. □

We adopt the terminology of [6] and call a centre function *centripedal* if it interchanges ω and $\bar{\omega}$, and *centrifugal* if it fixes them. A centre function is centripedal if and only if its conjugate is centrifugal, so the inverse of a centripedal centre in the circumcircle of the base triangle is centrifugal and vice versa.

5. Cevian centres

In addition to allowing near-mechanical proofs of theorems about special points, triangle functions may be used to provide more theoretical insights into the relations between triangle centres and the points involved in their construction. (See also [5] for a different theoretical perspective on triangle centres.) Example 4.2, for instance, relates the functions of associated points on the sides of the base triangle to that of their Miquel point.

For simplicity, we restrict our attention to one particular type of centre: *cevian* centres. Suppose we have a triangle $\triangle\mathbf{abc}$ and some construction or definition which produces a point $\mathbf{p}$ "opposite" vertex $\mathbf{a}$. We may "cycle" this same construction around the triangle to produce points $\mathbf{q}$ and $\mathbf{r}$ opposite vertices $\mathbf{b}$ and $\mathbf{c}$. If the construction is a "good" one, then for any $\triangle\mathbf{abc}$, the cevians $\mathbf{ap}$, $\mathbf{bq}$ and $\mathbf{cr}$ are concurrent at some special point. (A cevian is any line through a vertex other than a side.) The point is bilaterally symmetric if the original construction was, and is automatically cyclically symmetric.

In triangle coordinate terms, $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$ are a triple of associated points (Definition 2.2) corresponding to some triangle function $A: (\mathbb{C}\backslash\mathbb{R}) \to \mathbb{C}_\infty$ which embodies their definition or construction. The problem of finding those constructions which produce concurrent cevians then becomes the problem of finding appropriate conditions on the functions A.

If we know their *side-points* (the points where each cevian meets the opposite side) a complex version of Ceva's theorem tells us when the cevians are concurrent.

COMPLEX CEVA'S THEOREM (CCEV) [8]. *Suppose that cevians through vertices* $\mathbf{a}$, $\mathbf{b}$ *and* $\mathbf{c}$ *of* $\triangle\mathbf{abc}$ *have side-points* $\mathbf{s}$, $\mathbf{t}$ *and* $\mathbf{u}$ *respectively. Then these cevians meet at some point* $\mathbf{m} \in \mathbb{C}_\infty$ *if and only if* $\mathbf{s}_\triangle \mathbf{t}_{\triangle'} \mathbf{u}_{\triangle''} = 1$, *in which case* $\mathbf{m}_\triangle = \mathbf{t}_\triangle \mathbf{u}_\triangle$. *(We include the case* $\mathbf{m} = \infty$, *when the cevians are parallel.)*

Note that any point $\mathbf{m}$ not on a side of the triangle is always the intersection of cevians; from Corollary 5.1 of [8], the corresponding side-points are given by

$$\mathbf{s}_\triangle = \frac{\mathbf{m}_\triangle}{\tilde{\mathbf{m}}_\triangle}, \qquad \mathbf{t}_{\triangle'} = \frac{\mathbf{m}_{\triangle'}}{\tilde{\mathbf{m}}_{\triangle'}}, \qquad \mathbf{u}_{\triangle''} = \frac{\mathbf{m}_{\triangle''}}{\tilde{\mathbf{m}}_{\triangle''}}.$$

We now translate these statements into triangle function form.

THEOREM 5.1 (CCEV FOR SPECIAL POINTS ON SIDES). *Suppose* $S: (\mathbb{C}\backslash\mathbb{R}) \to \mathbb{C}_\infty$ *is the triangle function for some special point on the first side of the base triangle. Then the cevians through the associated triple of points corresponding to* S *are concurrent if and only if*

$$S(z)S(z')S(z'') \equiv 1.$$

In this case, the point of concurrence has triangle function

$$M(z) := \{S(z')\}''\{S(z'')\}'$$

and is cyclically symmetric. If S is bilaterally symmetric, then so is M.

Conversely, let $M: (\mathbb{C}\backslash\mathbb{R}) \to \mathbb{C}_\infty$ *be any centre function; then for any triangle* $\triangle$**abc**, *the corresponding centre* **m** *is the intersection of the cevians* **ap**, **bq** *and* **cr**, *where* **p**, **q** *and* **r** *are side-points with triangle functions* $S(z) = M(z)/\tilde{M}(\bar{z})$, $\{S(z')\}''$ *and* $\{S(z'')\}'$.

Proof. The formulas are direct translations of those of CCEV and the corollary cited above for $z = \triangle$, using $\mathbf{s}_\triangle = S(\triangle)$, $\mathbf{t}_{\triangle'} = S(\triangle')$ and $\mathbf{u}_{\triangle''} = S(\triangle'')$. The symmetry properties are those of Example 4.3. □

For instance, the median from vertex **a** has side-point with function $S(z) := -z$ (Example 2.1, (b)), so since $S(z)S(z')S(z''') = (-z)(-z')(-z'') = 1$, the medians are concurrent at the point with function $(-z')''(-z'')' = G(z)$, i.e. at the centroid.

Note that the second part of Theorem 5.1 implies that every centre is cevian, even though it may not have been defined that way originally.

As in Theorem 2.1 of [8], point **p** lies on side **bc** of $\triangle$**abc** if and only if $\mathbf{p}_\triangle/\triangle = [\infty, \mathbf{p}; \mathbf{c}, \mathbf{b}]$ is real. The triangle function S of any point on the first side of the base triangle must thus satisfy $S(z)/z \in \mathbb{R}$ for all $z \in \mathbb{C}\backslash\mathbb{R}$.

THEOREM 5.2. *Let* $S: (\mathbb{C}\backslash\mathbb{R}) \to \mathbb{C}_\infty$ *be a triangle function with* $\sigma(z) := S(z)/z \in \mathbb{R}$ *for all* $z \in \mathbb{C}\backslash\mathbb{R}$. *Then for all* $z \in \mathbb{C}\backslash\mathbb{R}$, *if* $\sigma(z) \neq 0, \infty$,

- $\sigma(z) = \sigma(\bar{z})$,
- $S(z^{-1}) = \{S(z)\}^{-1}$ *if and only if* $\sigma(z^{-1}) = \{\sigma(z)\}^{-1}$,
- $S(z)S(z')S(z'') = 1$ *if and only if* $\sigma(z)\sigma(z')\sigma(z'') = 1$.

Furthermore, when all three conditions hold, the centre function $M: (\mathbb{C}\backslash\mathbb{R}) \to \mathbb{C}_\infty$ *given by* $M(z) := \{S(z')\}''\{S(z'')\}'$ *takes the form*

$$M(z) = \frac{1 + (1-z)\sigma(1-z)}{z\sigma(z) - (1-z)\sigma(1-z)}\, z\sigma(z).$$

Proof. The proofs are straightforward calculations from the definition of σ. □

The new formula for M makes it easier to find some centre functions.

EXAMPLE 5.1: THE NAGEL POINT. The Nagel point is the intersection of the cevians from each vertex to the point of tangency of the opposite side and opposite excircle. From Example 2.4 of [8], the point of tangency has triangle coordinate

$$\left(\frac{1-e^{-iB}}{1-e^{iC}}\right)^2,$$

so, using the relations $e^{iB} = z'/|z'|$ and $e^{iC} = z''/|z''|$ as in Example 2.2, this point has function

$$\frac{1-|z|-|1-z|}{1-|z|+|1-z|}\,z,$$

whence

$$\sigma(z) = -\frac{1-|z|-|1-z|}{1-|z|+|1-z|}, \qquad \sigma(1-z) = -\frac{1-|z|-|1-z|}{1+|z|-|1-z|}.$$

A straightforward calculation verifies that $\sigma(z)\sigma(z')\sigma(z'') = 1$, so the cevians intersect, and another calculation gives the function

$$\text{NAG}(z) := \frac{z|1-z|-(2-z)|z|-z}{(1-2z)-|z|+|1-z|}\,z. \qquad \square$$

In light of Theorem 5.2, the problem of determining all the bilaterally symmetric side-points which produce concurrent cevians reduces to the following functional equations problem: characterize all functions $\sigma\colon (\mathbb{C}\backslash\mathbb{R}) \to \mathbb{R}\backslash\{0\}$ which satisfy

- $\sigma(z) = \sigma(\bar{z})$,
- $\sigma(z^{-1}) = \{\sigma(z)\}^{-1}$,
- $\sigma(z)\sigma(z')\sigma(z'') = 1$.

There are many functions satisfying this condition, most with no obvious geometric interpretation. We give a few examples.

EXAMPLES 5.2.

- Define $\sigma\colon (\mathbb{C}\backslash\mathbb{R}) \to \mathbb{R}\backslash\{0\}$ by $\sigma(z) := |z|^\alpha$ for any real α.
- For any $g\colon (\mathbb{C}\backslash\mathbb{R}) \to \mathbb{R}\backslash\{0\}$ with $g(z) \equiv g(\bar{z})$ and $g(z^{-1}) \equiv \{g(z)\}^{-1}$, define $\sigma\colon (\mathbb{C}\backslash\mathbb{R}) \to \mathbb{R}\backslash\{0\}$ by $\sigma(z) := g(z')/g(z'')$.

Note that the functions satisfying the three conditions above form an abelian group under multiplication; the identity is $\sigma(z) \equiv 1$, corresponding to the centroid.

This raises some interesting questions about interrelations among centres; for instance, given two such functions corresponding to known centres, what centre corresponds to their product, and how is it related geometrically to the first two? For example, it is not difficult to show that if $\sigma_1(z)\sigma_2(z) \equiv |z|^2$, then the centres corresponding to σ_1 and σ_2 are isogonal conjugates of each other, furthermore, reciprocal σ's correspond to *isotomic* conjugates (see the definition in [1]).

In general, cevians are defined by special points not on the sides of the base triangle. When this is the case, we may find their side-points by the side-point formulae [8]: if the cevian **ap** intersects side **bc** of $\triangle$**abc** at **s**, then

$$\mathbf{s}_\triangle = \frac{\mathbf{p}_\triangle}{\tilde{\mathbf{p}}_\triangle} = \frac{\operatorname{Im}[\triangle, 0; 1, \mathbf{p}_\triangle]}{\operatorname{Im}[\triangle, \infty; 1, \mathbf{p}_\triangle]} \frac{1}{\bar{\triangle}}.$$

Thus if **p** is a special point with function $P\colon (\mathbb{C}\backslash\mathbb{R}) \to \mathbb{C}_\infty$, the side-point of cevian **ap** has function

$$S(z) := \frac{P(z)}{\tilde{P}(\bar{z})} = \frac{\operatorname{Im}[z, 0; 1, P(z)]}{\operatorname{Im}[z, \infty; 1, P(z)]} \frac{1}{\bar{z}}.$$

Example 5.3: Kiepert's hyperbola. On each side of the base triangle, construct isosceles triangles with apex angle θ. The cevians from each vertex to the apex of the isosceles triangle on the opposite side are then known to be concurrent. We verify this statement, and find the function for the intersection.

Assume that $\operatorname{Im}(z) > 0$; then for $\lambda := e^{i\theta}$, the apex of the isosceles triangle opposite vertex **a** has function $P(z) := \lambda z$. The corresponding side-point then has function

$$\begin{aligned} S(z) &= \frac{\operatorname{Im}[z, 0; 1, \lambda z]}{\operatorname{Im}[z, \infty; 1, \lambda z]} \frac{1}{\bar{z}} \\ &= -\frac{(z - \bar{z}) - (\lambda z - \bar{\lambda}\bar{z}) + (\lambda - \bar{\lambda})}{(z - \bar{z}) - (\lambda z - \bar{\lambda}\bar{z}) + (\lambda - \bar{\lambda})z\bar{z}} z, \end{aligned}$$

so

$$\sigma(z) = \frac{(z - \bar{z}) - (\lambda z - \bar{\lambda}\bar{z}) + (\lambda - \bar{\lambda})}{(z - \bar{z}) - (\lambda z - \bar{\lambda}\bar{z}) + (\lambda - \bar{\lambda})z\bar{z}}.$$

A direct calculation now gives $\sigma(z^{-1}) = \{\sigma(z)\}^{-1}$,

$$\sigma(z') = -\frac{(z-\bar{z}) - (\lambda z - \bar{\lambda}\bar{z}) + (\lambda - \bar{\lambda})z\bar{z}}{(z-\bar{z}) + \lambda\bar{z} - \bar{\lambda}z}$$

and

$$\sigma(z'') = \frac{(z-\bar{z}) + \lambda\bar{z} - \bar{\lambda}z}{(z-\bar{z}) - (\lambda z - \bar{\lambda}\bar{z}) + (\lambda - \bar{\lambda})},$$

so $\sigma(z)(z')\sigma(z'') = 1$ and the cevians intersect. Further calculation using the fomula of Theorem 5.2 then gives the function of the point of intersection:

$$K_\theta(z) := \frac{(2-\lambda-\bar{\lambda}) - (1-\bar{\lambda})z}{(2-\lambda-\bar{\lambda})z - (1-\lambda)} \cdot \frac{(z-\bar{z}) - (\lambda z - \bar{\lambda}\bar{z}) + (\lambda - \bar{\lambda})}{(z-\bar{z}) - (\lambda z - \bar{\lambda}\bar{z}) + (\lambda - \bar{\lambda})z\bar{z}}\, z,$$
$$(\lambda = e^{i\theta}, \operatorname{Im}(z) > 0).$$

The points of intersection above are known to lie on a rectangular hyperbola, called *Kiepert's hyperbola.* (See [1] for a proof and further interesting discussion.) Several values of θ give well-known triangle centres: for example $\theta \to 0$, $\theta = \pi$ and $\theta = \pm\pi/3$ give the orthocentre, the centroid and the Fermat points respectively. Kiepert's hyperbola also contains the *Napoleon points*: erect equilateral triangles "outward" on the sides of the base triangle, then the cevians through each vertex to the centre of the equilateral triangle on the opposite side meet at the first Napoleon point. We now have $\theta = 2\pi/3$, so $\lambda = e^{2\pi i/3} = -\bar{\omega}$ and direct calculation gives

$$\text{NAP}_1(z) := K_{2\pi/3}(z) = \frac{\sqrt{3}i + \bar{\omega}z}{\sqrt{3}iz - \omega} \cdot \frac{\omega z + \bar{\omega}\bar{z} - 1}{\omega z + \bar{\omega}\bar{z} - z\bar{z}}, \qquad (\operatorname{Im}(z) > 0).$$

If the equilateral triangles are directed "inward", we get the second Napoleon point. This time $\theta = -2\pi/3$, and we calculate that

$$\text{NAP}_2(z) := K_{-2\pi/3}(z) = \frac{\sqrt{3}i - \omega z}{\sqrt{3}iz + \bar{\omega}} \cdot \frac{\bar{\omega}z + \omega\bar{z} - 1}{\bar{\omega}z + \omega\bar{z} - z\bar{z}}\, z, \qquad (\operatorname{Im}(z) > 0). \qquad \square$$

We may use the side-point formulas to generalize Theorem 5.1.

COROLLARY 5.1 (CCEV FOR SPECIAL POINTS). *Suppose P: $(\mathbb{C}\backslash\mathbb{R}) \to \mathbb{C}_\infty$ is the triangle function for some special point. Then the cevians through the associated triple* *of points corresponding to P are concurrent if and only if*

$$P(z)P(z')P(z'') \equiv \tilde{P}(\bar{z})\tilde{P}(\bar{z}')\tilde{P}(\bar{z}''),$$

or equivalently, if and only if

$$\frac{Im[z, 0; 1, P(z)]}{Im[z, \infty; 1, P(z)]} \frac{Im[z', 0; 1, P(z')]}{Im[z', \infty; 1, P(z')]} \frac{Im[z'', 0; 1, P(z'')]}{Im[z'', \infty; 1, P(z'')]} \equiv -1.$$

In this case, the point of concurrence has triangle function

$$M(z) := \left\{\frac{P[z']}{\tilde{P}(\bar{z}')}\right\}'' \left\{\frac{P(z'')}{\tilde{P}(\bar{z}'')}\right\}'$$

and is cyclically symmetric. It is bilaterally symmetric if P is.

Proof. The statements are direct consequences of Theorem 5.1 and the side-point formulas. □

Thus the problem of characterizing constructions which give intersecting cevians reduces to a much more complicated functional equations problem when the special points defining the cevians no longer lie on the sides.

To conclude our discussion of triangle functions, we tabulate all the functions derived in this paper.

Table 1. Triangle functions for associated points

Associated triangle	Triangle function $A(z)$ of first vertex
base	0
medial	$-z$
tangential	$-\bar{z}$
anticomplementary	z^2
orthic	$\dfrac{z + \bar{z} - 2}{2z\bar{z} - z - \bar{z}} z$
excentral	$\dfrac{z\lvert 1 - z\rvert - \lvert z\rvert(1 - z)}{\lvert 1 - z\rvert + (1 - z)}$

Table 2. Centre functions

Centre	Function
point at infinity	z
circumcentre	$O(z) = \bar{z}$
centroid	$G(z) = -\dfrac{z-2}{2z-1}z$
symmedian point	$L(z) = \dfrac{\bar{z}-2}{2\bar{z}-1}$
orthocentre	$H(z) = \dfrac{z+\bar{z}-2}{2z\bar{z}-z-\bar{z}}z^2$
incentre	$I(z) = \dfrac{z\lvert 1-z\rvert + \lvert z\rvert(1-z)}{\lvert 1-z\rvert - (1-z)}$
nine-point centre	$N(z) = \dfrac{z^2-2z+\bar{z}}{-z^2-\bar{z}+2z\bar{z}}z$
Spieker centre	$\mathrm{SPK}(z) = \dfrac{\lvert 1-z\rvert z - \lvert z\rvert + z(z-2)}{\lvert 1-z\rvert z - \lvert z\rvert(2z-1) - z^2}z$
Steiner point	$\mathrm{STN}(z) = \dfrac{z\bar{z}(z+\bar{z}-z\bar{z})}{z+\bar{z}-1}$
isodynamic points	$\mathrm{ISD}_1(z) := \bar{\omega}, \quad \mathrm{ISD}_2(z) := \omega, \quad (\mathrm{Im}(z) > 0)$
Fermat points	$\mathrm{FER}_1(z) = \dfrac{(1-z)+\bar{\omega}(1-\bar{z})}{-\omega\bar{z}(1-z)+\bar{\omega}z(1-\bar{z})}z, \quad (\mathrm{Im}(z) > 0)$
	$\mathrm{FER}_2(z) = \dfrac{(1-z)+\omega(1-\bar{z})}{-\bar{\omega}\bar{z}(1-z)+\omega z(1-\bar{z})}z, \quad (\mathrm{Im}(z) > 0)$
Nagel point	$\mathrm{NAG}(z) = \dfrac{z\lvert 1-z\rvert - (2-z)\lvert z\rvert - z}{(1-2z) - \lvert z\rvert + \lvert 1-z\rvert}z$
Napoleon points	$\mathrm{NAP}_1(z) = \dfrac{\sqrt{3}i+\bar{\omega}z}{\sqrt{3}iz-\omega}\cdot\dfrac{\omega z+\bar{\omega}\bar{z}-1}{\omega z+\bar{\omega}\bar{z}-z\bar{z}}z \quad (\mathrm{Im}(z) > 0)$
	$\mathrm{NAP}_2(z) = \dfrac{\sqrt{3}i-\omega z}{\sqrt{3}iz+\bar{\omega}}\cdot\dfrac{\bar{\omega}z+\omega\bar{z}-1}{\bar{\omega}z+\omega\bar{z}-z\bar{z}}z \quad (\mathrm{Im}(z) > 0)$
the centres on Kiepert's hyperbola	$K_\theta(z) = \dfrac{(2-\lambda-\bar{\lambda})-(1-\bar{\lambda})z}{(2-\lambda-\bar{\lambda})z-(1-\lambda)}\cdot\dfrac{(z-\bar{z})-(\lambda z-\bar{\lambda}\bar{z})+(\lambda-\bar{\lambda})}{(z-\bar{z})-(\lambda z-\bar{\lambda}\bar{z})+(\lambda-\bar{\lambda})z\bar{z}}z,$ $(\lambda = e^{i\theta}, \mathrm{Im}(z) > 0)$

Table 3. Miscellaneous triangle functions

Special point	Triangle function	
Brocard points	$\mathrm{BROC}_1(z) = \bar{z}' = \dfrac{1}{1-\bar{z}}$,	$\mathrm{BROC}_2(z) = \bar{z}'' = \dfrac{\bar{z}-1}{\bar{z}}$
intersection of first interior angle bisector and first side	$-\dfrac{z}{\lvert z\rvert}$	
intersection of first exterior angle bisector and first side	$\dfrac{z}{\lvert z\rvert}$	

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