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Three Theorems on ρ*-Mixing Random Fields

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This paper generalizes results by Bradley.[3] Suppose that for $1 = 1, 2,...$ $\{X_k^1: k \in \mathbb{Z}^d\}$ is a centered, weakly stationary ρ^*-mixing random field, and suppose $\lim_{l \to \infty} \text{Cov}(X_0^1, x_k^1)$ exists, any $k \in \mathbb{Z}^d$. Then the successive spectral densities converge uniformly to a continuous function. For a sequence of strictly stationary random fields that are uniformly ρ^*-mixing and satisfy a indeberg condition, a CLT is proved for sequences of sums from the fields. This result is then applied: given a centered strictly stationary ρ^*-mixing random field whose probability density and joint densities are continuous, then a kernel estimator for the probability density obeys the CLT.

KEY WORDS: Stationary random fields; ρ^*-mixing, spectral density; Central Limit Theorem.

1. INTRODUCTION

Let d be a positive integer. A d-dimensional discrete field of complex random variables $\{X_k: k \in \mathbb{Z}^d\}$ will be called "centered" if $EX_k = 0$ for all k. The field is "weakly stationary" if $E|X_0|^2 < \infty$ and $EX_k\bar{X}_j$ depends only on the vector $k-j$. The field is "weakly stationary of fourth degree" if it is weakly stationary and also satisfies: $E|X_0|^4 < \infty$ and $EX_aX_b\bar{X}_c\bar{X}_d = EX_0X_{b-a}\bar{X}_{c-a}\bar{X}_{d-a}$. A field is "strictly stationary" if for any finite set $S \subset \mathbb{Z}^d$ and vector $v \in \mathbb{Z}^d$, the set of r.v.'s $\{X_k: k \in S\}$ has the same joint distribution as the set $\{X_{k+v}: k \in S\}$.

CCWS will mean centered, complex, and weakly stationary. CRSS will mean centered, real, and strictly stationary. CCSS will mean centered, complex, and strictly stationary. CCWFS will mean centered, complex, and weakly stationary of the fourth degree. "almost all k" will mean "all but finitely many k." The symbol χ will mean the indicator function of a set.

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Let T denote $\{z \in \mathbb{C}: |z| = 1\}$. For $t \in T^d$, let λ be the vector in $(-\pi, \pi]^d$ such that $t = (\exp i\lambda_1, ..., \exp i\lambda_d)$. Let μ_T denote the normalized Lebesgue measure $(2\pi iz)^{-1}\, dz$ on T, and let μ_T^d be $\mu_T x \cdots x \mu_T$, the d-dimensional product measure on T^d.

If $\{X_k: k \in \mathbb{Z}^d\}$ is a CCWS random field, then a nonnegative Borel integrable function f is a "spectral density" for the field if for any $k \in \mathbb{Z}^d$,

$$EX_k \bar{X}_0 = \int_{T^d} e^{ik \cdot \lambda} f(t)\, d\mu_T^d(t)$$

Here $k \cdot \lambda$ denotes the dot product. If $k \in \mathbb{Z}^d$, then $\|k\|$ is the Euclidean norm. For two nonempty disjoint sets $S, T \subset \mathbb{Z}^d$, we define $\mathrm{dist}(S, T)$ to be $\min\{\|j-k\|: j \in S, k \in T\}$. Let $F(S) = \{\sum_{k \in S} a_k X_k: a_k \in \mathbb{C}$, and $a_k = 0$ for almost all $k\}$, and $F(T) = \{\sum_{k \in T} a_k X_k: a_k \in \mathbb{C}$, and $a_k = 0$ for almost all $k\}$. Let $\sigma(S)$ be the σ-field generated by $\{X_k: k \in S\}$, and define $\sigma(T)$ similarly.

We now define three measures of dependence of the sets S and T.

$$r(S, T) := \sup\{|Ef\bar{g}|/\|f\|_2\, \|g\|_2: f \in F(S),\ g \in F(T)\}$$

$$\rho(S, T) := \sup\{|E[(f - Ef)(g - Eg)]|/\|f - Ef\|_2\, \|g - Eg\|_2: f \in L_2(\sigma(S)),\ g \in L_2(\sigma(T))\}$$

$$\alpha(S, T) := \sup\{|P(A \cap B) - P(A)\, P(B)|: A \in \sigma(S),\ B \in \sigma(T)\}$$

From these we obtain mixing coefficients. For any real number $s \geqslant 1$, define

$$r^*(s) := \sup\{r(S, T): \mathrm{dist}(S, T) \geqslant s\}$$

$$\rho^*(s) := \sup\{\rho(S, T): \mathrm{dist}(S, T) \geqslant s\}$$

$$\alpha^*(s) := \sup\{\alpha(S, T): \mathrm{dist}(S, T) \geqslant s\}$$

For any $s \geqslant 1$, $r^*(s) \leqslant \rho^*(s)$, so the condition $\lim_{s \to \infty} r^*(s) = 0$ is weaker than $\lim_{s \to \infty} \rho^*(s) = 0$. When a Lindeberg condition is needed, as in Sections 4 and 5, we must have a truncation of some X_k's; $r^*(s)$ only concerns linear combinations of X_k's so we use $\lim_{s \to \infty} \rho^*(s) = 0$. Theorem 3.1 and the preliminaries in Section 2, however, need only $r^*(s)$ to go to 0 as $s \to \infty$.

$\alpha(S, T)$ and measures based on it appears often in mixing research (see Bradley[2] for a survey). $\alpha(S, T) \leqslant \rho(S, T)$ for all S and T, but Bradley[5] showed that if X is strictly stationary, then $\alpha(s) \leqslant \rho^*(s) \leqslant 2\pi\alpha^*(s)$; so Theorems 4.1 and 5.1 could be stated with the condition: $\lim_{s \to \infty} \alpha^*(s) = 0$. Since the proof of Theorem 4.1 used the definition of $\rho^*(s)$, that is the mixing coefficient used here.

In this paper, Theorem 3.1 applies Theorem 2.1 (in Preliminaries) to the case of a sequence of random fields with convergent covariances. Falk[6] proved a similar result; but in that paper it was assumed that for the lth field, $\sum_{m=0}^{\infty} r_l(2^m)$ converges, and that the convergence is uniform over all l. In Theorem 3.1, it is assumed that $\sup_l r_l^*(s)$ goes to zero as $s \to \infty$, but not that the r^*-coefficients are summable in some way.

Theorem 4.1 resembles Theorem 4 in Bradley[3]; here, however, the successive block sums are drawn from a sequence of random fields, not one random field.

Theorem 5.1 is similar to Theorem 3.1(i) in Bradley.[1] There it was assumed that $\sum_{n=1}^{\infty} \rho(2^n) < \infty$ and that $b_n^{-1} = o(n^{\beta})$ for some $\beta \in (0, 1)$. Theorem 5.1 assumed instead that $\rho^*(s) \to 0$ as $s \to \infty$ with no summability condition on the ρ^* coefficients or growth condition on b_n^{-1}.

It might be noted that by restricting the cardinalities of S and T, we can define a large class of mixing coefficients. If j, k are positive integers or ∞, define

$$\rho_{j,k}^*(s) = \sup\{\rho(S, T): \operatorname{dist}(S, T) \geqslant s, \operatorname{card} S \leqslant j, \operatorname{card} T \leqslant k\}$$

$\alpha^*(s)$ and $\rho^*(s)$ can be defined likewise (see Bradley[4]). $\rho^*(s)$ of this paper is $\rho_{\infty,\infty}^*(s)$ in this terminology. Tran[9] obtained results similar to those of Section 5 of this paper, using the condition

$$\sup_{j,k \geqslant 1} \frac{\alpha_{j,k}^*(s)}{j, k} \to 0 \qquad \text{as} \quad s \to \infty \tag{1.1}$$

This is weaker than the condition $\lim_{s \to \infty} \rho^*(s) = 0$, but an exponential rate of convergence was assumed for Eq. (1.1) in Ref. 9.

2. PRELIMINARIES

We now quote the following results.

Lemma 2.1. (Lemma 1 of Bradley[3]): Suppose $0 < r < 1$. Suppose $X_1, ..., X_n$ is a family of centered complex random variables such that $\|X\|_2 < \infty$ for all $j \in \{1, ..., n\}$, and such that: for any two disjoint nonempty subsets $S, T \subset \{1, ..., n\}$, we have $|E(\sum_{k \in S} X_k)(\sum_{k \in T} \bar{X}_k)| \leqslant r \cdot \|\sum_{k \in S} X_k\|_2 \cdot \|\sum_{k \in T} X_k\|_2$. Then

$$\frac{(1-r)}{(1+r)} \sum_{k=1}^{n} E|X_k|^2 \leqslant E\left|\sum_{k=1}^{n} X_k\right|^2 \leqslant \frac{(1+r)}{(1-r)} \sum_{k=1}^{n} E|X_k|^2$$

Lemma 2.2. (Lemma 2 of Bradley[3]): If $q := \{q_m\}$ is a nonincreasing sequence in [0, 1] such that $\lim_{m\to\infty} q_m < 1$, then there exists a positive constant $A = A(q)$ such that: If $\{X_k: k \in \mathbb{Z}^d\}$ is a CCWS random field for which $r^*(m) \leqslant q_m$, $\forall m \geqslant 1$, then for any finite set $S \subset \mathbb{Z}^d$,

$$E\left|\sum_{k\in S} X_k\right|^2 \leqslant A\cdot(\text{card } S)E\,|X_0|^2 \tag{2.1}$$

For a random field $X := \{X_k: k \in \mathbb{Z}^d\}$, $S(X:m)$ will denote the sum $\sum X_k$, with the sum taken over all $k := (k_1,..., k_d)$ such that $1 \leqslant k_S \leqslant m$, for $s = 1,..., d$. $F(X:m)$ will denote $m^{-d}E\,|S(X:m)|^2$, and $G(X:m)$ will be $m^{-2d}E\,|S(X:m)|^4$.

Lemma 2.3. (Lemma 3 of Bradley[3]): If $q := \{q_m\}$ is a nonincreasing sequence in [0, 1] such that $\lim_{m\to\infty} q_m = 0$, then for any $\varepsilon > 0$ there exists a positive integer $M(q, \varepsilon)$ such that: If $\{X_k: k \in \mathbb{Z}^d\}$ is a CCWS random field for which $r^*(m) \leqslant q_m$ $\forall m \geqslant 1$, then for any positive integer $M \geqslant M(q, \varepsilon)$, and any positive integer n,

$$|F(X:M) - F(X:nM)| \leqslant \varepsilon\cdot E\,|X_0|^2 \tag{2.2}$$

Further, $M(q, \varepsilon)$ can be chosen as follows: Let A be the constant $A(q)$ of Lemm 2.2, and let $L(q, \varepsilon)$ be a positive integer so large that $q_{L(q,\varepsilon)} < (\varepsilon/6A)^2$. Then $M(q, \varepsilon)$ need only be sufficiently large that $(1 + L(q, \varepsilon)/M(q, \varepsilon))^d - 1 < (\varepsilon/6A)^2$.

Lemma 2.4. (Lemma 4 of Bradley[3]): If $X := \{X_k: k \in \mathbb{Z}^d\}$ is a CCWS random field for which $r^*(m) \to 0$ as $m \to \infty$, then $\lim_{m\to\infty} F(X:m)$ exists in $[0, \infty)$.

For any $t \in T^d$, and any random field $\{X_k: k \in \mathbb{Z}^d\}$, $X_k^{(t)}$ will denote $e^{-ik\cdot\lambda}\cdot X_k$.

Lemma 2.5. (Lemma 18.4.1 in Ref. 7): If $\{\zeta_j\}_{j=1}^{\infty}$, $\{\xi_j\}_{j=1}^{\infty}$, and ζ are all random variables, and if ζ_j converges weakly to ζ as $j \to \infty$, and $\xi_j \to 0$ in probability as $j \to \infty$, then $\zeta_j + \xi_j$ converge weakly to ζ as $j \to \infty$.

Theorem 2.1. (Theorem 1 of Ref. 3): If $\{X_k: k \in \mathbb{Z}^d\}$ is a CCWS random field such that $r^*(m) \to 0$ as $m \to \infty$, then $\{X_k\}$ has a continuous spectral density $f(t)$ on T^d. Define $X_k^{(t)} = e^{-ik\cdot\lambda}X_k$ and $X^{(t)} := \{X_k^{(t)}: k \in \mathbb{Z}^d\}$; then $f(t) = \lim_{m\to\infty} F(X^{(t)}:m)$.

Proposition 2.1. Let $q := \{q_m\}$ be a nonincreasing sequence in [0, 1] such that $\lim_{m\to\infty} q_m = 0$. For $\varepsilon > 0$, let $M(q, \varepsilon)$ be the constant of

Lemma 2.3. Suppose $\{X_k: k \in \mathbb{Z}^d\}$ is a CCWS random field such that $r^*(m) \leqslant q_m$ for all positive integers m. By Theorem 2.1, $\{X_k: k \in \mathbb{Z}^d\}$ has a continuous nonnegative spectral density f.

Corollary of Lemma 2.3 and Theorem 2.1. Let $\{q_m\}$ and $\{X_k: k \in \mathbb{Z}^d\}$ be as in Proposition 2.1. Then for any integer $N \geqslant M(q, \varepsilon)$, and any $t \in T^d$, we have $|F(X^{(t)}: N) - f(t)| \leqslant \varepsilon$.

Proof. $\{X_k: k \in \mathbb{Z}^d\}$ is a CCWS random field with the same r-coefficients as $\{X_k: k \in \mathbb{Z}^d\}$. Therefore Lemma 2.3 applies: Since $N \geqslant M(q, \varepsilon)$, we have that for any positive integer k, $|F(X^{(t)}: N) - F(X^{(t)}: kN)| \leqslant \varepsilon$. $f(t) = \lim_{n \to \infty} F(X^{(t)}: n) = \lim_{k \to \infty} F(X^{(t)}: kN)$, so $|F(X^{(t)}: N) - f(t)| = \lim_{k \to \infty} |F(X^{(t)}: N) - F(X^{(t)}: kN)| \leqslant \sup_{k \geqslant 1} |F(X^{(t)}: N) - F(X: kN)| \leqslant \varepsilon$.

3. CONVERGENCE OF SPECTRAL DENSITIES

Suppose $\{\{X_{l,k}: k \in \mathbb{Z}^d\}: l = 1, 2,...\}$ is a sequence of CCWS random fields, and n a positive integer, then let $S(X_l: n)$ and $F(X_l: n)$ denote $S(X: n)$ and $F(X: n)$ for the lth field $\{X_{l,k}: k \in \mathbb{Z}^d\}$. For any real number $m \geqslant 1$, let $r_l^*(m)$ and $\rho_l^*(m)$ denote $r^*(m)$ and $\rho^*(m)$ for $\{X_{l,k}: k \in \mathbb{Z}^d\}$. If $\{X_{l,k}: k \in \mathbb{Z}^d\}$ has a continuous spectral density function, let this density be denoted $f_l(t)$.

Theorem 3.1. Suppose $\{\{X_{l,k}: k \in \mathbb{Z}^d\}: l = 1, 2,...\}$ is a sequence of CCWS random fields. Suppose that $\sup_l r_l^*(m) \to 0$ as $m \to \infty$, and that $\lim_{l \to \infty} EX_{l,0}\bar{X}_{l,k}$ exists for all $k \in \mathbb{Z}$. Then there exists a continuous nonnegative function g on T^d such that $\sup_{t \in T^d} \|g(t) - f_l(t)\| \to 0$ as $l \to \infty$.

Proof. For each $l \in \{1, 2,...\}$, the nonnegative continuous spectral density f_l of $\{X_{l,k}: k \in \mathbb{Z}^d\}$ exists by Theorem 2.1, and $f_l(t) = \lim_{n \to \infty} F(X_l^{(t)}: n)$. Also, $E|X_{l,0}^{(t)}|^2 = E|X_{l,0}|^2$, $\forall t \in T^d$ and $l \geqslant 1$, and also $\lim_{l \to \infty} E|X_{l,0}|^2$ exists, so there is a constant C such that $E|X_{l,0}^{(t)}|^2 < C$ for any $t \in T^d$ and $l \geqslant 1$. Let $q_m = \sup r_l^*(m)$. For each $l \geqslant 1$ and $t \in T^d$, the random field $\{X_{l,k}^{(t)}: k \in \mathbb{Z}^d\}$ satisfies the conditions of Lemma 2.3. Choose $\varepsilon > 0$ and let $M = M(q, \varepsilon/C)$. Then

$$\lim_{n \to \infty} F(X_l^{(t)}: nM) = \lim_{m \to \infty} F(X_l^{(t)}: m) = f_l(t), \qquad \forall t \in T^d, \quad \forall l \geqslant 1 \quad (3.1)$$

By Eqs. (2.2) and (3.1),

$$|F(X_l^{(t)}: M) - f_l(t)| \leqslant \sup_{n \in N} |F(X_l^{(t)}: M) - F(X_l^{(t)}: nM)|$$
$$\leqslant (\varepsilon|C) \cdot E|X_{l,0}^{(t)}|^2 < \varepsilon, \qquad \forall t \in T^d, \quad \forall l \geqslant 1 \quad (3.2)$$

Let V_k denote $\lim_{l \to \infty} EX_{l,0}\bar{X}_{l,k}$. Define $S(M)$ to be $\{k \in \mathbb{Z}^d: 1 \leqslant k_p \leqslant M,\ p = 1,..., d\}$ and define $g_\varepsilon(t) = M^{-d} \cdot \sum_{j,k \in S(M)} e^{i(k-j)\cdot\lambda} V_{k-j}$. There exists a positive integer $l(\varepsilon)$ such that $|EX_{l,0}\bar{X}_{l,k} - V_k| < M^{-d} \cdot \varepsilon$ if $l \geqslant l(\varepsilon)$, for $\forall k \in S(M)$. Then for an integer $l \geqslant l(\varepsilon)$ and $t \in T^d$,

$$|F(X_l^{(t)} : M) - g_\varepsilon(t)| \leqslant M^{-d} \left| \sum_{j,k \in S(M)} e^{i(k-j)\cdot\lambda}(EX_{l,j}\bar{X}_{l,k} - V_{k-j}) \right|$$
$$\leqslant M^{-d} \cdot M^{2d} \cdot M^{-d} \cdot \varepsilon = \varepsilon \tag{3.3}$$

By Eqs. (3.2) and (3.3),

$$|f_l(t) - g_\varepsilon(t)| < 2\varepsilon, \qquad \forall t \in T^d, \qquad \forall l \geqslant l(\varepsilon) \tag{3.4}$$

By Eq. (3.4),

$$\varlimsup_{l \to \infty} f_l(t) - \varliminf_{l \to \infty} f_l(t) = \varlimsup_{l \to \infty} f_l(t) - g_\varepsilon(t) + g_\varepsilon(t) - \varliminf_{l \to \infty} f_l(t)$$
$$\leqslant 2 \cdot \varlimsup_{l \to \infty} |f_l(t) - g_\varepsilon(t)| < 4\varepsilon, \qquad \text{any } t \in T^d$$

This is true for any $\varepsilon > 0$, so $\lim_{l \to \infty} f_l(t)$ exists for any $t \in T^d$. Let $g(t)$ denote $\lim_{l \to \infty} f_l(t)$.

For any $\varepsilon > 0$, if $l \geqslant l(\varepsilon)$ and $t \in T^d$, by Eq. (3.4)

$$|g(t) - f_l(t)| \leqslant |g(t) - g_\varepsilon(t)| + |g_\varepsilon(t) - f_l(t)|$$
$$= \lim_{j \to \infty} |f_j(t) - g_\varepsilon(t)| + |g_\varepsilon(t) - f_l(t)|$$
$$\leqslant \sup_{j \geqslant l(\varepsilon)} |f_j(t) - g_\varepsilon(t)| + |g_\varepsilon(t) - f_l(t)| \leqslant 2\varepsilon + 2\varepsilon = 4\varepsilon$$

Hence $\sup_{t \in T^d} |g(t) - f_l(t)| \leqslant 4\varepsilon$ if $l \geqslant l(\varepsilon)$, and $\{f_l: l = 1, 2,...\}$ converges uniformly to g. Since f_l is nonnegative and continuous for each l, g is also.

4. A CLT FOR A SEQUENCE OF RANDOM FIELDS

Theorem 4.1. Suppose $\{X_l := \{X_{l,k}: k \in \mathbb{Z}^d\},\ l = 1, 2,...\}$ is a sequence of CRSS random fields such that $\sup_l \rho_l^*(m) \to 0$ as $m \to \infty$, $\lim_{l \to \infty} f_l(1) > 0$, $0 < \inf_l EX_{l,0}^2$, and $\sup_l EX_{l,0}^2 < \infty$. If $\{n_l\}$ is a sequence of positive integers such that (i) $n_l \to \infty$ as $l \to \infty$ and (ii) $\lim_{l \to \infty} EX_{l,0}^2 \cdot \chi\{|X_{l,0}| > \varepsilon n_l^{d/2}\} = 0$ for any $\varepsilon > 0$, then $S(X_l : n_l)/\|S(X_l : n_l)\|_2 \Rightarrow N(0, 1)$ as $l \to \infty$.

Proof. Without loss of generality we may assume that $EX_{l,0}^2 = 1$, all

$l \geqslant 1$. Let $\rho^*(m)$ denote $\sup_l \rho_l^*(m)$ and let $q_m = \rho^*(m)$. Let A be the constant $A(q)$ of Lemma 2.2. Since $r_l^*(m) \leqslant \rho_l^*(m)$ for every $m \geqslant 1$, A satisfies Eq. (2.1) for the random field X_l, each $l \geqslant 1$. There exists $C > 0$ such that $f_l(1) > 4C$ for almost all l. Choose any ε in $(0, C)$. Let M be the integer $M(q, \varepsilon)$ of Lemma 2.3. By condition (i), $n_l > M$ for almost all l; so by the Corollary to Lemma 2.3 and Theorem 2.1,

$$\begin{aligned}|F(X_l : n_l)| &\geqslant f_l(1) - |F(X_l : n_l) - f_l(1)| \\ &> 4C - \varepsilon > 3C\end{aligned}$$

That is,

$$E\,|S(X_l : n_l)|^2 \geqslant 3Cn_l^d, \qquad \text{for almost all } l \tag{4.1}$$

Now define: $X'_{l,k} = X_{l,k} \cdot \chi\{|X_{l,k}| \leqslant n^{d/2}\} - EX_{l,k} \cdot \chi\{|X_{l,k}| \leqslant n_l^{d/2}\}$, and $X''_{l,k} = X_{l,k} - X'_{l,k}$, for $\forall l \geqslant 1$, $\forall k \in \mathbb{Z}^d$. For any random variable Y, $E(Y - EY)^2 \leqslant EY^2$, and $E(Y-EY)^4 \leqslant EY^4 + 4\,|EY^3 EY| + 6(EY^2)^2 + 3(EY)^4 \leqslant 16EY^4$. Therefore

$$\begin{aligned}&\text{(a)} \quad E(X'_{l,k})^2 \leqslant EX_{l,k}^2 \cdot \chi\{|X_{l,k}| \leqslant n_l^{d/2}\} \leqslant EX_{l,k}^2 = 1 \\ &\text{(b)} \quad E(X''_{l,k})^2 \leqslant EX_{l,k}^2 \cdot \chi\{|X_{l,k}| \leqslant n_l^{d/2}\} \\ &\text{(c)} \quad E(X'_{l,k})^4 \leqslant 16EX_{l,k}^4 \cdot \chi\{|X_{l,k}| \leqslant n_l^{d/2}\}\end{aligned} \tag{4.2}$$

For each $l = 1$, the random fields $X'_l := \{X_{l,k} : k \in \mathbb{Z}^d\}$ and $X''_l := \{X''_{l,k} : k \in \mathbb{Z}^d\}$ have ρ-coefficients no greater than those of X_l. Hence A satisfies Eq. (2.1) for X'_l and X''_l. By Eq. (2.1), Eq. (4.1), condition (ii), and Eq. (4.2)(b),

$$\begin{aligned}\lim_{l \to \infty} \frac{E\,|S(X''_l : n_l)|^2}{E\,|S(X_l : n_l)|^2} &\leqslant \lim_{l \to \infty} \frac{A \cdot n_l^d \cdot E\,|X''_{l,0}|^2}{3Cn_l^d} \\ &\leqslant (A/3C) \lim_{l \to \infty} EX_{l,0}^2 \cdot \chi\{|X_{l,0}| \leqslant n_l^{d/2}\} = 0\end{aligned} \tag{4.3}$$

This implies:

$$S(X''_l : n_l)/\|S(X_l : n_l)\|_2 \to 0 \text{ in probability} \tag{4.4}$$

$S(X_l : n_l) = S(X'_l : n_l) + S(X''_l : n_l)$, so Eq. (4.4) and Lemma 2.5 give

$$\begin{aligned}&\frac{S(X_l : n_l)}{\|S(X_l : n_l)\|_2} \Rightarrow N(0, 1) \qquad \text{as} \quad l \to \infty \quad \text{if and only if} \\ &\frac{S(X'_l : n_l)}{\|S(X_l : n_l)\|_2} \Rightarrow N(0, 1) \qquad \text{as} \quad l \to \infty\end{aligned} \tag{4.5}$$

By Eqs. (2.1) and (4.2)(a),

$$\begin{aligned}
&|F(X_l : n_l) - F(X'_l : n_l)| \\
&\quad = n_l^{-d} \,|\, \|S(X_l : n_l)\|_2^2 - \|S(X'_l : n_l)\|_2^2| \\
&\quad = n_l^{-d} \,|\, \|S(X_l : n_l)\|_2 - \|S(X'_l : n_l)\|_2| \cdot (\|S(X_l : n_l)\|_2 + \|S(X'_l : n_l)\|_2) \\
&\quad \leqslant n_l^{-d} \, \|S(X_l : n_l) - S(X'_l : n_l)\|_2 \cdot [(An_l^d EX_{l,0}^2)^{1/2} + (An_l^d E(X'_{l,0})^2)^{1/2}] \\
&\quad = n_l^{-d} \, \|S(X''_l : n_l)\|_2 \cdot 2(An_l^d EX_{l,0}^2)^{1/2} \\
&\quad \leqslant 2n_l^{-d} [An_l^d E(X''_{l,0})^2]^{1/2} (An_l^d)^{1/2} = 2A \, \|X''_{l,0}\|_2 \qquad (4.6)
\end{aligned}$$

By Eq. (4.6), condition (ii), and Eq. (4.2)(c),

$$\begin{aligned}
&\lim_{l \to \infty} |F(X_l : n_l) - F(X'_l : n_l)| \\
&\quad \leqslant 2A \lim_{l \to \infty} \|X''_{l,0}\|_2 \\
&\quad \leqslant 2A \lim_{l \to \infty} [EX_{l,0}^2 \cdot \chi\{|X_{l,0}| \leqslant n_l^{d/2}\}]^{1/2} = 0 \qquad (4.7)
\end{aligned}$$

By Eq. (4.1) and (4.7),

$$E\,|S(X'_l : n_l)|^2 \geqslant E\,|S(X_l : n_l)|^2 - n_l^d \,|F(X_l : n_l) - F(X'_l : n_l)| \geqslant 3Cn_l^d - o(n_l^d)$$

so

$$E\,|S(X'_l : n_l)|^2 > 2Cn_l^d \qquad \text{for almost all } l \qquad (4.8)$$

Equation (4.3) also implies that

$$\lim_{l \to \infty} \frac{E\,|S(X'_l : n_l)|^2}{E\,|S(X_l : n_l)|^2} = 1$$

so that Eq. (4.4) is equivalent to

$$S(X'_l : n_l)/\|S(X'_l : n_l)\|_2 \Rightarrow N(0, 1) \qquad \text{as} \quad l \to \infty \qquad (4.9)$$

We now obtain a bound on $E(X')$. By condition (ii), there exists a nonincreasing sequence $\{h_l\} \subset [0, 1]$ such that $h_l \to 0$ as $l \to \infty$ and $\lim_{l \to \infty} EX_{l,0}^2 \cdot \chi\{|X_{l,0}| > h_l n_l^{d/2}\} = 0$. Then

$$E(X'_{l,0})^4 \leqslant 16E[X_{l,0}^4 \cdot \chi\{|X_{l,0}| \leqslant h_l n_l^{d/2}\}]$$
$$+ 16E[X_{l,0}^4 \cdot \chi\{h_l n_l^{d/2} < |X_{l,0}| \leqslant n_l^{d/2}\}]$$
$$\leqslant 16h_l^2 n_l^d \cdot E[X_{l,0}^2 \cdot \chi\{|X_{l,0}| \leqslant h_l n_l^{d/2}\}]$$
$$+ 16n_l^d \cdot E[X_{l,0}^2 \cdot \chi\{|X_{l,0}| > h_l n_l^{d/2}\}]$$
$$\leqslant 16h_l^2 n_l^d \cdot EX_{l,0}^2 + 16n_l^d E[X_{l,0}^2 \cdot \chi\{|X_{l,0}| > h_l n_l^{d/2}\}]$$
$$= 16n_l^d(h_l^2 + E[X_{l,0}^2 \cdot \chi\{|X_{l,0}| > h_l n_l^{d/2}\}])$$

The expressions in parentheses in the last line are $o(1)$, so

$$E(X'_{l,0})^4 = o(n_l^d) \tag{4.10}$$

We can introduce two nondecreasing sequences of positive integers $\{m_l\}$ and $\{b_l\}$ such that

(a) $\lim_{l \to \infty} m_l = \lim_{l \to \infty} b_l = \infty$;

(b) $m_l \leqslant b_l$ all $l \geqslant 1$;

(c) $\lim_{l \to \infty} m_l \cdot \rho^*(b_l) = 0$;

(d) $\lim_{l \to \infty} m_l \cdot b_l / n_l = 0$.

Let p_l be the smallest integer such that

$$m_l(p_l + b_l - 1) \leqslant m_l(p_l + b_l) \qquad \text{for each} \quad l \geqslant 1 \tag{4.11}$$

Now define "blocks" of random variables as follows: for each $l \geqslant 1$,

$$W_{l,j} = \sum \{X'_{l,k}: (j-1)(p_l + b_l) < k_1 \leqslant j(p_l + b_l) - b_l, \text{ and } 1 \leqslant k_s \leqslant n_l \text{ for } s = 2,..., d\}$$

for $j = 1,..., m_l$;

$$V_{l,j} = \sum \{X'_{l,k}: j(p_l + b_l) - b_l < k_1 \leqslant j(p_l + b_l); 1 \leqslant k_s \leqslant n_l \text{ for } s = 2,..., d\}$$

for $j = 1,..., m_l - 1$;

$$U_l = \{X'_{l,k}: m_l(p_l + b_l) - b_l < k_1 \leqslant n_l; 1 \leqslant k_s \leqslant n_l, \text{ for } s = 2,..., d\}$$

By Eq. (4.10), $0 \leqslant n_l - m_l(p_l + b_l) + m_l \leqslant n_l - m_l(p_l + b_l) + b_l \leqslant b_l$, so the number of $X'_{l,k}$'s in U_l is $[n_l - m_l(p_l + b_l) + b_l]\, n_l^{d-1} \leqslant b_l n_l^{d-1}$. Each $V_{l,j}$ is the sum of $b_l n_l^{d-1} X'_{l,k}$'s. Hence, by Eqs. (2.1) and (4.2)

$$E\left|\sum_{j=1}^{m(l)-1} V_{l,j} + U_l\right|^2 \leqslant [(m_l - 1)\, b_l n_l^{d-1} + b_l n_l^{d-1}] \cdot A \cdot E(X'_{l,0})^2$$
$$\leqslant m_l b_l n_l^{d-1} \cdot A \tag{4.12}$$

By Eqs. (4.8) and (4.12), and condition (d),

$$\lim_{l\to\infty} \frac{E\,|\sum_{j=1}^{m(l)-1} V_{l,j} + U_l|^2}{E\,|S(X'_l : n_l)|^2} \leqslant \lim_{l\to\infty} \frac{m_l b_l n_l^{d-1}}{2Cn_l^d} = 0 \tag{4.13}$$

$S(X'_l : n_l) = (\sum_{j=1}^{m(l)} W_{l,j}) + (\sum_{j=1}^{m(l)-1} V_{l,j}) + U_l$, so Eq. (4.12) implies

$$\left(\sum_{j=1}^{m(l)-1} V_{l,j} + U_l\right)\Big/ \|S(X'_l : n_l)\|_2 \to 0 \text{ in probability as } l \to \infty \tag{4.14}$$

Equation (4.14) and Lemma 2.5 mean that

$$\frac{\sum_{j=1}^{m(l)} W_{l,j}}{\|S(X'_l : n_l)\|_2} \Rightarrow N(0, 1) \qquad \text{as} \quad l \to \infty \text{ if and only if}$$
$$\frac{S(X'_l : n_l)}{\|S(X'_l : n_l)\|_2} \Rightarrow N(0, 1) \qquad \text{as} \quad l \to \infty \tag{4.15}$$

Equation (4.14) also implies that

$$\lim_{l\to\infty} \frac{E\,|\sum_{j=1}^{m(l)} W_{l,j}|^2}{E\,|S(X'_l : n_l)|^2} = 1 \tag{4.16}$$

By Eqs. (4.16) and (4.8),

$$E\left|\sum_{j=1}^{m(l)} W_{l,j}\right|^2 \geqslant Cn_l^d \qquad \text{for almost all } l \tag{4.17}$$

By Eqs. (4.15) and (4.16), Eq. (4.9) is equivalent to

$$\left(\sum_{j=1}^{m(l)} W_{l,j} \Big/ \left\|\sum_{j=1}^{m(l)} W_{l,j}\right\|_2\right) \Rightarrow N(0, 1) \qquad \text{as} \quad l \to \infty \tag{4.18}$$

If Eq. (4.18) is true, then, the theorem holds.

The $W_{l,j}$'s are at least b_l distant from each other, for l fixed. By Lemma 2.1, for any $l \geqslant 1$,

$$E\left|\sum_{j=1}^{m(l)} W_{l,j}\right|^2 = d_l \sum_{j=1}^{m(l)} E\,|W_{l,j}|^2$$

$$= d_l m_l E\,|W_{l,0}|^2 \qquad \text{where} \quad d_l \in \left(\frac{1-\rho^*(b_l)}{1+\rho^*(b_l)}, \frac{1+\rho^*(b_l)}{1-\rho^*(b_l)}\right) \tag{4.19}$$

For each l, let $Y_{l,j} = (W_{l,j}/\|\sum_{j=1}^{m(l)} W_{l,k}\|_2)$, for $j \in \{1,..., m_l\}$, and let $\{Y'_{l,1},..., Y'_{l,m(l)}\}$ be i.i.d. random variables, each with the distribution of $Y_{l,1}$. Then $E\,|\sum_{j=1}^{m(l)} Y_{l,j}|^2 = 1$. For any $t \in \mathbb{R}$ and $m \in \{2,..., m_l\}$,

$$\left|E\exp\left(it\sum_{j=1}^{m} Y_{l,j}\right) - E\exp\left(it\sum_{j=1}^{m-1} Y_{l,j}\right) E\exp(itY_{l,m})\right| \leqslant \rho^*(b_l) \tag{4.20}$$

By Eq. (4.20) and condition (c),

$$\lim_{l\to\infty}\left|E\exp\left(it\sum_{j=1}^{m(l)} Y_{l,j}\right) - \prod_{j=1}^{m(l)} E\exp(itY_{l,j})\right| = \lim_{l\to\infty} m_l\rho^*(b_l) = 0 \tag{4.21}$$

$E\exp(it\sum_{j=1}^{m(l)} Y'_{l,j}) = \prod_{j=1}^{m(l)} E\exp(itY_{l,j})$, so Eq. (4.21) means that at each $t \in \mathbb{R}$, the characteristic function of $\sum_{j=1}^{m(l)} Y_{l,j}$ converges to the same limit (if any) as the c.f.of $\sum_{j=1}^{m(l)} Y'_{l,j}$. Pointwise convergence of c.f.'s is equivalent to weak convergence, by the Continuity Theorem for probability measures; therefore Eq. (4.18) is equivalent to

$$\sum_{j=1}^{m(l)} Y'_{l,j} \Rightarrow N(0, 1) \qquad \text{as} \quad l \to \infty \tag{4.22}$$

By the Lyapunov version of the CLT, Eq. (4.22) is true if

$$\lim_{l\to\infty}\left(\sum_{j=1}^{m(l)} \|Y'_{l,j}\|_4^4 \Big/ \left\|\sum_{j=1}^{m(l)} Y'_{l,j}\right\|_2^4\right) = 0 \tag{4.23}$$

By Eq. (4.19) and the independence of the $Y'_{l,j}$'s,

$$\left\|\sum_{j=1}^{m(l)} Y'_{l,j}\right\|_2^2 = m_l E(Y'_{l,0})^2 = \frac{m_l E\,|W_{l,0}|^2}{\|\sum_{j=1}^{m(l)} W_{l,j}\|_2^2} = 1/d_l \tag{4.24}$$

By Eqs. (2.3), (4.19), and (4.24), and Lemma 2.5,

$$\lim_{l\to\infty}\left(\sum_{j=1}^{m(l)}\|Y'_{l,j}\|_4^4\Big/\left\|\sum_{j=1}^{m(l)}Y'_{l,j}\right\|_2^4\right)$$

$$=\overline{\lim_{l\to\infty}}\ m_l\cdot\left(\|W_{l,0}\|_4^4\Big/\left\|\sum_{j=1}^{m(l)}W_{l,j}\right\|_2^4\right)\cdot d_l^2$$

$$\leqslant \lim_{l\to\infty} m_l\cdot(B[(p_l n_l^{d-1})\cdot\|X'_{l,0}\|_4^4+(p_l n_l^{d-1})^2\,\|X'_{l,0}\|_2^4]/C^2n_l^{2d})\cdot d_l^2$$

$$\leqslant (B/C^2)\cdot\lim_{l\to\infty} d_l^2\cdot[\lim_{l\to\infty}(m_l p_l/n_l)\cdot\lim_{l\to\infty}(\|X'_{l,0}\|_4^4/n_l^d)+\lim_{l\to\infty}(m_l p_l^2/n_l^2)]$$

By condition (c), $\lim_{l\to\infty}\rho^*(b_l)=0$, so $d_l\to 1$ as $l\to\infty$. $\lim_{l\to\infty}(m_l p_l/n_l)=1$, and $\lim_{l\to\infty}(p_l/n_l)=0$. $\lim_{l\to\infty}(\|X'_{l,0}\|_4^4/n_l^d)=0$ by Eq. (4.9). Hence Eq. (4.22) is true and the theorem is proved.

5. A KERNEL DENSITY ESTIMATOR

Proposition 5.1. Suppose that $X:=\{X_k: k\in\mathbb{Z}^d\}$ is a CRSS random field, and suppose that this field has a continuous marginal probability density function $p(x)$ on $\mathbb{R}$. Also suppose that for each $k\in\mathbb{Z}^d$ the joint probability density p_k of $\{X_0, X_k\}$ is continuous. Let $\{b_l\}$ be a sequence of positive numbers such that $\lim_{l\to\infty} b_l=0$, and $\lim_{l\to\infty} l^d\cdot b_l=\infty$. Let w be a real nonnegative measurable function on $\mathbb{R}$ such that:

(a) $\int_{-\infty}^{\infty} w(u)\,du=1$;
(b) $\exists U>0$ such that $w(u)=0$ if $|u|>U$;
(c) $\int_{-\infty}^{\infty}(w(u))^{2+\delta}\,du<\infty$ for some $\delta>0$.

Finally, for each $l\in\{1, 2,...\}$, $k\in\mathbb{Z}^d$, and $z\in\mathbb{R}$, define

$$g_{l,k}(z)=b_l^{-1/2}\left[w\left(\frac{X_k-z}{b_l}\right)-Ew\left(\frac{X_k-z}{b_l}\right)\right]$$

For any real number z and positive integers n, l,

$$S(g_l(z):n)=\sum g_{l,k}(z)$$

with the sum over the set $\{k\in\mathbb{Z}^d;\ 1\leqslant k_s\leqslant n \text{ for } s=1,..., d\}$.

Theorem 5.1. Let $\{X_k: k\in\mathbb{Z}^d\}$, $\{b_l\}$, w, and $\{g_{l,k}: k\in\mathbb{Z}^d,\ l=1,2,...\}$ be as in Proposition 5.1, and suppose that for the field of X_k's, $\rho^*(m)\to 0$ as $m\to\infty$. If $\{z_1,..., z_N\}$ is a finite collection of distinct real numbers such that $p(z_j)>0$ for $j=1,\dots,N$, then let Σ be the $N\times N$ matrix with entries $\sigma_{ij}=\delta_{ij}\,p(z_j)\int w^2(u)\,du$. Also let 0_N denote the zero vector of N coordinates. Then as $l\to\infty$, the random vector $l^{-d/2}S(g_l(z_j):l))_{j=1}^N$ converges weakly to the multivariate normal $N(0_N,\Sigma)$ distribution.

Proof. Let $\{r_1,\dots,r_N\}$ be arbitrary real numbers. Define the random field $X_l:=\{X_{l,k}\colon k\in\mathbb{Z}^d\}$ by $X_{l,k}=\sum_{j=1}^N r_j\,g_{l,k}(z_j)$ and $\sigma^2=\sum_{j=1}^N r_j^2 p(z_j)\cdot\int w^2(u)\,du$. To prove the theorem, it suffices to show that

$$l^{-d/2}S(X_l:l)\Rightarrow N(0,\sigma^2)\qquad\text{as}\quad l\to\infty \tag{5.1}$$

Lemma 5.1. For any $a,b\in\mathbb{Z}$ and $z,v\in\mathbb{R}$, we have that $\lim_{l\to\infty}Eg_{l,a}(z)\,g_{l,b}(v)=\delta_{ab}\delta_{zv}\,p(z)\cdot\int_{-\infty}^{\infty}w^2(u)\,du$.

Proof. This follows from standard calculations involving the definitions of $g_{l,k}(z)$ and integrals with marginal and joint probability densities; it is left to the reader.

By the lemma,

$$\begin{aligned}\lim_{l\to\infty}EX_{l,0}X_{l,k}&=\lim_{l\to\infty}\sum_{j,m=1}^{N}r_jr_m\,Eg_{l,0}(z_j)\,g_{l,k}(z_m)\\&=\sum_{j,m=1}^{N}r_jr_m\delta_{0k}\delta_{jm}\,p(z_j)\int w^2(u)\,du\\&=\sum_{j=1}^{N}r_j^2\delta_{0k}\,p(z_j)\int w^2(u)\,du=\delta_{0k}\sigma^2\end{aligned}\tag{5.2}$$

The random fields $X_l:=\{\{X_{l,k}\colon k\in\mathbb{Z}^d\},\ l=1,2,\dots\}$ have ρ^*-coefficients no greater than the ρ^*-coefficients of $X_l:=\{X_k\colon k\in\mathbb{Z}^d\}$. Their r^* coefficients are also bounded by ρ^* for $X_l:=\{X_k\colon k\in\mathbb{Z}^d\}$. Therefore, by Theorem 2.1, $X_l:=\{X_{l,k}\colon k\in\mathbb{Z}^d\}$ has a nonnegative continuous spectral density f_l on T^d, for each $l\geqslant 1$. Since the ρ^*-coefficients of the sequence $\{X_{l,k}\colon k\in\mathbb{Z}^d\}$ converge to zero uniformly, and since, by Eq. (5.2), $\lim_{l\to\infty}EX_{l,0}X_{l,k}$ exists for all $k\in\mathbb{Z}^d$, Theorem 3.1 applies; there is a continuous function g on T such that $\sup_{T^d}\|f_l-g\|\to 0$ as $l\to\infty$. $g(t)$ has a Fourier series $\sum_{k\in\mathbb{Z}^d}a_ke^{ik\cdot\lambda}$. By Eq. (5.2),

$$\begin{aligned}a_k&=\int_{T^d}e^{-ik\cdot\lambda}g(t)\,\mu_T^d(t)=\lim_{l\to\infty}\int_{T^d}e^{-ik\cdot\lambda}f_l(t)\,\mu_T^d(t)\\&=\lim_{l\to\infty}EX_{l,0}X_{l,k}=\delta_{0k}\sigma^2\end{aligned}$$

so that $g(t)$ is identically σ^2. Therefore

$$\lim_{l\to\infty} f_l(1)=g(1)=\sigma^2 \tag{5.3}$$

Let $q_m=\rho^*(m)$, the ρ^*-coefficient for $\{X_k: k\in\mathbb{Z}^d\}$; then Lemma 2.3 applies to $\{X_{l,k}: k\in\mathbb{Z}^d\}$, for every $l\geqslant 1$. Choose $\varepsilon>0$ and let M be the $M(q,\varepsilon)$ of Lemma 2.3. By the corollary of Lemma 2.3 and Theorem 2.1,

$$|F(X_l:l)-f_l(1)|<\varepsilon \qquad \text{if} \quad l\geqslant M \tag{5.4}$$

Equations (5.3) and (5.4) means that

$$|F(X_l:l)-\sigma^2|<2\varepsilon \qquad \text{for almost all } l \tag{5.5}$$

Equation (5.5) is true for any $\varepsilon>0$. So

$$\lim_{l\to\infty} F(X_l:l)=\lim_{l\to\infty}(E\,|S(X_l:l)|^2)\cdot l^{-d}=\sigma^2 \tag{5.6}$$

Equation (5.1) is therefore equivalent to

$$(S(X_l:l)/\|S(X_l:l)\|_2)\Rightarrow N(0,1) \qquad \text{as} \quad l\to\infty \tag{5.7}$$

For the sequence of random fields $\{\{X_{l,k}: k\in\mathbb{Z}^d\},\ l=1,2,...\}$ it has been shown that: the ρ^*-coefficients converge to zero uniformly as $m\to\infty$; $\lim_{l\to\infty} f_l(1)=\sigma^2>0$; and $EX_{l,0}^2\to\sigma^2$, so $EX_{l,0}^2>0$ for almost all l. Of course $l\to\infty$, so Theorem 4.1 will apply if condition (ii) of that theorem is satisfied.

Claim. For any $\varepsilon>0$, $\lim_{l\to\infty} X_{l,0}^2\cdot\chi\{|X^{l,0}|>\varepsilon l^{d/2}\}=0$.

Proof of Claim. First,

$$E(w((X_0-z)/b_l))=\int w((x-z)/b_l)\cdot p(x)\,dx$$

$$=b_l\cdot\int w(y)\,p(b_l y+z)\,dy \tag{5.8}$$

Since $J_l'(z)\cdot\int w(y)\,dy\leqslant\int w(y)\,p(b_l y+z)\,dy\leqslant J_l(z)\cdot\int w(y)\,dy$, Eq. (5.8) implies that $\lim_{l\to\infty} b_l^{-1/2}E(w(X_0-z)/b_l)=0$, $\forall z\in\mathbb{R}$. There is then a positive integer $l(1)$ such that $\sum_{j=1}^{N}|r_j Ew((X_0-z_j)/b_l)|<b_l^{1/2}$ if $l\geqslant l(1)$. Let $H_{l,j}=|r_j w((X_0-z_j)/b_l)|$. Then

$$|X_{l,0}| = b_l^{-1/2} \left| \sum_{j=1}^{N} r_j [w((X_0 - z_j)/b_l) - Ew((X_0 - z)/b_l)] \right|$$
$$\leqslant b_l^{-1/2} \left(b_l^{1/2} + \sum_{j=1}^{N} H_{l,j} \right) \tag{5.9}$$

Choose $\delta \in (0, 1)$ such that $\int |w(u)|^{2+\delta}\, du < \infty$. If $a_1,..., a_N$ are positive numbers, then when $b_l < 1$,

$$\left(b_l^{1/2} + \sum_{j=1}^{N} a_j \right)^{2+\delta} \leqslant \left(b_l^{1/2} + \sum_{j=1}^{N} a_j \right)^2 \left(1 + \sum_{j=1}^{N} a_j^\delta \right)$$
$$< \left(\sum_{j,k,m=1}^{N} a_j a_k a_m^\delta \right) + \sum_{j,k=1}^{N} (2a_j a_k^\delta + a_j a_k)$$
$$+ \sum_{j=1}^{N} (2a_j + a_j^\delta) + b_l \tag{5.10}$$

By Eqs. (5.9) and (5.10), and the fact that $b_l \to 0$ as $l \to \infty$, we have

$$E\, |X_{l,0}|^{2+\delta} \leqslant b_l^{-(2+\delta)/2} E \left(b_l^{1/2} + \sum_{j=1}^{N} H_{l,j} \right)^{2+\delta}$$
$$\leqslant b_l^{-(2+\delta)/2} \left[\sum_{j,k,m=1}^{N} EH_{l,j} H_{l,k} H_{l,m}^\delta \right.$$
$$+ \sum_{j,k=1}^{N} (EH_{l,j} H_{l,k} + 2EH_{l,j} H_{l,k}^\delta)$$
$$\left. + \sum_{j=1}^{N} (2EH_{l,j} + EH_{l,j}^\delta) + b_l \right] \quad \text{for almost all } l \tag{5.11}$$

There are $(N+1)^3 - 1$ integrals in the expression in brackets, as well as b_l. Let $j \in \{1,..., N\}$, and $\gamma \in \{\delta, 1, 1+\delta, 2, 2+\delta\}$. Then

$$\lim_{l \to \infty} b_l^{-1} EH_{l,j}^\gamma = \lim_{l \to \infty} b_l^{-1} \int |r_j w((x - z_j)/b_l)|^\gamma\, p(x)\, dx$$
$$= \lim_{l \to \infty} \int |r_j w(y)|^\gamma\, p(b_l y + z_j)\, dy$$
$$= p(z_j) \int |r_j w(y)|^\gamma\, dy < \infty.$$

Hence there exists a constant $C > 1$ such that

$$EH_{l,j}^\gamma \leqslant Cb_l \qquad \text{for} \quad j \in \{1,..., N\} \cdot \gamma \in \{\delta, 1, 1+\delta, 2, 2+\delta\},\ l \geqslant 1 \tag{5.12}$$

By Eq. (5.12) and Hölder's inequality, all the expectations in the brackets in Eq. (5.11) are bounded by $C \cdot b_l$. Therefore

$$E\,|X_{l,0}|^{2+\delta} \leqslant b_l^{-(2+\delta)/2}[((N+1)^3-1)\cdot Cb_l + b_l] \leqslant 8b_l^{-\delta/2}N^3C \quad (5.13)$$

By Eq. (5.18), $\lim_{l\to\infty} EX_{l,0}^2 \cdot I\{|X^{l,0}| \geqslant \varepsilon l^{d/2}\} \leqslant \lim_{l\to\infty} (\varepsilon l^{d/2})^{-\delta} \cdot EX_{l,0}^{2+\delta} = \lim_{l\to\infty} \varepsilon^{-\delta}(l^d b_l)^{-\delta/2} \cdot 8N^3C = 0$, and the claim is proved.

By the claim, (ii) of Theorem 4.1 is true, so Eq. (5.7) is true, and the theorem is proved.

By Theorem 5.1, $l^{-d/2}(S(g_l(z_j):l))_{j=1}^N \Rightarrow N(0_N, \Sigma)$ as $l\to\infty$. Also, $(b_l l^d)^{-1/2}$ converges to zero. Therefore, for any real number z, $b_l^{-1/2} l^{-d}(S(g_l(z):l))$ converges to zero in probability. Now

$$b_l^{-1/2} l^{-d}(S(g_l(z):l)) = b_l^{-1} l^{-d} \sum_{k\in\{1,\dots,l\}^d} \left[w\left(\frac{X_k - z}{b_l}\right) - Ew\left(\frac{X_k - z}{b_l}\right)\right]$$

and $b_l^{-1}Ew((X_k - z)/b_l)$ converges to $\rho(z)$ as $l\to\infty$, independently of k; so $(b_l l^d)^{-1}\sum_{k\in\{1,\dots,l\}^d} w((X_k - z)/b_l)$ must converge to $\rho(z)$ in probability. Therefore we may use $(b_l l^d)^{-1}\sum_{k\in\{1,\dots,l\}^d} w((X_k - z)/b_l)$ as an estimator $\hat{p}(X)$ for the kernel density $p(X)$.

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