Forward-looking variables in deterministic control

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In this paper, we adapt the Fair and Taylor [4] method for forward-looking variables in simulation models to control theory models. In particular, we develop a procedure for solving quadratic linear control models when there are forward-looking variables in the system equations. The simplest way to do this for deterministic problems would be to stack up the variables for all time periods using Theil's procedure [9], as suggested by Hughes-Hallet and Rees [5] for simulation models and done by Becker and Rustem [7] for perfect foresight problems. However, we plan to continue from the current paper and develop similar procedures for passive and active learning control problems, and the stacking procedure does not seem as natural for those problems. Therefore, we will use the Fair–Taylor approach here and adapt it for deterministic quadratic linear problems.

1. Problem statement

The standard quadratic linear tracking problem is written as

$$\text{find } (u_k)_{k=0}^{N-1}$$

to minimize the criterion

$$J = \frac{1}{2}[x_N - \tilde{x}_N]' W_N [x_N - \tilde{x}_N] + \frac{1}{2}\sum_{k=0}^{N-1}\left([x_k - \tilde{x}_k]' W_k [x_k - \tilde{x}_k] + [u_k - \tilde{u}_k]'\Lambda_k[u_k - \tilde{u}_k]\right), \quad (1.1)$$

where

x_k = n element state vector for period k,
u_k = m element control vector for period k,

$\tilde{x}_k$ = desired vector for state vector in period k,
$\tilde{u}_k$ = desired vector for control vector in period k,
W_k = penalty matrix on deviations of state variables from desired paths,
Λ_k = penalty matrix on control variables for deviations from desired paths,

subject to the system equations

$$x_{k+1} = Ax_k + Bu_k + Cz_k \quad k = 0, \ldots, N-1 \tag{1.2}$$

and the initial conditions

$$x_0 \text{ given,} \tag{1.3}$$

where

$A = n \times n$ coefficient matrix,
$B = n \times m$ coefficient matrix,
$C = n \times \ell$ coefficient matrix,
$z_k = \ell$ vector of exogenous variables.

A slightly different version of the system equations (1.2) is used here than is contained in Kendrick [6]. The modification is made to accommodate exogenous variables. This is done by replacing the exogenous vector c_k of constant terms in the previous notation with a coefficient matrix C and a vector z_k of exogenous variables, such that $c_k = Cz_k$.

The typical structure of C is a first column which contains the constant terms from the system equations, with the remainder of the matrix containing coefficients which multiply exogenous variables in the system equations. Consistent with this structure, the z_k vector usually has a one in first position, followed by a column of the exogenous variables which vary over time.

Now consider a quadratic linear tracking problem in which the outcome of the state variable for period $k+1$ is a function not only of the state and control variables in period k, but also of the expected value of the state variables in periods $k+1$ and $k+2$. This can be called a state equation with *forward-looking* variables. In this case, the system equations (1.2) is written as

$$x_{k+1} = Ax_k + Bu_k + Cz_k + D_1 x^e_{k+1|k} + D_2 x^e_{k+2|k}. \tag{1.4}$$

Thus, for instance, the variable

$$x^e_{k+1|k}$$

is the expected value of the state variable at period $k+1$ as projected from period k. Under the rational expectations hypothesis, the following condition holds:

$$x^e_{k+1|k-1} = x_{k+1}.$$

This means that it is assumed that economic agents have perfect foresight about future states. In a stochastic environment rather than the deterministic environment of this paper, the above equation would be transformed into

$$x^e_{k+1|k-1} = E_{k-1}x_{k+1},$$

i.e. the symbol E_{k-1} means the mathematical expectation operator at time $k-1$.

The Fair–Taylor procedure for determining the expected value of the forward-looking variables in simulation models is an iterative scheme where values of these variables are the same as the solution from the model. This requires choosing some nominal path for the expected values for the first iteration and then solving the model repeatedly. After each iteration, the expected values are updated to be the same as the solution values for the corresponding state variable in the previous solution. This process is continued until convergence is obtained. We adopt a similar procedure here for optimization models.

This procedure corresponds to the rather stringent assumption of perfect foresight in a deterministic model, meaning that economic agents can make the right prediction of the future states of the economy. In a stochastic model, as in Amman and Kendrick [1] and Amman et al. [2], this assumption would be replaced by the use of expected values. As there is no uncertainty in the deterministic case, expectations will be met perfectly.

The iterative scheme for solving the model will be outlined here using the notation

$$x^{ev}_{k+1|k}$$

to represent the expected value of the state variable at iteration v.

A number of different procedures could be used to get a starting path for the state and control variables. The simplest is to set

$$x^{e0}_{k+1|k} = x^0_{k+1} = 0 \quad \text{for all } k.$$

Thus, the iterative scheme used here is begun by setting

$$D_1 = D_2 = 0$$

and solving the resulting control problem. In this case, there are no forward-looking variables and the standard solution procedure for quadratic-linear tracking problems can be used. Call the optimal state variables for this solution

$$x^{NL},$$

i.e. the no-lead solution. Then set the expected values of the forward-looking variables equal to the no-lead solution for the first iteration:

$$x^{e1}_{k+1|k} = x^{NL}_{k+1} \quad \text{and} \quad x^{e1}_{k+2|k} = x^{NL}_{k+2} \quad \text{for all } k. \tag{1.5}$$

Then the system equation (1.4) on the first iteration becomes

$$x_{k+1}^1 = Ax_k^1 + Bu_k^1 + Cz_k + D_1 x_{k+1|k}^{e1} + D_2 x_{k+2|k}^{e1}. \tag{1.6}$$

However, the terms

$$Cz_k + D_1 x_{k+1|k}^{e1} + D_2 x_{k+2|k}^{e1} \tag{1.7}$$

are all known after the no-lead solution, so the system equations can be rewritten as

$$x_{k+1}^1 = Ax_k^1 + Bu_k^1 + \tilde{C}\tilde{z}_k^1, \tag{1.8}$$

where

$$\tilde{C} = [C \quad D_1 \quad D_2] \tag{1.9}$$

and

$$\tilde{z}_k^1 = \begin{bmatrix} z_k \\ x_{k+1|k}^{e1} \\ x_{k+2|k}^{e1} \end{bmatrix}. \tag{1.10}$$

Then the system equation (1.8) is of the same form as the original system equation (1.2) and the standard quadratic-linear tracking model can be used to solve the model at iteration one, cf. Kendrick [6, chap. 2]. Call the solution to this problem

$$(x_k^{*1}, u_k^{*1}) \quad \text{for all } k.$$

Begin iteration two by updating the expected values of the forward-looking state variables with

$$x_k^{e2} = x_k^{*1} \quad \text{for all } k. \tag{1.11}$$

Then the system equations on the second iteration become

$$x_{k+1}^2 = Ax_k^2 + Bu_k^2 + Cz_k + D_1 x_{k+1|k}^{e2} + D_2 x_{k+2|k}^{e2}. \tag{1.12}$$

Once again, the terms

$$Cz_k + D_1 x_{k+1|k}^{e2} + D_2 x_{k+2|k}^{e2} \tag{1.13}$$

are known before the problem is solved, so the system equations at iteration two can be written

$$x_{k+1}^2 = Ax_k^2 + Bu_k^2 + \tilde{C}\tilde{z}_k^2, \tag{1.14}$$

where

$$\tilde{C} = [C \quad D_1 \quad D_2] \tag{1.15}$$

and

$$\tilde{z}_k^2 = \begin{bmatrix} z_k \\ x_{k+1|k}^{e2} \\ x_{k+2|k}^{e2} \end{bmatrix}. \tag{1.16}$$

Equation (1.14) is now in the form of the system equations for the standard quadratic-linear tracking problem and can be solved again with that code.

The iterations as described above are then repeated until convergence is obtained, i.e. until

$$(x_k^{e,v+1} - x_k^{ev}) < \varepsilon \quad \text{for all } k, \tag{1.17}$$

where ε is the tolerance of convergence. At this stage, we do not know what conditions must be met to guarantee convergence of this procedure; however, our experience with the numerical example here was that only a small number of iterations were required.

The procedure described above is used for all time periods except for period $N-1$, which is the last period for the system equations. In that period, the value of $x_{k+2|k}^e$ will not be defined since, as $x_{N+1|N-1}^e$, it will be beyond the terminal period. A way to deal with this terminal condition is to set $x_{N+1|N-1}^e$ equal to x_N. The way Fair and Taylor deal with this issue is to extend the simulation period N to $N+s$. If s is taken sufficiently large, the impact of the boundary condition on the simulation period N, posed by the forward-looking variables, will generally be small. In this way, this boundary condition will have little impact on the solution of the optimal controls in the periods 0 to $N-1$.

2. A simple example: The MacRae problem

A simple example can be constructed from MacRae's [10] single-state single-control two-period stochastic control model. It was chosen because it is the simplest possible control problem and is well known in the literature. A deterministic version of the MacRae model, modified to cover five instead of two periods and to include a single forrward-looking variable for $k+2|k$, is

find $(u_0, u_1, u_2, u_3, u_4)$ to minimize

$$J = \frac{1}{2} w_2 x_2^2 + \frac{1}{2} \sum_{k=0}^{5} (w_k x_k^2 + \lambda_k u_k^2) \tag{2.1}$$

subject to $x_{k+1} = a x_k + b u_k + c z_k + d x_{k+2|k}^e$ for $k = 0,1,2,3,4,$ (2.2)

$$x_0 = 0. \tag{2.3}$$

The parameter values used are

$$\begin{aligned} &a = 0.7, \quad b = -0.5, \quad c = 3.5, \quad d = 0.2, \\ &z_k = 1 \ \forall k, \quad w_k = 1 \ \forall k, \quad \lambda_k = 1 \ \forall k. \end{aligned} \tag{2.4}$$

Also the desired paths in (2.1) are implicitly set to zero, so

$$\tilde{x}_k = 0, \ \tilde{u}_k = 0 \quad \forall k. \tag{2.5}$$

When this problem is solved using the forward-looking version of the QLP[1)] software and the algorithm of the previous section, it converges within only a few iterations. Figure 1 shows that the value of the state variable in the last time period x_5 was just under five in the first iteration. Recall that the solution for that iteration ignores the effect of the forward-looking variable. Then the solution quickly converges so that within only a few iterations, x_5 homes in on 5.69.

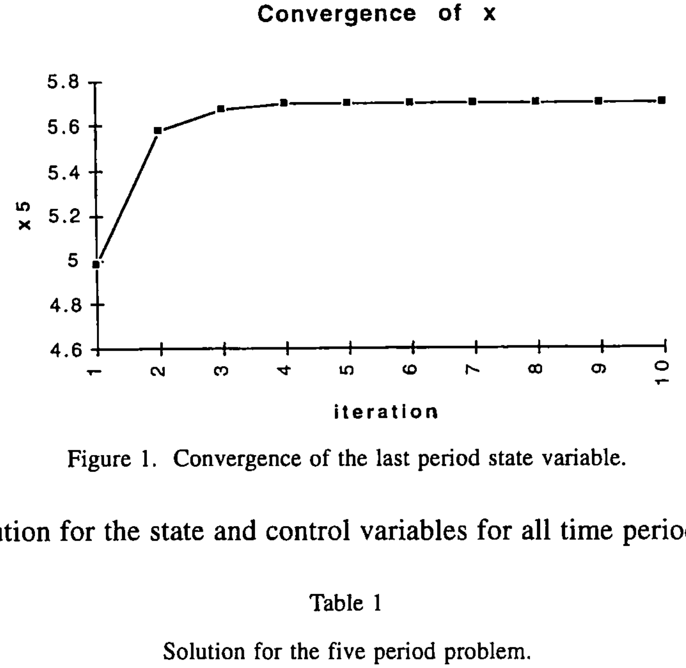

Figure 1. Convergence of the last period state variable.

The solution for the state and control variables for all time periods is shown in table 1.

Table 1

Solution for the five period problem.

Period	Control variable	State variable
0	4.53	0.00
1	5.16	1.84
2	5.22	3.01
3	4.58	4.03
4	2.85	5.17
5	N/A	5.69

3. A macroeconomic example: The Sargent and Wallace model

The Sargent and Wallace [8] model provides a macroeconomic example with three state variables, two control variables and two forward-looking variables. The two forward-looking variables are prices one and two periods ahead.

1) The QLP software may be obtained on request from the authors.

In the following, we will convert the Sargent and Wallace model from the form presented in their paper into a state space model. Begin with their model with four equations: aggregate supply, aggregate demand (IS curve), aggregate money demand (LM curve), and productive capacity.

Aggregate supply

$$y_t = a_1 k_{t-1} + a_2 (p_t - p^e_{t|t-1}); \tag{3.1}$$

IS curve

$$y_t = b_1 k_{t-1} + b_2 [r_t - (p^e_{t+1|t-1} - p^e_{t|t-1})]; \tag{3.2}$$

LM curve

$$m_t = p_t + c_1 y_t + c_2 r_t; \tag{3.3}$$

Productive capacity

$$k_t = d_1 k_{t-1} + d_2 [r_t - (p^e_{t+1|t-1} - p^e_{t|t-1})] + d_3 g_t, \tag{3.4}$$

where all variables are in natural logarithms unless otherwise noted. We switch here to using the subscript t for time periods and the letter k is used for the capital stock and not for the time index. Further,

y_t = output,
k_t = productive capacity,
p_t = price level,
m_t = money stock,
g_t = government expenditures,
r_t = interest rate,
$p^e_{t+s|t}$ = expected price in period $t+s$ as projected from period t.

There is one notational difference and one substantive difference in the above model from the Sargent and Wallace model. The notational difference is in the expected price variables. The equivalence is

Amman and Kendrick	Sargent and Wallace
$p^e_{t+1\|t-1}$	${}_{t+1}p^*_{t-1}$

Since we plan to do deterministic control experiments with this model, we have used the notation which is widely used in the control literature. The substantive difference is that Sargent and Wallace have an explicit monetary policy variable but place all fiscal policy variables in an exogenous vector which is driven by a first-order process. We have chosen here to take a single one of these fiscal policy variables, namely government expenditure, and include it explicitly as a control variable in the model. This was done for economic rather than computational reasons, that is, we wanted to emphasize the role of fiscal policy.

As discussed above, under the rational expectations hypothesis the following condition holds:

$$p^e_{t+1|t-1} = p_{t+1}$$

for the forward-looking variables. This means that it is assumed that economic agents have perfect foresight about future prices.

Next, the model is reduced from the four equations above to three equations. This is done by solving the LM equation for r_t and substituting it into the IS and productive capacity equations. Then the aggregate supply equation is inverted so it becomes a price equation. This is done in appendix A. In addition, the three equations are converted to state space form so that they can be written as

$$x_t = A_0 x_t + A_1 x_{t-1} + B_1 u_{t-1} + C_1 z_{t-1} + \hat{D}_1 x^e_{t|t-1} + \hat{D}_2 x^e_{t+1|t-1}, \tag{3.5}$$

with coefficient matrices

$$A_0 = \begin{bmatrix} -\frac{b_2 c_1}{c_2} & 0 & -\frac{b_2}{c_2} \\ -\frac{d_2 c_1}{c_2} & 0 & -\frac{d_2}{c_2} \\ \frac{1}{a_2} & 0 & 0 \end{bmatrix}, \tag{3.6}$$

$$A_1 = \begin{bmatrix} 0 & b_1 & 0 \\ 0 & d_1 & 0 \\ 0 & -\frac{a_1}{a_2} & 0 \end{bmatrix}, \tag{3.7}$$

$$B_1 = \begin{bmatrix} \frac{b_2}{c_2} & 0 \\ \frac{d_2}{c_2} & d_3 \\ 0 & 0 \end{bmatrix}, \qquad C_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \tag{3.8}$$

$$\hat{D}_1 = \begin{bmatrix} 0 & 0 & b_2 \\ 0 & 0 & d_2 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \hat{D}_2 = \begin{bmatrix} 0 & 0 & -b_2 \\ 0 & 0 & -d_2 \\ 0 & 0 & 0 \end{bmatrix}, \tag{3.9}$$

with

$$x_t = \begin{bmatrix} y_t \\ k_t \\ p_t \end{bmatrix}, \quad u_t = \begin{bmatrix} m_t \\ g_t \end{bmatrix}, \quad z_k = [1]. \tag{3.10}$$

As is discussed in appendix A, the control variables in (3.10) are lagged values of the control variables in the original model.

As will be discussed later, the structure of the matrix A_1 plays an important role in the response of the model. The first and third columns of this matrix consist entirely of zeroes so that perturbations to either output or prices do not have an effect on the model.

A stylized version of this model can be developed by using some rough estimates of the parameter values, which are

$$\begin{aligned} a_1 &= 0.33, & a_2 &= 0.05, \\ b_1 &= 0.30, & b_2 &= -0.15, \\ c_1 &= 0.60, & c_2 &= -0.01, \\ d_1 &= 0.70, & d_2 &= -0.12, & d_3 &= 0.16. \end{aligned} \tag{3.11}$$

The sign conventions of the Sargent and Wallace model are observed here, with the exception that d_3 is not explicitly used in their model. The magnitudes of a_1 and b_1 are about that of output–capital ratios. The magnitudes of c_1 and c_2 are in rough accord with similar parameters for a rational expectations model in Fair [3, p. 400]. Also, d_1 is a coefficient for a lagged dependent variable. No experience was available in choosing the magnitude of the other coefficients, so they were selected to be roughly the same size as the other coefficients but otherwise arbitrarily.

From these parameter values one can compute the numerical value of the coefficients in the matrices (3.6)–(3.9). These calculations are in appendix B.

While some control theory software permit model input using the Pindyck form of the model as described above, most require the collection of all x_t terms on the left-hand side of the system equations. This can be obtained from (3.5) as

$$x_{t+1} = Ax_t + Bu_t + Cz_t + D_1 x^e_{t+1|t} + D_2 x^e_{t+2|t}, \tag{3.12}$$

where

$$\begin{aligned} A &= (I - A_0)^{-1} A_1, \\ B &= (I - A_0)^{-1} B_1, \\ C &= (I - A_0)^{-1} C_1, \\ D_1 &= (I - A_0)^{-1} \hat{D}_1, \\ D_2 &= (I - A_0)^{-1} \hat{D}_2. \end{aligned} \tag{3.13}$$

Also, the time subscripts in (3.5) have been advanced by one period to obtain (3.12), which is in the traditional control theory form. To distinguish (3.5) from (3.12), we have call (3.5) the "I minus A" form, or simply the IA form, and (3.12) the I form of the system equations. The coefficients for the matrices in (3.13) are computed using the results from appendix B. The results of those calculations are discussed below.

The values of the coefficients in the matrices of (3.12) provide some insight into the dynamic response of the model, so they are reproduced below.

$$A = \begin{bmatrix} 0 & 0.32 & 0 \\ 0 & 0.72 & 0 \\ 0 & -0.19 & 0 \end{bmatrix}. \tag{3.14}$$

This shows that increases in the capital stock k_t have a positive effect on future output and capital stock and a negative effect on future prices.

$$B = \begin{bmatrix} 0.048 & 0.0 \\ 0.039 & 0.16 \\ 0.967 & 0.0 \end{bmatrix}. \tag{3.15}$$

Increases in the money stock decision variable m_t have a positive effect on future output, capital stocks and prices. Also, increases in the government expenditure decision variable g_t have a positive effect on capital stocks but no first-round effects on output or prices. Output and prices are not affected until the second period, through the impact of g_t on the capital stock.

$$C = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{3.16}$$

There are no constant terms and no exogenous variables in the equations of the Sargent and Wallace model, and this is reflected in the coefficients in C.

$$D_1 = \begin{bmatrix} 0 & 0 & -0.049 \\ 0 & 0 & -0.039 \\ 0 & 0 & 0.023 \end{bmatrix}. \tag{3.17}$$

This shows that increases in expected prices in the next period $p^e_{t+1|t}$ have a negative effect on output and capital stock, but a positive effect on prices in that period.

$$D_2 = \begin{bmatrix} 0 & 0 & 0.0005 \\ 0 & 0 & 0.0004 \\ 0 & 0 & 0.0097 \end{bmatrix}. \tag{3.18}$$

It is interesting to note by comparing D_1 and D_2 that the effects of expected price increases in period $t+2$ have an order of magnitude less effect on output, capital stock and prices than expected price increases in period $t+1$. Also, expected price increases in the next period have a negative effect on output and the capital stock, while expected price increases two periods hence have a positive effect on output and capital stock.

4. Experiments with the Sargent and Wallace model

A useful set of experiments to perform with this model is to make a perturbation of one of the initial values of the state variables and to analyze the model's response. In the original model, all of the initial values of the state variables are zero and the variables in the model are interpreted not as levels, but as deviations from steady-state equilibrium. The model has three state variables (output, capital stock and price level) and two control variables (money stock and government expenditure). Also, the desired values of the state variables are zero, so the model is used to return to the steady-state path from any perturbation in the initial state variables. With this in mind, three experiments were performed by making perturbations in the initial values of the income, price and capital stock variables.

Values of these state variables, as taken from the Economic Report of the President in February 1990, are shown in the first column of table 2. However, the capital stock data is not available in that source and is calculated here as being three times the level of output. The price level variable is the Consumer Price Index for all items from page 359. The output data is from page 305.

Table 2

State variables and perturbations.

	Level in billions of 1982 dollars	Five percent increase	Ratio of new to equilibrium state	Natural log of ratio of new to equilibrium state
Output, y	2,732	136.6	1.05	0.049
Capital stock, k	8,196	409.8	1.05	0.049
Price level, p	82.4	4.12	1.05	0.049

In the first experiment, the value of output in period zero was set to 0.049 and the model was solved. The result was that all the optimal states and controls were zero except for that initial value of output. This occurs because lagged output does not have any effect on the model, as can be ascertained by an examination of (3.1)–(3.4). Also, this assumption is apparent in the structure of the A_1 matrix in (3.7), where the first column of that matrix, which corresponds to y_{t-1}, is zero in every row.

Skipping the second experiment for the moment, consider the third, where the results were the same as for the first experiment and for the same reason. In this experiment, the initial price level was perturbed by 0.049, but the result was a zero solution for all states and controls except for the perturbation of prices in the initial period. The reason for this result is that lagged prices have no effect on the solution, as is seen in (3.1)–(3.4) and in the structure of the A_1 matrix in (3.7), where the third column of that matrix, which corresponds to p_{t-1}, is zero in every row.

So, in the Sargent and Wallace model an upturn in output or prices is met with no policy response. This occurs because these changes have an effect only in the quarter in which they occur. Therefore, no policy response is necessary to restore the economy to equilibrium.

In the second experiment, the results are different. A perturbation was made in the initial capital stock by changing it from zero to 0.049, as is indicated in table 2. The policy response to this change is shown in table 3.

Table 3

Policy response to an increase in the initial capital stock.

Time period	m_t	g_i
0	0.0042	−0.0014
1	0.0031	−0.0009
2	0.0022	−0.0006
3	0.0016	−0.0003
4	0.0012	0.0000

In order to properly interpret these results, it is useful to briefly review the nature of variables in models which are in log deviation form. Consider first a function of the form

$$Y_t = AX_t^{\beta} Z_t^{\gamma} \tag{4.1}$$

and the function with equilibrium values

$$Y^* = A(X^*)^{\beta}(Z^*)^{\gamma}. \tag{4.2}$$

The relationship in percent deviation form is the ratio of the above two functions,

$$\frac{Y_t}{Y^*} = \frac{AX_t^{\beta} Z_t^{\gamma}}{A(X^*)^{\beta}(Z^*)^{\gamma}} = \left(\frac{X_t}{X^*}\right)^{\beta}\left(\frac{Z_t}{Z^*}\right)^{\gamma}. \tag{4.3}$$

In logarithmic form, (4.3) is written as

$$\ln Y_t - \ln Y^* = \beta(\ln X_t - \ln X^*) + \gamma(\ln Z_t - \ln Z^*). \tag{4.4}$$

In log deviation form, (4.4) becomes

$$y_t = \beta x_t + \gamma z_t, \tag{4.5}$$

where

$$y_t = \ln Y_t - \ln Y^* = \ln\left(\frac{Y_t}{Y^*}\right),$$

$$x_t = \ln X_t - \ln X^* = \ln\left(\frac{X_t}{X^*}\right), \tag{4.6}$$

$$z_t = \ln Z_t - \ln Z^* = \ln\left(\frac{Z_t}{Z^*}\right).$$

From (4.6), it is apparent that when

$$Y_t > Y^* \Rightarrow \frac{Y_t}{Y^*} > 1 \Rightarrow \ln\left(\frac{Y_t}{Y^*}\right) = y_t > 0, \tag{4.7}$$

so that when the numbers in table 3 are positive, it means that the policy variable is above the equilibrium values. Also when

$$Y_t < Y^* \Rightarrow \frac{Y_t}{Y^*} < 1 \Rightarrow \ln\left(\frac{Y_t}{Y^*}\right) = y_t < 0, \tag{4.8}$$

Therefore, negative numbers in table 3 mean that the policy variable is below equilibrium.

This is further illustrated in figure 2, which provides a plot of the natural log of ratios. The ratios are shown on the horizontal axis and the natural log of the ratios are shown on the vertical axis.

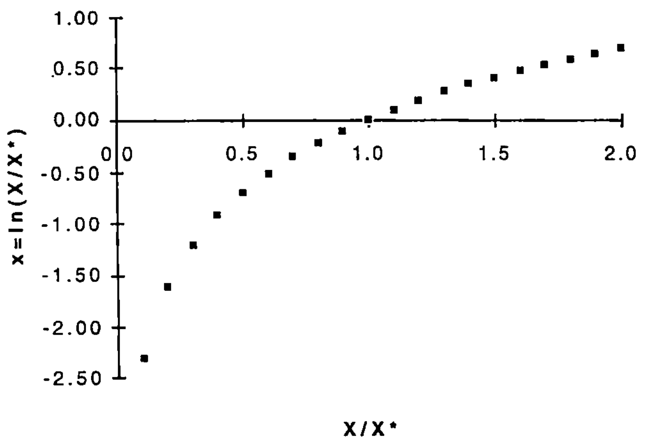

Figure 2. Natural log of ratios.

In the Sargent and Wallace model, all variables are natural logs of ratios. For example, their output variable, y_t, is the natural log of the ratio of actual to equilibrium output. Therefore, when this variable is above zero it means that the ratio of actual to equilibrium output, Y_t/Y^*, is above one. Conversely, when a log deviate variable such as price, p_t, is below zero it means that the ratio of actual to equilibrium price, P_t/P^*, is below one.

Moreover, a careful examination of figure 2 in the range where X_t/X^* is between 1.00 and 1.50 shows that the function is approximately linear and that it maps to the values between 0.00 and 0.50. Thus, in the Sargent and Wallace model, when the solution value for y_t is 0.0162, this maps to a ratio for Y_t/Y^* of 1.0162. This in turn means that output is about 1.6 percent above its equilibrium value.

The same procedure can be used for solution values in the Sargent and Wallace model which are negative, but it is necessary to subtract the result from one. Therefore, the government expenditure variable of − 0.0014 in table 3 means that government expenditures were roughly (1.00 − 0.0014) = 0.9986, or slightly more than one tenth of one percent below equilibrium.

Thus, the policy response in table 3 is for the money supply to be pushed about 0.42 percent above equilibrium and then to slowly return to equilibrium. Similarly, government expenditures are cut to a value about a tenth of one percent (0.14 percent) below equilibrium and then slowly rise to equilibrium. Therefore, the response in table 3 is to raise the money supply to decrease the interest rate and to lower government expenditures slightly to mitigate the output effect. This is necessary to keep the prices fixed. Prices cannot be directly influenced by government expenditures, as is apparent from the B matrix.

Table 4

State variable results of an initial capital stock increase.

Time period	y_t	k_t	p_t
0	0.0000	0.0490	0.0000
1	0.0162	0.0353	− 0.0055
2	0.0116	0.0254	− 0.0040
3	0.0084	0.0183	− 0.0029
4	0.0060	0.0132	− 0.0020
5	0.0044	0.0096	− 0.0014

The results for the state variables are shown in table 4. Output and the capital stock are pushed above equilibrium by the initial positive shock to the capital stock and then slowly drift down toward equilibrium. Prices, on the other hand, are shifted below equilibrium by the initial increase in the capital stock and then move upward towards equilibrium over time.

5. Conclusions

We have presented a methodology of dealing with forward-looking variable in a quadratic-linear control model framework by adapting the Fair–Taylor procedure from simulation models for use with control models. The method we have presented is applied first to the MacRae model then to the Sargent and Wallace model. In contrast to Sargent and Wallace, our procedure focuses on the short and mid-term effects of forward-looking variables on economic policy. Our method can be implemented easily and can be applied to a broad class of economic models which contain forward-looking variables.

Appendix A: Derivation of the system equations of the Sargent and Wallace model

In this appendix, the model is reduced from four equations (3.1)–(3.4) to three equations in state space form. This is done by solving the LM equation for r_t and substituting it into the IS and productive capacity equations. Then the aggregate supply equation is inverted so that it becomes a price equation. For ease of reference, equations (3.1)–(3.4) are reproduced here. As is indicated in the text, the variables are all defined as natural logarithms unless otherwise noted.

Aggregate supply

$$y_t = a_1 k_{t-1} + a_2 (p_t - p^e_{t|t-1}); \tag{3.1}$$

IS curve

$$y_t = b_1 k_{t-1} + b_2 [r_t - (p^e_{t+1|t-1} - p^e_{t|t-1})]; \tag{3.2}$$

LM curve

$$m_t = p_t + c_1 y_t + c_2 r_t; \tag{3.3}$$

Capital stock

$$k_t = d_1 k_{t-1} + d_2 [r_t - (p^e_{t+1|t-1} - p^e_{t|t-1})] + d_3 g_t. \tag{3.4}$$

Begin by solving the LM curve (3.3) for r_t to obtain

$$\begin{aligned} r_t &= \frac{1}{c_2} m_t - \frac{c_1}{c_2} y_t - \frac{1}{c_2} p_t \\ &= \alpha m_t - \beta y_t - \alpha p_t, \end{aligned} \tag{A.1}$$

where

$$\alpha = \frac{1}{c_2}, \quad \beta = \frac{c_1}{c_2}.$$

Then substitute (A.1) into (3.2) and (3.4) to obtain

$$y_t = b_1 k_{t-1} + b_2 [\alpha m_t - \beta y_t - \alpha p_t - (p^e_{t+1|t-1} - p^e_{t|t-1})] \tag{A.2}$$

and

$$k_t = d_1 k_{t-1} + d_2 [\alpha m_t - \beta y_t - \alpha p_t - (p^e_{t+1|t-1} - p^e_{t|t-1})] + d_3 g_t. \tag{A.3}$$

Next, invert the aggregate supply equation (3.1) so that is becomes a price equation:

$$a_2(p_t - p^e_{t|t-1}) = y_t - a_1 k_{t-1} \tag{A.4}$$

or

$$p_t = \frac{1}{a_2} y_t - \frac{a_1}{a_2} k_{t-1} + p^e_{t|t-1} \tag{A.5}$$

or

$$p_t = \eta y_t - \upsilon k_{t-1} + p^e_{t|t-1}, \tag{A.6}$$

where

$$\eta = \frac{1}{a_2}, \quad \upsilon = \frac{a_1}{a_2}.$$

The model now consists of three equations (A.2), (A.3) and (A.6) in the state vector

$$x_t = \begin{bmatrix} y_t \\ k_t \\ p_t \end{bmatrix}$$

and the control vector

$$u_t = \begin{bmatrix} m_t \\ g_t \end{bmatrix}.$$

The model can be written in state space form beginning with (A.2) by collecting the time period t variables on the left-hand side

$$(1 + b_2\beta)y_t + b_2\alpha p_t = b_1 k_{t-1} + b_2\alpha m_t - b_2 p^e_{t+1|t-1} + b_2 p^e_{t|t-1}. \tag{A.7}$$

Then transform (A.3) in the same way to obtain

$$d_2\beta y_t + k_t + d_2\alpha p_t = d_1 k_{t-1} + d_2\alpha m_t - d_2 p^e_{t+1|t-1} + d_2 p^e_{t|t-1} + d_3 g_t. \tag{A.8}$$

Also, transform (A.6) in the same way to obtain

$$\eta y_t - p_t = \upsilon k_{t-1} - p^e_{t|t-1}. \tag{A.9}$$

Next, write the state equations in a form similar to the form which is used by Pindyck [11], i.e.

$$x_t = A_0 x_t + A_1 x_{t-1} + B_1 u_{t-1} + C_1 z_{t-1}, \tag{A.10}$$

where z is a vector of exogenous variables and C_1 is the appropriate matrix of parameters. Placing the state variable for each of the equations (A.7), (A.8) and (A.9) alone on the left-hand side and arranging the other variables in the proper order yields

$$y_t = -b_2\beta y_t - b_2\alpha p_t + b_1 k_{t-1} + b_2\alpha m_t - b_2 p^e_{t+1|t-1} + b_2 p^e_{t|t-1}, \tag{A.11}$$

$$k_t = -d_2\beta y_t - d_2\alpha p_t + d_1 k_{t-1} + d_2\alpha m_t + d_3 g_t - d_2 p^e_{t+1|t-1} + d_2 p^e_{t|t-1}, \tag{A.12}$$

$$p_t = \eta y_t - \upsilon k_{t-1} + p^e_{t|t-1}. \tag{A.13}$$

Equations (A.11)–(A.13) are not yet in the Pindyck form, but rather in the form

$$x_t = A_0 x_t + A_1 x_{t-1} + B_1 \hat{u}_t + C_1 z_{t-1} + \hat{D}_1 x^e_{t|t-1} + \hat{D}_2 x^e_{t+1|t-1}, \tag{A.14}$$

where

$x^e_{t|t-1}$ = the expected value of the state vector at time t as projected from time period $t-1$.

The differences between (A.14) and the Pindyck form are that the control vector $\hat{u}_t$ is not lagged and that there are two forward-looking variables which are expected values of the state variables. The lag in the control variables is created by defining decision variables which precede money stock and government expenditure variables by one period, i.e.

$$\hat{u}_t = u_{t-1}, \tag{A.15}$$

where

$$u_{t-1} = \begin{bmatrix} m_{t-1} \\ g_{t-1} \end{bmatrix}, \tag{A.16}$$

(m = decision variable for the money stock, g = decision variable for government expenditures).

Substitution of (A.15) into (A.14) leaves

$$x_t = A_0 x_t + A_1 x_{t-1} + B_1 u_{t-1} + C_1 z_{t-1} + \hat{D}_1 x^e_{t|t-1} + \hat{D}_2 x^e_{t+1|t-1}, \tag{A.17}$$

that is, the same as (3.5) in the main text of the paper.

Then, comparing (A.17) to (A.11)–(A.13) and using the coefficient definitions after (A.1) and (A.6), one obtains the matrices

$$A_0 = \begin{bmatrix} -b_2\beta & 0 & -b_2\alpha \\ -d_2\beta & 0 & -d_2\alpha \\ \eta & 0 & 0 \end{bmatrix} = \begin{bmatrix} -\frac{b_2 c_1}{c_2} & 0 & -\frac{b_2}{c_2} \\ -\frac{d_2 c_1}{c_2} & 0 & -\frac{d_2}{c_2} \\ \frac{1}{a_2} & 0 & 0 \end{bmatrix}, \tag{A.18}$$

$$A_1 = \begin{bmatrix} 0 & b_1 & 0 \\ 0 & d_1 & 0 \\ 0 & -\upsilon & 0 \end{bmatrix} = \begin{bmatrix} 0 & b_1 & 0 \\ 0 & d_1 & 0 \\ 0 & -\frac{a_1}{a_2} & 0 \end{bmatrix}, \tag{A.19}$$

$$B_1 = \begin{bmatrix} b_2\alpha & 0 \\ d_2\alpha & d_3 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{b_2}{c_2} & 0 \\ \frac{d_2}{c_2} & d_3 \\ 0 & 0 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \tag{A.20}$$

$$\hat{D}_1 = \begin{bmatrix} 0 & 0 & b_2 \\ 0 & 0 & d_2 \\ 0 & 0 & 1 \end{bmatrix}, \quad \hat{D}_2 = \begin{bmatrix} 0 & 0 & -b_2 \\ 0 & 0 & -d_2 \\ 0 & 0 & 0 \end{bmatrix}, \tag{A.21}$$

with

$$x_t = \begin{bmatrix} y_t \\ k_t \\ p_t \end{bmatrix}, \quad u_t = \begin{bmatrix} m_t \\ g_t \end{bmatrix}, \quad z_k = [1]. \tag{A.22}$$

These matrices and vectors are included as (3.6)–(3.10) in the main text of the paper.

Appendix B: Derivation of coefficient matrices in system equations for the Sargent and Wallace model

Using the parameter values in (3.11) and the definitions of the coefficient matrices in (3.6)–(3.9), one obtains

$$A_0 = \begin{bmatrix} -\dfrac{b_2 c_1}{c_2} & 0 & -\dfrac{b_2}{c_2} \\ -\dfrac{d_2 c_1}{c_2} & 0 & -\dfrac{d_2}{c_2} \\ \dfrac{1}{a_2} & 0 & 0 \end{bmatrix} = \begin{bmatrix} -\dfrac{(-0.15)(0.6)}{-0.01} & 0 & -\dfrac{-0.15}{-0.01} \\ -\dfrac{(-0.12)(0.6)}{-0.01} & 0 & -\dfrac{-0.12}{-0.01} \\ \dfrac{1}{0.05} & 0 & 0 \end{bmatrix} = \begin{bmatrix} -9 & 0 & -15 \\ -7.2 & 0 & -12 \\ 20 & 0 & 0 \end{bmatrix}, \tag{B.1}$$

$$A_1 = \begin{bmatrix} 0 & b_1 & 0 \\ 0 & d_1 & 0 \\ 0 & -\dfrac{a_1}{a_2} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0.30 & 0 \\ 0 & 0.70 & 0 \\ 0 & -\dfrac{0.33}{0.05} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0.30 & 0 \\ 0 & 0.70 & 0 \\ 0 & -6.6 & 0 \end{bmatrix}, \tag{B.2}$$

$$B_1 = \begin{bmatrix} \dfrac{b_2}{c_2} & 0 \\ \dfrac{d_2}{c_2} & d_3 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \dfrac{-0.15}{-0.01} & 0 \\ \dfrac{-0.12}{-0.01} & 0.16 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 15.0 & 0.0 \\ 12.0 & 0.16 \\ 0.0 & 0.0 \end{bmatrix}, \tag{B.3}$$

$$\hat{D}_1 = \begin{bmatrix} 0 & 0 & b_2 \\ 0 & 0 & d_2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -0.15 \\ 0 & 0 & -0.12 \\ 0 & 0 & 1.00 \end{bmatrix}, \tag{B.4}$$

$$\hat{D}_2 = \begin{bmatrix} 0 & 0 & -b_2 \\ 0 & 0 & -d_2 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0.15 \\ 0 & 0 & 0.12 \\ 0 & 0 & 0.00 \end{bmatrix}. \tag{B.5}$$

These matrices are further transformed using $(I - A_0)^{-1}$ in (3.13) to convert the system equations from the IA (Pindyck) form to the I (standard) form. The resulting values of the A, B, C, D_1 and D_2 matrices are presented in the main text of the paper in (3.14)–(3.18).

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