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Optimizing a Linear Function over an Efficient Set[1]

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Abstract. The problem (P) of optimizing a linear function $d^T x$ over the efficient set for a multiple-objective linear program (M) is difficult because the efficient set is typically nonconvex. Given the objective function direction d and the set of domination directions D, if $d^T \pi \geqq 0$ for all nonzero $\pi \in D$, then a technique for finding an optimal solution of (P) is presented in Section 2. Otherwise, given a current efficient point $\hat{x}$, if there is no adjacent efficient edge yielding an increase in $d^T x$, then a cutting plane $d^T x = d^T \hat{x}$ is used to obtain a multiple-objective linear program ($\bar{\text{M}}$) with a reduced feasible set and an efficient set $\bar{E}$. To find a better efficient point, we solve the problem (I_i) of maximizing $c_i^T x$ over the reduced feasible set in ($\bar{\text{M}}$) sequentially for i. If there is a $x^i \in \bar{E}$ that is an optimal solution of (I_i) for some i and $d^T x^i > d^T \hat{x}$, then we can choose x^i as a current efficient point. Pivoting on the reduced feasible set allows us to find a better efficient point or to show that the current efficient point $\hat{x}$ is optimal for (P). Two algorithms for solving (P) in a finite sequence of pivots are presented along with a numerical example.

Key Words. Multiple-objective linear programming, efficient sets, domination cones, nonconvex optimization.

1. Introduction

In a multiple-objective linear program, a convex polyhedron X is given over which several linear objectives are to be maximized. Throughout

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this paper, we will assume that X is a bounded nondegenerate convex polyhedron.

Suppose that we have the feasible set in standard form,

$$X = \{x \in R^n \mid (A, I)x = b, x \geqq 0\},$$

where $(A, I) \in R^{m \times n}$ and $b \in R^m$, $b > 0$. Let

$$C = \begin{bmatrix} c_1^T \\ c_2^T \\ \vdots \\ c_k^T \end{bmatrix}$$

where $c_i \in R^n$ for $i = 1, \ldots, k$. With this understanding, a multiple-objective linear program is formulated as follows:

$$\text{(M)} \quad \max \quad Cx, \qquad \text{s.t.} \quad x \in X.$$

Definition 1.1. A point $x^0 \in X$ is an efficient point if there is no $x \in X$ such that $Cx \geqq Cx^0$ and $Cx \neq Cx^0$.

Throughout this paper, we will use the following notation for vectors. For $x, y \in R^n$, $x \geq y$ means $x \geqq y$ and $x \neq y$.

Let E be the set of efficient points for (M), and let X_{ex} be the set of extreme points for X. The basic problem (P) that we investigate is to maximize a linear function $d^T x$ over the set E,

$$\text{(P)} \quad \max \quad d^T x, \qquad \text{s.t.} \quad x \in E.$$

Associated with (P), the relaxed problem is

$$\text{(R)} \quad \max \quad d^T x, \qquad \text{s.t.} \quad x \in X.$$

The importance and motivation of problem (P) have been discussed extensively in the literature; see for example the presentations of Philip (Ref. 1), Isermann and Steuer (Ref. 2), Benson (Refs. 3–7), Dessouky, Ghiassi, and Davis (Ref. 8), and Weistroffer (Ref. 9). In particular, Benson shows in Ref. 3 that, in some modeling problems involving multiple objectives, models of the form (P) are more realistic and appropriate than the more usual multiple objective linear programs. Also, solving (P) with the efficient set defined implicitly avoids the computational difficulties of enumerating all efficient extreme points. The production planning example given by Benson in Ref. 3 illustrates these points nicely. We should also mention that, as special cases of problem (P), we can find upper and lower bounds for the individual objectives $c_i^T x$ over the efficient set.

In Ref. 1, Philip first studied problem (P) and suggested an algorithm for solving it. Later, Isermann and Steuer (Ref. 2) outlined a similar procedure for solving (P) where the objective function $d^T x$ is one of the multiple objectives $c_i^T x$ in (M). These methods use a cutting hyperplane and try to find an efficient point on the cutting face that has an adjacent efficient edge yielding an increase in the objective function $d^T x$. As discussed by Benson, who studied in Ref. 6 more general cases of problem (P), including the case where X is a general convex set, neither of these procedures explain explicitly how to find a point on the cutting face that has an adjacent efficient edge yielding an increase in $d^T x$. In Refs. 4 and 5, Benson suggests two methods for solving (P) using different approaches from the Philip method. These methods guarantee that at least one extreme point optimal solution of (P) can be found in a finite number of steps. One step of the Benson algorithms in Refs. 4 and 5 calls for finding, if one exists, a feasible solution to a nonlinear system of equations and inequalities. This step is implemented by solving a sequence of linear programming problems. Benson (Ref. 6) gave a number of algebraic and geometric results concerning the existence and nature of global optimal solutions to the problem (P).

In our paper, we develop methods for solving (P) that involve only pivoting on the feasible set for (M) or a reduced problem ($\bar{\text{M}}$) to overcome the local optimality problem that arises when, in an efficient searching procedure, an efficient extreme point is encountered having no efficient edges yielding an increase in $d^T x$. We conclude this section by giving an overview of our approach, which uses the Philip approach (Ref. 1) and makes it implementable and practical by explaining exactly how to use cutting hyperplanes to overcome the local optimality problem.

Given a current efficient point $\hat{x}$, a pivoting technique for finding adjacent efficient edges at $\hat{x}$ is discussed in Section 3. If there is no adjacent efficient edge yielding an increase in $d^T x$, which indicates that $\hat{x}$ is a local optimal solution of (P), then we add a cutting hyperplane $d^T x = d^T \hat{x}$ to the feasible set X and we consider the following multiple objective linear program:

$$(\bar{\text{M}}) \quad \max \quad Cx,$$
$$\text{s.t.} \quad x \in \bar{X},$$
$$\bar{X} = \{x \in X \mid d^T x \geqq d^T \hat{x}\}.$$

We let $\bar{E}$ denote the set of efficient points for ($\bar{\text{M}}$). Let (I_i) denote the following linear program:

$$(\text{I}_i) \quad \max \quad c_i^T x,$$
$$\text{s.t.} \quad x \in \bar{X}.$$

In our algorithm, we check whether or not there is an optimal solution x^i of (I_i) such that $d^T x^i > d^T \hat{x}$ and $x^i \in \bar{E}$ for some i. If there is such a point x^i, then we can choose x^i as a new current efficient point and keep searching for an adjacent efficient edge yielding an increase in $d^T x$. If no such point x^i exists, we show how to pivot to obtain a better efficient point or show that the current efficient point $\hat{x}$ is an optimal solution of (P). We begin in Section 2 by deriving some preliminary results.

2. Differences between Problems (R) and (P)

The following definition of domination cone D is given in Ref. 10.

Definition 2.1. Let D be the semipositive polar cone generated by the gradients of the k objective functions in (M), that is,

$$D = \{\pi \in R^n \mid C\pi \geq 0\} \cup \{0\}.$$

We call D the domination cone.

The following well-known theorem provides another description for an efficient point.

Theorem 2.1. $x^0 \in E$ if and only if there is no nonzero feasible domination direction π in X at x^0.

Proof. Notice that there is no nonzero feasible domination direction π in X at x^0 if and only if there is no $x \in X$ such that $x - x^0 \in D$, which means there is no $x \in X$ such that $Cx \geq Cx^0$. This completes the proof. □

If an optimal solution to the relaxed problem (R) is efficient, such a solution is optimal for (P). Also, if (P) and (R) have a common optimal solution, then an optimal solution to (P) is one of the multiple optimal solutions to the linear program (R), and so problem (P) is again easy to solve.

Lemma 2.1. If $d^T \pi > 0$ for all nonzero $\pi \in D$, then every optimal solution of (R) is an optimal solution of (P).

Proof. Let x^* be an optimal solution of (R). We need to show that $x^* \in E$. Suppose that $x^* \notin E$. Then, there is a nonzero $\pi \in D$ and π is a feasible direction at x^*, so there is a point $x^* + \alpha\pi \in X$ for some $\alpha > 0$. Then,

$$d^T(x^* + \alpha\pi) = d^T x^* + \alpha d^T \pi > d^T x^*.$$

But this inequality contradicts x^* being an optimal solution of (R). Therefore, $x^* \in E$, and so x^* solves (P). □

The following is a special case of a result of Benson; see Theorem 4.7 in Ref. 6.

Lemma 2.2. If $d^T\pi \geqq 0$ for all nonzero $\pi \in D$, then at least one optimal solution of (R) is also an optimal solution of (P).

Proof. We will give a short proof as follows. Suppose that x^* is an optimal solution of (R). If $x^* \in E$, then x^* is an optimal solution of (P). If $x^* \notin E$, then as in Refs. 11 and 12, the linear program

$$(\mathrm{P}_{x^*}) \quad \max \quad e^T s,$$
$$\text{s.t.} \quad Cx = Is + Cx^*,$$
$$x \in X,$$
$$s \geqq 0,$$

where e is a vector with each component equal to one, has an efficient solution $\bar{x}$ with $C\bar{x} \geq Cx^*$ or equivalently $C(\bar{x} - x^*) \geq 0$. Thus, $\pi = \bar{x} - x^* \in D$. But $d^T\pi \geqq 0$, so $d^T\bar{x} \geqq d^T x^*$. Since x^* is an optimal solution of (R), we have $d^T\bar{x} = d^T x^*$, and so $\bar{x}$ is an optimal solution of (R) and is efficient. Thus, $\bar{x}$ is an optimal solution of (P). □

The above two lemmas show that it is easy to find a solution of (P) when $d^T\pi > 0$ or $d^T\pi \geqq 0$ for each nonzero $\pi \in D$. Consider the following cases:

Case 1. $d^T\pi > 0$, for all nonzero $\pi \in D$.

Case 2. $d^T\pi \geqq 0$, for all nonzero $\pi \in D$.

Case 3. $d^T\pi < 0$, for all nonzero $\pi \in D$.

Case 4. None of the above cases holds.

In Section 4 below, we show that if Case 3 holds, then the cutting face

$$F = \{x \in X \mid d^T x = d^T \hat{x}\}$$

is efficient for the reduced problem $\bar{X}$. This allows us to show how to pivot to get a better efficient point or show that the current efficient point $\hat{x}$ is an optimal solution of (P). If Case 4 holds, then the cutting face F may not be efficient for $\bar{X}$, but again we show how to pivot to get a better efficient point or show that $\hat{x}$ is an optimal solution of (P).

Given d, we will determine which of the above four cases holds. Consider the following two linear programs:

$$\text{(A)} \quad \max \quad d^T\pi, \qquad \text{s.t.} \quad C\pi \leqq 0;$$

$$\text{(B)} \quad \max \quad d^T\pi, \qquad \text{s.t.} \quad C\pi \geqq 0.$$

Lemma 2.3.

(i) The zero vector 0 is the unique optimal solution of (A) if and only if $d^T\pi > 0$ for all nonzero $\pi \in D$ and there is no feasible and nonzero π' for (A) such that $C\pi' = 0$ and $d^T\pi' = 0$.

(ii) If 0 is an optimal solution of (A), then $d^T\pi \geqq 0$ for all nonzero $\pi \in D$.

(iii) The zero vector 0 is the unique optimal solution of (B) if and only if $d^T\pi < 0$ for all nonzero $\pi \in D$ and there is no feasible and nonzero π' for (B) such that $C\pi' = 0$ and $d^T\pi' = 0$.

Proof.

(i) Suppose that 0 is the unique optimal solution of (A), and let π be a nonzero vector in D. We know that $C\pi \geqq 0$ and so $C(-\pi) \leqq 0$. Thus, $-\pi$ is feasible for (A) and $d^T(-\pi) < 0$. Hence, $d^T\pi > 0$ for all nonzero $\pi \in D$. Also, there is no feasible π such that $C\pi = 0$ and $d^T\pi = 0$, because the zero vector is the unique optimal solution of (A).

Now, suppose that $d^T\pi > 0$ for all nonzero $\pi \in D$. Since every nonzero $\pi \in D$ satisfies $C(-\pi) \leqq 0$, so $-\pi$ is feasible for (A) and $d^T(-\pi) < 0$. We will show that D does not include any nonzero π^2 such that $C\pi^2 = 0$. If there is π^2 such that $C\pi^2 = 0$ and $d^T\pi^2 > 0$, then for any nonzero $\pi^1 \in D$, there is a positive $\alpha > 0$ such that

$$d^T\{\alpha\pi^1 + (1-\alpha)(-\pi^2)\} = 0.$$

Since

$$C\{\alpha\pi^1 + (1-\alpha)(-\pi^2)\} \geq 0,$$

so

$$\alpha\pi^1 + (1-\alpha)(-\pi^2) \in D.$$

By the above,

$$d^T\{\alpha\pi^1 + (1-\alpha)(-\pi^2)\} > 0,$$

which is a contradiction. Thus, there is no π^2 such that $C\pi^2 = 0$ and $d^T\pi^2 > 0$. Thus, $d^T(-\pi) < 0$ for all nonzero feasible $-\pi$ of (A). But the zero vector is feasible for (A), so it is the unique optimal solution (A).

(ii) We proceed by contrapositive statement. Suppose that $d^T\pi < 0$ for some nonzero $\pi \in D$. Since $-\pi$ is feasible of (A) and $d^T(-\pi) > 0$, the zero vector is not an optimal solution of (A).

(iii) The proof follows by rewriting (B) as

$$\text{(B)} \quad \max \quad d^T\pi,$$
$$\text{s.t.} \quad C(-\pi) \leqq 0,$$

and applying (i) to (B). □

By solving (A), we can determine if $d^T\pi > 0$ for all nonzero $\pi \in D$ or if $d^T\pi \geqq 0$ for all nonzero $\pi \in D$. If $d^T\pi > 0$ for all nonzero $\pi \in D$, then by Lemma 2.1, every optimal solution of (R) is an optimal solution of (P). If $d^T\pi \geqq 0$ for all nonzero $\pi \in D$, then choose any optimal solution x^* of (R), and by solving (P_{x^*}), we can get an optimal solution of (P). If we have Case 3 or 4, then we will show how to find an optimal solution of (P) in Section 4.

3. Pivoting Techniques to Find Efficient Edges

In this section, we assume that we have an initial efficient extreme point x^0 for (M). Finding such a starting point is not difficult; see Refs. 11–12 and Refs. 16–17 for example. By appropriate pivots, we represent this efficient vertex x^0 and problem (M) by the following tableau T:

	x^N	x^B
f	$-C$	0
b	A	I

Here, x^N denotes the variables which are nonbasic at the basic feasible solution x^0 and x^B denotes the variables which are basic. In this section, we summarize the results of Refs. 14–16 that show how to find all edges incident to x^0 that are efficient. To simplify the discussion, we assume here that T is a nondegenerate tableau. For details on the degenerate case, see Ref. 16.

Suppose that x^0 is an efficient extreme point. In order to determine which edges incident to x^0 are efficient, we use the following theorem.

Theorem 3.1. Suppose that the given tableau T is nondegenerate and its associated basic feasible solution x^0 is efficient. Consider the linear program

$$(\mathrm{Q}^j) \quad \max \quad z = e^T s, \quad \text{s.t.} \quad Cu = s + C^j, \quad u \geqq 0, \quad s \geqq 0,$$

where C^j denotes the jth column of C in T and e is a vector with each component equal to one. Let F^j be the edge incident to x^0 obtained by increasing the nonbasic variable x_j^N. Then,

$$F^j \subseteq E, \qquad \text{iff } z_{\max} = 0 \text{ in } (\mathrm{Q}^j).$$

Proof. See Theorem 1 in Ref. 16. □

The dual of (Q^j) is

$$\min \quad -C^{jT} y, \quad \text{s.t.} \quad C^T y \geqq 0, \quad -y \geqq e.$$

Let $y = -(v + e)$, where the vector v is the variable vector and e is a vector with each component equal to one. We can formulate the dual of (Q^j) as the following program:

$$(\mathrm{R}^j) \quad \min \quad -e^T C^j - v^T C^j, \quad \text{s.t.} \quad C^T v \leqq -C^T e, \quad v \geqq 0.$$

By duality, (Q^j) has optimal value zero if and only if the optimal value of (R^j) is zero. As in Ref. 16, this implies that the optimal value of (R^j) is zero if and only if

$$S = \{(v, w^N) \geqq 0 \mid C^T v + w^N = -C^T e, w_j^N = 0\} \neq \varnothing. \tag{1}$$

We will use the set S to determine which edges incident to x^0 are efficient. Let the tableau S_T be defined as follows:

	v	w^N
$-C^Te$	C^T	I

.

We assume that $-C^Te \geqq 0$, otherwise simplex pivots can be performed on S_T to obtain a nonnegative constant column. A key observation is that, if w_j^N can be made nonbasic by pivoting on S_T while maintaining a nonnegative constant column, then F^j is an efficient edge. Typically, many edges incident to x^0 can be identified as efficient by this observation. In the worst case, one could minimize w_j^N over S_T to determine if $F^j \subseteq E$.

To solve (P), we will use the above result to develop a pivoting technique for finding an efficient edge which gives a direction of increase for the objective function d^Tx, if one exists. We will illustrate how the Philip algorithm (Ref. 1) works with using the above efficient searching technique. Figure 1 is helpful in describing how to proceed in our pivoting technique if an efficient point $\hat{x}$ is obtained having no efficient edges that yield an increase in d^Tx. In Fig. 1, the efficient set E is indicated by the bold edges and x^0 is the initial efficient extreme point. By using Theorem 3.1 and the tableau S_T, we would find two efficient edges incident to x^0. Given the objective function d^Tx in Fig. 1, suppose that we increase d^Tx by pivoting to $\hat{x}$. From $\hat{x}$, further increases in d^Tx along efficient edges cannot be obtained, but $\hat{x}$ is not a solution to (P).

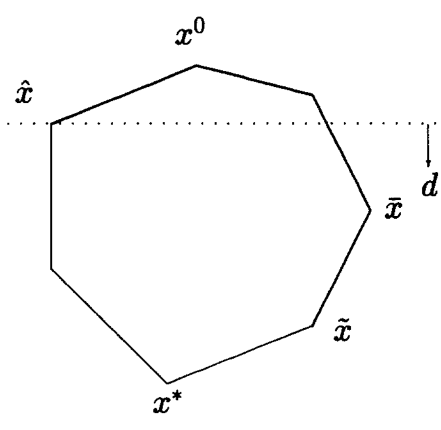

Fig. 1. Original feasible set and maximization direction d.

To proceed further, we need to determine if there is a point $\bar{x} \in E$ with $d^T\bar{x} > d^T\hat{x}$ as is the case in Fig. 1. We will show how this can be done by cutting the feasible set X through the extreme point $\hat{x}$ by the hyperplane $d^Tx = d^T\hat{x}$. In the next section, we derive some important results that allow us to construct an algorithm for finding better efficient points, if any exist.

4. Efficient Points in the Reduced Problem

Given a point $\hat{x} \in E \cap E_{ex}$ with no efficient edges yielding an increase in d^Tx [i.e., $\hat{x}$ is a local optimal solution of (P)], we consider the reduced problem,

$$(\bar{\mathrm{M}}) \quad \max \quad Cx,$$
$$\text{s.t.} \quad x \in \bar{X},$$
$$\bar{X} = \{x \in X \mid d^Tx \geqq d^T\hat{x}\},$$

with $\bar{E}$ denoting the set of efficient points for $(\bar{\mathrm{M}})$.

Lemma 4.1. If $x \in \bar{X} \cap E$, then $x \in \bar{E}$.

Proof. Suppose that $x \notin \bar{E}$. Then, there exists a point $y \in \bar{X}$ such that $Cy \geq Cx$. But $y \in X$. So, $x \notin E$ completes the proof. □

Notice that the converse of Lemma 4.1 does not hold.

Theorem 4.1. If $\hat{x} \in E \cap X_{ex}$ and there is no efficient edge $e = [\hat{x}, z]$ in E such that $d^Tz > d^T\hat{x}$, then not exclusively, either $\hat{x}$ is an optimal solution of (P) or there is an efficient edge $\bar{e} = [\hat{x}, y]$ in $\bar{E}$.

Proof. Suppose that $\hat{x}$ is not an optimal solution of (P). Since $\hat{x}$ is not an optimal solution of (P), there exists $\tilde{x} \in E \cap X_{ex}$ such that $d^T\tilde{x} > d^T\hat{x}$ and $\tilde{x}$ is not connected to $\hat{x}$ by an efficient edge in E. By the previous Lemma 4.1, $\tilde{x} \in \bar{E}$. But $\bar{E}$ is connected, so there must exist a path of efficient edges in $\bar{E}$ from $\tilde{x}$ to $\hat{x}$. Thus, $\hat{x}$ has an adjacent efficient edge in $\bar{E}$. This completes the proof. □

Notice by Theorem 4.1 that, if $\hat{x}$ is not an optimal solution of (P), but it is a local optimal solution of (P), then there is an efficient edge $\bar{e} = [\hat{x}, y]$ in $\bar{E}$, that is, on the hyperplane $\{x \in X \mid d^T\hat{x} = d^Tx\}$.

We use $\bar{X}_{ex}$ to denote the set of extreme points of $\bar{X}$.

Theorem 4.2. Suppose that $\hat{x} \in E \cap X_{ex}$. If there is a point $\bar{x} \in \bar{E}$ such that $d^T\bar{x} > d^T\hat{x}$, then $\bar{x} \in E$.

Proof. Suppose that $\bar{x} \notin E$. Then, there exists $w \in X$ such that $C(w - \bar{x}) \geq 0$. Notice that $w \notin \bar{X}$, because the above inequality would contradict $\bar{x} \in \bar{E}$. Also since $w \in X$, it follows by the definition of $\bar{X}$ that $d^Tw < d^T\hat{x}$. Also $d^T\hat{x} < d^T\bar{x}$, so $d^Tw < d^T\hat{x} < d^T\bar{x}$. Therefore, there is an $\epsilon > 0$ such that

$$\epsilon w + (1 - \epsilon)\bar{x} = \epsilon(w - \bar{x}) + \bar{x} \in \bar{X}.$$

Thus,

$$C(\epsilon(w - \bar{x}) + \bar{x}) = \epsilon C(w - \bar{x}) + C\bar{x} \geq C\bar{x},$$

since $\epsilon > 0$ and $C(w - \bar{x}) \geq 0$. But

$$\epsilon(w - \bar{x}) + \bar{x} \in \bar{X},$$

so this contradicts $\bar{x} \in \bar{E}$. Thus, $\bar{x} \in E$ and this completes the proof. □

The following theorem establishes an important property of the cutting face

$$F = \{x \in X \mid d^Tx = d^T\hat{x}\}$$

when d satisfies Case 3 in Section 2.

Theorem 4.3. If $d^T\pi < 0$ for all nonzero $\pi \in D$, then the cutting face F is efficient for the reduced problem, that is, $F \subseteq \bar{E}$.

Proof. For any interior point z of the cutting face F, we need to show that $z \in \bar{E}$. Notice that the set of feasible direction in $\bar{X}$ at z is $\{\pi \in X \mid d^T\pi \geqq 0\}$. But there is no nonzero feasible direction $\pi \in D$ satisfying $d^T\pi \geqq 0$. Therefore, $z \in \bar{E}$, which completes the proof. □

If $\hat{x} \in E \cap X_{ex}$ and there is no efficient edge $e = [\hat{x}, z]$ in E such that $d^Tz > d^T\hat{x}$, then as discussed above, we cut the feasible set X by the hyperplane $d^Tx = d^T\hat{x}$. As indicated in Theorem 4.2 and the contrapositive of Lemma 4.1, we need to check whether or not there is a point $\bar{x} \in \bar{E}$ satisfying $d^T\bar{x} > d^T\hat{x}$. To check whether or not there is such a point $\bar{x}$, consider the following linear programs for each $i = 1, \ldots, k$:

$$(\mathrm{I_i}) \quad \max \ c_i^Tx, \quad \text{s.t.} \ x \in \bar{X};$$

$$(\mathrm{PI_i}) \quad \max \ c_i^Tx, \quad \text{s.t.} \ x \in \bar{E}.$$

By Lemma 2.2, there is a common optimal solution for (I_i) and (PI_i), because $\pi^T c_i \geqq 0$ for all $\pi \in D$. By solving (I_i) sequentially for $i = 1, \ldots, k$, let x^i be an optimal solution of (I_i). Notice that, if x is feasible for (I_i), then $d^T x \geqq d^T \hat{x}$. Consider the following cases for x^i:

(i) $d^T x^i > d^T \hat{x}$ and $x^i \in \bar{E}$;
(ii) $d^T x^i > d^T \hat{x}$ and $x^i \notin \bar{E}$;
(iii) $d^T x^i = d^T \hat{x}$.

If (ii) holds, then there is a point $\bar{x} \in \bar{E}$ such that $C\bar{x} \geq Cx^i$. Notice that $c_i^T \bar{x} = c_i^T x^i$ because x^i solves (I_i). So $\bar{x}$ is also an optimal solution of (I_i). For case (ii), we consider two subcases:

(iia) $d^T \bar{x} > d^T \hat{x}$,
(iib) $d^T \bar{x} = d^T \hat{x}$.

In case (i), $x^i \in E$ by Theorem 4.2, so we can use x^i as a new current efficient point and seek an adjacent efficient edge yielding an increase in $d^T x$. In case (iia), $\bar{x} \in E$ by Theorem 4.2, so we can use $\bar{x}$ as a new current efficient point and seek an adjacent efficient edge yielding an increase in $d^T x$. In cases (iib) and (iii), there is an efficient optimal solution of (I_i) in the cutting face. We now solve (I_{i+1}) and check to see if x^{i+1} satisfies (i), (iia), (iib), or (iii). If x^{i+1} satisfies (i) or (iia), we can continue with x^{i+1} as a new current efficient point. If x^{i+1} satisfies (iib) or (iii), we continue as above and solve (I_{i+2}). If we continue until (I_k) and never obtain (i) and (iia), we then show below in Theorem 4.4 that the current efficient point $\hat{x}$ solves (P) provided that there is no efficient point in $\bar{E}$ on the cutting face F which has an adjacent efficient edge yielding an increase in $d^T x$.

We illustrate this process by continuing the example in Fig. 1. Recall from Fig. 1 that $\hat{x}$ is the current efficient point. In Fig. 2, the unique optimal solution of (I_1) is $x^1 = \hat{x}$, which is on the cutting face. So, we solve (I_2) and obtain $x^2 = \tilde{x}$. Since $d^T \tilde{x} > d^T \hat{x}$ and $\tilde{x} \in \bar{E}$, we are in case (i). So, we take $\tilde{x}$ as a new current efficient point.

With this new current efficient point, we can seek an adjacent efficient edge yielding an increase in $d^T x$. In this example, no adjacent efficient edge yielding an increase in $d^T x$ exists. We therefore cut the feasible set by $d^T x = d^T \tilde{x}$; see Fig. 3. Solving (I_1), we obtain y which is on the cutting face. So, we now solve (I_2) and obtain $\tilde{x}$ which is also on the cutting face. We have reached (I_2) and are in case (iii). We now need to determine if there is a better efficient point in the current reduced feasible set $\bar{X}$.

Notice that $d^T \pi < 0$ for all nonzero $\pi \in D$ implies $F \subseteq \bar{E}$ by Theorem 4.3. So, all edges on F are efficient in $\bar{X}$. Therefore, we need to pivot on F to determine whether or not there is a point y on F which has an adjacent efficient edge yielding an increase in $d^T x$. If $\hat{x}$ is a current efficient point,

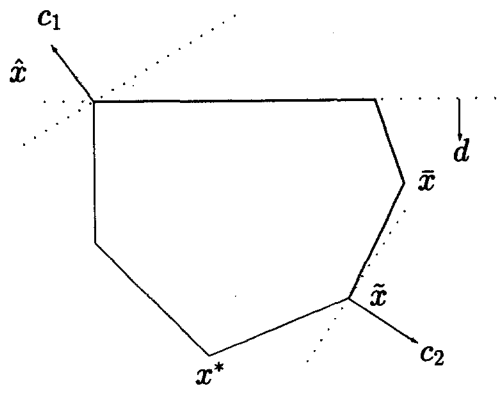

Fig. 2. Reduced feasible set for problem ($\tilde{\mathrm{M}}$).

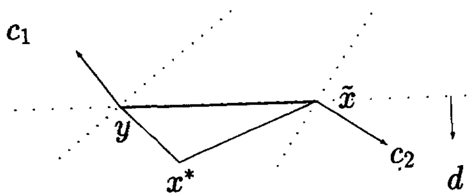

Fig. 3. Reduced feasible set for problem ($\bar{\mathrm{M}}$).

and there is no efficient edge yielding an increase in $d^T x$, and if $S_i \cap F \neq \varnothing$, where S_i is the set of efficient optimal solutions of (I_i) for any i, and there is no efficient edge yielding an increase in $d^T x$ at $x^i \in S_i$, then it is necessary to pivot on the cutting face F to determine if there is an efficient point $y \in F$ such that y has an adjacent efficient edge yielding an increase in $d^T x$. So, in the worst case, we would need to generate all efficient extreme points of $\bar{E}$ on F using, for example, the method in Ref. 15.

Theorem 4.4. If all points in $F \cap \bar{E}$ connected to $\hat{x}$ by efficient edges of $\bar{E}$ in F have no adjacent efficient edge yielding an increase in $d^T x$, then $F \cap \bar{E}$ is connected.

Proof. Let V be the subset of $F \cap \bar{E}$ where all the points in V are connected to $\hat{x}$ by efficient edges in F. Suppose that $y \in F \cap \bar{E}$ and y is not connected to $\hat{x}$ by a path of efficient edges in F. Because $\bar{E}$ is connected, there is an edge $[y, \bar{x}] \subseteq \bar{E}$ with $d^T \bar{x} > d^T y = d^T \hat{x}$. Also, there is a path of efficient edges from $\bar{x}$ to $\hat{x}$. Suppose that this path reenters V at an efficient extreme point $\tilde{x}$. Then, $\tilde{x} \in V$ has an adjacent efficient edge yielding an increase in $d^T x$. This contradiction shows that $F \cap \bar{E}$ must be connected. □

Corollary 4.1. If all points in $F \cap \bar{E}$ connected to $\hat{x}$ by efficient edges of $\bar{E}$ in F have no adjacent efficient edge yielding an increase in $d^T x$, then the current efficient point $\hat{x}$ is an optimal solution of (P).

Proof. This follows immediately from Theorem 4.4 because, in this case, $F \cap \bar{E}$ is connected. □

5. Algorithms for Solving (P)

The following is an algorithm for solving problem (P).

Algorithm 5.1.

Step 0. Find an optimal solution x^* of (R). If $x^* \in E$, then x^* is an optimal solution of (P). Else, find an optimal solution $\tilde{x}$ of (P_{x^*}). If $d^T \tilde{x} = d^T x^*$, then $\tilde{x}$ solves (P). Otherwise, go to Step 1.

Step 1. Start with an efficient point $\tilde{x}$ which is an optimal solution of (P_{x^*}).

Step 2. At the current efficient extreme point, say $\hat{x}$, find an adjacent efficient edge which gives a direction of increase in the objective function value. If there is no efficient edge which yields an increase in objective function value, then go to Step 3. Otherwise, pivot to the next extreme point $\hat{x}$ of the chosen efficient edge and do another iteration of Step 2.

Step 3. Add the row which represents the cutting hyperplane and a slack variable x_{n+1} to obtain a tableau $\bar{T}_{\hat{x}}$ which represents $\hat{x}$ as an extreme point in $\bar{X}$.

Step 4. See Steps 4.1 to 4.4 below.

Step 4.1. Set $i = 1$.

Step 4.2a. Find an optimal solution x^i of (I_i) which is an efficient extreme point in $(\bar{M})$.

Step 4.2b. If $d^T x^i > d^T \hat{x}$, then use x^i as a new current efficient point. Adjust the current tableau for representing x^i as an extreme point in X. Go to Step 2.

Step 4.2c. If $d^T x^i = d^T \hat{x}$ and there is an adjacent efficient edge $[x^i, \bar{x}]$ yielding an increase in $d^T x$, then use $\bar{x}$ as a new current efficient point. Go to Step 2.

Step 4.2d. If $i \leq k - 1$, do another iteration for $i \leftarrow i + 1$. Go to Step 4.2.

Step 4.3a. If $d^T \pi < 0$ for all nonzero $\pi \in D$, recall that the cutting face $F \subseteq \bar{E}$. Pivot on F seeking a point y which has an adjacent efficient edge yielding an increase in d^Tx. If no such y exists, then the current point $\hat{x}$ is an optimal solution of (P).

Step 4.3b. If $d^T\pi \not< 0$ for some nonzero $\pi \in D$, then not all of the cutting face F need to be in $\bar{E}$. Pivot on F following efficient edges on F seeking a point y which has an adjacent efficient edge yielding an increase in d^Tx. If no such y exists, then the current point $\hat{x}$ is an optimal solution of (P).

Step 4.4. If there is such an edge $[y, \bar{x}] \subseteq \bar{E}$, then use $\bar{x}$ as a new current efficient point. Go to Step 2.

We can modify Algorithm 5.1 to obtain an algorithm that does not necessarily need to follow efficient edges. Given a current efficient point $\hat{x}$, the following algorithm does not seek adjacent efficient edges yielding an increase in d^Tx. Instead, at the current efficient point $\hat{x}$, which is not necessarily a local optimal solution of (P), we immediately generate a cutting hyperplane and maximize the individual objectives over the reduced set seeking a better efficient point.

Algorithm 5.2.

Step 0. Find an optimal solution x^* of (R). If $x^* \in E$, then x^* is an optimal solution of (P). Else, find an optimal solution $\tilde{x}$ of (P_{x^*}). If $d^T\tilde{x} = d^Tx^*$, then $\tilde{x}$ solves (P_{x^*}). Otherwise, go to Step 1.

Step 1. Start with an efficient point $\tilde{x}$ which is an optimal solution of (P_{x^*}).

Step 2. Add the row which represents the cutting hyperplane and a slack variable x_{n+1} to obtain the tableau $\bar{T}_{\hat{x}}$ which represents $\hat{x}$ as an extreme point in $\bar{X}$.

Step 3. See Steps 3.1 to 3.4 below.

Step 3.1. Set $i = 1$.

Step 3.2a. Find an optimal solution x^i of (I_i), which is an efficient extreme point in $(\bar{M})$.

Step 3.2b. If $d^Tx^i > d^T\hat{x}$, then use x^i as a new current efficient point. Adjust the current tableau for representing x^i as an extreme point in X. Go to Step 2.

Step 3.2c. If $d^Tx^i = d^T\hat{x}$ and there is an adjacent efficient edge $[x^i, \bar{x}]$ yielding an increase in d^Tx, then use $\bar{x}$ as a new current efficient point. Go to Step 2.

Step 3.2d. If $i \le k - 1$, do another iteration for $i \leftarrow i + 1$. Go to Step 3.2a.

Step 3.3a. If $d^T\pi < 0$ for all nonzero $\pi \in D$, recall that the cutting face $F \subseteq \bar{E}$. Pivot on F seeking a point y which has an adjacent efficient edge yielding an increase in d^Tx. If no such y exists, then the current point $\hat{x}$ is an optimal solution of (P).

Step 3.3b. If $d^T\pi \not< 0$ for some nonzero $\pi \in D$, then not all of the cutting face F need be in $\bar{E}$. Pivot on F following efficient edges on F seeking a point y which has an adjacent efficient edge yielding an increase in d^Tx. If no such y exists, then the current point $\hat{x}$ is an optimal solution of (P).

Step 3.4. If there is such an edge $[y, \bar{x}] \subseteq \bar{E}$, then use $\bar{x}$ as a new current efficient point. Go to Step 2.

The authors are currently developing an implementation for computational experiments showing how the algorithms work in practice. In the following section, we illustrate the algorithms with an example.

6. Example

Consider the multiple-objective problem

$$\max \begin{bmatrix} x_1 - 3x_2 \\ x_1 + 3x_2 \end{bmatrix},$$

$$\text{s.t.} \quad x \in X,$$

$$X = \{x \in R^2 \mid x_1 + 2x_2 \leqq 8, 2x_1 + x_2 \leqq 7, x_1 - 2x_2 \leqq 1, x_1, x_2 \geqq 0\}.$$

Figure 4 gives the feasible set X with the efficient set indicated by the bold eges.

The problem (P) that we want to solve is

$$\text{(P)} \quad \max \quad (-3x_1 - 2x_2),$$

$$\text{s.t.} \quad x \in E.$$

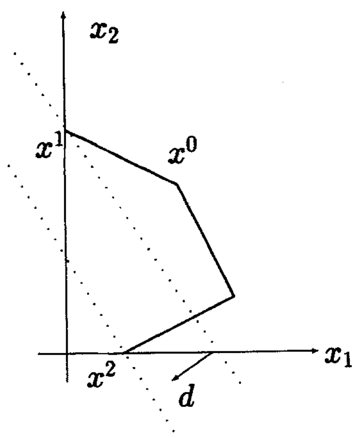

Fig. 4. Feasible set for the example.

Solving (R) gives the unique point (0, 0). Since $(0, 0) \notin E$, $d^T\pi < 0$ for some nonzero $\pi \in D$. Also, as we explained in Section 2, solving (B) gives the unique optimal solution 0, so from Lemma 2.3, $d^T\pi < 0$ for all nonzero $\pi \in D$. By Theorem 4.3, cutting faces will be efficient in the reduced problems.

We now show how Algorithm 5.1 works.

Solving (P) with Algorithm 5.1. Suppose that we start with the initial efficient extreme point $x^0 = (2, 3)^T$ with tableau T_0,

	x_1	x_2	x_3	x_4	x_5
−12	0	0	−1/3	−4/3	0
−7	0	0	−7/3	5/3	0
11	0	0	5/3	−1/3	0
3	0	1	2/3	−1/3	0
2	1	0	−1/3	2/3	0
5	0	0	5/3	−4/3	1

In Step 2, we need to check which edges adjacent to x^0 are efficient. Forming the tableau S_{T_0} and pivoting by the subproblem technique (see Ref. 18) to get a nonnegative constant column, tableau S_{T_0} becomes

	v_3	v_4	w_3	w_4
18/3	−18/3	0	1	5
4	−5	1	0	3

Because w_4 is nonbasic, increasing x_4 gives (as discussed in Section 3) an efficient edge adjacent to x^0. Also from tableau T_0, we see that increasing x_4 gives an increase in objective function d^Tx. So in Step 2, we choose x_4 to enter the basis and we pivot to the new efficient extreme point $x^1 = (0, 4)^T$ with tableau T_1,

	x_1	x_2	x_3	x_4	x_5
−8	2	0	−1	0	0
−12	−5/2	0	−3/2	0	0
12	1/2	0	3/2	0	0
4	1/2	1	1/2	0	0
3	3/2	0	−1/2	1	0
9	2	0	1	0	1

Again, we need to find which edges adjacent to x_1 are efficient. The edge adjacent to x^1 obtained by increasing x_1 in tableau T_1 is efficient. But increasing x_1 yields a decrease in $d^T x$. Thus, there is no adjacent efficient edge which yields an increase in $d^T x$. So, in Step 3, we have to cut the feasible set X through x^1 by the hyperplane $d^T x = d^T x^1$, that is, $-3x_1 - 2x_2 = -8$. We add the cutting hyperplane as a constraint to the tableau T_1 to obtain the tableau $\bar{T}_1$,

	x_1	x_2	x_3	x_4	x_5	x_6
−8	2	0	−1	0	0	0
−12	−5/2	0	−3/2	0	0	0
12	1/2	0	3/2	0	0	0
4	1/2	1	1/2	0	0	0
3	3/2	0	−1/2	1	0	0
9	2	0	1	0	1	0
0	2	0	−1	0	0	1

To solve (I_1), maximize the first objective function $c_1^T x$ over $\bar{T}_1$. After a couple of pivots, we have the following tableau $\bar{T}_{\bar{x}}$:

	x_1	x_2	x_3	x_4	x_5	x_6
−3	0	8	0	0	−3	0
1	0	1	0	0	1	0
1	0	−5	0	0	1	0
5	0	8	0	0	−3	1
5	0	5	0	1	−2	0
7	0	4	1	0	−1	0
1	1	−2	0	0	1	0

The optimal solution of (I_1) is $\bar{x}$ and $d^T \bar{x} > d^T x^1$. In Step 4, we need to adjust the tableau. Deleting the cutting hyperplane constraint (by deleting the row and column corresponding to x_6), we obtain the following tableau T_2 representing $\bar{x}$ as an efficient extreme point in X:

	x_1	x_2	x_3	x_4	x_5
−3	0	8	0	0	−3
1	0	1	0	0	1
1	0	−5	0	0	1
5	0	5	0	1	−2
7	0	4	1	0	−1
1	1	−2	0	0	1

We then return to Step 2 using $x^2 = \bar{x}$ as the current efficient extreme point.

We could see that there is no efficient edge adjacent to x^2 which yields an increase in $d^T x$. Again, we could cut the feasible set by the hyperplane $-3x_1 - 2x_2 = -3$ yielding the following tableau $\bar{T}_2$ for the reduced problem:

	x_1	x_2	x_3	x_4	x_5	x_6
−3	0	8	0	0	−3	0
1	0	1	0	0	1	0
1	0	−5	0	0	1	0
5	0	5	0	1	−2	0
7	0	4	1	0	−1	0
1	1	−2	0	0	1	0
0	0	8	0	0	−3	1

The optimal solution of (I_1) is x^2 from the tableau $\bar{T}_2$. We now solve (I_2) to obtain the point $y = (0, 3/2)^T$ with the following tableau $\bar{T}_y$:

	x_1	x_2	x_3	x_4	x_5	x_6
−3	0	0	0	0	0	−1
−9/2	−11/2	0	0	0	0	−3/2
9/2	7/2	0	0	0	0	3/2
11/2	1/2	0	0	1	0	−1/2
5	−2	0	1	0	0	−1
4	4	0	0	0	1	1
3/2	3/2	1	0	0	0	1/2

We know that all the optimal solutions of (I_1) and (I_2) are on the cutting face and there is no efficient point on the cutting face that has an adjacent efficient edge yielding an increase in $d^T x$; by Theorem 4.4, the current efficient point $x^2 = (1, 0)^T$ is an optimal solution for (P).

We now show how Algorithm 5.2 works.

Solving (P) with Algorithm 5.2. The given initial efficient extreme point is $x^0 = (2, 3)^T$ with tableau T_0,

	x_1	x_2	x_3	x_4	x_5
-12	0	0	$-1/3$	$-4/3$	0
-7	0	0	$-7/3$	$5/3$	0
11	0	0	$5/3$	$-1/3$	0
3	0	1	$2/3$	$-1/3$	0
2	1	0	$-1/3$	$2/3$	0
5	0	0	$5/3$	$-4/3$	1

In Step 2, we reduce the feasible set by adding the constraint $d^T x^0 \leqq d^T x$; we obtain tableau $\bar{T}_0$,

	x_1	x_2	x_3	x_4	x_5	x_6
-12	0	0	$-1/3$	$-4/3$	0	0
-7	0	0	$-7/3$	$5/3$	0	0
11	0	0	$5/3$	$-1/3$	0	0
3	0	1	$2/3$	$-1/3$	0	0
2	1	0	$-1/3$	$2/3$	0	0
5	0	0	$5/3$	$-4/3$	1	0
0	0	0	$-1/3$	$-4/3$	0	1

Maximizing the first objective function over $\bar{T}_0$ gives the point $x^2 = (1, 0)^T$ with tableau $\bar{T}_1$,

	x_1	x_2	x_3	x_4	x_5	x_6
−3	0	8	0	0	−3	0
1	0	1	0	0	1	0
1	0	−5	0	0	1	0
5	0	5	0	1	−2	0
7	0	4	1	0	−1	0
1	1	−2	0	0	1	0
3	0	8	0	0	−3	1

Now, we adjust the tableau $\bar{T}_1$ for the reduced feasible set by changing the constant column entry in the last row to 0 as in the following tableau $\bar{T}_1$:

	x_1	x_2	x_3	x_4	x_5	x_6
−3	0	8	0	0	−3	0
1	0	1	0	0	1	0
1	0	−5	0	0	1	0
5	0	5	0	1	−2	0
7	0	4	1	0	−1	0
1	1	−2	0	0	1	0
0	0	8	0	0	−3	1

The optimal solution of (I_1) is x^2 and the optimal solution of (I_2) is $z=(0, 3/2)^T$. Since all the optimal solutions are on the cutting face and there is no efficient point on the cutting face that has an adjacent efficient edge yielding an increase in d^Tx, we know by Theorem 4.4 that the current efficient point $x^2=(1, 0)^T$ is an optimal solution of (P).

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