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On the Asymptotic Normality of Sequences of Weak Dependent Random Variables[1]

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The aim of this paper is to investigate the asymptotic normality for strong mixing sequences of random variables in the absense of stationarity or strong mixing rates. An additional condition is imposed to the coefficients of interlaced mixing. The results are applied to linear processes of strongly mixing sequences. The class of applications include filters of certain Gaussian sequences.

KEY WORDS: Central limit theorem; mixing sequences of random variables.

1. INTRODUCTION

Let (Ω, K, P) be a probability space and let $\mathscr{A}, \mathscr{B}$ be two sub σ-algebras of K. Define the strong mixing coefficient by

$$\alpha(\mathscr{A}, \mathscr{B}) = \sup_{A \in \mathscr{A}, B \in \mathscr{B}} |P(AB) - P(A)\,P(B)| \tag{1.1}$$

and the maximal coefficient of correlation

$$\rho(\mathscr{A}, \mathscr{B}) = \sup_{f \in L_2(\mathscr{A}),\, g \in L_2(\mathscr{B})} |\mathrm{corr}(f, g)| \tag{1.2}$$

A strictly stationary sequence $\{X_i\}_{i \in Z}$ is called α-mixing if $\alpha_n \to 0$ where

$$\alpha_n = \alpha(\sigma(X_i, i \leqslant 0), \sigma(X_i, i \geqslant n))$$

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It should be noted that in order for the central limit theorem to hold for a strictly stationary strong mixing sequence of random variables it is required the existence of moments of order strictly higher than two in combination with a polynomial mixing rate. (See Peligrad,[15] for a survey and Doukhan *et al.*,[7] for a recent result.)

In many situations the mixing rates are hard to estimate. Therefore it is interesting to replace the strong mixing rates by sufficient conditions imposed to some other dependence cofficients that might be in certain situations easier to verify. Papers by Bradley,[2] and Peligrad[16–17] are steps in this direction. Also because of the applications in statistics and in other fields it is interesting to have a CLT for nonstationary strong mixing sequences.

In this paper we prove several central limit theorems for strongly mixing sequences satisfying the Lindeberg condition and an additional assumption imposed to an interlaced mixing coefficient. For some other nonstationary weak dependent sequences we mention among other results papers by Utev,[18] Peligrad and Utev.[14]

For a stationary sequence $\{X_k\}_{k \in Z}$ denote by $\mathscr{F}_T = \sigma(X_i, i \in T)$ where T is a finite family of integers. Define

$$\alpha_n^* = \sup \alpha(\mathscr{F}_T, \mathscr{F}_S) \tag{1.3}$$

$$\rho_n^* = \sup \rho(\mathscr{F}_T, \mathscr{F}_S) \tag{1.4}$$

where these sup are taken over all pairs of nonempty finite sets S, T of Z such that $\text{dist}(S, T) \geqslant n$.

According to Bradley[4] for every $n \geqslant 1$

$$\alpha_n^* \leqslant \rho_n^* \leqslant 2\pi\alpha_n^*$$

Bradley[4] proved in the context of strictly stationary random fields that the condition $\alpha_n^* \to 0$ as $n \to \infty$ contains enough information to assume the CLT without any additional rate or moments higher than 2. Miller[12] analyzed the fourth moment of partial sums of such a random field not necessarily stationary, and proved the CLT for some estimators of spectral density for strictly stationary random fields. One of the results in Miller[13] is a central limit theorem for block sums from sequences of strictly stationary random fields satisfying a Lindeberg condition and uniformly satisfying $\rho_n^* \to 0$ with no assumption of a mixing rate and no assumption of higher order moments. Bryc and Smolenski[6] found bounds for the moments of partial sums for sequences of random variables satisfying $\rho_1^* < 1$.

In this paper we shall investigate the validity of CLT for strongly mixing sequences satisfying either $\rho_1^* < 1$ or $\lim_{n\to\infty} \rho_n^* < 1$. By the Remark 3 in Bryc and Smolenski[6] we know that these conditions do not necessarily imply $\lim_{n\to\infty} \rho_n^* = 0$.

Moreover in some situations these coefficients, or closely related ones are easy to estimate. With the same notation as in Bradley[3] we denote by

$$r_n^* = \sup |\text{corr}(V, W)| \tag{1.5}$$

where the supremum is taken over all finite subsets S, T of Z such that $\text{dist}(S, T) \geqslant n$ and over all the linear combinations $V = \sum_{i\in S} a_i X_i$ and $W = \sum_{i\in T} b_i X_i$.

According to the proof of Theorem 2 in Bradley[3] and the Remark 3 in Bryc and Smolenski[6] one can see that if $\{X_k\}_{k\in Z}$ has a bounded positive spectral density, i.e. $0 < m < f(t) < M$ for every t one has $r_1^* \leqslant 1 - m/M < 1$.

For stationary Gaussian sequences the coefficients ρ_n^* and r_n^* are identical (Kolmogorov and Rozanov[11]).

As a consequence our results are easily applicable to filters

$$\xi_{ni} = f_n(X_i, X_{i+1},..., X_{i+m_n})$$

where the underlying sequence $\{X_i\}$ is stationary, strongly mixing Gaussian sequence which has a bounded spectral density which stays away from 0. When $m_n = 0$ for every n such a sequence satisfies $\rho_n^* < 1$ and when $\sup_n m_n < \infty$ we have $\lim_{n\to\infty} \rho_n^* < 1$.

The strong mixing property for a Gaussian sequence can also be expressed in terms of the form of the spectral density (Ibragimov and Rozanov,[10] Chapters 4 and 5).

Some of our results, Theorems (2.1) and (2.2) do not assume stationarity and they deal with triangular arrays of random variables, $\{\xi_{ni}, 1 \leqslant i \leqslant k_n\}$ where $k_n \to \infty$.

In this context we shall define

$$\bar{\alpha}_{nk} = \sup_{s\geqslant 1} \alpha(\sigma(\xi_{ni}, i \leqslant s), \sigma(\xi_{nj}, j \geqslant s+k)) \tag{1.6}$$

and $\bar{\alpha}_k = \sup_n \bar{\alpha}_{nk}$.

The array will be called strongly mixing if $\lim_{k\to\infty} \bar{\alpha}_k = 0$. Similarly we define

$$\bar{\rho}_{nk}^* = \sup_{k} \rho(\sigma(\xi_{ni}, i \in T), \sigma(\xi_{nj}, j \in S)) \tag{1.7}$$

where T, $S \subset \{1, 2,..., k_n\}$ are nonempty subsets such that $\text{dist}(T, S) \geqslant k$ and

$$\bar{\rho}_k^* = \sup_n \bar{\rho}_{nk}^* \tag{1.8}$$

2. RESULTS

Our first two theorems refer to triangular arrays of strongly mixing random variables which satisfy the Lindeberg condition. No mixing rate is required or the existence of moments of order higher than 2. An additional condition is imposed to $\bar{\rho}_n^*$ which cannot be deleted from the theorems not even in the strictly stationary case (see Peligrad[15] for a survey and Doukhan *et al.*[7]).

Theorem 2.1. Let $\{\xi_{ni};\ 1 \leqslant i \leqslant k_n\}$ be a triangular array of centered random variables, which is strongly mixing and have finite second moments. Assume $\lim_{n \to \infty} \bar{\rho}_n^* < 1$. Denote by $\sigma_n^2 = \text{var}(\sum_{i=1}^{k_n} \xi_{ni})$ and assume

$$\sup_n \frac{1}{\sigma_n^2} \sum_{i=1}^{k_n} E\xi_{ni}^2 < \infty \tag{2.1}$$

and for every $\varepsilon > 0$

$$\frac{1}{\sigma_n^2} \sum_{i=1}^{k_n} E\xi_{ni}^2 I(|\xi_{ni}| > \varepsilon\sigma_n) \to 0 \qquad \text{as} \quad n \to \infty \tag{2.2}$$

Then

$$\frac{\sum_{i=1}^{k_n} \xi_{ni}}{\sigma_n} \xrightarrow{\mathscr{D}} N(0, 1) \qquad \text{as} \quad n \to \infty \tag{2.3}$$

Theorem 2.2. Assume $\{\xi_{ni};\ 1 \leqslant i \leqslant k_n\}$ is a triangular array of centered random variables which is strongly mixing and satisfies the Lindeberg condition in Eq. (2.2). Assume in addition $\bar{\rho}_1^* < 1$. Then Eq. (2.3) holds.

Motivated by the asymptotic behavior of linear processes we give next two corollaries to Theorems 2.1 and 2.2, respectively, where the main part of nonstationarity comes from nonrandom normalizers.

Corollary 2.1. Suppose $\{X_k\}$ is a strongly mixing sequence of random variables which is centered and $\{X_k^2\}$ is a uniformly integrable family. Consider the triangular array of random variables $\{a_{nk} X_k,\ 1 \leqslant k \leqslant n\}$

where a_{nk} are numerical constants and denote $\sigma_n^2 = \text{var}(\sum_{i=1}^n a_{ni} X_i)$. Assume

$$\max_{1 \leqslant k \leqslant n} \frac{|a_{nk}|}{\sigma_n} \to 0 \qquad \text{as} \quad n \to \infty \tag{2.4}$$

and

$$\sup_n \sigma_n^{-2} \sum_{k=1}^n a_{nk}^2 < \infty \tag{2.5}$$

Assume in addition $\lim_{n \to \infty} \bar{\rho}_n^* < 1$. Then

$$\frac{1}{\sigma_n} \sum_{k=1}^n a_{nk} X_k \xrightarrow{\mathscr{D}} N(0, 1) \qquad \text{as} \quad n \to \infty \tag{2.6}$$

Corollary 2.2. Assume $\{X_k\}$ is a strongly mixing sequence centered such that $\{X_k^2\}$ is a uniformly integrable family. Assume Eq. (2.4) holds, and in addition $\bar{\rho}_1^* < 1$ and $\inf_k EX_k^2 > 0$. Then Eq. (2.6) holds.

In the strictly stationary case Theorems 2.1 and 2.2 give also new results:

Corollary 2.3. Suppose $\{X_k\}$ is a strongly mixing strictly stationary sequence of random variables which is centered and has finite second moments. Assume $\lim_{n \to \infty} \rho_n^* < 1$ and $\sigma_n^2 \to \infty$. Then

$$\liminf \sigma_n^2 / n > 0 \tag{2.7}$$

and

$$\frac{\sum_{k=1}^n X_k}{\sigma_n} \xrightarrow{\mathscr{D}} N(0, 1) \qquad \text{as} \quad n \to \infty \tag{2.8}$$

Corollary 2.4. Suppose $\{X_k\}$ is a strongly mixing strictly stationary sequence of random variables which is centered and has finite second moments. Assume $\rho_1^* < 1$. Then Eqs. (2.7) and (2.8) hold.

3. PROOFS

The proof of Theorem 2.1 uses the following two lemmas. The first lemma gives bounds for the variance of partial sums in terms of a cofficient based on the correlation of sums. It is Lemma 1 in Bradley.[3]

Lemma 3.1. Suppose $0<r<1$. Suppose $\{X_1, X_2,..., X_n\}$ is a family of centered L_2-integrable random variables such that for any nonempty subset

$$S \subset \{1, 2,..., n\} \qquad S^* = \{1, 2,..., n\} - S$$

$$\left|\operatorname{corr}\left(\sum_{k \in S} X_k, \sum_{k \in S^*} X_k\right)\right| \leqslant r$$

Then

$$\frac{(1-r)}{(1+r)} \sum_{k=1}^{n} EX_k^2 \leqslant E\left(\sum_{k=1}^{n} X_k\right)^2 \leqslant \frac{1+r}{1-r} \sum_{k=1}^{n} EX_k^2$$

The next lemma gives estimates of higher moments of partial sums. It is Lemma 3 in Bryc and Smolenski.[6]

Lemma 3.2. Assume $\{X_1, X_2,..., X_n\}$ is a family of centered random variables which are integrable in L_q for q a fixed real, $2 \leqslant q \leqslant 4$. Denote by $\tilde{\rho} = \sup_S \rho(\mathscr{F}_S, \mathscr{F}_{S^*})$ where $S \subset \{1, 2,..., n\}$ and $S^* = \{1, 2,..., n\} - S$, and assume $\tilde{\rho} < 1$. Then there is a positive constant C depending only on q and $\tilde{\rho}$ such that

$$E\left(\left|\sum_{k=1}^{n} X_k\right|^q\right) \leqslant C\left(\sum_{k=1}^{n} E|X_k|^q + \left(\sum_{k=1}^{n} EX_k^2\right)^{q/2}\right) \tag{3.1}$$

In the next text we are going to use a variant of Lemma 3.2, namely

Lemma 3.3. Assume $\{X_1, X_2,..., X_n\}$ are centered random variables in L_q, $2 \leqslant q \leqslant 4$. Assume that there is a positive number p, $1 \leqslant p \leqslant n$ such that $\bar{\rho}_p^* < 1$ where $\bar{\rho}_p^*$ is defined by Eq. (1.8). Then we can find a constant C depending only on p, q and $\bar{\rho}_p^*$ such that Eq. (3.1) holds for this C.

Proof. The proof follows by Lemma 3.2 after a standard reduction procedure. Denote by k the integer part of n/p and write

$$\sum_{i=1}^{n} X_i = \sum_{j=0}^{p-1} Y_j + Y_p$$

where for every $0 \leqslant j \leqslant p-1$, $Y_j = \sum_{l=0}^{k} X_{lp+j}$ and $Y_p = \sum_{i=kp+1}^{n} X_i$.

Now we apply Lemma 3.2 to each Y_j, $0 \leqslant j \leqslant p-1$. Notice that for each j fixed the variables added in Y_j satisfy $\tilde{\rho} \leqslant \bar{\rho}_p^* < 1$ and Lemma 3.2 applies. Also Y_p contains at most p terms. The result follows now by standard arguments.

3.1. Proof of Theorem 2.1

The proof of this theorem requires several steps.

3.1.1. Normalization and Truncation

Denote by $\zeta_{ni}=\xi_{ni}/\sigma_n$ and notice that $\mathrm{var}(\sum_{i=1}^{k_n}\zeta_{ni})=1$. With this notation the conditions Eqs. (2.1) and (2.2) can be replaced by:

$$\sum_{i=1}^{k_n} E\zeta_{ni}^2 I(|\zeta_{ni}|>\varepsilon)\to 0 \quad \text{as} \quad n\to\infty \text{ for every } \varepsilon>0 \tag{3.2}$$

and

$$\sup_n \sum_{i=1}^{k_n} E\zeta_{ni}^2<\infty \tag{3.3}$$

Because of Eq. (3.2) we can construct a sequence of positive numbers $\varepsilon_n\to 0$ such that

$$\sum_{i=1}^{k_n} E\zeta_{ni}^2 I(|\zeta_{ni}|>\varepsilon_n)\to 0 \quad \text{as} \quad n\to\infty \tag{3.4}$$

We truncate now at the level ε_n.

Define

$$\eta_{ni}=\zeta_{ni}I(|\zeta_{ni}|\leqslant\varepsilon_n)-E\zeta_{ni}I(|\zeta_{ni}|\leqslant\varepsilon_n)$$

and

$$\gamma_{ni}=\zeta_{ni}I(|\zeta_{ni}|>\varepsilon_n)-E\zeta_{ni}I(|\zeta_{ni}|>\varepsilon_n)$$

Because $\lim_n \bar{\rho}_n^*<1$ we can find a positive integer p such that $\bar{\rho}_p^*<1$. By Lemma 3.3 applied with $q=2$ we have for some positive constant C which does not depend on n

$$\mathrm{var}\left(\sum_{i=1}^{k_n}\gamma_{ni}\right)\leqslant C\sum_{i=1}^{k_n}\mathrm{var}(\gamma_{ni})$$
$$\leqslant 2C\sum_{i=1}^{k_n} E\zeta_{ni}^2 I(|\zeta_{ni}|>\varepsilon_n)$$

which converges to 0 when $n\to\infty$ by Eq. (3.4).

Therefore the problem is reduced now to prove the central limit theorem for a triangular array of random variables $\{\eta_{ni}, i \leqslant 1 \leqslant k_n\}$ which is centered, and satisfies

$$|\eta_{ni}| \leqslant 2\varepsilon_n \qquad \text{where} \quad \varepsilon_n > 0, \quad \varepsilon_n \to 0 \tag{3.5}$$

$$\operatorname{var}\left(\sum_{i=1}^{k_n} \eta_{ni}\right) \to 1 \qquad \text{as} \quad n \to \infty \tag{3.6}$$

and

$$\sup_n \sum_{i=1}^{k_n} \operatorname{var} \eta_{ni} < \infty \tag{3.7}$$

3.1.2. Blocking Procedure

At this step we divide the variables in big blocks and small blocks, the sum of the variables in big blocks will then be approximated by a sum of independent random variables while the sum of variables in small blocks are negligible for the convergence in distribution. This is a variation of Bernstein's method of dealing with weak dependent random variables.

In the next text we denote $\alpha_n = \bar{\alpha}(n)$ and $[x]$ denotes the integer part of x. With $\{\varepsilon_n\}$, $\varepsilon_n \to 0$, as $n \to \infty$ constructed at the Step 1 fixed, we construct now a sequence of integers, $\{q_n\}$ such that the following three convergences hold simultaneously

$$q_n \to \infty \qquad \text{as} \quad n \to \infty \tag{3.8}$$

$$q_n \varepsilon_n \to 0 \qquad \text{as} \quad n \to \infty \tag{3.9}$$

$$q_n \bar{\alpha}([\varepsilon_n^{-1}]) \to 0 \qquad \text{as} \quad n \to \infty \tag{3.10}$$

This is possible because $\varepsilon_n \to 0$ and $\bar{\alpha}_n \to 0$. Notice also that by Eqs. (3.5), (3.6), and Lemma 3.3 without any loss of generality we can assume $k_n \to \infty$. As a matter of fact we can find a constant C such that $k_n \geqslant C\varepsilon_n^{-2}$.

For each positive integer n, we define recurrently the integers:

$$m_0 = 0$$

For $j = 0, 1, 2,\ldots$ put

$$m_{2j+1} = \min\left\{m;\, m > m_{2j},\ \sum_{i=m_{2j}+1}^{m} \operatorname{var} \eta_{ni} \geqslant q_n^{-1}\right\} \tag{3.11}$$

$$m_{2j+2} = m_{2j+1} + [\varepsilon_n^{-1}] \tag{3.12}$$

We denote now the consecutive sets of indexes

$$I_j = \{k\colon\ m_{2j} < k \leqslant m_{2j+1}\} \tag{3.13}$$

$$J_j = \{k\colon m_{2j+1} < k \leqslant m_{2(j+1)}\} \tag{3.14}$$

for $j = 0, 1, 2, \ldots$. It should be kept in mind that the integers m_j and the sets I_j and J_j depend on n but the dependence will be suppressed in the notations.

Notice that because $\sup_n \sum_{i=1}^{k_n} \operatorname{var} \eta_{ni} < \infty$, this procedure is going to produce a finite number of blocks of indexes I_j. Denote their number by ℓ. The construction ends when either J_ℓ or $I_{\ell+1}$ cannot be constructed, i.e., when either the number of the remaining variables after constructing I_ℓ is inferior to $[\varepsilon_n^{-1}]$ or after constructing J_ℓ the remaining variables satisfy $\sum_{j=m_{2\ell}+1}^{k_n} \operatorname{var} \eta_{nj} < q_n^{-1}$. After constructing I_ℓ we put all the remaining variables, if any, into a last block denoted by J_ℓ. Denote now by

$$Y_{nj} = \sum_{i \in I_j} \eta_{ni}$$

and

$$Z_{nj} = \sum_{i \in J_j} \eta_{ni}$$

for $1 \leqslant j \leqslant \ell$.

Notice that ℓ depends on n. Put $\ell = \ell_n$. By Eqs. (3.13), (3.11), and (3.7) we can find a constant C_1 which does not depend on n such that

$$C_1 > \sum_{i=1}^{k_n} \operatorname{var} \eta_{ni} \geqslant \sum_{j=1}^{\ell_n} \sum_{i \in I_j} \operatorname{var} \eta_{ni} \geqslant \ell_n q_n^{-1}$$

Therefore the number of blocks, ℓ_n, satisfies:

$$\ell_n \leqslant C_1 q_n \tag{3.15}$$

We estimate now the variance of $\sum_{j=1}^{\ell_n} Z_{nj}$. Notice first that because of the construction of the last block we can find (by using Lemma 3.3 with $q = 2$) a constant C_2 independent on n such that

$$\operatorname{var} Z_{n\ell_n} \leqslant C_2([\varepsilon_n]^{-1} \max_{1 \leqslant i \leqslant k_n} E\eta_{ni}^2 + q_n^{-1})$$

From this estimate we obtain by Eqs. (3.5) and (3.8)

$$\operatorname{var} Z_{n\ell_n} \to 0 \qquad \text{as} \quad n \to \infty \tag{3.16}$$

By Lemma 3.3 with $q=2$, applied twice and by Eqs. (3.12) and (3.14) we find some constants C_3 and C_4 such that

$$\operatorname{var}\left(\sum_{j=1}^{\ell_n-1} Z_{nj}\right) \leqslant C_3 \sum_{j=1}^{l_n-1} \operatorname{var} Z_{nj}$$
$$\leqslant C_4 \sum_{j=1}^{\ell_n-1} \sum_{i\in J_j} \operatorname{var} \eta_{ni} \leqslant C_4 \ell_n \varepsilon_n^{-1} \max_{1\leqslant i\leqslant k_n} (\operatorname{var} \eta_{ni})$$

whence by Eqs. (3.15), (3.5), and (3.9) this variance converges to 0 as $n\to\infty$. This remark in combination with Eq. (3.16) gives

$$\operatorname{var}\left(\sum_{j=1}^{\ell_n} Z_{nj}\right) \to 0 \qquad \text{as} \quad n\to\infty \tag{3.17}$$

This shows that $\sum_{j=1}^{\ell_n} Z_{nj}$ is negligible for the convergence in distribution. Moreover by Eqs. (3.6) and (3.17) we get

$$\lim_{n\to\infty} \operatorname{var}\left(\sum_{j=1}^{\ell_n} Y_{nj}\right) = 1 \tag{3.18}$$

Notice that by Lemma 3.1 since $\lim_{n\to\infty} \bar{\rho}_n^* < 1$ and the variables Y_{nj} are spaced apart by $[\varepsilon_n^{-1}]$ variables we can find by Eq. (3.18) two constants $0 < K_1 < K_2$ such that for every n sufficiently large: $(n \geqslant n_0)$ we have

$$0 < K_2 < \sum_{j=1}^{\ell_n} \operatorname{var} Y_{nj} < K_1 \tag{3.19}$$

Denote by $a_n = (\sum_{j=1}^{\ell_n} \operatorname{var} Y_{nj})^{1/2}$. Now by a standard argument based on recurrence and the definition of the strong mixing coefficients for every t we have the following estimate:

$$\left| E\exp\left(ita_n^{-1} \sum_{j=1}^{\ell_n} Y_{nj}\right) - \Pi_{j=1}^{\ell_n} E\exp(ita_n^{-1} Y_{nj})\right| \leqslant 16\ell_n \bar{\alpha}([\varepsilon_n^{-1}]) \tag{3.20}$$

which converges to 0 as $n\to\infty$ by Eqs. (3.10) and (3.15). Therefore the problem now is reduced to study the asymptotic behavior of a triangular array $\{Y_{nj}^*, 1\leqslant j\leqslant \ell_n\}$ of independent random variables such that for each n and j the variable Y_{nj}^* is distributed as Y_{nj} and satisfies Eq. (3.19).

3.1.3. The Proof of the Central Limit Theorem

At this step we prove that the triangular array $\{a_n^{-1} Y_{nj}^*;\ 1\leqslant j\leqslant \ell_n\}$ constructed at the Step 2 satisfies the C.L.T.

Because $\operatorname{var}(\sum_{j=1}^{\ell_n} a_n^{-1} Y_{nj}^*) = 1$ we have only to establish the Lindeberg condition, namely:

For every $\varepsilon > 0$

$$\sum_{j=1}^{\ell_n} EY_{nj}^2 I(|Y_{nj}| > \varepsilon) \to 0 \qquad \text{as} \quad n \to \infty \tag{3.21}$$

In order to establish Eq. (3.21) we shall estimate $\sum_{j=1}^{\ell_n} EY_{nj}^4$.

For every j fixed, $1 \leqslant j \leqslant \ell_n$ we apply Lemma 3.3 with $q = 4$ to each Y_{nj} and we get, for a certain positive constant C_5 which does not depend on n

$$EY_{nj}^4 \leqslant C_5 \left(\sum_{i \in I_j} E\eta_{ni}^4 + \left(\sum_{i \in I_j} E\eta_{ni}^2 \right)^2 \right) \tag{3.22}$$

By Eqs. (3.11), (3.13), and (3.5) we have:

$$\sum_{i \in I_j} E\eta_{ni}^2 \leqslant q_n^{-1} + \max_{1 \leqslant i \leqslant k_n} E\eta_{ni}^2 \leqslant q_n^{-1} + 4\varepsilon_n^2 \tag{3.23}$$

Once again by Eq. (3.5) we have

$$E\eta_{ni}^4 \leqslant 4\varepsilon_n^2 E\eta_{ni}^2 \tag{3.24}$$

Now by adding the relations in Eq. (3.22) and by taking into account Eqs. (3.23), (3.24), and (3.7) we can find a constant K_1, independent on n such that

$$\sum_{j=1}^{\ell_n} EY_{nj}^4 \leqslant K_1(\varepsilon_n^2 + q_n^{-1})$$

which approaches 0 as $n \to \infty$ by the construction of ε_n and Eq. (3.8). Therefore Eq. (3.21) is proved and as a consequence

$$a_n^{-1} \sum_{i=1}^{\ell_n} Y_{ni}^* \xrightarrow{\mathscr{D}} N(0, 1) \qquad \text{as} \quad n \to \infty$$

This convergence together with Eqs. (3.20), (3.17), and (3.19) gives

$$a_n^{-1} \sum_{i=1}^{k_n} \eta_{ni} \xrightarrow{\mathscr{D}} N(0, 1) \qquad \text{as} \quad n \to \infty \tag{3.25}$$

and the theorem will be proved if we show $\lim_{n \to \infty} a_n$ exists and equals 1.

3.1.4. The Identification of a_n

At this step we prove that $\{(\sum_{i=1}^{k_n} \eta_{ni})^2\}$ is a uniformly integrable family, which together with Eqs. (3.6) and (3.25) will imply by the convergence of the moments in the Central Limit Theorem that $\lim_{n\to\infty} a_n^{-1}=1$, which will complete the proof of Theorem 2.1. In order to prove this we shall estimate $E(\sum_{i=1}^{k_n}\eta_{ni})^4$. By Lemma 3.3 applied with $q=4$ for a constant C_1 independent on n we have

$$E\left(\sum_{i=1}^{k_n}\eta_{ni}\right)^4 \leqslant C_1\left[\left(\sum_{i=1}^{k_n}E\eta_{ni}^4\right)+\left(\sum_{i=1}^{k_n}E\eta_{ni}^2\right)^2\right]$$

and by Eqs. (3.24) and (3.7) we can find another constant C_2 such that

$$\sup_n E\left(\sum_{i=1}^{k_n}\eta_{ni}\right)^4 < C_2$$

and the result follows.

Proof of Theorem 2.2. By Lemma 2.1 and the fact that $\bar{\rho}_1^* < 1$ the condition Eq. (2.1) of Theorem 2.1 is satisfied. Therefore we apply Theorem 2.1 and we have the desired result.

Proof of Corollary 2.1. In order to prove this corollary we apply the Theorem 2.1 to the triangular array

$$\xi_{ni}=q_{ni}X_i, \qquad 1\leqslant i\leqslant n$$

By Eq. (2.5) and the uniform integrability of $\{X_i^2\}$ the condition Eq. (2.1) of Theorem 2.1 is satisfied. In order to verify Eq. (2.2) we estimate

$$\frac{1}{\sigma_n^2}\sum_{i=1}^{n} a_{ni}^2 EX_i^2 I(|X_i| > \varepsilon\, |a_{ni}|^{-1}\sigma_n)$$
$$\leqslant \max_{1\leqslant i\leqslant n} EX_i^2 I(|X_i| > \varepsilon\, |a_{ni}|^{-1}\sigma_n) \sum_{i=1}^{n} a_{ni}^2/\sigma_n^2$$

which is convergent to 0 by Eq. (2.4), Eq. (2.5) and the uniform integrability of $\{X_i^2\}$. Therefore Theorem 2.1 applies and we have the desired result.

Proof of Corollary 2.2. Under the condition $\rho_1^* < 1$, we can apply Lemma 2.1 and as a consequence Eq. (2.5) is satisfied. Therefore Corollary 2.2 follows now from Corollary 2.1.

Proof of Corollaries 2.3 and 2.4. By the proof of Theorem 3 of Bradley[3] with $\lambda=0$, one can easily see that under the conditions of Corollary 2.3, the relation in Eq. (2.7) follows. The conclusion in Eq. (2.8) follows by stationarity from Theorem 2.1. Corollary 2.4 is a consequence of Lemma 3.1 and Theorem 2.2.

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